Excess of Loss Reinsurance Under Joint Survival Optimality

by

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Abstract

Explicit expressions for the probability of joint survival up to time x of the cedent and the reinsurer, under an excess of loss reinsurance contract with a limiting and a retention level are obtained, under the reasonably general assumptions of any non-decreasing premium income function, Poisson claim arrivals and continuous claim amounts, modelled by any joint distribution. By stating appropriate optimality problems, we show that these results can be used to set the limiting and the retention levels in an optimal way with respect to the probability of joint survival. Alternatively, for fixed retention and limiting levels, the results yield an optimal split of the total premium income between the two parties in the excess of loss contract. This methodology is illustrated numerically on several examples of independent and dependent claim severities. The latter are modelled by a copula function. The effect of varying its dependence parameter and the marginals, on the solutions of the optimality problems and the joint survival probability, has also been explored.

Keywords: excess of loss reinsurance, probability of non-ruin, Appell polynomials, joint survival of cedent and reinsurer, dependent claim severities, copula functions

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1. Introduction

Several approaches to optimal reinsurance have been attempted in the actuarial literature, based on risk theory, economic game theory and stochastic dynamic control. Examples of research in each of these directions are the papers by Dickson and Waters (1996, 1997), Centeno (1991, 1997), Andersen (2000), Krvavych (2001), by Aase (2002), Suijs, Borm and De Waegenaere (1998), and by Schmidli (2001, 2002), Hipp and Vogt (2001), Taksar and Markussen (2003). A common feature of most of the quoted works is that optimality is considered with respect to the interest of solely the direct insurer, minimizing his (approximated) ruin probability, under the classical assumptions of linearity of the premium income function and independent, identically distributed claim severities.

Recently, a different reinsurance optimality model, which takes into account the interests of both the cedent and the reinsurer, has been considered by Ignatov, Kaishev and Krachunov (2004). As a joint optimality criterion they introduce the direct insurer's and the reinsurer's probability of joint survival up to a finite time horizon. Under this model, a volume of risks is insured by a direct insurer, who is entitled to receiving certain premium income in return for the obligation to cover individual claims. The latter are assumed to have any discrete joint distribution and Poisson arrivals. It is further assumed that the cedent is seeking to share claims and premium income with a reinsurer under a simple excess of loss contract with a retention level M, taking integer values. In their paper, Ignatov, Kaishev and Krachunov (2004) have derived expressions for the probability of joint survival of the cedent and the reinsurer and have demonstrated its applicability in the context of optimal reinsurance.

Catastrophic events in recent years have caused insurance and reinsurance losses of increasing frequency and severity. As a result, some reinsurance companies have been downgraded with respect to their credit rating while others, such as the 6-th largest reinsurer worldwide Gerling Global Re, even became insolvent and went out of business. The latter developments have motivated even stronger the proposed idea of considering reinsurance not solely from the point of view of the direct insurer, but taking into account the contradicting interests of the two parties, by jointly measuring the risk they share.

Our aim in this paper is to generalize the joint survival optimality reinsurance model, introduced by Ignatov, Kaishev and Krachunov (2004). We extend it here by considering an excess of loss (XL) contract in which the reinsurer covers each individual claim in excess of a retention level M, but up to a limiting level L and individual claim severities are not discrete but are modelled by continuous (dependent) random variables, with any joint distribution. Under these reasonably general assumptions we give closed form expressions for the probability of joint survival of the cedent and the reinsurer up to a fixed future moment in time. Based on these expressions, we state two optimality problems, according to which optimal values of M and L or alternatively, an optimal split of the total premium income, maximizing the probability of joint survival, can be obtained. These problems have been solved numerically, due to the infeasibility of their analytical solution. The derived joint survival probability formulae, conveniently allow the use of copula functions in modelling the dependency between claim severities. We have shown how varying the degree of dependence through the copula parameter(s) affects the optimal choice of the retention and the limiting levels, the optimal sharing of the premium income and also the probability of joint survival.

The results presented in this paper comprise an extension of the model considered by Ignatov, Kaishev and Krachunov (2004), to the practically more important case of continuous, dependent claim severities. In addition, the more general XL contract considered here gives a refined control over the optimal structure of this risk sharing arrangement. For further details on XL contracts with one or more layers, see e.g. Bugmann (1997).

The paper is organized as follows. In Section 2 we introduce the XL contract and the related joint survival probability model, considered further. Our main results are stated in Section 3 and illustrated numerically in Section 4, where we have introduced the copula approach to modelling dependence of consecutive claim severities under reinsurance. The final Section 5 provides some concluding remarks and indicates questions for further research.

2. The XL contract.

We will consider an insurance portfolio, generating claims with inter-occurrence times $\tau_1, \tau_2, \ldots.,$ assumed identically, exponentially distributed r.v.s with parameter λ. Denote by $T_1 = \tau_1$, $T_2 = \tau_1 + \tau_2$, ... the sequence of random variables representing the consecutive moments of occurrence of the claims. Let $N_t = \#\{i : T_i \le t\}$, where # is the number of elements of the set $\{.\}$. The claim severities are modeled by the non-negative continuous r.v.s. $W_1, W_2, \ldots, W_k, \ldots$, with joint density function $\psi(w_1, \ldots, w_k)$. It will be convenient to introduce the random variables $Y_1 = W_1$, $Y_2 = W_1 + W_2, \ldots$ representing the partial sums of consecutive claim severities.

The r.v.s $W_1, W_2, \ldots,$ are assumed to be independent of N_t. Then, the risk (surplus) process R_t , at time t, is given by $R_t = h(t) - Y_{N_t}$, where $h(t)$ is a nonnegative, non-decreasing, real function, defined on $\mathbb{R}_+$, representing the aggregated premium income up to time t, to be received for carrying the risk associated with the entire portfolio. The function $h(t)$ may be continuous or not. If $h(t)$ is discontinuous we will assume that $h^{-1}(y) = \inf\{z : h(z) \ge y\}$. Clearly, $h(t)$ represents a rather general class of functions and the classical case, $h(t) = u + c\,t$, with initial reserve u and premium rate c, is of course included. We will assume that the premium has been determined in such a way that the premium income defined by the function $h(t)$ adequately corresponds to the aggregate claim amount, generated by the portfolio up to time t. For the purpose, the various premium rating principles (see e.g., Gerber, 1979 and Wang, 1995) or other practical rating techniques can be used.

Without reinsurance, explicit formulae for the probability of non-ruin (survival) $P(T > x)$ of the direct insurer, in a finite time interval $[0, x]$, $x > 0$, with the time T of ruin, defined as

$$T := \inf\{t : t > 0,\ R_t < 0\}, \tag{1}$$

were derived by Ignatov and Kaishev (2004) and by Kaishev and Dimitrova (2003).

Here, we will be concerned with the case when the direct insurer wishes to reinsure his portfolio of risks by concluding an XL contract with a retention level M and a limiting level L, $M \geq 0$, $L \geq M$. In other words, the cedent reinsures the part of each claim which hits the layer $m = L - M$, i.e., each individual claim W_i is shared between the two parties so that $W_i = W_i^c + W_i^r$ $i = 1, 2, \ldots$ where W_i^c and W_i^r denote the parts covered respectively by the cedent and the reinsurer. Clearly, we can write

$$W_i^c = \min(W_i, M) + \max(0, W_i - L)$$

and

$$W_i^r = \min(L - M, \max(0, W_i - M)).$$

Denote by $Y_1^c = W_1^c$, $Y_2^c = W_1^c + W_2^c$, ... and by $Y_1^r = W_1^r$, $Y_2^r = W_1^r + W_2^r$, ... the consecutive partial sums of claims to the cedent and to the reinsurer, respectively. Under our XL reinsurance model, the total premium income $h(t)$ is also divided between the two parties so that $h(t) = h_c(t) + h_r(t)$, where $h_c(t)$, $h_r(t)$ are the premium incomes of the cedent and the reinsurer, assumed also non-negative, non-decreasing functions on $\mathbb{R}_+$. As a result, the risk process, R_t, can be represented as a superposition of two risk processes, that of the cedent

$$R_t^c = h_c(t) - Y_{N_t}^c \tag{2}$$

and of the reinsurer

$$R_t^r = h_r(t) - Y_{N_t}^r \tag{3}$$

i.e., $R_t = R_t^c + R_t^r$.

There are two alternative optimization problems which may be stated in connection with such an XL contract. The first is, given M and m are fixed, how should then the premium income $h(t)$ be divided between the two parties, so as to optimize a certain criterion measuring their joint risk or performance. And alternatively, if the total premium income $h(t)$ is divided in an agreed way between the cedent and the reinsurer, i.e., $h_c(t)$ and $h_r(t) = h(t) - h_c(t)$ are fixed, how should the parameters M and L of the XL contract be optimally set so as to minimize (maximize) the chosen joint risk or performance criterion.

3. The probability of joint survival optimality.

In this section we will introduce some risk measures, assuming both the cedent and the reinsurer jointly survive up to time x.

Define the moments, T^c and T^r, of ruin of correspondingly the cedent and the reinsurer as in (1), replacing R_t with R_t^c and R_t^r respectively. Clearly, the two events $(T^c > x)$ and $(T^r > x)$, of survival of the cedent and the reinsurer are dependent since the two risk processes R_t^c and R_t^r are dependent through the common claim arrivals and the claim severities W_i, $i = 1, 2, \ldots$ as seen from (2) and (3). Hence, as has been proposed in Ignatov, Kaishev and Krachunov (2004), it is meaningful to consider the probability of joint survival, $P(T^c > x, T^r > x)$, as a measure of the risk the two parties share and jointly carry. The two optimization problems we have stated can now be formulated more precisely as follows.

Problem 1. For fixed $h(t)$, $h_c(t)$, $h_r(t)$ such that $h(t) = h_c(t) + h_r(t)$, find

$$\max_{L,M} P(T^c > x, T^r > x) \ .$$

Problem 2. For fixed M, L and $h(t)$, find

$$\max_{\substack{h_c(t), \\ h(t)=h_c(t)+h_r(t)}} P(T^c > x, T^r > x) \ .$$

Problems 1 and 2 may be given the following interpretation. In Problem 1, the ceding company may wish to retain a certain fixed part, $h_c(t)$, of the premium income, $h(t)$, and then to find values for M and L, defining the corresponding optimal portion of the risk it would need to accept, so as to have maximum chances of joint with the reinsurer survival, up to a finite time x. Alternatively, the values M and L may be fixed, according to the ceding company's risk aversion and/or according to decisions, driven by negotiations with the reinsurer or other market conditions, after which the optimal split of $h(t)$, between the two parties would need to be defined, solving Problem 2. To explore Problems 1 and 2, next we will derive closed form expressions for the probability $P(T^c > x, T^r > x)$.

Theorem 1. *The probability of joint survival of the cedent and the reinsurer up to a finite time x under an XL contract with a retention level M and a limiting level L is*

$$P(T^c > x, T^r > x) =$$

$$e^{-\lambda x}\left(1 + \sum_{k=1}^{\infty} \lambda^k \int_0^{h(x)} \int_0^{h(x)-w_1} \cdots \int_0^{h(x)-w_1-\ldots-w_{k-1}} A_k(x\,;\tilde{v}_1, \ldots, \tilde{v}_k)\ \psi(w_1, \ldots, w_k)\, dw_k \ldots dw_2\, dw_1\right) \quad (4)$$

where

$\tilde{v}_j = \min(\tilde{z}_j, x)$, $\tilde{z}_j = \max(h_c^{-1}(y_j^c), h_r^{-1}(y_j^r))$, $y_j^c = \sum_{i=1}^{j} w_i^c$, $y_j^r = \sum_{i=1}^{j} w_i^r$, $j = 1, \ldots, k$,

$w_i^c = \min(w_i, M) + \max(0, w_i - L)$, $w_i^r = \min(L - M, \max(0, w_i - M))$, *and*

$A_k(x\,;\tilde{v}_1,\ ...,\tilde{v}_k)$, $k = 1, 2, \ ...$ *are the classical Appell polynomials* $A_k(x)$ *of degree* k, *defined by*

$$A_0\,(x) = 1, \quad A_k^{'}\,(x) = A_{k-1}\,(x), \quad A_k\,(\tilde{v}_k) = 0.$$

Remark 1. Appell polynomials were introduced by P.E. Appell (1880) and up to a normalization, contain many classical sequences of polynomials, among which the Bernoulli, Hermite and Laguerre polynomials. The sequence of Appell polynomials $\{A_k(x):\ k = 0, 1, \ ...\}$ are alternatively defined by a generating function

$$f(y)\,e^{x\,y} = \sum_{k=0}^{\infty} A_k(x)\,(y^k/k!),$$

where $f(y) = \sum_{k=0}^{\infty} A_k(0)\,(y^k/k!)$, $(f(0) \neq 0)$. and the values $A_k(0)$, $k = 0, 1, \ ...$ uniquely determine $\{A_k(x):\ k = 0, 1, \ ...\}$.

Clearly, Theorem 1 establishes a promising link of the survival probability $P(T^c > x,\ T^r > x)$ to the wide and important class of Appell polynomials. This link, worth further exploration, may give new insights into the properties of formula (4), and in particular may lead to a substantial improvement of its numerical efficiency. For a more detailed account on Appell polynomials we refer to Kaz'min (2002).

Proof of Theorem 1. The event of joint survival $\{T^c > x,\ T^r > x\}$ can be expressed as

$$P(T^c > x,\ T^r > x) = \sum_{k=0}^{\infty} P(N_x = k)\,P(T^c > x,\ T^r > x \mid N_x = k)$$

$$\begin{aligned}\{T^c > x,\ T^r > x\} &= \bigcap_{j=1}^{\infty}\,[\{(h_c^{-1}(Y_j^c) < T_j) \bigcup (h_r^{-1}(Y_j^r) < T_j)\} \bigcup \{x < T_j\}] \\ &= \bigcap_{j=1}^{\infty}\,[\{\max(h_c^{-1}(Y_j^c),\ h_r^{-1}(Y_j^r)) < T_j\} \bigcup \{x < T_j\}]\end{aligned} \tag{5}$$

Noting that $\Omega = \bigcup_{k=0}^{\infty}\{N_x = k\}$, applying the partition theorem we have

$$P(T^c > x,\ T^r > x) = \sum_{k=0}^{\infty} P(N_x = k)\,P(T^c > x,\ T^r > x \mid N_x = k)$$

$$= \sum_{k=0}^{\infty} \frac{(\lambda\,x)^k}{k!}\,e^{-\lambda\,x}\,P(T^c > x,\ T^r > x \mid \{T_k \leq x\} \bigcap \{T_{k+1} > x\}) \tag{6}$$

In (6), we have used the fact that the event $\{N_x = k\} \equiv \{T_k \leq x\} \bigcap \{T_{k+1} > x\}$.

If we now express $\{T^c > x,\ T^r > x\}$ in (6) using its representation given by (5) we obtain

$$\begin{aligned}P(T^c > x,\ T^r > x) &= \sum_{k=0}^{\infty} \frac{(\lambda\,x)^k}{k!}\,e^{-\lambda\,x} \\ &\quad P(\bigcap_{j=1}^{\infty}\,[\{\max(h_c^{-1}(Y_j^c),\ h_r^{-1}(Y_j^r)) < T_j\} \bigcup \{x < T_j\}] \mid \{T_k \leq x\} \bigcap \{T_{k+1} > x\}) \\ &= \sum_{k=0}^{\infty} \frac{(\lambda\,x)^k}{k!}\,e^{-\lambda\,x} \\ &\quad P((\bigcap_{j=1}^{\infty}\,[\{\max(h_c^{-1}(Y_j^c),\ h_r^{-1}(Y_j^r)) < T_j\} \bigcup \{x < T_j\}]) \bigcap \{T_k \leq x\} \bigcap \{T_{k+1} > x\} \mid \\ &\qquad \{T_k \leq x\} \bigcap \{T_{k+1} > x\})\end{aligned} \tag{7}$$

where in the last equality we have used that $P(A \mid B) = P(A \bigcap B \mid B)$. Applying some algebraic manipulations on the event in (7) it can be shown that

$$\left(\bigcap_{j=1}^{\infty}\left[\{\max(h_c^{-1}(Y_j^c),\, h_r^{-1}(Y_j^r)) < T_j\} \bigcup \{x < T_j\}\right]\right) \bigcap \{T_k \le x\} \bigcap \{T_{k+1} > x\}$$
$$= \left(\bigcap_{j=1}^{k}\{\max(h_c^{-1}(Y_j^c),\, h_r^{-1}(Y_j^r)) < T_j\}\right) \bigcap \{T_k \le x\} \bigcap \{T_{k+1} > x\} \tag{8}$$

Substituting (8) back in (7) leads to

$$P(T^c > x,\, T^r > x)$$
$$= \sum_{k=0}^{\infty} \frac{(\lambda x)^k}{k!} e^{-\lambda x} P\left(\bigcap_{j=1}^{k}\left[\{\max(h_c^{-1}(Y_j^c),\, h_r^{-1}(Y_j^r)) < T_j\} \bigcap \{T_k \le x\} \bigcap \{T_{k+1} > x\}\right] \,\Big|\, \{T_k \le x\} \bigcap \{T_{k+1} > x\}\right)$$
$$= \sum_{k=0}^{\infty} \frac{(\lambda x)^k}{k!} e^{-\lambda x} P\left(\bigcap_{j=1}^{k}\{\max(h_c^{-1}(Y_j^c),\, h_r^{-1}(Y_j^r)) < T_j\} \,\Big|\, \{T_k \le x\} \bigcap \{T_{k+1} > x\}\right) \tag{9}$$

It is known that (see Karlin and Taylor, 1981)

$$P(T_1 \le t_1,\ \ldots,\ T_k \le t_k \mid \{T_k \le x\} \bigcap \{T_{k+1} > x\}) = P(\tilde{T}_1 \le t_1,\ \ldots,\ \tilde{T}_k \le t_k) \tag{10}$$

where $\tilde{T}_1 \le \ldots \le \tilde{T}_k$ are the order statistics of k independent, uniformly distributed random variables in the interval $(0, x)$. From the independence of the two sequences of random variables Y_j^c, Y_j^r, $j = 1, 2, \ldots$ and T_k, $k = 1, 2, \ldots$ and applying (10) we can rewrite (9) as

$$P(T^c > x,\, T^r > x) = \sum_{k=0}^{\infty} \frac{(\lambda x)^k}{k!} e^{-\lambda x} P\left(\bigcap_{j=1}^{k} \max(h_c^{-1}(Y_j^c),\, h_r^{-1}(Y_j^r)) < \tilde{T}_j\right) \tag{11}$$

The random variables $\tilde{T}_1 \le \ldots \le \tilde{T}_k$ have a joint density (see Karlin and Taylor, 1981)

$$f_{\tilde{T}_1,\ldots,\tilde{T}_k}(t_1,\ \ldots,\ t_k) = \begin{cases} \frac{k!}{x^k} & \text{if} \quad 0 \le t_1 \le \ldots \le t_k \le x \\ 0 & \text{otherwise} \end{cases}$$

hence, introducing the notation

$$\mathcal{D}_k \equiv \begin{pmatrix} 0 \le w_1,\ \ldots,\ 0 \le w_k \\ w_1 +\ \ldots\ + w_k \le h(x) \end{pmatrix},$$

we can express the probability on the right-hand side of (11) as

$$P\left(\bigcap_{j=1}^{k} \max(h_c^{-1}(Y_j^c),\, h_r^{-1}(Y_j^r)) < \tilde{T}_j\right)$$
$$= \int \cdots \int_{\mathcal{D}_k} \psi(w_1,\ \ldots,\ w_k) \int \cdots \int_{\substack{\min[\max(h_c^{-1}(y_1^c), h_r^{-1}(y_1^r)), x] < t_1 < x \\ \cdots \\ \min[\max(h_c^{-1}(y_k^c), h_r^{-1}(y_k^r)), x] < t_k < x \\ t_1 \le \ldots \le t_k}} \frac{k!}{x^k}\, d t_k \cdots d t_1\, d w_k \cdots d w_1 \tag{12}$$

where $\min[\max(h_c^{-1}(y_j^c),\, h_r^{-1}(y_j^r)),\, x]$, $j = 1, 2,\ \ldots,\ k$ appear as lower limits of integration since $\max(h_c^{-1}(y_j^c),\, h_r^{-1}(y_j^r))$ can in general exceed x for some value $y_j = y_j^c + y_j^r = w_1^c + \ldots + w_j^c + w_1^r + \ldots + w_j^r = w_1 + \ldots + w_j$, $j = 1, 2,\ \ldots,\ k$. In this case $\min[\max(h_c^{-1}(y_j^c),\, h_r^{-1}(y_j^r)),\, x] = x$, i.e., the integral in (11) vanishes as is necessary, since such trajectories $t \mapsto y_j$ cause ruin of at least one of the parties and therefore should not contribute to the probability of their joint survival. To simplify notation, we let $\tilde{\nu}_j = \min[\tilde{z}_j, x]$, $\tilde{z}_j = \max(h_c^{-1}(y_j^c), h_r^{-1}(y_j^r))$, $j = 1, 2, ..., k$ and use (12) to rewrite (11) as

$$P(T^c > x, T^r > x)$$

$$= e^{-\lambda x} \sum_{k=0}^{\infty} \frac{(\lambda x)^k}{k!} \int \cdots \int_{\mathcal{D}_k} \psi(w_1, ..., w_k) \int \cdots \int_{\substack{\tilde{\nu}_1 < t_1 < x \\ .... \\ \tilde{\nu}_k < t_k < x \\ t_1 \le ... \le t_k}} \frac{k!}{x^k} \, d t_k \cdots d t_1 \, d w_k \cdots d w_1$$

$$= e^{-\lambda x} \sum_{k=0}^{\infty}$$

$$\frac{(\lambda x)^k}{k!} \int \cdots \int_{\mathcal{D}_k} \psi(w_1, ..., w_k) \frac{k!}{x^k} \int_{\tilde{\nu}_1}^{x} \int_{\max[\tilde{\nu}_2, t_1]}^{x} \cdots \int_{\max[\tilde{\nu}_k, t_{k-1}]}^{x} d t_k \cdots d t_2 \, d t_1 \, d w_k \cdots d w_1$$

$$= e^{-\lambda x} \sum_{k=0}^{\infty} \lambda^k \int \cdots \int_{\mathcal{D}_k} \psi(w_1, ..., w_k) \, A_k(x ; \tilde{\nu}_1, ..., \tilde{\nu}_k) \, d w_k \cdots d w_1 \tag{13}$$

where we have set

$$A_k(x ; \tilde{\nu}_1, ..., \tilde{\nu}_k) = \int_{\tilde{\nu}_1}^{x} \int_{\max[\tilde{\nu}_2, t_1]}^{x} \cdots \int_{\max[\tilde{\nu}_k, t_{k-1}]}^{x} d t_k \cdots d t_2 \, d t_1 .$$

It can be seen directly that $A_k(x ; \tilde{\nu}_1, ..., \tilde{\nu}_k)$ is a polynomial of degree k with a coefficient at the highest degree $1/k!$. Moreover, applying similar reasoning as in Theorem 1 of Ignatov and Kaishev (2004) it can be shown that $A_k(x ; \tilde{\nu}_1, ..., \tilde{\nu}_k)$, $k = 1, 2, ...$ are the classical Appell polynomials.

The asserted joint survival probability formula now follows, appropriately rewriting the multiple integral in (13).□

An alternative formula for $P(T^c > x, T^r > x)$ is provided by the following

Theorem 2. *The probability of joint survival is*

$$P(T^c > x, T^r > x) = e^{-\lambda x} \left(\sum_{k=1}^{\infty} \int_0^{h(x)} \int_0^{h(x)-w_1} ... \int_0^{h(x)-w_1-...-w_{k-2}} \int_{h(x)-w_1-...-w_{k-1}}^{\infty} B_l(\tilde{z}_1, ..., \tilde{z}_{l-1}, x) \, \psi(w_1, ..., w_k) \, d w_k \, d w_{k-1} ... d w_2 \, d w_1 \right) \tag{14}$$

where

$B_l(\tilde{z}_1, ..., \tilde{z}_{l-1}, x) = \sum_{j=0}^{l-1} (-\lambda)^j \, b_j(\tilde{z}_1, ..., \tilde{z}_j) \left(\sum_{m=0}^{l-j-1} \frac{(x\lambda)^m}{m!} \right)$, with $B_0(\cdot) \equiv 0$, $B_1(\cdot) = 1$, *l is such that* $\tilde{z}_1 \le ... \le \tilde{z}_{l-1} \le x < \tilde{z}_l$,

$b_j(\tilde{z}_1, ..., \tilde{z}_j) = \sum_{i=1}^{j} (-1)^{j+i} \frac{\tilde{z}_j^{j-i+1}}{(j-i+1)!} \, b_{i-1}(\tilde{z}_1, ..., \tilde{z}_{i-1})$, *with* $b_0 \equiv 1$,

$\tilde{z}_j$ and $\psi(w_1, \ ..., w_k)$ are defined as in Theorem 1.

Proof of Theorem 2. The probability of survival of the cedent without reinsurance (see Kaishev and Dimitrova, 2003) is given by

$$P(T > x) = \sum_{k=1}^{\infty} \int_0^{h(x)} \int_0^{h(x)-w_1} \cdots \int_0^{h(x)-w_1-\ldots-w_{k-2}} \int_{h(x)-w_1-\ldots-w_{k-1}}^{\infty} P(T > x \mid W_1 = w_1, \ ..., W_{k-1} = w_{k-1}; W_k \geq h(x) - w_1 - \ldots - w_{k-1}) \times \psi(w_1, \ ..., w_k)\, dw_k\, dw_{k-1} \ldots dw_2\, dw_1 \tag{15}$$

where

$$P(T > x \mid W_1 = w_1, \ ..., W_{k-1} = w_{k-1}; W_k \geq h(x) - w_1 - \ldots - w_{k-1}) = e^{-\lambda x} B_k(z_1, \ ..., z_{k-1}, x) \tag{16}$$

and $z_j = h^{-1}(w_1 + \ldots + w_j)$, provided that $h^{-1}(w_1 + \ldots + w_{k-1}) \leq x < h^{-1}(w_1 + \ldots + w_k)$.

By analogy with the reasoning in deriving (15) we can write

$$P(T^c > x, T^r > x) = \sum_{k=1}^{\infty} \int_0^{h(x)} \int_0^{h(x)-w_1} \cdots \int_0^{h(x)-w_1-\ldots-w_{k-2}} \int_{h(x)-w_1-\ldots-w_{k-1}}^{\infty} P(T^c > x, T^r > x \mid W_1 = w_1, \ ..., W_{k-1} = w_{k-1}; W_k \geq h(x) - w_1 - \ldots - w_{k-1}) \psi(w_1, \ ..., w_k)\, dw_k\, dw_{k-1} \ldots dw_2\, dw_1 \tag{17}$$

Following equality (10) of Ignatov, Kaishev and Krachunov (2004), it is possible to show that

$$P(T^c > x, T^r > x \mid W_1 = w_1, \ ..., W_{k-1} = w_{k-1}; W_k \geq h(x) - w_1 - \ldots - w_{k-1}) = P\left(\bigcap_{j=1}^{k-1} \{\max(h_c^{-1}(y_j^c), h_r^{-1}(y_j^r)) \leq T_j\} \bigcap \{T_k > x\}\right) \tag{18}$$

From (16) and (18) it can be concluded that

$$P\left(\bigcap_{j=1}^{k-1} \{\max(h_c^{-1}(y_j^c), h_r^{-1}(y_j^r)) \leq T_j\} \bigcap \{T_k > x\}\right) = e^{-\lambda x} B_k(\tilde{z}_1, \ ..., \tilde{z}_{k-1}, x) \tag{19}$$

where $\tilde{z}_j = \max(h_c^{-1}(y_j^c), h_r^{-1}(y_j^r)), \ j = 1, \ ..., k$. It is not difficult to see that there should exist an index $1 \leq l \leq k$, such that $\tilde{z}_1 \leq \ldots \leq \tilde{z}_{l-1} \leq x < \tilde{z}_l$ and since we consider the events of ruin of the cedent and the reinsurer up to time x only, hence we can rewrite (19) as

$$P\left(\bigcap_{j=1}^{k-1} \{\tilde{z}_j \leq T_j\} \bigcap \{T_k > x\}\right) = e^{-\lambda x} B_l(\tilde{z}_1, \ ..., \tilde{z}_{l-1}, x) \tag{20}$$

Formula (14) now follows from (18), (20) and (17) which completes the proof of Theorem 2.□

The use of formulae (4) and (14) to compute $P(T^c > x, T^r > x)$ is discussed in Section 4 where the case of independent and dependent claim severities are thoroughly explored.

4. Computational considerations and results.

In this section we demonstrate that using the results of Theorem 1 and 2, one can successfully find solutions to Problems 1 and 2, stated in Section 3, and optimally determine the parameters of an XL contract. A quick analysis of formulae (4) and (14) reveals that an attempt to use them in solving the optimization Problems 1 and 2 analytically is confronted with considerable difficulties. For example formula (4) requires the maximization of a complex functional with respect to the function $h_c(t)$, with the constraint $h(t) = h_c(t) + h_r(t)$, and under the additional assumption of invertibility of $h_c(t)$ and $h_r(t)$. This is a task which is hardly feasible, at least under the rather general definitions of $h(t)$, $h_c(t)$ and $h_r(t)$ assumed here. For this reason, in what follows we will use (4) and (14) to solve Problems 1 and 2 numerically.

Formulae (4) and (14) have been implemented in *Mathematica* in the case of any joint distribution of the original claims and linear premium income function $h(t) = u + c\,t$, where u is the total initial reserve and c is the total premium rate. Thus, Problems 1 and 2 have been solved with different joint distributions for the claim amounts and different choices for the rest of the model parameters. In the independent case, results for Exponential, Pareto and Weibull claim amount distributions are presented and the effect of their varying tail behavior on the probability of joint survival is assessed. In order to model dependence between claim severities, copula functions have been successfully used. The copula approach has allowed us to study how the assumption of dependence affects the solutions to Problems 1 and 2 and the probability of joint survival. For the purpose, a combination of Rotated Clayton copula with Weibull marginals has been implemented.

In general, our experience has shown that expression (4) is computationally more efficient than (14) since it converges faster with respect to k, i.e., a small number of terms is required in the summation in order to reach a desired accuracy of the result. The multiple integration is less computationally involved and hence faster, since all limits of integration in (4) are finite whereas in (14) the inner most integral is infinite. However, it should be noted that the derived expressions for $P(T^c > x, T^r > x)$ are rather general and that in each particular case, when the input parameters are fixed, both formulae could be simplified and of course, depending on the software used for the implementation, the computational efficiency may turn to be in favour of (14).

4.1 Independent claim severities.

Here, we have assumed that claim amounts are independent and have three alternative distributions: lighter tailed Exponential and heavier tailed Pareto and Weibull distributions. The optimization Problems 1 and 2 have been solved in each of these cases and the effect of the different tail behaviour of the claim distributions on the optimal solutions have been studied. Sensitivity results with respect to the choice of other model parameters are also presented.

The solution of the optimization Problem 2 in the case of exponentially distributed claim severities with parameter $\alpha = 1$, Poisson intensity $\lambda = 1$, finite time interval $x = 2$ and $h(t) = u + c\,t$, with total initial reserve $u = 0$ and premium rate $c = 1.55$, is illustrated in Fig 1. For fixed combinations of values of the levels M and L, an optimal reinsurance premium rate, c_r, is found, which maximizes $P(T^c > x, T^r > x)$, given that $h(t) = h_c(t) + h_r(t) = (1.55 - c_r)\,t + c_r\,t$. This is achieved by varying the proportion, $h_r(t) = c_r\,t$, of the premium income, given to the reinsurer from 1% to 99%, i.e., c_r is varied from 0.1 to 1.5 with a step 0.1. In the left panel of Fig. 1 we present results for the case of an XL contract without a limiting level, i.e. $L = \infty$, while the right panel refers to a retention level M and a limiting level $L = M + 0.5$. In both cases, the optimal premium rate c_r decreases when the retention level M increases. This complies well with the market principle that a smaller reinsurance premium should be charged for a smaller proportion of the risk, taken by the reinsurer. Comparing the two cases $L = \infty$ and $L = M + 0.5$, it can be seen that, in the latter case, the optimal solutions for c_r are shifted to the left, since there is a fixed non-zero layer $m = L - M = 0.5$, covered by the reinsurer.

From both panels of Fig. 1 it can also be seen that each curve has a single global maximum of the joint survival probability. This suggests that the optimization Problem 2 has a unique solution, at least for the classical linear $h(t)$. The proof of this interesting conjecture is hindered by the complexity of formulae (4) and (14) and in particular of the definitions of $\tilde{v}_j$, $\tilde{z}_j$, w_i^c, w_i^r, and is a subject of current investigation.

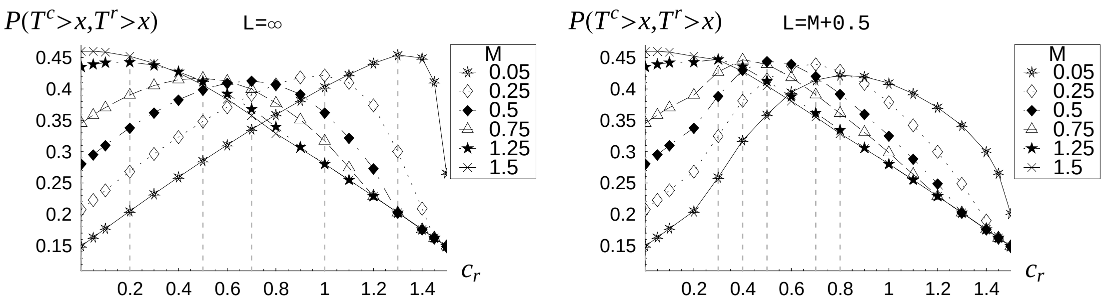

Fig. 1. Solutions to the optimality Problem 2: independent claim severities, Exp(1) distributed, $\lambda = 1$, $x = 2$, $h(t) = h_c(t) + h_r(t) = (1.55 - c_r)\,t + c_r\,t$.

Problem 2 has also been solved for different choices of the total initial reserve u and the initial reserves of the cedent, u_c and the reinsurer, u_r. The impact of different initial reserves on $P(T^c > x, T^r > x)$ and hence on the optimal value of c_r is illustrated in the left panel of Fig 2, for fixed levels $M = 0.5$, $L = \infty$ and parameters as in Fig 1, i.e., Exp(1) distributed claim severities, $\lambda = 1$ and $x = 2$. For this set of parameters, an optimal value, c_r, is found, which maximizes $P(T^c > x, T^r > x)$, given that $h(t) = u + c\,t$,

$h_c(t) = u_c + (1.55 - c_r)\, t$, $h_r(t) = u_r + c_r\, t$, with $u = u_c + u_r$ and $c = c_c + c_r = (1.55 - c_r) + c_r$. Five curves are given in the left panel of Fig 2 which correspond to five different choices of the pair of values u_c, u_r, for which the total reserve $u = u_c + u_r$ is correspondingly equal to $0.0, 1.0, 0.5, 1.0, 1.0$. There are two effects which can be observed. First, with the increase of the total reserve u, given $u_c = u_r$, (see curves corresponding to $(u_c, u_r) = \{(0, 0), (0.25, 0.25), (0.5, 0.5)\}$), the probability of joint survival increases as can be expected. The second effect is that, for fixed value of the total reserve $u = 1$, the optimal reinsurance premium c_r is lower if $u_c < u_r$, increases when $u_c = u_r$, and goes further up if $u_c > u_r$. Hence, the conclusion is that, if a direct insurance company wants to pay less in reinsurance premium and at the same time wants to maximize its and the reinsurer's chances of survival, the company should seek for a reinsurer with initial reserves higher than its own reserves, which is a practically meaningful business strategy. In the alternative case, $u_c > u_r$, the optimal reinsurance premium is much higher, since given the direct insurance company wants a maximum probability of joint survival, it has to pay much more in order to compensate the lower level of reserves kept by the reinsurer. But this clearly is not in favour of the direct insurer and is not what reinsurance is about.

In the right panel of Fig 2, we illustrate the impact of the time horizon x on the probability of joint survival and c_r. As can be seen, $P(T^c > x, T^r > x)$ decreases for longer time horizons, which is natural to expect. On the other hand, increasing x from 0.5 to 3 results in higher reinsurance premium, whereas further increase of x does not affect c_r. This can be explained with the higher possibility of arrival of large claims to the reinsurer as x initially goes up.

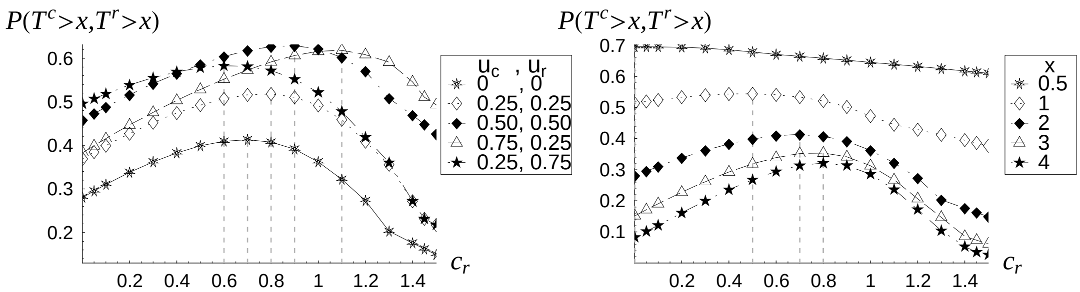

Fig. 2. Solutions to the optimality Problem 2: independent claim severities, Exp(1) distributed, $\lambda = 1$, $x = 2$, $c = 1.55$, $L = \infty$, $M = 0.5$; Left panel: $u \geq 0$, Right panel: $u = u_c = u_r = 0$, $x = 0.5, 1, 2, 3, 4$.

The solution of the optimization Problem 1 has been performed in the case of exponentially and Pareto distributed claim severities, both with unit mean, $\lambda = 1$, $x = 2$ and $h(t) = 1.55\, t$. Thus, in Fig. 3 two 3D plots are given, which illustrate the behaviour of the probability of joint survival as a function of M and $m = L - M$ when the premium income is equally shared, i.e. $h_c(t) = h_r(t)$ for any $t \geq 0$. The left panel of Fig. 3 refers to the case of exponentially distributed claim amounts, W_i, $i = 1, 2, \ldots$ with mean and variance $E(W) = V(W) = 1$, whereas the plot in the right panel is for Pareto claims with $E(W) = 1$ and $V(W) = 3$. As seen from both panels of Fig. 3, $P(T^c > x, T^r > x)$ has a single global maximum with respect to M and m. As with Problem 2, the existence of a unique solution of Problem 1 can be conjectured, but the proof is related with similar difficulties.

Solutions of Problem 1 for different choices of c_r, i.e., for different proportions in which the total premium income is shared, are summarized in Table 1. As can be seen, giving higher proportion of $h(t)$ to the reinsurer causes the optimal retention level, M, to drop and the optimal limiting level, m, to increase. The latter is not surprising as the cedent's retained risk should decrease when the premium income, passed on to the reinsurer, increases.

Table 1. Optimal values of M and m, maximizing $P(T^c > x, T^r > x)$ in the case of independent claim severities, Exp(1) distributed, with $\lambda = 1$, $x = 2$, $h(t) = h_c(t) + h_r(t) = (1.55 - c_r)\, t + c_r\, t$.

$\max_{M,m} P(T^c > x, T^r > x)$	$c_r = 0.25$	$c_r = 0.50$	$c_r = 0.775$	$c_r = 1.00$	$c_r = 1.25$
M	0.4	0.3	0.3	0.2	0.001
m	0.1	0.3	0.7	1.2	> 1.5

As can also be seen from Fig. 3, although the implemented Exponential and Pareto distributions have different variance and imply lighter and heavier tails of the claim severities, the two surfaces are very similar and the optimal values of M and m, which maximize $P(T^c > x, T^r > x)$ in each case, are very close. This is explained by the similarity in the shape of the Exponential and Pareto densities, as can be seen from the left panel of Fig. 4, since all other model parameters are the same. We have also implemented Weibull distributed claims, which does not affect the form of the surface as well. It is interesting to note that the probability of joint survival is higher for Pareto distributed claim amounts, compared with the exponential case, given that other model parameters coincide. The probability $P(T^c > x, T^r > x)$ is even higher if the claim size follows Weibull distribution with the same mean, $E(W) = 1$, and $V(W) = 2.2$. An illustration of the latter phenomenon is given in the right panel of Fig. 4. It can be explained by the fact that the time interval, $[0, 2]$, is relatively short and $P(T^c > x, T^r > x)$ is affected most significantly by the distribution of the smaller but more probable claims rather than by the less probable extreme claims in the tail. This is in compliance with the order of the probabilities 0.955, 0.940, 0.917, computed as $P(W \le h(2)) = P(W \le 3.1)$ correspondingly for exponentially, Pareto and Weibull distributed claims. The shape of the three densities, given in the left panel of Fig. 4, are also in support of this explanation. Our experience shows that for higher x the tail behaviour is of more importance for $P(T^c > x, T^r > x)$ and the order may reverse.

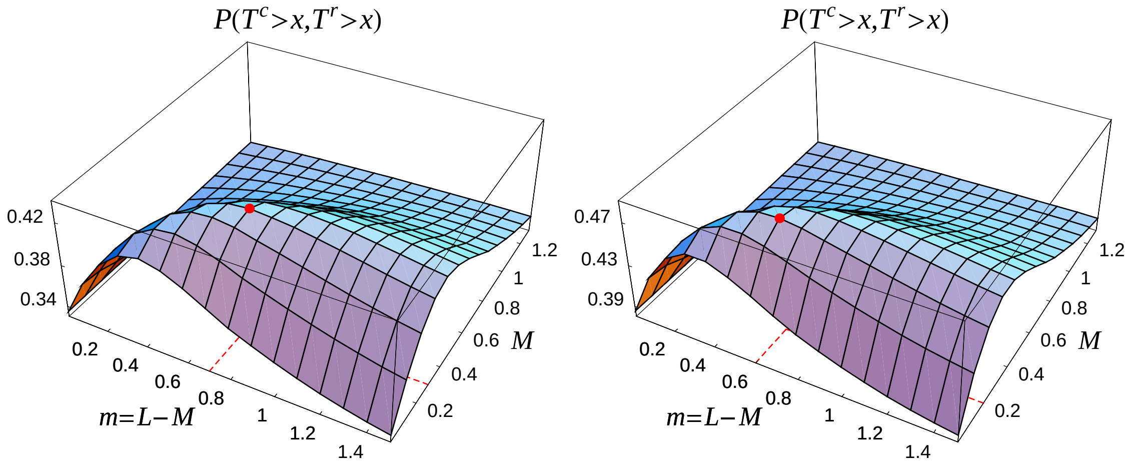

Fig. 3. Solutions to the optimality Problem 1: independent claim severities, $\lambda = 1$, $x = 2$, $h(t) = h_c(t) + h_r(t) = (1.55 - c_r)\,t + c_r\,t$, $c_r = 0.775$. Left panel - exponentially distributed, $E(W) = V(W) = 1$; Right panel - Pareto distributed, $E(W) = 1$, $V(W) = 3$.

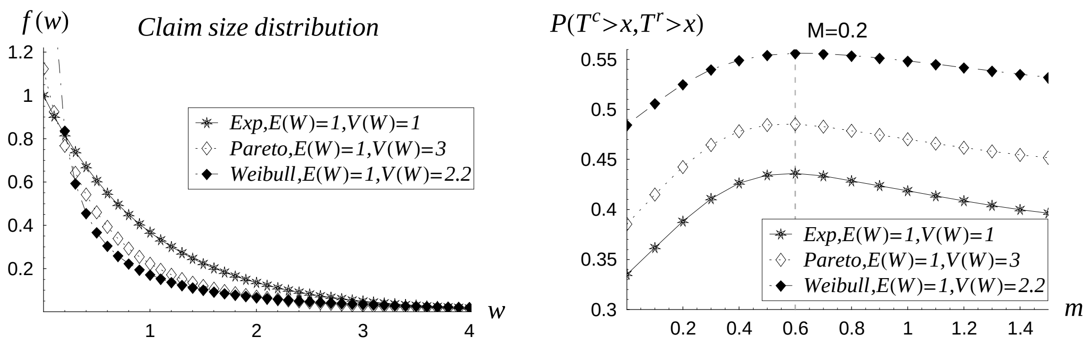

Fig. 4. Left panel - assumed probability density functions for the claim amounts W_i, $i = 1, 2, \ldots$; Right panel - $P(T^c > x, T^r > x)$ as a function of the layer m, $\lambda = 1$, $x = 2$, $h(t) = h_c(t) + h_r(t) = (1.55 - c_r)\,t + c_r\,t$, $c_r = 0.775$.

The general conclusion based on these examples is that $P(T^c > x, T^r > x)$ is a relevant reinsurance risk optimization criterion, which complies with some basic principles driving reinsurance risk assessment and pricing decisions.

4.2 Dependent claim severities.

In what follows, we provide some very interesting results for the probability of joint non-ruin and the solutions of Problems 1 and 2, assuming dependence between the claim severities $W_1, W_2, \ldots$. We show how this dependence could be modelled, using copula functions. The effect on $P(T^c > x, T^r > x)$ of the degree of dependence, modelled by the underlying copula parameter, and of the choice of the marginals, is also studied.

A difficulty, related to the copula approach is that, in general, a large number of consecutive claims may arrive at the insurance company and modelling their joint distribution will require highly multivariate copulas. The curse of dimensionality is overcome here due to the fast convergence of formula (4), for which only the first few terms in the summation with respect to k are needed, in order to compute $P(T^c > x, T^r > x)$ with a reasonable accuracy. This allows us to use up to a five-variate copula in the numerical examples presented here.

Let H denote the k-dimensional distribution function of the random vector of consecutive claim amounts $(W_1, ..., W_k)$ with continuous marginals $F_1, ..., F_k$. Then, one can use the well-known Sklar's theorem to represent H through a k-dimensional copula $C(u_1, ..., u_k)$, $0 \le u_j \le 1$, which depends on a set of parameters $\boldsymbol{\theta}$, as $H(w_1, ..., w_k) = C(F_1(w_1), ..., F_k(w_k))$. By changing the values of $\boldsymbol{\theta}$ within a specified range, one can control the degree of dependence, in general, from extreme negative, through independence, to extreme positive dependence. To measure the dependence in the tails of the distributions of two consecutive claims W_1 and W_2, one can use the upper and lower tail dependence coefficients, defined as

$$\lambda_L = \lim_{u \to 0^+} C(u, u)/u$$

$$\lambda_U = \lim_{u \to 1^-} (1 - 2u + C(u, u))/(1 - u)$$

where $\lambda_L \in (0, 1]$, $\lambda_U \in (0, 1]$. The copula C has no upper (lower) tail dependence iff $\lambda_U = 0$ ($\lambda_L = 0$). For example, in our context, $\lambda_U > 0$ would mean that extremely large insurance losses are likely to occur jointly. For further properties of copulas and related dependence measures we refer to Joe (1997). An extensive account on some actuarial applications of copulas can be found in Frees and Valdez (1998).

It should be noted that dependence between the components of the random vector $(W_1, ..., W_k)$ implies dependence between the components of the random vector $(W_1^c, ..., W_k^c)$ and also between the components of $(W_1^r, ..., W_k^r)$, since $W_i = W_i^c + W_i^r$. So, the two risk processes, R_t^c and R_t^r, which implicitly define $P(T^c > x, T^r > x)$, also incorporate dependent claims, namely $(W_1^c, ..., W_k^c)$ and $(W_1^r, ..., W_k^r)$. However, since formulae (4) and (14) involve the joint density function $\psi(w_1, ..., w_k)$ of the random vector $(W_1, ..., W_k)$, in order to compute $P(T^c > x, T^r > x)$ under dependence, we express this density through the copula function as

$$\begin{aligned}\psi(w_1, ..., w_k) &= \frac{\partial^k C(F_1(w_1), ..., F_k(w_k))}{\partial w_1 \ldots \partial w_k} \\ &= \frac{\partial^k C(u_1, ..., u_k)}{\partial u_1 \ldots \partial u_k} \prod_{i=1}^{k} \frac{\partial F_i(w_i)}{\partial w_i} = c(F_1(w_1), ..., F_k(w_k)) \prod_{i=1}^{k} f_{W_i}(w_i)\end{aligned} \tag{21}$$

where $c(u_1, ..., u_k)$ is the density of the copula C and $f_{W_i}(w_i)$, $i = 1, ..., k$ are the marginal density functions. As can be seen from (21), the copula approach to modelling dependence between claim amounts is very convenient since it separates the dependence structure, incorporated into the copula, from the marginals. Thus, one can independently choose the copula and its parameter(s), and the marginals, and study separately the effect of these two choices on $P(T^c > x, T^r > x)$ and on the solutions of the optimality Problems 1 and 2. For the purpose, we have chosen C to be the k-dimensional Rotated Clayton copula, C^{RCl}, and $F_1, \ ..., F_k$ to be identical Weibull(α, β) marginals.

Clayton and Rotated Clayton copulas are suitable for modelling dependence between claim severities. To see this, let us first introduce the Clayton copula, which is an Archimedean copula, with generator $\phi(t) = t^{-\theta} - 1$, $\theta > 0$, defined as

$$C^{\mathrm{Cl}}(u_1, \ ..., u_k; \theta) = \left(\textstyle\sum_{i=1}^{k} u_i^{-\theta} - k + 1\right)^{-1/\theta},$$

where $0 \le u_i \le 1$, $i = 1, \ ..., k$ and $\theta \in (0, \infty)$ is a parameter. Its density is given by

$$c^{\mathrm{Cl}}(u_1, \ ..., u_k; \theta) = \theta^k \frac{\Gamma(1/\theta+k)}{\Gamma(1/\theta)} \left(\textstyle\prod_{i=1}^{k} u_i^{-\theta-1}\right) \left(\textstyle\sum_{i=1}^{k} u_i^{-\theta} - k + 1\right)^{-1/\theta-k}.$$

As $\theta \to 0$, the Clayton copula converges to the product copula with density $c(u_1, \ ..., u_k) = 1$, which, as seen from (21), corresponds to independent claim amounts. The degree of dependence increases as θ increases. Further properties of the Clayton copula and its application in finance can be found in Cherubini et al. (2004).

In the general insurance context, it is of interest to consider the case in which the occurrence of large claims is highly correlated with the emergence of further large claims. Hence, it is meaningful to use a copula with upper tail dependence. However, the Clayton copula has lower tail dependence with coefficient $\lambda_L = 2^{-1/\theta}$, which makes it convenient for modeling dependence in the left tails of the marginal distributions, i.e. between very small claims. A typical example would be the joint occurrence of a large number of small motor insurance claims caused by a common (catastrophic) event, e.g. hail or bad driving conditions.

Based on the Clayton copula, one can model upper tail dependence using the multivariate Rotated Clayton copula, defined as

$$C^{\mathrm{RCl}}(u_1, \ ..., u_k; \theta) = \textstyle\sum_{i=1}^{k} u_i - k + 1 + \left(\sum_{i=1}^{k} (1-u_i)^{-\theta} - k + 1\right)^{-1/\theta}, \tag{22}$$

with density $c^{\mathrm{RCl}}(u_1, \ ..., u_k; \theta) = c^{\mathrm{Cl}}(1-u_1, \ ..., 1-u_k; \theta)$ and $\theta \in (0, \infty)$. The value $\theta = 0$ corresponds to independence as for C^{Cl}. A two dimensional version of (22) has been considered by Patton (2004). The Rotated Clayton copula has upper tail dependence with coefficient $\lambda_U = 2^{-1/\theta}$ and is suitable for modeling dependence between extreme insurance losses. The dependence structure, defined by a Rotated Clayton copula with parameter $\theta = 5$, is illustrated in the left panel of Fig. 5 through a random sample of 500 simulated pairs (u_1, u_2). In the right panel, we give the corresponding simulated claim amounts with joint distribution function $H(w_1, w_2) = C^{\mathrm{RCl}}(F_1(w_1), F_2(w_2); \theta)$ and identical Weibull$(1, 1)$ marginals. The presence of positive dependence, determined by $\theta = 5$, and of upper tail dependence, $\lambda_U = 2^{-1/5}$, are clearly visible.

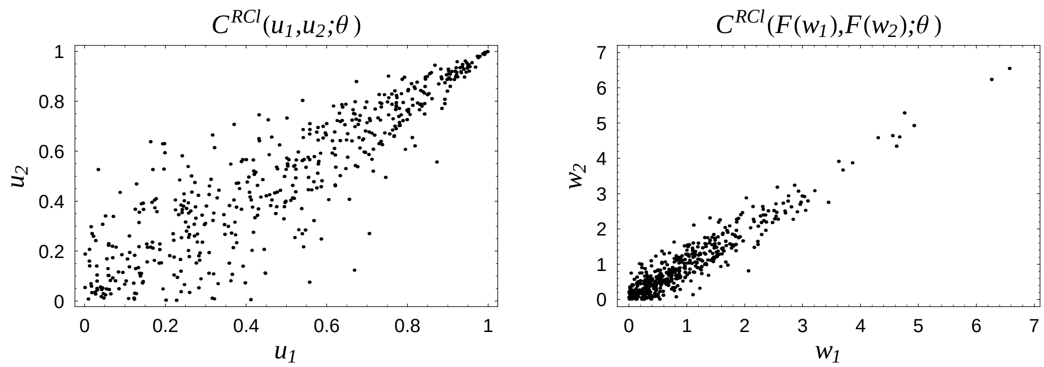

Fig. 5. A random sample of 500 simulations from a bivariate Rotated Clayton copula, with dependence parameter $\theta = 5$, marginals $F \equiv$ Weibull(1, 1) $\equiv$ Exp(1).

With the increase of θ, the solution of the optimality Problem 2 does not change, as illustrated in the left panel of Fig. 6 for fixed Weibull marginals with unit mean and variance. It can also be seen that, for any c_r, $P(T^c > x, T^r > x)$ goes up as θ deviates from zero. This may seem unexpected but it should be mentioned that, as θ increases, not only the tail dependence increases but so does the dependence throughout the whole range of claim amounts. As a result of this, jointly small claims occur with higher probability and through the risk processes, R_t^c and R_t^r, affect more significantly $P(T^c > x, T^r > x)$ than the occurrence of jointly large claims.

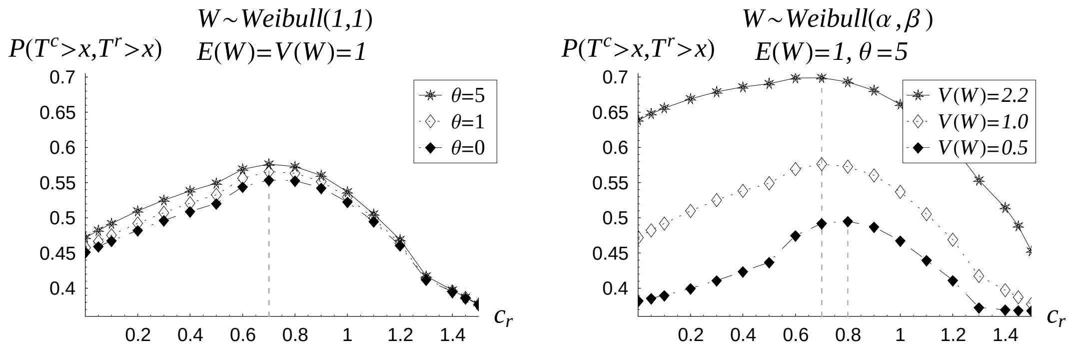

Fig. 6. Solutions to the optimality Problem 2: dependent claim severities, $C^{\mathrm{RCl}}(F(w_1), ..., F(w_k); \theta)$ distributed, marginals $F \equiv$ Weibull(α, β), $\lambda = 1$, $x = 1$, $h(t) = h_c(t) + h_r(t) = (1.55 - c_r)\, t + c_r\, t$, $M = 0.25$, $L = M + 0.5$.

The solution of the optimality Problem 2 for Weibull marginals with mean 1 and increasing variance is given in the right panel of Fig. 6. As can be seen, the optimal value for c_r slightly decreases as the variance increases. This is meaningful, since the variance of the cedent's claims increases with the variance of the original claims more significantly than that of the reinsurer and hence, the reinsurance premium should decrease. The latter effect is due to the fact that the reinsurer's liability is limited within the layer m. It can also be seen from the right panel of Fig. 6 that $P(T^c > x, T^r > x)$ increases as the variance increases which is a phenomenon, similar to the one illustrated in Fig. 4 and can be explained applying similar reasoning.

5. Conclusions and comments.

In this paper, we have demonstrated that the optimal retention and limiting levels and the optimal sharing of the premium income, obtained by maximizing the probability of joint survival of the cedent and the reinsurer in an excess of loss contract, assuming continuous claim severities, are sensible. It will be instructive to test this joint optimality criterion on real claim data.

An interesting finding is the presence of unique solutions to Problems 1 and 2 in the examples of Section 4.1. Proofs of such conjectures are a subject of ongoing research.

We have also demonstrated that formulae (4) and (14), through their reasonable generality, conveniently allow to implement copulas in modelling dependence between consecutive claim severities. These are only first steps in this important new direction of research and a variety of open problems arrises. For example, it is interesting to explore how the solutions of Problems 1 and 2, and also $P(T^c > x, T^r > x)$, will be affected by different dependence structures. In particular, will the upper and lower Fréchet bounds lead to upper and lower bounds for $P(T^c > x, T^r > x)$?

Finally, viewing $P(T^c > x, T^r > x)$ as a risk measure, one could define a performance measure based on the expected profits, at the end of the time horizon x, of the insurer and the reinsurer and consider an optimality criterion which combines these measures and could be used to optimally set the parameters of a reinsurance contract. The latter is a subject of future investigation.

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