Less is more: increasing retirement gains by using an upside terminal wealth constraint

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Abstract

We solve a portfolio selection problem of an investor with a deterministic savings plan who aims to have a target wealth value at retirement. The investor is an expected power utility-maximizer. The target wealth value is the maximum wealth that the investor can have at retirement.

By constraining the investor to have no more than the target wealth at retirement, we find that the lower quantiles of the terminal wealth distribution increase, so the risk of poor financial outcomes is reduced. The drawback of the optimal strategy is that the possibility of gains above the target wealth are eliminated.

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1 Introduction

Investing for retirement is usually characterized by a period of savings followed by a period of consumption. The question of how to invest the savings before retirement has been considered widely in the academic literature. We consider the problem of how to invest an initial wealth and periodic amounts in order to reach some target capital at a fixed time horizon that represents the intended retirement date. This is a different formulation of one of the problems described in Dhaene et al. (2005, page 277), in which an investor wishes to find the optimal constant-proportion portfolio that attains the highest target capital with a fixed probability. We constrain the investor to have *at most* the target capital at the time of retirement, whereas Dhaene et al. (2005) ensure that at least the target capital is attained with maximum probability. Since our focus is on a broad analysis of following the optimal strategy, we assume throughout this paper a simple continuous-time complete market model. Wealth can be invested in a risky asset and in a risk-free asset. Our discussion is about the strategies regarding the amount invested in each of those. As the investment period is long, we are interested in the long-run outcome, namely the distribution of the terminal wealth, rather than in the fluctuations of wealth during the savings phase.

Our paper is about the reduction of the risk of terminal wealth being too large and too low. We assume that investors are willing to accept that gains may not be too large in the long-run, if there is a higher chance that terminal wealth is not too low. We consider this problem to be of crucial importance to consumers, who do not want the accumulated value of their retirement savings to be insufficient for their retirement needs. Our approach differs from Gerrard et al. (2014) who examined the lowest part of the terminal wealth distribution after savings and consumption. Here we study only the savings phase and we rather fix an upper target wealth, which should not be exceeded at the terminal time point. This is what we call a constrained strategy. As a return for the sacrifice of profits, the terminal wealth distribution is more concentrated in the values below the target wealth than in the constrained plan, so that the probability of small values is lower than under the pure unconstrained investment strategy.

We find an optimal strategy for investors in the current framework. This result follows from maximizing the expected utility of terminal wealth plus designing a call option on the fixed target. Moreover, we also find that there is an optimal target level of wealth to be chosen, which provides a larger difference in the rate of return to the investor compared with the optimal unconstrained strategy, in which the investor holds a fixed proportion of his wealth in the risky asset.

We do not look at portfolio selection when there is more than one risky asset available, like in Van Weert et al. (2010), but we do take into consideration risk aversion through a utility function in addition to the constraint on the terminal wealth values. We also permit a dynamic asset allocation strategy. Our results can easily be extended to the case where investors have both an upper and a lower target in the terminal wealth distribution.

We should mention here several recent works on dynamic asset allocation strategies. Some authors do not formally specify the investor's problem and simply propose an investment strategy. Basu et al. (2011) look at performance relative to a target return and suggest a contrarian strategy of switching the investor's asset allocation between 100% of wealth invested in stocks, and 80% of wealth in stocks and the remainder in bonds according to whether the cumulative target return is attained or not. Similar strategies are compared in Basu and Drew (2009). Both papers show that defensiveness towards the end of the investor's time horizon, through diminishing the investment in the risky assets (a so-called lifecycle strategy), is costly in terms of the overall return (Guillén et al. 2013 arrives at a similar conclusion using a different methodology).

Another approach in the literature is to specify the investor's problem within a model and then determine the optimal investment strategy. Typically, the investor's core problem is to maximize the expected utility of terminal wealth subject to specified constraints being satisfied. Grossman and Zhou (1996) impose the constraint that the terminal wealth must be at least some fraction of the initial wealth. Korn and Trautmann (1995) impose a constraint on the expected value of the terminal wealth. Other authors impose the constraint that the investor's terminal wealth is at least a minimum value with a certain probability. (This is similar to the problem in Dhaene et al. (2005, page 277) except that the latter maximise the minimum value directly and do not use a utility function.) In Boyle and Tian (2007), the minimum value is a random variable that models a benchmark strategy. Bouchard et al. (2010) prove a viscosity solution characterization of the value function in a very general setting when there are terminal wealth constraints. De Franco and Tankov (2011) and Gaibh et al. (2009) use a risk-measure constraint that is applied only to terminal losses that are worse than a fixed level.

Browne (1999) solves a similar problem to the one that we consider, except that he maximises directly the probability of reaching the target retirement wealth. The formulation is attractive, since it requires only a target wealth to be specified by the investor; calibrating utility functions to individual investors is complicated (von Gaudecker et al., 2011). In other words, Browne (1999) does not capture explicitly the investor's emotional responses to investment gains and losses, as we do here very simply with a power utility function or as Jin and Zhou (2008) do by applying prospect theory. However, the consequence is that following the optimal strategy in Browne (1999) results in an "all-or-nothing" terminal wealth: either the target wealth is attained or the terminal wealth is zero. We believe that such binary outcomes would be disagreeable to most retirement investors. Indeed, the experiments of Benartzi and Thaler (1999) suggest that investors are highly sensitive to the distribution of terminal wealth.

We are not aware of another paper that considers constraining the terminal wealth to be at most some target capital. Setting a retirement savings wealth goal is in line with advice given to individuals by financial advisers (Greninger et al., 2000). Although it may appear to be rather unambitious to aim at or below a target value rather than above it, we find that there are very appealing consequences. The probability of attaining the target is higher than under the optimal unconstrained strategy. This may be more reassuring to the retirement investor. The quantiles below the target are higher than those for the optimal unconstrained strategy, and they are higher by a constant ratio that can be calculated in advance. In summary, the investor increases their chances of attaining their desired target retirement wealth and, even if they fail to reach it, they still have a higher wealth than if they had no such target.

This paper is organized as follows. Section 2 presents the market model and the investor generic savings behaviour with deterministic cash flows. Our setting can be generalized, but we do not consider random cash flows for simplicity. Section 3 provides the solution to the unconstrained case, where terminal wealth is not bounded. The constrained optimal strategy is shown in Section 4. Section 5 discusses the choice of a target level in the terminal wealth distribution. A numerical illustration and a discussion conclude the paper.

2 Notation and model assumptions

2.1 Market model

We assume investment in a continuous-time market model over a finite time horizon $[0, T]$ for an integer $T > 0$. We refer to T as the *terminal time*.

The market consists of one risky stock and one risk-free bond. The price of the stock is driven by an 1-dimensional, standard Brownian motion $W = \{W(t); t \in [0, T]\}$ defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The risk-free bond has price process $\{S_0(t); t \in [0, T]\}$ and the risky stock has price process $\{S_1(t); t \in [0, T]\}$ with dynamics

$$\mathrm{d}S_0(t) = rS_0(t)\,\mathrm{d}t, \qquad \mathrm{d}S_1(t) = S_1(t)\left(\mu dt + \sigma dW(t)\right), \tag{2.1}$$

with $\sigma > 0$, $S_0(0) = 1$, a.s. and $S_1(0) = 1$, a.s. We assume that $\mu > r \geq 0$. Define the constant market price of risk

$$\theta := \frac{\mu - r}{\sigma}.$$

The information available to investors is represented by the filtration

$$\mathcal{F}_t := \sigma\{W(s), s \in [0, t]\} \vee \mathcal{N}(\mathbb{P}), \quad \forall t \in [0, T],$$

where $\mathcal{N}(\mathbb{P})$ denotes the collection of all $\mathbb{P}$-null events in the probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

2.2 Investor

An investor starts with a fixed non-random initial wealth $x_0 > 0$ and plans to make a sequence of known future savings. Define $C(t)$ to be the undiscounted sum from time 0 to time t of the investor's planned savings. The idea is that the investor has decided at time 0 how much money they will save towards their retirement. The *savings plan* $g : [0, T] \to [0, \infty)$ of the investor, i.e. the discounted sum of the future savings to be made by the investor, is given by

$$g(t) := \int_t^T e^{-r(s-t)} dC(s), \quad \forall t \in [0, T]. \tag{2.2}$$

A portfolio process $\pi = \{\pi(t); t \in [0, T]\}$ is a $\mathbb{R}$-valued, square-integrable, $\{\mathcal{F}_t\}$-progressively measurable process. The investor follows a self-financed strategy, investing at each instant $t \in [0, T]$ a monetary amount $\pi(t)$ in the stock such that the $\pi = \{\pi(t); t \in [0, T]\}$ is a portfolio process.

The wealth process $X^\pi = \{X^\pi(t); t \in [0,T]\}$ corresponding to a portfolio process π and savings plan g is the $\{\mathcal{F}_t\}$-adapted process given by the *wealth equation*

$$dX^\pi(t) = \left(r\left(X^\pi(t) + g(t)\right) + \pi(t)\sigma\theta\right)dt + \pi(t)\sigma\, dW(t) - dg(t), \quad X^\pi(0) = x_0 \text{ a.s.} \tag{2.3}$$

The set of *admissible portfolios* for the investor's initial wealth $x_0 > 0$ is defined to be

$$\mathcal{A} := \{\pi : \Omega \times [0,T] \to \mathbb{R} : X^\pi(0) = x_0, \text{ a.s. and } X^\pi(t) + g(t) \geq 0, \forall t \in (0,T] \text{ a.s.}\}.$$

We say that a portfolio process π is *admissible* if $\pi \in \mathcal{A}$.

Define the state price density process H as $H(t) := \exp\left(-\left(r + \frac{1}{2}\theta^2\right)t - \theta W(t)\right)$, for each $t \in [0,T]$. A portfolio π must satisfy the *budget constraint* that

$$\mathbb{E}\left(H(T)X^\pi(T)\right) \leq x_0 + g(0). \tag{2.4}$$

The utility function of the investor is the power utility function

$$U(x) := \frac{1}{\gamma}x^\gamma, \qquad x > 0,$$

for a fixed constant $\gamma \in (-\infty, 1) \setminus \{0\}$. The investor seeks to maximise the expected utility of their terminal wealth, subject to constraints on the range of values of the terminal wealth.

3 Unconstrained problem

Before solving the constrained problem, we give the solution to the corresponding unconstrained problem. The unconstrained solution provides the foundation for the solution to the constrained problem. To avoid confusion caused by subtle differences in notation, we use $u^\star$ to denote the optimal portfolio for the unconstrained problem.

Problem 3.1 (Unconstrained problem)**.** The unconstrained problem is to find $u^\star \in \mathcal{A}$ such that

$$\mathbb{E}\left(U(X^{u^\star}(T))\right) = \sup_{u \in \mathcal{A}}\{\mathbb{E}\left(U(X^u(T))\right)\}.$$

As the solution to the above problem can be found in, for example, Gerrard et al. (2014, Section 3) or Korn and Krekel (2002, Theorem 4), we simply state it. However, first we introduce some notation that we use throughout the paper.

Define the constant

$$A := \frac{\theta}{\sigma(1-\gamma)}$$

and the process

$$Z(t) = \exp\left(\left(r + \theta\sigma A - \frac{1}{2}\sigma^2 A^2\right)t + \sigma A W(t)\right), \quad \forall t \in [0,T]. \tag{3.1}$$

The optimal investment strategy for Problem 3.1 is to invest in the risky stock the monetary amount

$$u^\star(t) := u^\star(t; x_0, g) := A\left(x_0 + g(0)\right)Z(t). \tag{3.2}$$

The more risk averse the investor, the smaller the value of the constant A and the less that the investor puts in the risky stock. The corresponding wealth process is

$$X^{u^\star}(t) = (x_0 + g(0))\, Z(t) - g(t), \tag{3.3}$$

in which $Z(t)$ is defined by (3.1) and $g(t)$ by (2.2). Thus we can write $u^\star(t) = A\left(X^{u^\star}(t) + g(t)\right)$, so that the optimal amount invested in the risky asset at time t depends on the investor's current wealth and the discounted value of their future savings.

Remark 3.2. If the investor does not plan to make any future savings, so that $g \equiv 0$, then the optimal investment strategy for Problem 3.1 is to invest in the risky stock the monetary amount $Ax_0Z(t)$ at each time $t \in [0, T]$. Comparing this amount to that for an investor who has a non-trivial savings plan $g \neq 0$, the investor with a savings plan puts more in the risky stock at each time. Effectively, the latter investor is borrowing against their future savings to do this. Indeed, as $u^\star(t; x_0, g) = u^\star(t; x_0 + g(0), 0)$, an investor with initial wealth x_0 and a savings plan g invests in the risky stock as if he had initial wealth $x_0 + g(0)$ and no savings plan (although the wealth process would be adjusted to take account of the future savings).

This result is well known in the literature (for example, see Bodie et al. 1992). The message is that the investor should today borrow against their future human capital, i.e. they should take more risk now, in order to maximize their expected utility of wealth at their retirement date. In practice, most investors will only invest at most the value of their current wealth in the risky stock. Various papers encompass such portfolio or borrowing constraints (for example, Cuoco 1997 and Zariphopoulou 1994), which we do not consider here.

Nonetheless, we leave in the savings plan to emphasize that it should be considered in the asset allocation decision. It is particularly important for highly risk-averse investors who would invest less than the current value of their wealth in the risky stock if they did not have a savings plan.

4 Constrained problem

Next we introduce the constrained problem, in which the investor seeks to maximize the expected utility of their terminal wealth, subject to the terminal wealth being bounded above by a *target wealth* $K > 0$. Although the wealth is not constrained to hit the target, but rather to be no greater than the target, there is a high probability of the wealth hitting the target. Naturally, the probability of attaining the target wealth K at the terminal time increases as K decreases.

The motivating idea for introducing the target wealth constraint is that the investor gives up the "upside" (i.e. there is zero probability of having a wealth in excess of the target wealth) in exchange for a reduction in the "downside" (i.e. an increased probability of having a particular wealth that is below the target wealth). Furthermore, the extent to which the "downside" quantiles of the constrained terminal wealth are higher than the corresponding quantiles for the unconstrained terminal wealth can be explicitly calculated in our chosen model.

The security of more certainty about the range of values for the terminal wealth may be attractive to an individual. However, the disadvantage of giving up potentially high values of wealth is that the constrained investor has a lower expected terminal wealth compared to an unconstrained investor.

In this section, we show that the solution to the constrained problem is to invest in line with the unconstrained optimal portfolio, i.e. the solution to the unconstrained problem (3.1), but with a different initial wealth value, while simultaneously selling a synthetic European call option that is written on the corresponding unconstrained optimal wealth process. The solution is detailed in Proposition 4.5.

Problem 4.1 (Constrained problem)**.** The constrained problem is to find $\pi^\star \in \mathcal{A}$ such that

$$\mathbb{E}\left(U(X^{\pi^\star}(T))\right) = \sup_{\pi\in\mathcal{A}}\{\mathbb{E}\left(U(X^{\pi}(T))\right)\},$$

and $X^{\pi^\star}(T) \in [0, K]$, a.s.

In order to avoid the uninteresting case that the investor can immediately be assured of maximizing the terminal utility, we assume that

Assumption 4.2. $K > (x_0 + g(0))e^{rT}$.

4.1 The optimal terminal wealth

The next proposition is the key result that helps us to determine the optimal investment strategy for Problem 4.1.

Proposition 4.3. *Define*

$$X^\star(T) := (z_0 + g(0))Z(T) - \max\{0, (z_0 + g(0))Z(T) - K\}, \qquad (4.1)$$

with the shadow wealth $z_0 > 0$ *chosen so that the budget constraint (2.4) is satisfied with equality by* $X^\star(T)$*. Then* $\sup_{\pi\in\mathcal{A}} \mathbb{E}\left(U(X^\pi(T))\right) \le \mathbb{E}\left(U(X^\star(T))\right)$.

Proof. Adapt Grossman and Zhou (1996, Proof of Lemma 2). For the investor's utility function, the first derivative $U'(x) = x^{\gamma-1}$, which is a strictly decreasing function, has a strictly decreasing inverse I with

$$I(y) := y^{\frac{1}{\gamma-1}}, \quad y > 0.$$

After some algebra, we find that for the constant $y^\star := (z_0+g(0))^{\gamma-1}e^{\left(\gamma r+\frac{1}{2}\frac{\gamma}{1-\gamma}\theta^2\right)T}$, we have

$$(z_0 + g(0))Z(T) = I(y^\star H(T)).$$

We work with $I(y^\star H(T))$ in the proof, rather than with $(z_0 + g(0))Z(T)$ due to the properties of $I(x)$ and $U'(x)$: they are both strictly decreasing functions.

Let $X(T) \in [0, K]$, a.s. be any attainable terminal wealth (i.e. there exists a portfolio process $\pi \in \mathcal{A}$ that replicates $X(T)$) with $\mathbb{E}\left(H(T)X(T)\right) \le x_0 + g(0)$. We show that

$$\mathbb{E}\left(U(X(T))\right) \le \mathbb{E}\left(U(X^\star(T))\right).$$

Then by arbitrary choice of X, $\sup_{\pi\in\mathcal{A}} \mathbb{E}\left(U(X^\pi(T))\right) \le \mathbb{E}\left(U(X^\star(T))\right)$.

From equation (4.1) and using the fact that U' is a strictly decreasing function,

$$X^\star(T) = \begin{cases} I(y^\star H(T)) & \text{if } I(y^\star H(T)) < K \\ K & \text{if } I(y^\star H(T)) \ge K \end{cases} = \begin{cases} I(y^\star H(T)) & \text{if } y^\star H(T) > U'(K) \\ K & \text{if } y^\star H(T) \le U'(K). \end{cases}$$

First, as U is a concave function then for any $a, b \in \mathbb{R}$, $U(a) - U(b) \le U'(b) \cdot (a - b)$. In particular,

$$U(X(T)) - U(X^\star(T)) \le U'(X^\star(T)) \cdot (X(T) - X^\star(T)), \quad \text{a.s.}$$

Take expectations in the above inequality to get

$$\begin{aligned}
&\mathbb{E}\left(U(X(T)) - U(X^\star(T))\right) \\
\le&\mathbb{E}\left(U'(X^\star(T)) \cdot (X(T) - X^\star(T))\right) \\
\le&\mathbb{E}\left(U'(X^\star(T)) \cdot (X(T) - X^\star(T)) \,|\, y^\star H(T) > U'(K)\right) \cdot \mathbb{P}\left[y^\star H(T) > U'(K)\right] \\
&+ \mathbb{E}\left(U'(X^\star(T)) \cdot (X(T) - X^\star(T)) \,|\, y^\star H(T) \le U'(K)\right) \cdot \mathbb{P}\left[y^\star H(T) \le U'(K)\right].
\end{aligned}$$

We consider the last two terms separately. Observe that on the event $[y^\star H(T) > U'(K)]$,

$$U'(X^\star(T)) = U'(I(y^\star H(T))) = y^\star H(T)$$

so that

$$\begin{aligned}
&\mathbb{E}\left(U'(X^\star(T)) \cdot (X(T) - X^\star(T)) \,|\, y^\star H(T) > U'(K)\right) \\
=&\mathbb{E}\left(y^\star H(T) \cdot (X(T) - X^\star(T)) \,|\, y^\star H(T) > U'(K)\right).
\end{aligned}$$

Next observe that on the event $[y^\star H(T) \le U'(K)]$, as $X(T) \in [0, K]$ a.s. and $X^\star(T) = K$ a.s, then

$$X(T) - X^\star(T) = X(T) - K \le 0$$

and

$$U'(X^\star(T)) = U'(K) \ge y^\star H(T).$$

Upon multiplying both sides of the inequality $U'(X^\star(T)) \ge y^\star H(T)$ by the negative random variable $X(T) - X^\star(T)$, we find that on the event $[y^\star H(T) \le U'(K)]$,

$$U'(X^\star(T)) \cdot (X(T) - X^\star(T)) \le y^\star H(T) \cdot (X(T) - X^\star(T)).$$

Thus

$$\begin{aligned}
&\mathbb{E}\left(U'(X^\star(T)) \cdot (X(T) - X^\star(T)) \,|\, y^\star H(T) \le U'(K)\right) \\
\le&\mathbb{E}\left(y^\star H(T) \cdot (X(T) - X^\star(T)) \,|\, y^\star H(T) \le U'(K)\right).
\end{aligned}$$

In summary,

$$\begin{aligned}
&\mathbb{E}\left(U(X(T)) - U(X^\star(T))\right) \\
\le&\mathbb{E}\left(y^\star H(T) \cdot (X(T) - X^\star(T)) \,|\, y^\star H(T) > U'(K)\right) \cdot \mathbb{P}\left[y^\star H(T) > U'(K)\right] \\
&+ \mathbb{E}\left(y^\star H(T) \cdot (X(T) - X^\star(T)) \,|\, y^\star H(T) \le U'(K)\right) \cdot \mathbb{P}\left[y^\star H(T) \le U'(K)\right] \\
\le&\mathbb{E}\left(y^\star H(T) \cdot (X(T) - X^\star(T))\right).
\end{aligned}$$

As both $X(T)$ and $X^\star(T)$ satisfy the budget constraint (2.4), from the last line in the above inequality

$$\mathbb{E}\left(y^\star H(T) \cdot (X(T) - X^\star(T))\right) \le y^\star \cdot ((x_0 + g(0)) - (x_0 + g(0))) = 0,$$

which means $\mathbb{E}\left(U(X(T)) - U(X^\star(T))\right) \le 0$, as required. $\square$

4.2 The synthetic European call option

If we can find an admissible portfolio that results in terminal wealth $X^{\star}(T)$, given by (4.1), then $\sup_{\pi\in\mathcal{A}}\mathbb{E}\left(U(X^{\pi}(T))\right)=\mathbb{E}\left(U(X^{\star}(T))\right)$, and the portfolio is the optimal one for the constrained strategy.

To this end, note that the expression for $X^{\star}(T)$ is the sum of the optimal terminal wealth for the unconstrained strategy, albeit with initial wealth z_0 rather than x_0, less the value of a synthetic European call option. The latter unconstrained strategy has value $Y(t)-g(t)$ at time t, with

$$Y(t):=\left(z_0+g(0)\right)Z(t).$$

The synthetic European call option has strike price K and is written on a basket of assets with value $Y(t)$ at time t. We begin by finding the pricing function of the call option.

Lemma 4.4. *The price at time $t\in[0,T]$ of a European call option with payoff* $\max\{0,(z_0+g(0))Z(T)-K\}$ *is given by $c(t,Y(t))$ with*

$$c(t,y):=y\Phi(d_+(t,y))-Ke^{-r(T-t)}\Phi(d_-(t,y)),$$

in which $\Phi(d)$ denotes the cumulative standard normal distribution function at $d\in\mathbb{R}$ and

$$d_{\pm}(t,y):=\frac{1}{\sigma A\sqrt{T-t}}\left(\ln\left(\frac{y}{K}\right)+\left(r\pm\frac{1}{2}\sigma^2A^2\right)(T-t)\right),\quad\forall y>0.\tag{4.2}$$

The replicating portfolio for the option is to hold the amount $\pi_c\left(t,Y(t)\right)$ in the risky stock at time $t\in[0,T]$, with

$$\pi_c\left(t,y\right):=Ay\,\Phi(d_+(t,y)),\quad\forall y>0,\tag{4.3}$$

and the remaining amount $c(t,Y(t))-\pi_c\left(t,Y(t)\right)$ in the risk-free bond.

Proof. We use risk-neutral pricing to calculate the pricing function $c(t,y)$ of the call option. Denote the risk-neutral pricing measure by $\mathbb{Q}$ and define $W^{\mathbb{Q}}(t):=W(t)+\theta t$, for $t\in[0,T]$. Then $W^{\mathbb{Q}}$ is a standard Brownian motion under $\mathbb{Q}$ and

$$e^{-rT}Y(T)=Y(0)\exp\left(-\frac{1}{2}\sigma^2A^2T+\sigma A\sqrt{T}L\right),$$

in which $L:=W^{\mathbb{Q}}(T)/\sqrt{T}\sim\mathcal{N}(0,1)$ under $\mathbb{Q}$. After some algebra, we find that

$$Y(T)>K\quad\Leftrightarrow\quad L>-d_-(0,Y(0)).$$

Using $\mathbb{1}\left[A\right]$ to denote the zero-one indicator function on the set $A\subset\Omega$,

$$\begin{aligned}&c(0,Y(0))\\=&\mathbb{E}_{\mathbb{Q}}\left(e^{-rT}\max\left\{0,Y(T)-K\right\}\right)\\=&\mathbb{E}_{\mathbb{Q}}\left(\left(Y(0)\exp\left(-\frac{1}{2}\sigma^2A^2T+\sigma A\sqrt{T}L\right)-Ke^{-rT}\right)\cdot\mathbb{1}\left[L>-d_-(0,Y(0))\right]\right).\end{aligned}$$

By integration over the probability density function of the standard normally distributed random variable L, we obtain

$$c(0,Y(0))=Y(0)\Phi\left(d_+(0,Y(0))\right)-Ke^{-rT}\Phi\left(d_-(0,Y(0))\right).$$

The general result follows by the usual argument: we re-start the Black-Scholes model at time $t \in (0,T)$. Then there are $T-t$ years until maturity, we replace $Y(0)$ by $Y(t)$, etc.

Next we calculate the replicating portfolio of the European call option. Differentiating the pricing function c, we get

$$c_t(t,y) = -\frac{y\phi(d_+(t,y))\sigma A}{2\sqrt{T-t}} - rKe^{-r(T-t)}\Phi(d_-(t,y)),$$

$$c_y(t,y) = \Phi(d_+(t,y)), \qquad c_{yy}(t,y) = \frac{\phi(d_+(t,y))}{y\sigma A\sqrt{T-t}}.$$

By Ito's formula,

$$dc(t,Y(t)) = c_t(t,Y(t))dt + c_y(t,Y(t))dY(t) + \frac{1}{2}c_{yy}(t,Y(t))d\left[Y\right](t),$$

in which

$$dY(t) = (r + \theta\sigma A)\, Y(t)dt + \sigma AY(t)dW(t).$$

Substituting for the derivatives of the pricing function c, the dynamics of Y and the candidate replicating portfolio $\pi_c\,(t,Y(t)) := AY(t)\,\Phi(d_+(t,Y(t)))$, we find that the dynamics of the pricing function c satisfy the wealth equation (with $g \equiv 0$). Hence $\pi_c\,(t,Y(t))$ is the amount to be invested in the risky stock at time t in order to replicate the payoff of the synthetic European call option. $\square$

4.3 An optimal strategy for Problem 4.1

Proposition 4.5. *An optimal investment strategy for Problem 4.1 is to invest the amount*

$$\pi^\star(t) := AY(t)\;[1 - \Phi(d_+\,(t,Y(t)))] \tag{4.4}$$

in the risky stock at time t, in which the function d_+ is defined by equation (4.2) and $Y(t) = (z_0 + g(0))\,Z(t)$.

The wealth process corresponding to this optimal investment strategy is

$$X^{\pi^\star}(t) = Y(t) - g(t) - c(t,Y(t)). \tag{4.5}$$

In particular, the shadow initial wealth z_0 is chosen to satisfy

$$x_0 = z_0 - c(0, z_0 + g(0)). \tag{4.6}$$

Proof. By substituting the portfolio $\pi^\star$ into the wealth equation (2.3), we find that the corresponding wealth process $X^{\pi^\star}$ agrees with (4.5). As $X^{\pi^\star}(t) + g(t) = Y(t) - c(t,Y(t)) \geq 0$, then $\pi^\star \in \mathcal{A}$.

Furthermore, as $g(T) = 0$, then $X^{\pi^\star}(T)$ is the same a.s. as $X^\star(T)$ defined by (4.1). It follows from Proposition 4.3 that $\sup_{\pi\in\mathcal{A}} \mathbb{E}\left(U(X^\pi(T))\right) = \mathbb{E}\left(U(X^{\pi^\star}(T))\right)$. As $X^{\pi^\star}(T) \in [0,K]$, a.s, we have shown that $\pi^\star$ is an optimal solution for Problem 4.1.

The equation to be satisfied by the shadow initial wealth z_0 follows from Proposition 4.3 and by evaluating the budget constraint (2.4) with equality. $\square$

Remark 4.6. As the target wealth K becomes larger and larger, the shadow initial wealth z_0 converges to the investor's actual initial wealth: from equation (4.6), since $\lim_{K\to\infty} c(0, z_0 + g(0)) = 0$, it follows that $\lim_{K\to\infty} z_0 = x_0$. Consequently, the optimal unconstrained strategy can be obtained from the optimal constrained strategy by letting the target wealth tend to infinity.

Remark 4.7. The amount invested in the risky stock is always positive which can be seen from the positivity of the terms on the right-hand side of equation (4.4) (note that under the assumption that $\mu > r$, it follows that $A > 0$). Thus the investor never short-sells the risky stock for the optimal strategy.

4.4 Interpretation of the shadow initial wealth z_0

The relative value of the shadow initial wealth z_0 over the investor's actual initial wealth x_0 has a concrete interpretation. For the p-quantiles of the constrained terminal wealth that fall below the target wealth K, it gives their uplift over those for the unconstrained terminal wealth.

To see this, we calculate the p-quantiles under both the constrained and the unconstrained strategies. For the constrained strategy, there is a probability mass at the target wealth K. For this reason we use the following generalised definition of the p-quantile.

Definition 4.8. Fix $p \in (0, 1)$. The p-quantile for a random variable X is

$$\mathcal{Q}_p(X) = \inf \{y \in \mathbb{R} : \mathbb{P}[X \leq y] \geq p\},$$

with the convention that $\inf \{\emptyset\} = \infty$.

For an investor with initial wealth x_0 and savings plan g, let $X^\star(t; x_0, g, \infty)$ be the optimal wealth (3.3) for the unconstrained problem. For the same investor who also chooses a target wealth K, denote by $X^\star(t; x_0, g, K)$ the optimal wealth (4.5) for the constrained problem. Then

$$X^\star(t; x_0, g, K) = X^\star(t; z_0, g, \infty) - c(t, Y(t))$$

and, following from Remark 4.6, $\lim_{K\to\infty} X^\star(t; x_0, g, K) = X^\star(t; x_0, g, \infty)$, a.s.

Lemma 4.9 (p-quantiles)**.** *Suppose an investor has initial wealth $x_0 > 0$ and follows the savings plan g. Fix $p \in (0, 1)$ and define the constant*

$$\beta_p := \sigma A \sqrt{T}\, \Phi^{-1}(p) + \left(r + \theta\sigma A - \frac{1}{2}\sigma^2 A^2 \right) T. \tag{4.7}$$

If the investor follows the optimal unconstrained strategy then the p-quantile of the investor's terminal wealth $X^\star(T; x_0, g, \infty)$ is

$$\mathcal{Q}_p(X^\star(T; x_0, g, \infty)) = (x_0 + g(0))e^{\beta_p}. \tag{4.8}$$

If the investor follows the optimal constrained strategy with a target wealth K then the p-quantile of the investor's terminal wealth $X^\star(T; x_0, g, K)$ is

$$\mathcal{Q}_p(X^\star(T; x_0, g, K)) = \min \left\{K, (z_0 + g(0))e^{\beta_p}\right\}, \tag{4.9}$$

in which z_0 satisfies equation (4.6).

Proof. Suppose first that the investor follows the optimal unconstrained strategy, resulting in a terminal wealth $X^\star(T; x_0, g, \infty) = (x_0 + g(0))\, Z(T)$. As there is no probability mass at the terminal time for the unconstrained strategy,

$$p = \mathbb{P}\left[(x_0 + g(0))\, Z(T) \leq \mathcal{Q}_p(X^\star(T; x_0, g, \infty))\right].$$

Substituting for $Z(T)$ from equation (3.1), using the fact that $W(T)/\sqrt{T} \sim \mathcal{N}(0, 1)$ under $\mathbb{P}$, we obtain the desired expression (4.8).

Next suppose that the investor follows the optimal constrained strategy, resulting in a terminal wealth $X^\star(T; x_0, g, K)$. We determine the value of

$$\mathcal{Q}_p(X^\star(T; x_0, g, K)) = \inf\left\{y \in \mathbb{R} : \mathbb{P}\left[X^\star(T; x_0, g, K) \leq y\right] \geq p\right\}.$$

It is useful to consider another investor who has the initial wealth z_0, with z_0 satisfying equation (4.6), savings plan g and follows the optimal unconstrained strategy. The wealth at time T of the second investor is

$$X^\star(T; z_0, g, \infty) = (z_0 + g(0))Z(T) - g(T) = Y(T).$$

Thus the terminal wealth of the constrained investor is related to that of the second unconstrained investor by

$$X^\star(T; x_0, g, K) = \begin{cases} X^\star(T; z_0, g, \infty) & \text{if } X^\star(T; z_0, g, \infty) \leq K \\ K & \text{if } X^\star(T; z_0, g, \infty) > K. \end{cases}$$

The desired expression (4.9) follows by consideration of the last expression. □

Define p_K to be the solution to $\mathcal{Q}_{p_K}(X^\star(T; x_0, g, K)) = K$. Then by Lemma 4.9,

$$\frac{\mathcal{Q}_p(X^\star(T; x_0, g, K))}{\mathcal{Q}_p(X^\star(T; x_0, g, \infty))} = \frac{z_0 + g(0)}{x_0 + g(0)}, \quad \text{for } p \in (0, p_K].$$

Furthermore, as $x_0 = z_0 - c(0, z_0 + g(0)) \leq z_0$, the quantiles of the constrained terminal wealth exceed those of the unconstrained terminal wealth, up to the point at which the target wealth K is attained by the constrained terminal wealth (see Leshno and Levy 2002 for the related concept of "almost stochastic dominance"), i.e.

$$\mathcal{Q}_p(X^\star(T; x_0, g, K)) \geq \mathcal{Q}_p(X^\star(T; x_0, g, \infty)), \quad \text{for } p \in (0, p_K].$$

The uplift $\frac{z_0+g(0)}{x_0+g(0)}$ in the lower quantiles represent the reduction in the "downside" risk obtained by giving up the possibility of a terminal wealth above the the target K.

Remark 4.10. The uplift $\frac{z_0+g(0)}{x_0+g(0)}$ decreases as the target wealth K gets larger. This is because the shadow initial wealth z_0 is a decreasing function of the target wealth K, which can be seen by differentiating equation (4.6) with respect to K to find

$$\frac{\partial z_0}{\partial K} = -\frac{e^{-rT}\Phi\left(d_-(0, z_0 + g(0))\right)}{\Phi\left(-d_+(0, z_0 + g(0))\right)} < 0. \tag{4.10}$$

5 Determining the target wealth K that maximises a chosen p-quantile

Here we look at choosing the target wealth in order to maximise a p-quantile of the constrained terminal wealth, for a chosen value of $p \in (0, 1)$. The impact of this maximal target wealth can be significant. For example, in the numerical illustrations, we find that choosing the target wealth that maximises the median terminal wealth gives an increase in the median annual return of around 0.5% per annum over that for the unconstrained strategy.

Proposition 5.1. *Fix $p \in (0, 1)$. The target wealth K_p that maximises the p-quantile for the constrained terminal wealth is*

$$K_p = \frac{x_0 + g(0)}{e^{-\beta_p}\left(1 - \Phi\left(\eta_p\right)\right) + e^{-rT}\Phi\left(\eta_p - \sigma A\sqrt{T}\right)}, \tag{5.1}$$

with the constant β_p given by equation (4.7) and the constant $\eta_p := -\Phi^{-1}(p) + (\sigma A - \theta)\sqrt{T}$.

Proof. As we are varying the target wealth in the proof, for the target wealth K denote the p-quantile for the constrained strategy by $y(G)$ and use $z_0(K)$ to represent the shadow initial wealth.

From equation (4.9),

$$y(K) = \min\left\{K, (z_0(K) + g(0))e^{\beta_p}\right\}.$$

The target wealth K_p given by (5.1) is the value of K at which the two terms in the above minimum are equal: set $z_0(K) + g(0) = Ke^{-\beta_p}$ in equation (4.6) and solve to find the explicit expression for K_p. Then

$$y(K_p) = K_p = (z_0(K_p) + g(0))e^{\beta_p}. \tag{5.2}$$

We show that $y(K)$ is maximised at $K = K_p$ by showing that $y(K) < y(K_p)$ for all $K \neq K_p$.

First suppose that $K < K_p$. As noted in Remark 4.10, the shadow initial wealth $z_0(K)$ is a decreasing function of the target wealth K. Thus $K < K_p$ implies that $z_0(K) > z_0(K_p)$. Hence

$$(z_0(K) + g(0))e^{\beta_p} > (z_0(K_p) + g(0))e^{\beta_p} = K_p > K.$$

Thus for the quantile

$$y(K) = \min\left\{K, z_0(K)e^{\beta_p}\right\} = K < K_p = y(K_p).$$

Next suppose that $K > K_p$. Following the same argument as above results in

$$(z_0(K) + g(0))e^{\beta_p} < K_p < K.$$

Hence for the quantile

$$y(K) = \min\left\{K, (z_0(K) + g(0))e^{\beta_p}\right\} = (z_0(K) + g(0))e^{\beta_p} < K_p = y(K_p).$$

We have now shown that the maximum p-quantile for the constrained terminal wealth occurs at K_p. $\square$

6 Numerical illustration

Here we investigate the optimal strategy for the constrained strategy. We use the unconstrained strategy as the benchmark strategy. We set the unit time period to be one year and fix the parameter values:

$$r = 0, \quad \mu = 0.0343, \quad \sigma = 0.1544, \quad A = 1, \quad T = 30, \quad g \equiv 0, \quad x_0 = 300.$$

The target wealth K is varied according to the illustration. Note that the choice of the parameters implies that the investor's risk aversion constant is $\gamma = -0.44$.

Figure 1 shows four sample paths of the proportion of wealth invested in the risky stock for an investor who follows the optimal constrained strategy with a target wealth of $K = 1038.57$. The impact of the target wealth constraint is seen in Figures 1(a)-1(b), as the amount in the risky asset declines to zero before the terminal time.

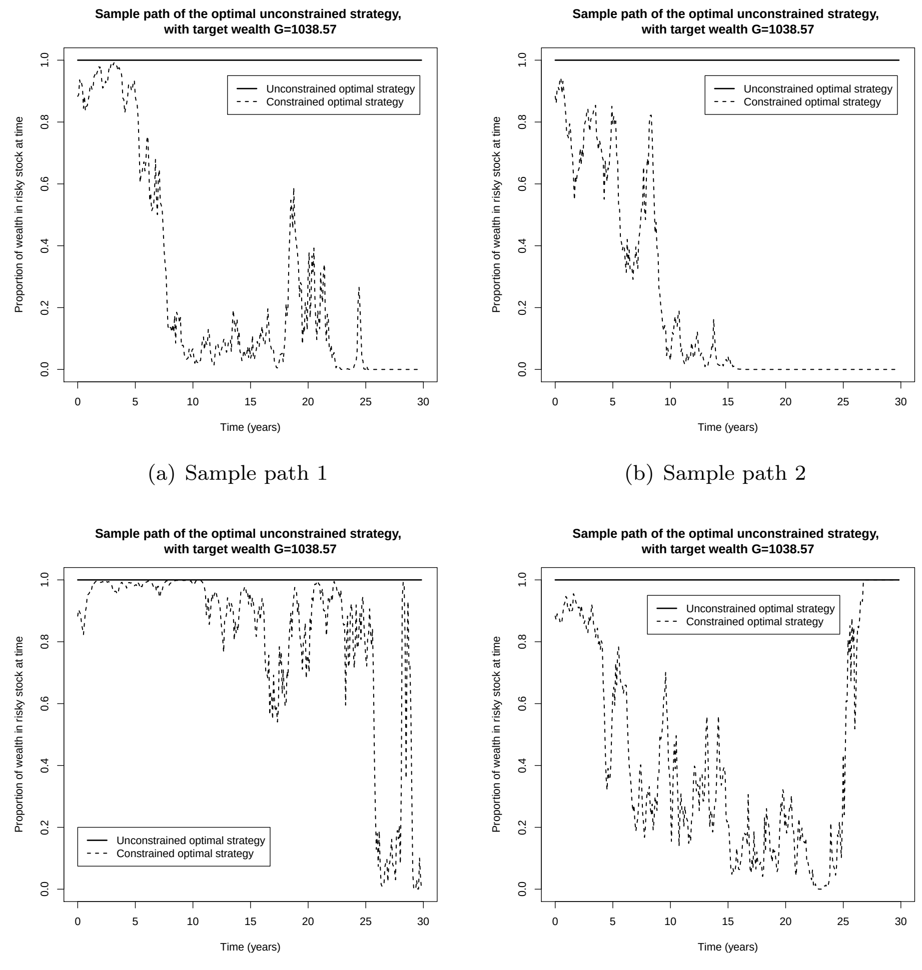
Table 1 shows part of the distribution of the terminal wealth under the constrained strategy for various choices of the target wealth. The distribution of the terminal wealth under the unconstrained strategy is also shown. The target wealths have been chosen to correspond to the 50%, 75% and 95% quantiles of the unconstrained strategy.

As the target wealth becomes larger, the probability of achieving it at the terminal time declines. Similarly, the degree by which the quantiles of the constrained terminal wealth exceed those of the unconstrained terminal wealth decreases as the target wealth increases. However, the uplift in the quantiles can be significant: it is 23% for the target wealth $K = 587.10$.

Table 1: Table showing the quantiles of the distribution of the terminal wealth for the unconstrained strategy and the constrained strategy, for various choices of the target wealth K.

p	**Unconstrained p-quantile** $\mathcal{Q}_p(X^\star(30; 300, 0, \infty))$	**Constrained p-quantile** $\mathcal{Q}_p(X^\star(30; 300, 0, K))$ **for** $K = 587.10$	$K = 1038.57$	$K = 2359.53$
5%	146.08	179.48	154.12	146.88
25%	331.88	407.77	350.14	333.69
50%	587.10	587.10	616.40	590.30
75%	1038.57	587.10	1038.57	1044.23
95%	2359.53	587.10	1038.57	2359.53
$\mathbb{P}[X^\star(30; 300, 0, K) = K]$	N/A	59.62%	27.05%	5.07%
z_0	N/A	368.59	316.50	301.63
Quantile uplift $z_0/300$	N/A	122.86%	105.50%	100.55%

The optimal constrained strategy results in a dynamic asset allocation, which changes as the price of the risky stock fluctuates (equation (2.1) gives the dynamics of the risky stock price S_1). Figure 2(a) shows the proportion of wealth invested in the risky stock plotted as a function of the risky stock price, at two different times. For low values of the stock price, the investor's wealth is below the target wealth and the optimal unconstrained strategy is to invest a high proportion of the investor's wealth in the risky stock. As the stock price increases,



(a) Sample path 1

(b) Sample path 2

(c) Sample path 3

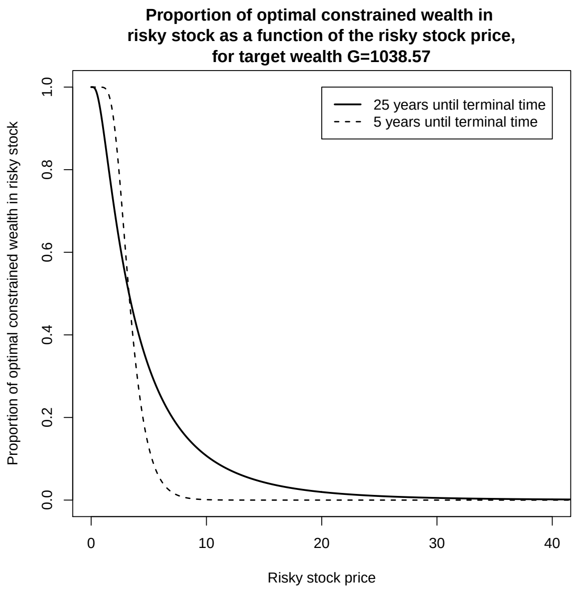
(d) Sample path 4

Figure 1: Sample paths of the proportion of wealth invested in the risky stock when following the optimal constrained strategy with a target wealth of $K = 1038.57$. The optimal unconstrained strategy, shown by the solid line in each figure, is to invest all of the wealth in the risky stock at all times. The sample paths are generated at monthly time intervals.

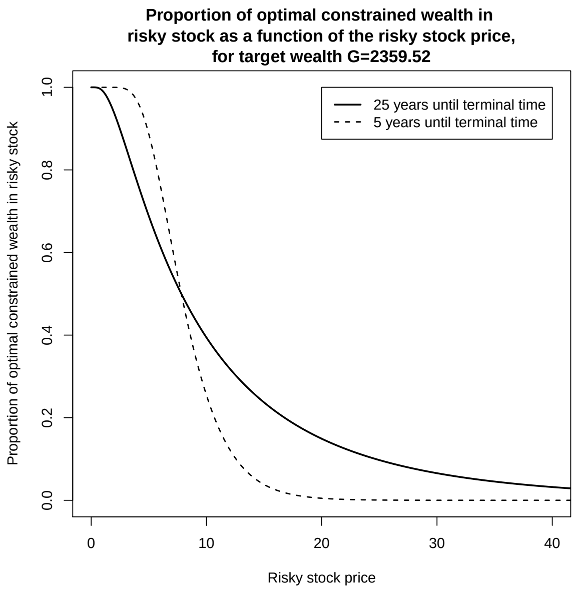
it is more likely that the investor's wealth achieves the target wealth, and so the proportion of wealth in the risky stock declines. These effects are magnified as the time remaining until the terminal time decreases. For example, if the price of the risky stock is low near to the terminal time, a greater proportion of the investor's wealth is invested in the risky stock in order to have a chance of attaining the target wealth.

Figure 2(b) shows the same plot but for a higher target wealth. In this case, a higher proportion of the investor's wealth is invested in the risky asset in order to attempt to achieve the higher target wealth.

As shown in Proposition 5.1, the target wealth can be chosen to maximize the



(a) $K = 1038.57$



(b) $K = 2359.53$

Figure 2: Proportion of wealth in the risky stock plotted as a function of the risky stock price, at two different times from maturity. Each figure is plotted for a different target wealth value.

value of a p-quantile of the constrained terminal wealth, for any chosen $p \in (0, 1)$. Table 2 shows the maximal target wealths for a selection of values of p. The corresponding p-quantiles of the terminal wealth for an investor who follows the

Table 2: Table illustrating how the maximal choice of the target wealth K can provide a higher equivalent annual return at a desired p-quantile. Note that a maximised target wealth K_p is equal to the p-quantile of the constrained terminal wealth. The maximal target wealths are calculated from equation (5.1).

p	**Unconstrained p-quantile** $\mathcal{Q}_p(X^\star(30; 300, 0, \infty))$	**Equivalent return on unconstrained p-quantile**	**Maximal target wealth** K_p	**Equivalent return on constrained p-quantile**
5%	146.08	-2.40% p.a.	343.29	0.45% p.a.
25%	331.88	0.03% p.a.	470.20	1.50% p.a.
50%	587.10	2.24% p.a.	679.83	2.73% p.a.
75%	1038.57	4.14% p.a.	1089.17	4.30% p.a.
95%	2359.53	6.87% p.a.	2372.17	6.89% p.a.

optimal unconstrained strategy are shown for comparison. Also shown – to ease the interpretation of the wealth values – is the equivalent annual continuously-compounded return: for a terminal wealth value $x > 0$, and assuming no future savings, this is calculated as $\frac{1}{T}\ln\left(\frac{x}{x_0}\right)$.

For example, an investor who wishes to maximize their median constrained terminal wealth (i.e. 50%-quantile) would choose the target wealth $K_{0.5} = 679.83$. This means that the investor has probability 50% ($= 1 - p$) of attaining a terminal wealth 679.83 by following the optimal constrained strategy (equivalent to an annual return of 2.73%). By comparison, the median unconstrained terminal wealth is the lower value of 587.10 (equivalent to an annual return of 2.24%). The difference in annual returns between the median constrained and unconstrained terminal wealth values is 0.49%, averaged over the 30–year time horizon of the investor.

Note that the uplift in the lower quantiles, i.e. those quantiles below the target, are not maximised by K_p. As discussed in Remark 4.10, the uplift decreases as the target wealth is increased. That means, for example, that the uplift in the lower quantiles for the optimal target wealth $K_{0.5} = 679.83$ is less than that for the target wealth $K = 587.10$ (in fact, the uplift is 115.80% for the former and 122.86% for the latter).

7 Conclusion and discussion

Investors face a number of possible strategies to maximize wealth at retirement. These investment strategies have traditionally been dominated by the maximization of expected returns, with some control on the risk when approaching the retirement age. Our method addresses the control of risk in the terminal wealth distribution from the start of the savings period and finds an optimal strategy which is advantageous compared to an unconstrained plan.

We intend to extend our results to dynamic strategies that can be implemented in practice and we also propose to explore strategies that allow only positive liabilities.

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