Entropy, Longevity and the Cost of Annuities

Steven Haberman, Marwa Khalaf-Allah and Richard Verrall

1. Introduction

Many demographers have analysed the changes in life expectancy over time. For example, Pollard (1982, 1988), Arriaga (1984), and Pressat (1985) have focused on discrete differences in life expectancy at two moments in time. In contrast, Keyfitz (1985) introduced the concept of entropy, and analysed this in the context of continuous changes in life expectancy. In particular, he related the entropy of a survival model to the time-derivative of life expectancy and used the entropy measure as an index to measure the effect of proportional change in the force of mortality on life expectancy. Thus, a higher level of entropy indicates that the life expectancy has a greater propensity to respond to a change in the force of mortality than a lower level of entropy. So a high level of entropy means that further reductions in mortality rates would have an impact on measures like life expectancy – conversely, a low level of entropy level means that life expectancy would be relatively "immune" to further reductions in mortality rates.

This approach taken by Keyfitz (1985) extends the definition of the entropy of a population that had been derived and applied in population biology by Demetrius (1976) to measure the variability of the contribution of the different age classes to the stationary age distribution. The concept of entropy has also been advocated as a measure of the rectangularization of the life table by a number of researchers – for example Goldman and Lord (1986), Nagnur (1986) and Nusselder and Mackenbach (1996).

In this paper, we extend the Keyfitz definition of entropy in order to measure the effect of any changes in the force of mortality on the discounted cost of a life annuity. We also investigate the properties of the measure of entropy for different scenarios of interest rates and levels of mortality improvements. One of the advantages of the concept of entropy is that it allows the sensitivity of the cost of a life annuity contract to changes in longevity to be summarized in a single figure index.

We focus initially on using period life tables to calculate both the expectation of life and annuity values. Calculated in this way, the entropy is a useful summary measure and has an intuitive interpretation, providing an objective means of comparison of trends between populations. It should be noted that the interpretation requires care since period-based expectations of life are calculated using a set of age-specific mortality rates for a given period, with no allowance for any future changes in mortality. We also consider cohort based expectations of life and annuity values which are calculated using a set of age specific mortality rates which allow for known or forecast changes in mortality rates at future ages (in future years). Cohort based indices may constitute a more theoretically satisfactory concept but they require forecasted mortality rates and as such make heavy data demands and are subject to substantial model risk and forecasting error (Pitacco et al 2009, Booth & Tickle 2008). Other examples of the use of entropy in this context can be found in Khalaf-Allah (2007).

The paper is organized as follows. In section 2, we consider the definition of entropy in demography and how it can be extended to measure the effect of proportional change in the force of mortality on the cost of life annuity. In section 3, numerical values for the entropy measure for life annuities over the whole age range are obtained and analysed for different interest rates using English life tables over the period from 1851 to 1991 for both males and females. In section 4, numerical values for the entropy measure for life annuities are derived using different mathematical models for mortality projection for both males and females as an attempt to gain a better understanding regarding the properties of the entropy measure. Section 5 deals with testing the sensitivity of the results with regard to the different factors that are likely to affect the value of the entropy measure: we focus on the effect of gender, assumed interest rate and the level of mortality improvement. In section 6, we provide some overall conclusions.

2. Entropy and Mortality

This section considers the formula for the entropy measure (*H*) for life expectancy at birth as obtained by Keyfitz (1985), and then derives a corresponding formula in the case of a life annuity. Section 2.1 contains a summary of the theory from Keyfitz (1985), and section 2.2 derives the new entropy measure for annuities.

2.1 Entropy and life expectancy at birth

Entropy as defined by Demetrius (1976) is a single figure index used to measure the effect on life expectancy at birth of a proportional change in the force of mortality over the whole age range. Suppose that the force of mortality μ_x at age *x* is multiplied by $(1+\varphi)$, where φ is a constant change in the force of mortality at all ages, and could be positive or negative. Then $\mu_x^* = \mu_x \times (1+\varphi)$ and the new probability of surviving till age *x* becomes

$$ {}_x p_0^* = \exp\left[-\int_0^x \mu_a^* da\right] = \exp\left[-\int_0^x (1+\varphi)\,\mu_a da\right] = \exp\left[-\int_0^x \mu_a da\right]^{1+\varphi} = \left({}_x p_0\right)^{1+\varphi} \qquad (1)$$

where ${}_x p_0 = \exp\left[-\int_0^x \mu_a da\right]$.

The new (period) expectation of life is

$$ \overset{o}{e}{}_0^* = \int_0^\omega \left({}_a p_0\right)^{1+\varphi} da \qquad (2)$$

where ω is the maximum age in the life table. In order to find the effect of small changes in φ on the (period) expectation of life, we consider the derivative of equation (2) with respect to φ:

$$ \frac{d\,\overset{o}{e}{}_0^*}{d\varphi} = \int_0^\omega \ln\left({}_a p_0\right)^{1+\varphi} \left({}_a p_0\right)^{1+\varphi} da \qquad (3)$$

The quantity (3) cannot be positive, since ${}_a p_0$ cannot be greater than 1 and ${}_a p_0^{1+\varphi}$ is always positive. In the neighbourhood of $\varphi = 0$ we have the following approximation based on a Taylor expansion:

$$\frac{\Delta \overset{o}{e}_0}{\overset{o}{e}_0} \cong \frac{\int_0^{\omega} \left[\ln {}_a p_0\right] {}_a p_0 da}{\int_0^{\omega} {}_a p_0 da} \varphi = -H\varphi . \tag{4}$$

The quantity *H* is known as the entropy measure:

$$H = -\frac{\int_0^{\omega} \left[\ln {}_a p_0\right] {}_a p_0 da}{\int_0^{\omega} {}_a p_0 da} .$$

It should be noted that *H* is a positive quantity, since the ratio of the integrals is always negative. *H* is effectively a weighted average of values of $\ln {}_a p_0$ as age ranges from 0 to ω and, as such, has no upper bound. As mortality rates improve over time, we would expect a larger fraction of deaths to occur at older ages, and it can be shown that this leads to a drop in the value of *H* so that it becomes closer to 0. Indeed, *H* would be 0 if all mortality were concentrated at a single age, and hence entropy can be regarded as an index of rectagularisation (Nusselder and Mackenbach, 1996) and it can also be interpreted as a measure of the heterogeneity of the population with respect to mortality at different ages: see Keyfitz (1985), Vaupel (1986).

Analyses in the literature (see, for example, Vaupel, 1986) point out the inverse relationship between the period expectation of life and the entropy measure *H.* As the force of mortality falls and life expectancy rises, it is observed that there is an upwards shift in the ages where further reductions in mortality would be most effective in increasing life expectancy (Vaupel, 1985, calculates that for the US, these ages have moved from 20-25 in 1900 to 80-85 as projected for 2000). This leads to life expectancy being less sensitive to further changes in the force of mortality, and so we would expect the entropy measure *H* to decrease over time as mortality improves. We will explore this relationship further in later sections.

The theory outlined in this section has considered the concept of entropy in relation to the expectation of life. We now show how to extend this in order to find the corresponding entropy value for annuities, also calculated on a period basis.

2.2 Entropy and Annuities

With the same assumption of a proportionate increase in the force of mortality as considered in section 2.1, the new value of a (period) life annuity at age *x* becomes

$$\bar{a}_x^* = \int_0^{\infty} \left({}_t p_x\right)^{1+\varphi} \exp\left[-\delta t\right] dt \tag{5}$$

where δ is the force of interest.

The effect of a small change in φ on the value of a life annuity at age *x* can be derived in the same manner as in section 2.1. Firstly, the derivative is

$$\frac{d}{d\varphi}\bar{a}_x^* = \int_0^\infty \left(\ln {}_t p_x\right)\left({}_t p_x\right)^{1+\varphi} \exp\left[-\delta t\right] dt \qquad (6)$$

and using a Taylor expansion, it is straightforward to show that

$$\frac{\Delta \bar{a}_x}{\bar{a}_x} = -H_x\left(\delta\right)\varphi$$

where $$H_x\left(\delta\right) = \frac{-\int_0^\infty \left(\ln {}_t p_x\right)\left({}_t p_x\right)\exp\left[-\delta t\right] dt}{\int_0^\infty {}_t p_x \exp\left[-\delta t\right] dt}$$

Here $H_x\left(\delta\right)$ is the entropy measure for an annuity value and can be thought of as minus the weighted average value of $\ln {}_t p_x$, weighted by ${}_t p_x \exp\left[-\delta t\right]$. Again, $H_x(\delta)$ is a positive quantity because the ratio of the integrals in (7) is always negative. As for the entropy measure in section 2.1, when mortality rates improve, we would expect a larger fraction of deaths to occur at older ages, resulting in a drop in the value of $H_x(\delta)$ so that it becomes closer to 0. As before, *H* would be 0 if all mortality were concentrated at a single age. In the case of improving mortality rates, the value of $H_x(\delta)$ increases, but the pace by which this happens is dependent on the concavity of the survivorship curve of the population in question. Also, as the value of δ increases, $H_x(\delta)$ is expected to decrease, as any change of the force of mortality is expected to have a lower impact on the cost of a life annuity at higher interest rates.

Using integration by parts,

$$H_x(\delta) = \frac{-\int_0^\infty \mu_{x+s}\ {}_s p_x e^{-\delta s}\bar{a}_{x+s} ds}{\bar{a}_x} \quad . \qquad (8)$$

Since ${}_s p_x \mu_{x+s}$ is the probability density function for the time of death for an individual aged x, equation (8) indicates why $H_x\left(\delta\right)$ is a measure of heterogeneity (as in the case when $\delta = 0$).

The theory developed in this section allows us to find the effect on the value of an annuity of a uniform proportional increase or decrease in the force of mortality over the whole age range. However, such a uniform change is unlikely to occur in practice as mortality rates have tended to change in different proportions at different ages. One way to deal with this problem would be to compute the uniform proportional change in the force of mortality that will have the same effect on the value of a life annuity as a set of step-wise changes at different ages caused by observed trends in mortality, so that the above theory could be used in a wider range of applications.

3 Entropy values using English Life Tables

In this section, the value of the entropy measure $H_x(\delta)$ is calculated using English Life mortality tables over the period from 1851 to 1991. This allows us to examine how the measure has behaved over a period where mortality rates have been decreasing. The measure is calculated for both males and females at different rates of interest (0%, 2%, 4%, 6% and 8%). Since the main area of interest for the application of the concept of entropy for annuities is for people over 60, the age ranges 60 – 110 and 70 – 110 are considered, as well as the whole age range (0 – 110). Hence we calculate $H_0(\delta)$, $H_{60}(\delta)$ and $H_{70}(\delta)$.

In order to calculate values for $H_x(\delta)$, it is necessary to evaluate the ratio of integrals in equation (7). We adopt a numerical approach proposed by Pollard (1982) and used, for example , by Khalaf-Allah et al (2006).

Let ${}_tQ_x = \int_0^t \mu_{x+u}\,du$ and ${}_tE_x = {}_tp_x \exp(-\delta t)$.

Then ${}_tQ_x$ can be calculated from:

$${}_tQ_x = -\ln\left(\frac{l_{x+t}}{l_x}\right) = -\ln {}_tp_x \quad .$$

Using the mean value theorem for integrals, the integrals in (7) can be replaced by sums of one-year integrals, leading to the following approximation:

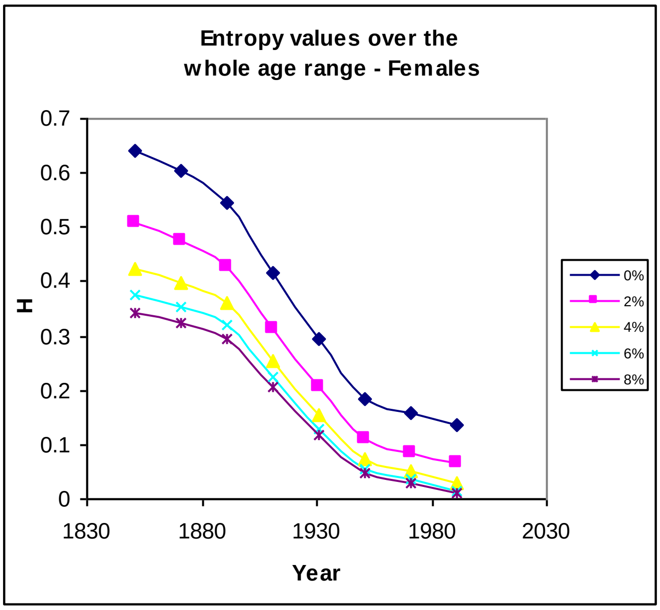
$$H_x(\delta) \approx \frac{\sum_{t=0}^{\omega} {}_{t+\frac{1}{2}}Q_x \; {}_{t+\frac{1}{2}}E_x}{\sum_{t=0}^{\omega} {}_{t+\frac{1}{2}}E_x} \qquad (9)$$

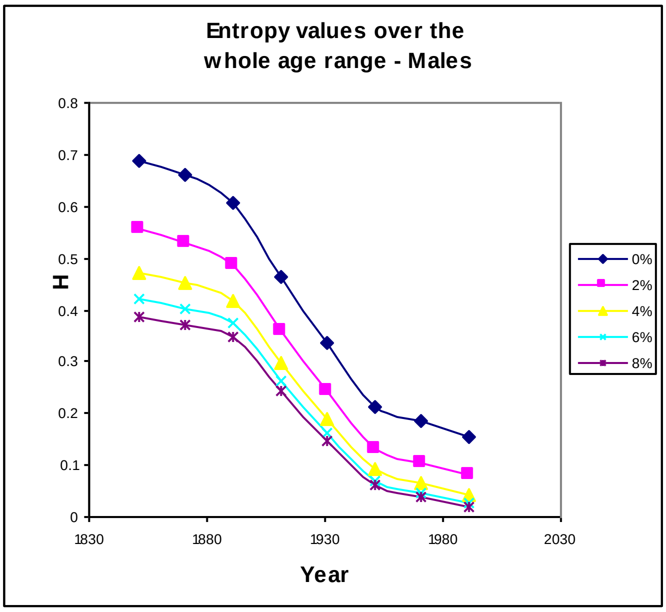
where ω is the maximum age in the tables. Equation (9) can be used directly to calculate the values of $H_x(\delta)$ for different values of *x*.

3.1 Numerical results for $H_0(\delta)$

We first consider the values of the entropy measure, $H_x(\delta)$, over the whole age range, $H_0(\delta)$. A number of different interest rates are considered, using English life tables over the period 1851-1991. The results are shown in figure 1. Tables of these values are contained in the Appendix. It can be seen that, as mortality rates have decreased, the value of $H_0(\delta)$ has decreased. This reflects the fact that any change in the force of mortality in a low mortality environment will have a smaller effect on the expectation of life and the cost of a life annuity as compared to the corresponding effect in a high mortality environment. Also, as expected, the effect is of lesser importance at higher rates of interest since the value of $H_0(\delta)$ decreases when the rate of interest increases. For each level of interest rates and each year, the value of $H_0(\delta)$ is lower for females than males, corresponding to the higher expectations of life for females in each year.

<u>Figure 1:</u> Entropy values for different English life tables at different rates of interest for both females and males at age 0





It can also be seen that the changes in the values of $H_0(\delta)$ fall into three distinct periods, for both males and females. In the earlier years, at higher levels of mortality, any change in the force of mortality does not markedly affect the entropy measure (the slope of the curves is not that steep). Moreover, this effect decreases with the increase in the interest rate so that, at high interest rates, the curve looks almost horizontal. The same applies in later years, when mortality is at a lower level. However, over the middle period, any change in the force of mortality has a stronger impact on the value of the entropy measure. We can express this feature by dividing the curve into three sections - one on the far left where the mortality level is high (A) , one on the far right where the mortality level is low (C) and the section in the middle where mortality is neither high or low (B). For both sections A and C the slope of the curve shows that the value of $H_0(\delta)$ is inelastic and changes in the force of mortality do not have a great effect on the value of $H_0(\delta)$. However, in section B the slope is steeper reflecting the fact that the value of *H* is more elastic, and changes in the force of mortality have a strong effect on the value of the entropy measure.

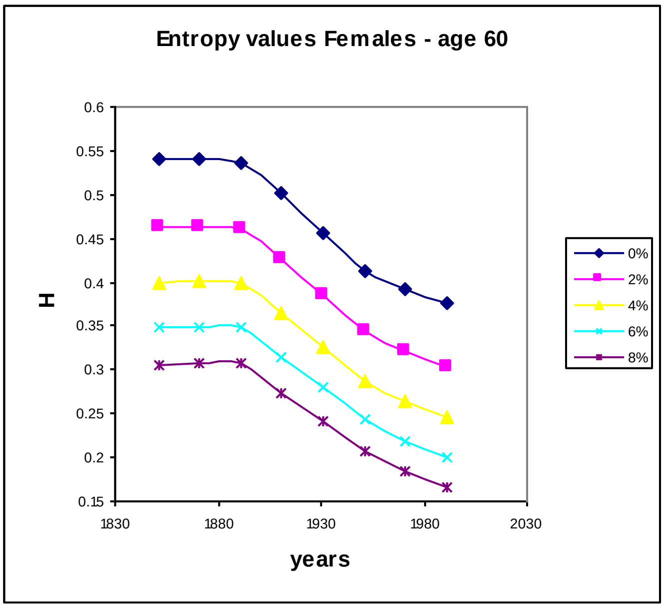
3.2 Numerical results for $H_x(\delta)$ at older ages

Since the main area of interest is likely to be the effect of any change of the force of mortality on the cost of a life annuity at age 60 and 70 at different rates of interest, this section considers the values for $H_x(\delta)$ at ages 60 and 70, for both males and females. Using the same data and interest rates as before, Figure 2 shows the values of the entropy measure $H_{60}(\delta)$ for females and males. The actual values are shown in the Appendix.

For females, it can be seen that the values of $H_{60}(\delta)$ are higher than the corresponding ones for females at age 0, $H_0(\delta)$, in section 3.1. This reflects the greater effect of a change in the force of mortality on the expectation of life and the cost of a life annuity in the higher age group (where mortality rates are higher). Again, the higher the interest rate the lower is the effect of changes in the force of mortality on the cost of life annuity, and hence the lower are the values of $H_{60}(\delta)$. It is expected that the value of $H_{60}(\delta)$ should decrease when levels of mortality are lower so that, for more recent life tables, the values of $H_{60}(\delta)$ should be lower at all rates of interest. This is true except for year 1871, where the values of *H* at all interest rates were slightly higher than the corresponding ones calculated using for year 1851. Overall, the results are consistent with those in section 3.1.

Similar features can be seen in the results for males. The values of $H_{60}(\delta)$ are higher than the values of $H_0(\delta)$, and they are also higher than the values for females. As before, the higher the interest rate, the lower is the effect of changes in the force of mortality on the cost of life annuity, and the lower are the values of $H_{60}(\delta)$. We expect the value of $H_{60}(\delta)$ to decrease when levels of mortality are lower so that, for more recent years the values of $H_{60}(\delta)$ should be lower at all rates of interest. There are a few anomalies in the results for males. The values of *H* have increased slightly from 1851 to 1871 and again from 1871 to 1891 for all rates of interest after which they decrease with the improvements in mortality. Further, at a rate of interest of 0%, $H_{60}(\delta)$ has increased from 1951 to 1971 and, at some low interest rates, the decrease in $H_{60}(\delta)$ from 1951 to 1971 has been quite small. Further, analysis shows that this latter feature is matched by a decrease in the modal age at death as we move from 1951 to 1971, and only a small increase in the expectation of life over this 20 year period.

<u>Figure 2:</u> Entropy values for different English life tables at different rates of interest for both females and males at age 60



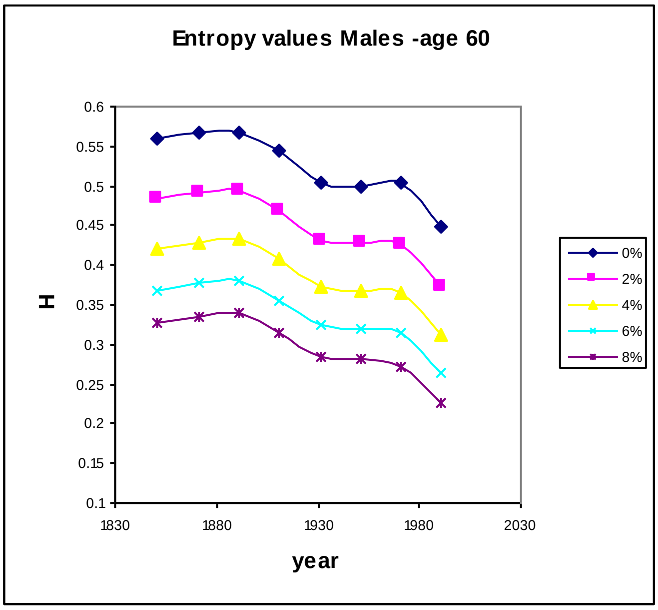
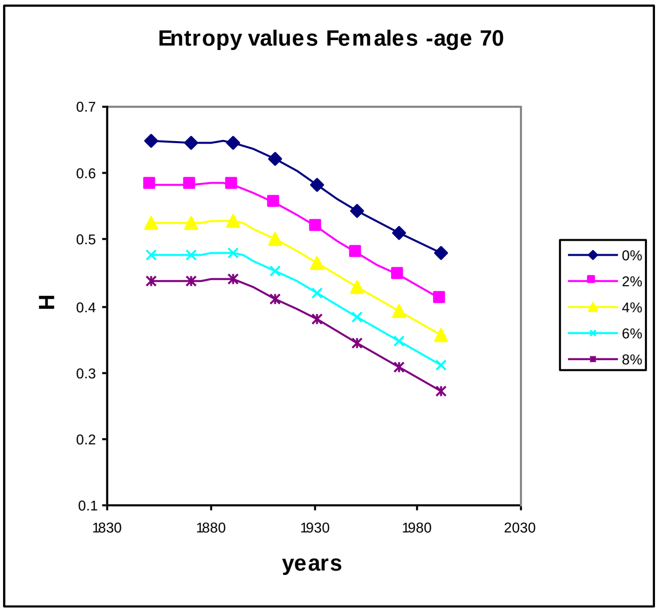


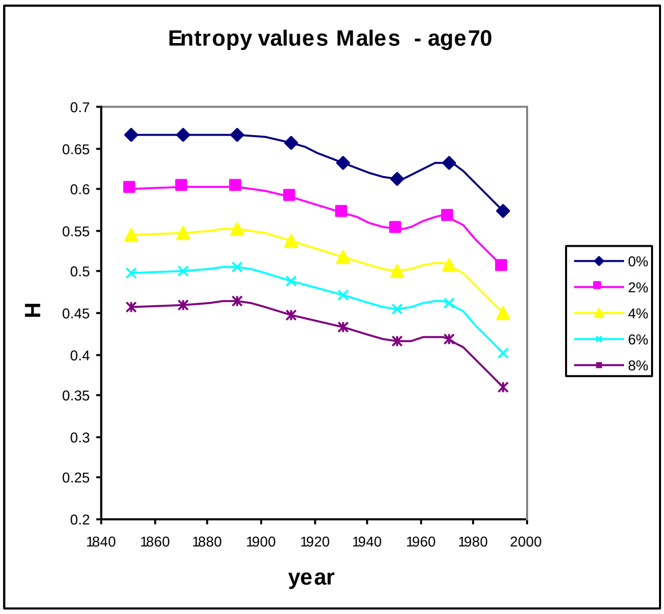
Figure 2 shows that the value of $H_{60}(\delta)$ does not always decrease when mortality improves. In Figure 1, $H_0(\delta)$ decreases as mortality improves, but very slowly at both very high and very low levels of mortality to the extent that that the curve looks almost horizontal. This feature is more marked in Figure 2 - for females at age 60, at a high level of mortality, a considerable reduction in mortality is needed to decrease the value of $H_x(\delta)$. Although the mortality rates for 1871 are lower than for 1851, the difference is not sufficient to decrease the value of $H_x(\delta)$. When the reduction in mortality becomes more considerable, the value of $H_x(\delta)$ starts to decrease. The same happens in the case of males at age 60 at both very high and very low levels of mortality, where improvements in mortality rates are not sufficient to be translated into a material decrease in the value of $H_x(\delta)$.

Figure 3 shows that values of $H_{70}(\delta)$ for both females and males. For females, at high levels of mortality, the value of $H_{70}(\delta)$ has increased as mortality has improved, for 1871 and 1891. It can be seen that the decreases in $H_{70}(\delta)$ occurs later as mortality improves, compared with $H_{60}(\delta)$. Also, the values of $H_{70}(\delta)$ are higher than the values of $H_{60}(\delta)$ and $H_0(\delta)$ at all interest rates.

The results for males are similar but we note that, as at age 60, an anomaly occurs as we move from 1951 to 1971: the value of $H_{70}(\delta)$ increases for all rates of interest. As noted above, this is linked to the downward shift in the modal age of death in this 20 year period.

<u>Figure 3:</u> Entropy values for different English life tables at different rates of interest for both females and males at age 70





A Gompertz model of the form $\mu_x = \exp(b+cx)$ is often used as the basis for the progression of mortality rates, particularly over ages (approximately) 50-90. Although other models may be more appropriate for modelling particular features of mortality rates (for example, at the oldest ages), the Gompertz model has the advantage of simplicity and familiarity and we use it to illustrate the application of entropy to annuity values. We consider the simple case of exponentially decreasing rates over time so that

$$\mu_{x+u}^{t} = \mu_{x+u}^{0} RF\left(t\right) \tag{10}$$

where μ_{x+u}^{0} is the mortality rate in the base year. The reduction factor for the force of mortality, $RF(t)$, is assumed to take the simple form, $\exp\left[-\alpha t\right]$. The behaviour of the period-based expectation of life, $\overset{o}{e}_x\left(t\right)$, and life annuity values as functions of time can then be explored. As discussed in Vaupel (1986), it is straightforward to demonstrate that

$$\frac{\partial}{\partial t}\overset{o}{e}_x\left(t\right) \approx \frac{\alpha}{c}\left[1-\mu_x^t\,\overset{o}{e}_x\left(t\right)\right] \tag{11}$$

so that, in the long run,

$$\frac{\partial}{\partial t}\overset{o}{e}_x\left(t\right) \approx \frac{\alpha}{c}$$

and is roughly constant. Thus, the rate of change over time in the expectation of life eventually reaches a stable level (and the same applies to annuity values within the constraints of the above model). This underlying tendency is one of the driving forces behind the changes that have been demonstrated in $H_x\left(\delta\right)$ from the historical perspective of the English life table calculations depicted in Figures 2 and 3.

4 Properties of entropy using a mathematical model for mortality and allowing for mortality improvements

In this section, the values of the entropy measure are calculated when mortality is assumed to follow a mathematical model instead of using published life tables. In this approach, we do not depend directly on observational data and this facilitates our adopting a cohort perspective rather than a period perspective. The approach allows us better to examine the different properties of entropy and how its value is affected by different factors. In section 4.1, mortality is assumed to follow the Gompertz model and improvements in mortality are allowed for using a reduction factor. For simplicity, it is assumed that the reduction factor depends only on the time, *t*, from the base year, *b*. In this context, the Continuous Mortality Investigation Bureau has highlighted the importance of the rate of improvement in mortality as well as the advantages of simple models (CMI 2009). In section 4.2, a more sophisticated model is used to model mortality improvements, based on Sithole et al (2000).

4.1 Gompertz law assumption

We assume that the force of mortality is such that it follows the Gompertz law, with parameters *b* and *c*, so that the baseline force of mortality (at time 0) at age $x+u$ can be expressed as

$$\mu_{x+u}^{0} = \exp\left[b + c\left(x+u\right)\right].$$

Assuming that (10) holds and that the reduction factor is of the form $\exp\left[-\alpha t\right]$, $\mu_{x+u}^{t} = \mu_{x+u}^{0} e^{-\alpha t}$. Then the probability that a life aged x to survives t years (calculated on a cohort basis) is

$$\begin{aligned} {}_t p_x^* &= \exp\left[-\int_0^t \mu_{x+u}^{u} du\right] \\ &= \exp\left[-\int_0^t \exp\left[b + c\left(x+u\right) - \alpha u\right] du\right] \\ &= \exp\left[-\mu_x^0 \left(\frac{e^{(c-\alpha)t} - 1}{c-\alpha}\right)\right] \qquad (12) \end{aligned}$$

Entropy is now calculated on a cohort basis and it is a function of the parameters *b* and *c*, α and δ, and so we use the notation $H_x\left(b,c,\alpha,\delta\right)$. Hence,

$$H_x\left(b,c,\alpha,\delta\right)=\frac{-\int_0^\infty \ln\left({}_t p_x^*\right){}_t p_x^* e^{-\delta t}dt}{\int_0^\infty {}_t p_x^* e^{-\delta t}dt}$$

$$=\frac{\mu_x^0}{c-\alpha}\left[\frac{\int_0^t \exp\left(-\frac{\mu_x^0}{c-\alpha}\left(e^{(c-\alpha)t}-1\right)\right)e^{(c-\alpha-\delta)t}dt}{\int_0^t \exp\left(-\frac{\mu_x^0}{c-\alpha}\left(e^{(c-\alpha)t}-1\right)\right)e^{-\delta t}dt}-1\right] \tag{13}$$

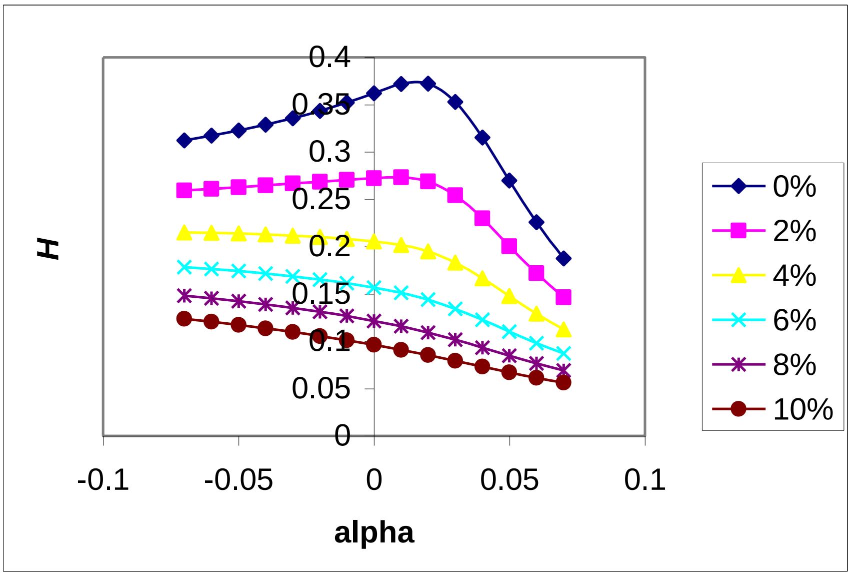
Clearly α is constrained to be less than *c*, otherwise the probability of survival in (12) will be greater than 1. $H_x\left(b,c,\alpha,\delta\right)$ can be calculated for different values of *b, c,* α and δ in order to test the effect of the base table used, the level of mortality improvements and the force of interest.

Each of the integrals in (13) can be written as an incomplete gamma function, by several changes of variables. Alternatively, numerical approximation for integrals can be used to evaluate the integrals, and this has been implemented using Mathematica.

In order to illustrate the behaviour of $H_x\left(b,c,\alpha,\delta\right)$, its value has been calculated using data from the CMI for insured female pensioners at ages 60 and over. The base mortality table used is the (1991-1994) mortality table. In order to perform the calculations, we need to have values for μ_{60}^0, *c,* α and δ. The value for μ_{60}^0 can be taken directly from the base mortality table and the Gompertz model has been fitted in order to estimate *c*. We then test the sensitivity of $H_{60}\left(b,c,\alpha,\delta\right)$ to changes in the values of α and δ. A range of values for α from -0.07 to 0.07, and for δ from 0% to 10% were used. The value for μ_{60}^0 using the 1991-1994 tables for females is 0.00552155 and the estimate for *c* is 0.085, obtained by a simple regression approach. The values of $H_{60}\left(b,c,\alpha,\delta\right)$ for different values of α and δ are shown in Figure 4. Note that positive values of α correspond to improving mortality rates over time and negative values correspond to deteriorating mortality rates over time.

Figure 4 shows that the value of $H_{60}\left(b,c,\alpha,\delta\right)$ decreases as the force of interest, δ, increases, reflecting the decreasing effect of mortality improvement on the cost of life annuities at high levels of interest rates. It can also be seen that at zero or low levels of interest rate, the trend in $H_{60}\left(b,c,\alpha,\delta\right)$ is peaked: the value increases with the improvement in mortality (as α increases) until a level where any further improvement in mortality has a decreasing effect on $H_{60}\left(b,c,\alpha,\delta\right)$ and hence on the expectation of life and the cost of a life annuity. Thereafter, the value of $H_{60}\left(b,c,\alpha,\delta\right)$ starts to decrease as mortality improves exhibiting a similar pattern to that of females aged 60 and 70 in section 3 (based on population life tables). On the other hand, when the rate of interest is high, $H_{60}\left(b,c,\alpha,\delta\right)$ decreases continuously as mortality improves.

<u>Figure 4:</u> Values for entropy when mortality follows the Gompertz law for different rates of interest (0% up to 10%) and different levels of mortality improvement (different values of α)



We can also test the effect on $H_{60}(b,c,\alpha,\delta)$ of the base mortality table, by calculating $H_{60}(b,c,\alpha,\delta)$ for different values of *c*. As an illustration, we explore the effect of using values of 0.08, 0.085 and 0.09, and Figure 5 shows the results for δ = 0% and 3%.

<u>Figure 5:</u> Values for $H_{60}(b,c,\alpha,\delta)$ for different values of c and α

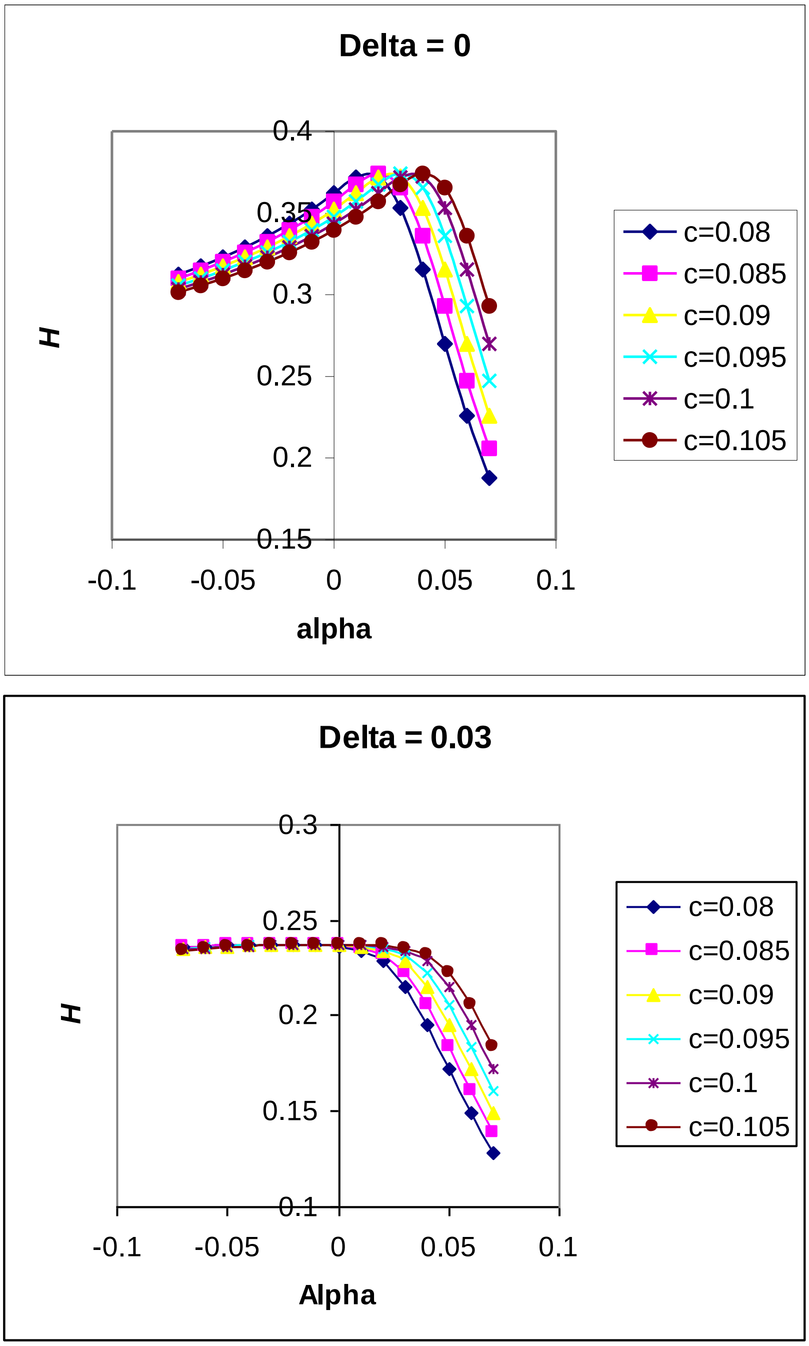


Figure 5 shows that, for the different values of *c* tested, when the force of interest is equal to zero (top panel), the profile of $H_{60}(b,c,\alpha,\delta)$ as a function of α is peaked (as in Figure 4). When *c* = 0.08 (assuming a lower mortality level for the base table), the values of $H_{60}(b,c,\alpha,\delta)$ are higher than the corresponding ones when *c* = 0.09 (assuming a higher mortality level for the base table) when the reduction factor is greater than 1 (α is negative) and the relationship is reversed when mortality rates are improving and the reduction factor is less than one (α is positive). This confirms our earlier conclusion that the expectation of life for the high mortality group (c = 0.09) would be less responsive (and hence have lower values for $H_{60}\left(b,c,\alpha,\delta\right)$) to an increasing trend in the force of mortality than when c = 0.08. At low levels of mortality, it is the other way round: the cost of a life annuity for the high mortality group (c = 0.09) would be more responsive (hence leading to higher values for $H_{60}\left(b,c,\alpha,\delta\right)$) to a decreasing trend in the force of mortality than when c = 0.08.

When the force of interest increases to 3% (lower panel), the effect of any change in the force of mortality on the cost of a life annuity is less dramatic, resulting in lower values for $H_{60}\left(b,c,\alpha,\delta\right)$ for the different scenarios of mortality base table. Also, the value for $H_{60}\left(b,c,\alpha,\delta\right)$ decreases with increases in the rate of improvement in mortality.

One advantage of using a mathematical model for the rates of mortality is that it enables us to investigate further the mathematical properties of the derivative of $H_{60}\left(b,c,\alpha,\delta\right)$ with respect to the parameters c,α and δ. This should help us to understand better the behaviour of the entropy measure. A fuller analysis reveals that the numerical values of the partial derivative of $H_{60}\left(b,c,\alpha,\delta\right)$ with respect to different levels of the improvement factor α are consistent with results that we have obtained so far, which confirms the conclusions that we have reached regarding the relationships between different variables.

4.2 Modelling mortality improvements using Sithole et al (2000)

In this section, mortality improvements are allowed for using the model of Sithole et al (2000), again considering the CMI insured female pensioners (1991-1994) as the base mortality table. The Sithole et al (2000) reduction factor can be written in the form

$$RF(x,t)=\exp\left[\left(-\alpha+\beta x\right)t\right] \tag{14}$$

Different levels of mortality improvement are considered by allowing α to vary. The estimated value of α in Sithole et al (2000) is 0.050651, and a range of values of α from 60% to 140% of this estimate were considered. Figure 6 shows the values of $H_{60}\left(b,c,\alpha,\beta,\delta\right)$ at different rates of interest and for different values of α.

<u>Figure 6:</u> Values for $H_{60}(b,c,\alpha,\beta,\delta)$ at different rates of interest and mortality improvements allowed for using the Sithole et al (2000) model

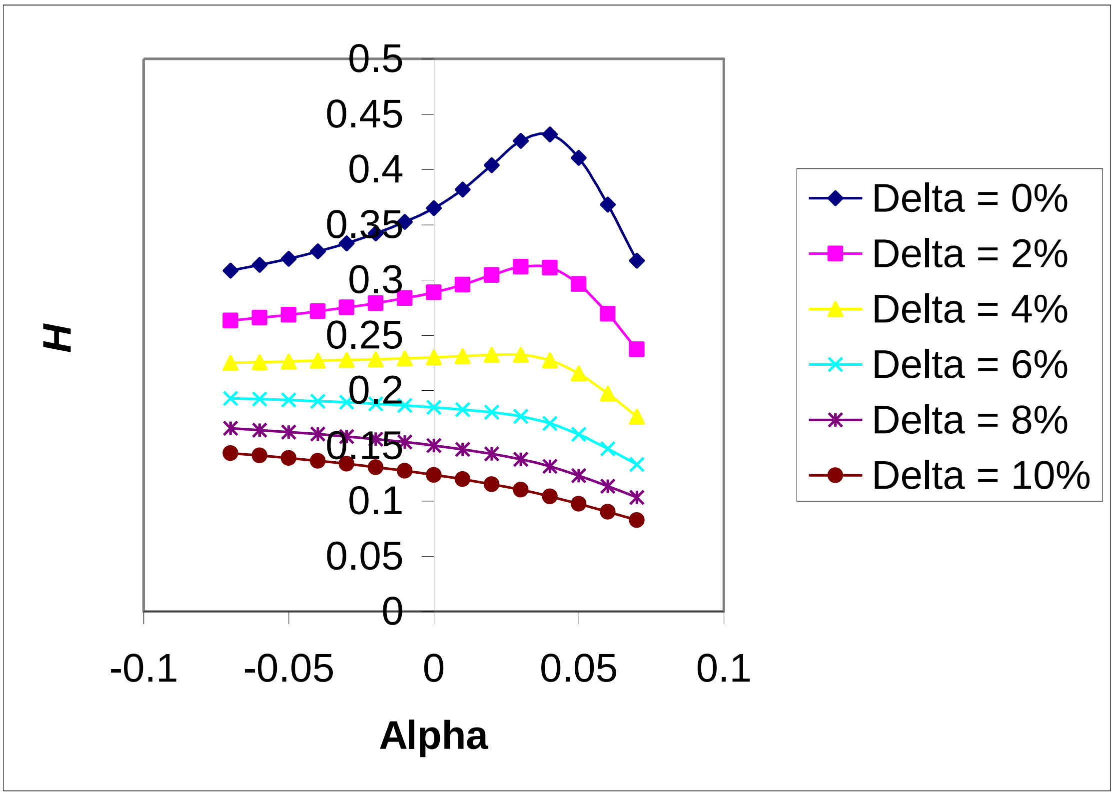


Figure 6 shows that the value of $H_{60}(b,c,\alpha,\beta,\delta)$ decreases when the rate of interest increases for all rates of interest. At higher rates of interest, the value of $H_{60}(b,c,\alpha,\beta,\delta)$ increases with decreases in the rate of mortality (α), while at lower rates of interest, it falls with the decreases in α until it reaches a point where the decreases in mortality rates have a greater effect on the cost of a life annuity. At this point, the value of $H_{60}(b,c,\alpha,\beta,\delta)$ starts to increase again. We note that the levels of the improvement factor α considered in this part of the analysis lead to very low values of mortality. The pattern of results shown is similar to that which we have obtained using the English Life Tables (section 3) as well as those obtained using the Gompertz law of mortality and the exponential reduction factor model (section 4.1).

5 Summary and remarks

In this paper, the entropy measure applied in population biology by Demetrius (1976) and in demography by Vaupel (1986) has been extended to measure the effect of any changes in the force of mortality on the cost of life annuity for different interest rate scenarios. This single figure index allows the effect of different sources of risk in a life annuity contract can be measured and analysed.

It is worth mentioning that there are some limitations related to the use of the entropy value to measure of the effect of change of mortality on the cost of life annuity. Namely, if the survival function is already very rectangular shaped (when entropy is close to zero), an expansion of the survival function will have a big impact on the cost of life annuities but this would not be captured by the entropy measure. Also, as we have noted, in the case of complete rectangularization of the survival function, entropy is zero regardless of the age at which all members of the population die.

The entropy measure has been calculated under different mortality and interest rates assumptions. Also, it has been calculated over the whole age range (from age 0) and at older ages (ages 60 and 70). For all cases, the lower the rate of interest, the higher is the value of entropy, indicating a higher effect of longevity risk on the present value of annuity payments. This reflects the importance of longevity risk in the context of life annuity portfolios especially in a low interest environment.

At very high or low levels of mortality, the numerical results suggest that the effect of mortality changes on the value of life annuity is of less importance. This reflects the fact that when mortality is already very high or very low, any change (whether an increase or a decrease in the force of mortality) is less likely to have a significant effect on the cost of life annuity. This means that in theory, even if mortality continues to improve, it will reach a level beyond which any more improvements would not markedly affect the discounted cost of survival benefits.

In terms of future work, it would be useful to investigate other models of mortality since in this paper we consider just the Gompertz model. It would also be of interest to investigate the mathematical properties of the derivative of the measure of entropy with respect to the underlying parameters, in order have a better understanding of the way it behaves. For example, it would enable us to answer questions such as: at a given level of interest rate, what would be the level of mortality that is considered too high or too low such that the cost of life annuity is relatively insensitive to further changes in the force of mortality?

The models in this paper could be extended to reflect the effect of the uncertainty of future returns on the cost of life annuity, by incorporating a dynamic model for the interest rate. Also, the scenario analysis used in this paper to test the sensitivity of results could be replaced by distributional assumptions for the parameters.

References

Arriaga, E. (1984), *Measuring and Explaining the Change in Life Expectancies.* Demography, vol. 21, p. 83-96.

Booth, H. and Tickle, L. (2008). *Mortality and forecasting: a review of methods*. Annals of Actuarial Science, vol 3, p. 3-43.

CMI (2009), *Working Paper 38: A Prototype mortality projections model: Part 1 – an outline of the proposed approach*. The Institute of Actuaries and Faculty of Actuaries.

Demetrius, L (1976), *Measures of variability in age-structured populations.* Journal of Theoretical Biology, vol. 63, p. 397-404.

Goldman, N. and Lord, G. (1986), *A New Look at Entropy and the Life Table*, Demography, Vol 23, p. 275-282.

Keyfitz, N. (1985). *Applied Mathematical Demography.* Second edition, Springer-Verlag, New York.

Khalaf-Allah, M. (2007). *Stochastic analysis of longevity and investment risk in the context of life annuities.* PhD Thesis, Cass Business School, City University London.

Khalaf-Allah, M., Haberman, S. and Verrall, R. (2006), *Measuring the effect of mortality improvements on the cost of annuities.* Insurance: Mathematics and Economics, vol. 39, p. 231-249.

Nagnur D. (1986). *Rectangularization of the survival curve and entropy: the Canadian experience, 1921-1981.* Canadian Studies in Population, vol 13, p. 83-102.

Nusselder, M.A. and Mackenbach J.P. (1996). *Rectangularization of the survival curve in the Netherlands, 1950-1992.* The Gerontologist, vol 36, p. 773-782.

Pitacco, E., Denuit, M., Haberman, S. and Olivieri, A-M. (2009). *Modelling Longevity Dynamics for Pensions and Annuity Business.* Oxford University Press. Oxford.

Pollard, J.H (1982), *The Expectation of Life and Its Relationship to Mortality*. Journal of the Institute of Actuaries, vol. 109, p.225-240.

Pollard, J. H. (1988), *On the Decomposition of Changes in Expectation of Life and Differentials in life Expectancy*. Demography, vol. 25, p. 265-276.

Pressat, R. (1985), *Contribution des Écarts de Mortalité par Âge à la Différence des Vies Moyennes*. Population, vol 4-5, p. 765-770.

Sithole, T.Z., Haberman, S. and Verrall, R.J. (2000). *An Investigation into Parametric Models For Mortality Projections, With Applications to Immediate Annuitants' and Life Office Pensioners' Data*. Insurance: Mathematics and Economics, vol.27, p.285-312.

Vaupel, J.W. (1986). *How change in age-specific mortality affects life expectancy.* Population Studies, vol 40, p.147-157.

Appendix

Table 1: Entropy values over the whole age range $H_0(\delta)$ – Females

English Life Table (Females)	Interest Rate				
	0%	2%	4%	6%	8%
1991	0.13763	0.06536	0.02948	0.01630	0.01124
1971	0.15825	0.08513	0.05013	0.03527	0.02906
1951	0.18431	0.10951	0.07200	0.05497	0.04722
1931	0.29546	0.20531	0.15584	0.13011	0.11620
1911	0.41575	0.31363	0.25583	0.22418	0.20592
1891	0.54386	0.42841	0.35996	0.32021	0.29591
1871	0.60370	0.47632	0.39873	0.35206	0.32248
1851	0.64237	0.50796	0.42498	0.37429	0.34171

Table 2: Entropy values over the whole age range $H_0(\delta)$ – Males

English Life Table (Males)	Interest Rate				
	0%	2%	4%	6%	8%
1991	0.15576	0.07926	0.04213	0.02602	0.01916
1971	0.18524	0.10556	0.06552	0.04725	0.03900
1951	0.21369	0.13309	0.09115	0.07111	0.06152
1931	0.33543	0.24201	0.18957	0.16149	0.14593
1911	0.46447	0.35892	0.29805	0.26396	0.24392
1891	0.60601	0.48756	0.41643	0.37442	0.34832
1871	0.66110	0.53117	0.45144	0.40293	0.37184
1851	0.68864	0.55498	0.47204	0.42093	0.38774

Table 3: Entropy values for Females $H_{60}(\delta)$ - age 60

English Life Table Females	Interest Rate 0%	2%	4%	6%	8%
1991	0.37663	0.30309	0.24542	0.20056	0.16575
1971	0.39191	0.32088	0.26432	0.21954	0.18415
1951	0.41319	0.34367	0.28772	0.24280	0.20675
1931	0.45649	0.38500	0.32696	0.27992	0.24176
1911	0.50177	0.42583	0.36423	0.31425	0.27361
1891	0.53708	0.46164	0.39980	0.34906	0.30732
1871	0.54043	0.46365	0.40088	0.34951	0.30735
1851	0.54042	0.46310	0.39999	0.34843	0.30618

Table 4: Entropy values for Males $H_{60}(\delta)$ - age 60

English Life Table (Males)	Interest Rate 0%	2%	4%	6%	8%
1991	0.44971	0.37382	0.31298	0.26434	0.22543
1971	0.50313	0.42641	0.36414	0.31361	0.27250
1951	0.49838	0.42714	0.36854	0.32038	0.28074
1931	0.50284	0.43135	0.37259	0.32428	0.28448
1911	0.54445	0.46890	0.40696	0.35610	0.31420
1891	0.56787	0.49358	0.43209	0.38113	0.33877
1871	0.56717	0.49114	0.42851	0.37686	0.33412
1851	0.55978	0.48323	0.42038	0.36871	0.32608