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The Impact of Heavy Tails and Comovements in Downside-Risk Diversification

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Abstract

This paper uncovers the factors influencing optimal asset allocation for downside-risk averse investors. These are comovements between assets, the product of marginal tail probabilities, and the tail index of the optimal portfolio. We measure these factors by using the Clayton copula to model comovements and extreme value theory to estimate shortfall probabilities. These techniques allow us to identify useless diversification strategies based on assets with different tail behaviour, and show that in case of financial distress the asset with heavier tail drives the return on the overall portfolio down. An application to financial indexes of *UK* and *US* shows that mean-variance and downside-risk averse investors construct different efficient portfolios.

JEL classification: C1, C2, G1.

Keywords: Comovements, Copulas, Downside-risk diversification, Expected shortfall, Heavy tails, Lower partial moments, Shortfall probability

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1 Introduction

The notion of diversification is age-old. It consists, roughly speaking, on diminishing risk by spreading resources in many different areas with very little interdependence. In Finance this is usually identified with investing in uncorrelated assets (proper diversification) or with investing in assets negatively correlated (hedging strategies).

However it was not until Markowitz (1952) that the concept of portfolio diversification was formalized. This author developed a sound theory to study diversification in an optimal asset allocation context. Markowitz showed that investors should choose assets as if they care only about the mean and variance of portfolio returns. By upholding the variance as the pertinent risk measure investors decide to penalize equally departures from expected wealth in both sides. The conclusion of this analysis is that investors use the variance to guide the trade-off between risk and return. This can be seen for example in Stiroh and Rumble (2005). Thus, investors employ an statistical moment used to gauge the presence of uncertainty to take decisions on asset allocation strategies. Knight (1921) was the first to note that uncertainty and risk were two different and separate concepts. According to this author uncertainty in contrast to risk was defined by the absence of knowledge about the likelihood of an event. In this way Knight concluded that insurance markets cannot develop under uncertainty but they do under risk given one can always put a fair price to the risk assumed.

The merits of Markowitz's theory are outstanding however. Investors construct optimal portfolios by minimizing a simple statistical measure identified with risk. Alternatively, Roy (1952) developed the concept of safety first portfolios. The aim of investors constructing these portfolios is to minimize an upper bound of the likelihood of a dread event. This is usually identified with the left tail of the distribution of returns on the portfolio. Roy also confined himself to distributions where only the first two moments are known. Building on this interpretation of risk Markowitz (1959) proposed the semivariance. This risk measure focused only on deviations below a threshold value determined by the expected return on the investment. The analysis of this measure however was fraught with difficulties arisen from non-differentiability problems. Hogan and Warren (1974), Bawa (1975), Arzac and Bawa (1977) or Bawa and Lindenberg (1977) continued on the idea of risk based on dread events introduced by Roy and proposed risk measures based on the chance of these events. Building on Roy's (1952) formulation of risk and extending the semivariance of Markowitz (1959) these authors introduced lower partial moments of the distribution of returns to describe risk. These include in its simplest version the shortfall probability and quantile measures as Value at Risk, or more involved measures as the expected shortfall or the semivariance. Bawa ((1975), (1976), (1978)) and later Harlow and Rao (1989) extended these models to asset pricing and developed financial portfolio theory for mean-downside-risk averse investors. Finally Ang, Chen and Xing (2006) revisit the problem and propose asset pricing models accounting for downside-risk but controlling for other cross sectional variables as coskewness, size effect and the book-to-market ratio.

Markowitz in his pioneering work assumed that the returns on the portfolio followed a multivariate normal distribution. In this framework the mean-variance methodology encloses downside-risk measures. However during the last forty years empirical analyses of the distribution of returns have been consistently rejecting this hypothesis and pointing towards heavy tailed distributions, see Fama (1965) or modern books on risk management and heavy tails as Embrechts (2000) or Malevergne and Sornette (2006). This stylized fact has gained further popularity during the last decade where more sophisticated statistical and probabilistic techniques have been developed to study heavy tails and extreme events, see Chavez-Demoulin, Embrechts, and Nešlehová (2006) in an operational risk context. The use of these techniques has also made possible the revival of portfolio theories based on downside-risk measures (Hyung and de Vries, 2005).

The first aim of this paper is to uncover the factors having an influence on asset allocation for downside-risk averse investors. In order to do this we analyze lower partial moments of order zero and one of the distribution of returns. By doing this we are able to decompose the shortfall probability - risk measure employed for safety first portfolios in Roy (1952) - into a probability function measuring the degree of comovements between the assets in the portfolio, and the product of marginal tail probabilities of each asset. We extend the analysis to the expected shortfall because it is a risk measure consistent with utility functions describing preferences of risk-averse investors (Harlow and Rao, 1989). This further decomposition shows that the tail index of the distribution of the portfolio and the downside variance have an outstanding role on diversification.

The previous findings contribute to *positive economics* in what they are an attempt to describe the optimal choices of fully rational individuals, while the second contribution of the paper introduced below is embodied in *normative economics*, that is, the desire to improve people's imperfect choices. In this aspect the paper contributes to the literature by proposing statistical techniques to measure properly downside-risk and to develop investment strategies to diminish it. We achieve this by using copula functions to model comovements in the tails; and by using extreme value theory techniques. These techniques permit to identify useless diversification strategies based on portfolios consisting of a large number of assets with different marginal tail behaviour. In this case the shortfall probability of the portfolio is driven by the shortfall probability of the asset with heavier tail. Therefore adding assets to the portfolio does not diminish risk but adds complexity to its management.

The paper is structured as follows. Section 2 describes investors' optimal asset allocation decision problem for mean-variance and mean-downside-risk averse investors. The section studies the factors having an influence on portfolio downside-risk measures by decomposing them in terms of commovement risk and marginal downside-risks specific of each asset. Section 3 introduces statistical techniques to measure properly these factors. The methodology includes the use of copula functions to measure the degree of asymptotic tail dependence in the portfolio and extreme value theory to gauge the probability of shortfall and expected shortfall of each asset and of the overall portfolio. The next section calculates the efficient portfolio frontier for portfolios simulated from a Student's-t family of distributions. The efficiency of these portfolios is assessed in terms of comovements and marginal and portfolio tail behaviour. Section 5 studies an example of diversification for data from economies with well developed financial markets. Finally, Section 6 concludes with the main findings of the paper.

2 Investors' Efficient Portfolio Frontier

Markowitz devised an economy consisting of mean-variance minimizing agents with m risky assets yielding returns R_i, $i = 1, \ldots, m$. The return on a portfolio P of these assets is

$$R_P = \sum_{j=1}^{m} x_j R_j, \tag{1}$$

with $\sum_j x_j = 1$, and $X = (x_1, \ldots, x_m)$ depicting share of investment on each risky asset. The efficient portfolio frontier for these investors is derived from minimizing

$$\min_{x_j} \sigma_p^2 = \sum_{i=1}^{m}\sum_{j=1}^{m} x_i x_j \sigma_{ij}, \tag{2}$$

with σ_{ij} standing for the covariance between returns and σ_j^2 for the variance.

If there exits a risk-free asset in the economy the efficient portfolio frontier is determined by a straight line of this form

$$E[R_j] - R_f = \beta_j (E[R_p] - R_f), \tag{3}$$

with $\beta_j = \frac{\sigma_{jp}}{\sigma_p^2}$ and R_f denoting the return on the risk-free asset.

This diversification strategy is limited however. Investors simply punish deviations from expected levels of wealth. This was pointed out by Hogan and Warren (1974), Bawa (1975) or Bawa and Lindenberg (1977) that propose to study Lower Partial Moments (LPM_n) of the distribution of returns as alternative risk measures to the variance of a portfolio. Bawa ((1975), (1976), (1978)) introduces the following family of utility functions consistent with these LPM_n risk measures, [1]

$$u(R_p; n, \tau) = a + bR_p - c(\tau - R_p)^n I(R_p \leq \tau), \tag{4}$$

where a,b, and c are constants, $I(\cdot)$ is an indicator function and τ denotes a target return. Investors with preferences described by these functions are denominated downside-risk averse investors.

While in the mean-variance framework investors maximize their expected utility by minimizing the variance of the return on P, downside-risk averse investors achieve that by minimizing LPM_n measures. Bawa and Lindenberg (1977) and Harlow and Rao (1989) show that downside-risk averse investors' optimal portfolio choice is the solution of the following,

$$\min_X LPM_n(\tau; X) = \int_{-\infty}^{\tau} (\tau - X'R)^n dF(R_p), \tag{5}$$

subject to $\sum_j x_j E[R_j] = \mu$, with μ denoting certain return level. This integral is computed on the probability measure of the variable R_p denoted by F.

The efficient portfolio frontier is the result of minimizing this objective function. If there exits a risk-free asset the set of optimal portfolios is given by a straight line as (3) with the following slope

$$\beta_j^{lpm_n(\tau)} = \frac{\int_{-\infty}^{\tau} \int_{-\infty}^{\infty} (\tau - R_p)^{n-1}(R_f - R_j) dF(R_j, R_p)}{\int_{-\infty}^{\infty} (\tau - R_p)^{n-1}(R_f - R_j) dF(R_p)}. \tag{6}$$

The index lpm_n stands for n-lower partial moment. For $n = 2$, $\tau = R_f$ and returns normally distributed both mean-variance and mean-downside-risk efficient portfolio frontiers coincide.

In the downside-risk framework there is no need to impose restrictive and unrealistic assumptions on the distribution of returns. The use of the variance to gauge risk usually requires assuming normal returns, whereas for LPM_n measures $F(R_p)$ can be any one of a class of distributions simply characterized by a location and a scale parameter (see Harlow and Rao, 1989). On the other hand this entails the difficulty of having to entertain other statistical moments in conjunction with mean and variance to decide how to allocate resources.

[1] By consistent utility function we mean that maximizing its expected utility is equal to minimizing LPM_n risk measures.

2.1 Shortfall probability as downside-risk measure

For illustration purposes we will confine ourselves first to study LPM_0. A well diversified portfolio will be the result of minimizing

$$P\{R_P \leq \tau\},$$

for certain expected return level. The parameter τ will be assumed to be known and determined exogenously. This value is usually identified in the literature with a zero return or with the return on the risk-free asset.

By Bayes' theorem portfolio's P shortfall probability can be written as

$$P\{R_p \leq \tau\} = P\{R_p \leq \tau | R_1 \leq \tau, \ldots, R_m \leq \tau\} P\{R_1 \leq \tau, \ldots, R_m \leq \tau\} +$$

$$+ P\{R_p \leq \tau | R_1 > \tau \; or \; \ldots or \; R_m > \tau\} P\{R_1 > \tau \; or \; \ldots or \; R_m > \tau\}.$$

This can be expressed as

$$P\{R_p \leq \tau\} = [p_c(R_p, \tau) - \widetilde{p}_c(R_p, \tau)] P\{R_1 \leq \tau, \ldots, R_m \leq \tau\} + \widetilde{p}_c(R_p, \tau)$$

with

$p_c(R_p, \tau) = P\{R_p \leq \tau | R_1 \leq \tau, \ldots, R_m \leq \tau\}$, and
$\widetilde{p}_c(R_p, \tau) = P\{R_p \leq \tau | R_1 > \tau \; or \; \ldots or \; R_m > \tau\}$.

Summing and substracting the product of each asset returns' marginal distribution this probability reads as

$$P\{R_p \leq \tau\} = [p_c(R_p, \tau) - \widetilde{p}_c(R_p, \tau)][cr(\tau) + p(R_1, \tau) \cdots p(R_m, \tau)] + \widetilde{p}_c(R_p, \tau), \tag{7}$$

where

$p(R_j, \tau) = P\{R_j \leq \tau\}$, and
$cr(\tau) = P\{R_1 \leq \tau, \ldots, R_m \leq \tau\} - P\{R_1 \leq \tau\} \cdots P\{R_m \leq \tau\}$.

The preceding formula simplifies if investors can only hold long positions. In this case $p_c(R_p, \tau) = 1$. Then under some simple algebra it is easy to see that

$$P\{R_p \leq \tau\} = [1 - \widetilde{p}_c(R_p, \tau)] \left[cr(\tau) + p(R_1, \tau) \cdots p(R_m, \tau) - 1\right] + 1. \tag{8}$$

The optimal allocation of risky assets is function of τ; of conditional probabilities depending on the shares invested on each asset: $p_c(R_p,\tau)$ and $\widetilde{p}_c(R_p,\tau)$; of the marginal distributions tail behaviour: $p(R_1,\tau)\cdots p(R_m,\tau)$; and finally is function of the degree of dependence between assets in the tails: $cr(\tau)$. Hence the degree of heaviness of the distributional tails of returns and the extent of tail dependence - comovements hereafter - between assets are fundamental for a downside-risk averse investor. These factors are modelled as follows.

A probability distribution $p(R_i,\tau)$ is exponentially decaying in the tails - determined by τ - if

$$p(R_i,\tau) = A_i \exp^{-B_i(-\tau)^{\beta_i}}[1+o(1)],\ A_i, B_i, \beta_i > 0, \tag{9}$$

as $\tau \to -\infty$. For the standard normal distribution $\beta_i = 2$ and $B_i = 1/2$. These distributions are characterized by having infinite bounded moments. On the contrary we will define heavy-tailed distributions as those with a polynomial tail decay. These probability functions satisfy

$$p(R_i,\tau) = A_i(-\tau)^{-\frac{1}{\xi_i}}[1+o(1)],\ \xi_i, -\tau, A_i > 0, \tag{10}$$

as $\tau \to -\infty$. Probability distributions satisfying this property are also denominated regularly varying. These are also defined by

$$p(R_i,\tau) = (-\tau)^{-\frac{1}{\xi}} L(-\tau),$$

with $\xi > 0$ and $\lim\limits_{-\tau\to\infty} \frac{L(-t\tau)}{L(-\tau)} = 1$, $\forall t > 0$. These distributions are characterized by bounded moments up to $1/\xi$.

For the analysis of comovements and tail dependence we use the concept of positive quadrant dependence (PQD) introduced by Lehman (1966). This author defined m random variables $\varepsilon_1,\ldots,\varepsilon_m$ as *PQD* if for all $(\tau,\ldots,\tau) \in \mathbb{R}^m$,

$$P\{\varepsilon_1 \leq \tau,\ldots,\varepsilon_m \leq \tau\} \geq P\{\varepsilon_1 \leq \tau\}\cdots P\{\varepsilon_m \leq \tau\}, \tag{11}$$

or equivalently if

$$P\{\varepsilon_1 > \tau,\ldots,\varepsilon_m > \tau\} \geq P\{\varepsilon_1 > \tau\}\cdots P\{\varepsilon_m > \tau\}.$$

Our definition of comovements is derived from the definition of positive quadrant dependence.

Definition 2.1. *There exists Comovement risk at level τ in a portfolio P consisting of m risky assets if $cr(\tau) > 0$, with*

$$cr(\tau) = P\{R_1 \leq \tau, \ldots, R_m \leq \tau\} - P\{R_1 \leq \tau\} \cdots P\{R_m \leq \tau\}, \tag{12}$$

as denoted before.

From expressions (9), (10) and (12) we can study in detail the optimal portfolio allocation problem consisting of minimizing (7) given some expected return on portfolio P.

Note the risk measure LPM_0 relevant for downside-risk averse investors consists of the same ingredients than the risk measure for the mean-variance diversification problem (2). The contribution to risk of measures of linear dependence between assets (covariances) is replaced now by a measure of comovements $cr(\tau)$. The counterpart of assets' variance (σ_j^2) is the marginal downside probability $p(R_j, \tau)$. The extra remaining terms are used to determine the optimal weights that minimize the shortfall probability of the portfolio.

2.2 Expected shortfall and semivariance as downside-risk measures

Utility functions consistent with shortfall probability (LPM_0) measures (see (4)) fail to describe any form of risk aversion relevant for the decision-making process. Two features of this measure are that investors' marginal utility is constant and that the risk measure assigns the same weight to each possible outcome of the return in the tail.

This is overcome by LPM measures involving higher moments. Simple and popular extensions of LPM_0 are the expected shortfall ($n = 1$) and the semivariance ($n = 2$). These risk measures are consistent with utility functions describing risk-averse investors' preferences. Moreover, as Harlow and Rao (1989) show, the two-fund separation theorem of Ross (1978) holds allowing to express the value of any asset in the economy in terms of the risk-free asset and an efficient risky portfolio.

In particular for $n = 1$ the optimization problem is

$$\min_X LPM_1(\tau; X) = \int_{-\infty}^{\tau} (\tau - R_p) dF(R_p), \tag{13}$$

subject to $\sum_j x_j E[R_j] = \mu$ and $x_0 + \sum_j x_j = 1$.

After some simple algebra the preceding equation becomes

$$\min_X LPM_1(\tau; X) = (\tau - E[R_p | R_p \leq \tau]) \, p(R_p, \tau). \tag{14}$$

These portfolios are held by investors with a higher degree of risk aversion than those simply minimizing shortfall probability or the variance of the portfolio. Negative returns far from the target are more penalized than exceedances near τ. The importance of comovements and marginal tail behaviour is stressed in these measures that put an extra weight on large negative returns.

The objective function (14) can be further refined by assuming τ is sufficiently large in absolute value to use extreme value theory techniques. Note the concept *sufficiently large* does not give much guidance about appropriate choices, see Embrechts, Klüppelberg and Mikosch (1997) or more recently Coles (2001) for a detailed review of these techniques. Then, for appropriate values of τ the expected value in the tail can be well approximated by the following

$$E[R_p|R_p \leq \tau] = \begin{cases} \tau - \frac{\sigma_{\tau,p}}{1-\xi_p}, & if \quad \xi_p \neq 0 \\ \tau - \sigma_{\tau,p}, & if \quad \xi_p = 0, \end{cases} \tag{15}$$

with ξ_p and $\sigma_{\tau,p}$ parameters of a Generalized Pareto distribution modelling the conditional distribution of returns below τ. The parameter ξ_p depicts the ratio of decay of the left tail of the distribution of R_p. The proof of this result is sketched as follows.

The conditional distribution of R_p for values less than τ is the conditional distribution of $-R_p$ for values greater than $-\tau$. Thereby

$$E[R_p|R_p \leq \tau] = -E[-R_p| - R_p > -\tau].$$

From extreme value theory we know that the conditional distribution of the upper tail converges to a Generalized Pareto distribution (*GPD*) as $-\tau$ goes to the right end point of the distribution. This result is the Pickands (1975), Balkema-de Haan (1974) theorem. The *GPD* takes the form

$$GPD_{\xi,\sigma_{-\tau,p}}(y) = \begin{cases} 1 - \left(1 + \xi_p \frac{y}{\sigma_{-\tau,p}}\right)^{-\frac{1}{\xi_p}} & \text{if } \xi_p \neq 0 \\ 1 - e^{\frac{-y}{\sigma_{-\tau,p}}} & \text{if } \xi_p = 0. \end{cases} \tag{16}$$

We further assume that the choice of τ is sufficiently low (high $-\tau$) for the *GPD* to approximate the conditional distribution of the upper tail. Then it is immediate to derive the conditional expected value of the exceedances of $-\tau$. This is

$$E[-R_p| - R_p > -\tau] = \begin{cases} -\tau + \frac{\sigma_{-\tau,p}}{1-\xi_p}, & if \quad \xi_p \neq 0 \\ -\tau + \sigma_{-\tau,p}, & if \quad \xi_p = 0. \end{cases}$$

Note that $\sigma_{-\tau,p} = \sigma_{\tau,p}$ by construction.

It follows then from (14) that the risk measure LPM_1 for τ sufficiently low is given by

$$\min_X LPM_1(\tau; X) = \frac{\sigma_{\tau,p}}{1-\xi_P} p(R_p, \tau) \tag{17}$$

for portfolios with heavy-tailed distributions, and

$$\min_X LPM_1(\tau; X) = \sigma_{\tau,p} p(R_p, \tau) \tag{18}$$

for portfolios with distributions exponentially decaying.

Formulas (7), (17) and (18) show that the risk profile of downside-risk averse investors depends on the tail index of portfolio P; on the downside variance of R_p ($\sigma_{\tau,p}$), and on the shortfall probability of the portfolio. Thereby the presence of comovements and the marginal tail behaviour of each asset have an important role in optimal asset allocation and diversification.

For $n = 2$ the optimal allocation problem becomes

$$\min_X LPM_2(\tau; X) = \Big(V[R_p|R_p \leq \tau] + (E[R_p|R_p \leq \tau] - \tau)^2 \Big) p(R_p, \tau), \tag{19}$$

where $V(\cdot)$ stands for the variance and hence LPM_2 is named a semivariance risk measure. The proof of this result is obtained by adding and substracting $E[R_p|R_p \leq \tau]$ into the integrand in (5).

It is interesting to observe that minimizing LPM_2 implies minimizing LPM_1. The same applies to LPM_1 and LPM_0. For increasing n risk is represented by higher moments of the conditional distribution of returns below the target. Thereby risk measures based on high n include extra penalization for heavy-tailed distributions. We will not study this risk measure further in the paper and concentrate on LPM_0 and LPM_1 given they can be identified with the most popular risk measures used nowadays in the risk management literature; these are Value at Risk and Expected Shortfall respectively.

3 The impact of Heavy Tails and Comovements

In portfolio theory any efficient portfolio has some share of every risky asset in the economy. The composition depends on the level of investor's risk aversion. In practice however, a professional investor, *e.g.* a fund manager, does not have free access or the possibility to observe the whole universe of assets trading in a financial market. These professionals specialize in a subset of these risky assets and construct diversified portfolios by choosing optimal weights within those assets.

Asset allocation and portfolio diversification consists of two different aspects; a choice of optimal assets in the sense of minimizing cross dependencies and a vector X of optimal proportions invested in each asset. This is not a sequential procedure in the sense the assets are selected first and then the optimal weights, for it can occur that assets exhibiting higher comovements have combinations displaying lower LPM_1. Therefore in order to construct optimal portfolios investors need on the one hand to compute the shortfall probability and the tail index of the efficient portfolio itself; and on the other hand to find optimal weights that minimize conditional probabilities of the type shown in (7). All these elements depend on the level of tail dependence between assets in the portfolio (comovements) and on marginal tail probabilities (heavy tails).

3.1 Heavy tails

Investors' optimal asset allocation depends on the shape of different distributional tails in two ways: the contribution of each asset to the risk in the portfolio given by marginal shortfall probabilities, and the tail index of the portfolio itself.

It is well known that if the returns on a portfolio are normally distributed and the joint distribution is also multivariate normal the distribution of R_p is normal and the tail index ξ_j of every asset and of the portfolio (ξ_p) is zero. In this case the downside-risk optimization problem boils down to study marginal and overall variances (mean-variance methodology). This result can be extended to portfolios where individual returns have distributions exponentially decaying and satisfy

$$\frac{1}{m}\sum_{i=1}^{m}\sum_{\substack{j=1\\ i\neq j}}^{m} Cov(R_i, R_j) \rightarrow \gamma, \tag{20}$$

with γ a constant value (see Lehman p.107, 1999). In this case the central limit theorem for dependent variables applies and the preceding nice results on diversification hold.

However, if returns do not exhibit an exponential decay in the tails - as empirical evidence on asset returns is suggesting since Fama (1965) - standard statistical results on diversification do not hold and one has to study the probability in the tails for they provide extra information not contained in the variance-covariance structure. These different tail behaviours determine some important properties for downside-risk averse investors when constructing well diversified portfolios (portfolios consisting of independent assets).

Some results.-

- The shortfall probability of returns is smaller for assets with exponentially decreasing distributions in the left tail.

- The shortfall probability of two returns with heavy tails is smaller for the asset with smaller tail index.
- The shortfall probability of a portfolio of assets exhibiting the same tail behaviour (same ξ) decreases as the number of assets in the portfolio increases. This phenomenon is more pronounced for exponentially decreasing distributions.

Similar findings are given in Danielsson et al. (2006) from the study of downside-risk measures for single assets with heavy-tailed distributions.

The proof of these properties is immediately derived from definitions (9) and (10) and from noting that if the marginal distributions are heavy tailed the tail index characterizing the distribution of the portfolio is also heavy tailed. Moreover, by applying the convolution theorem of Feller (1971, VIII.8), Dacorogna et al. (2001) and Hyung and de Vries (2005) find that the distribution of an equally weighted portfolio consisting of m independent risky assets with distributions regularly varying at infinity all at the same rate $\frac{1}{\xi}$ is regularly varying at a rate $\frac{1}{\xi}$. In particular they find that

$$P\{\frac{1}{m}\sum_{i=1}^{m} R_i \leq -x\} = m^{1-\frac{1}{\xi}} A x^{-\frac{1}{\xi}}[1+o(1)], \quad as \quad x \to \infty,$$

with ξ the common tail index and A a constant. It can be seen that these heavy-tailed portfolios also benefit from a higher number of assets (m larger) and from thinner tails (a lower ξ towards zero).

Note however that if the assets in the portfolio have different tail behaviour investors do not obtain a real benefit from diversification (in the sense of diminishing risk by aggregating elements to the portfolio). In order to see this we consider a portfolio of independent assets having each a regularly varying distribution with tail index ξ_j for $j = 1, \ldots, m$ where ξ_k can be different from ξ_l. It follows directly from the definition of regular varying that the sum of these m independent variables is also regularly varying at infinity with tail index the maximum of the tail indexes of the marginal distributions. This result is formulated as follows

$$P\{S_m \leq -x\} = A_m x^{-\min(\frac{1}{\xi_1},\ldots,\frac{1}{\xi_m})}[1+o(1)], \quad as \quad x \to \infty, \tag{21}$$

with $S_m = \sum_{i=1}^{m} R_i$ and A_m some constant. The proof of the preceding expression for $m = 2$ is immediate by observing that

$$P\{R_1 \leq -x\} + P\{R_2 \leq -x\} - P\{R_1 \leq -x, R_2 \leq -x\} \leq P\{S_m \leq -x\},$$

and for $\varepsilon > 0$,

$$P\{S_m \leq -x\} \leq P\{R_1 \leq -(1-\varepsilon)x, R_2 > -\varepsilon x\} + P\{R_1 > -\varepsilon x, R_2 \leq -(1-\varepsilon)x\} + P\{R_1 \leq -\varepsilon x, R_2 \leq -\varepsilon x\}.$$

If ξ^* denotes the tail index of the variable S_m and using that R_1 and R_2 are independent we have

$$\lim_{x\to\infty} inf \frac{P\{R_1 \leq -tx\} + P\{R_2 \leq -tx\}}{P\{S_m \leq -x\}} \leq \lim_{x\to\infty} \frac{P\{S_m \leq -tx\}}{P\{S_m \leq -x\}},$$

and

$$\lim_{x\to\infty} \frac{P\{S_m \leq -tx\}}{P\{S_m \leq -x\}} \leq \lim_{x\to\infty} sup \frac{P\{R_1 \leq -(1-\varepsilon)tx\}}{P\{S_m \leq -x\}} + \frac{P\{R_2 \leq -(1-\varepsilon)tx\}}{P\{S_m \leq -x\}}.$$

Then, using the concept of regular variation the preceding result reads as

$$\lim_{x\to\infty} inf \left(t^{-\frac{1}{\xi_1}} x^{(-\frac{1}{\xi_1}+\frac{1}{\xi^*})} + t^{-\frac{1}{\xi_2}} x^{(-\frac{1}{\xi_2}+\frac{1}{\xi^*})} \right) \leq \lim_{x\to\infty} \frac{P\{S_m \leq -tx\}}{P\{S_m \leq -x\}}, \tag{22}$$

and

$$\lim_{x\to\infty} \frac{P\{S_m \leq -tx\}}{P\{S_m \leq -x\}} \leq \lim_{x\to\infty} sup \left([(1-\varepsilon)t]^{-\frac{1}{\xi_1}} x^{(-\frac{1}{\xi_1}+\frac{1}{\xi^*})} + [(1-\varepsilon)t]^{-\frac{1}{\xi_2}} x^{(-\frac{1}{\xi_2}+\frac{1}{\xi^*})} \right). \tag{23}$$

Letting $\varepsilon \to 0$ and $t > 0$ fixed we observe that the ratio of probabilities converges for $\xi^* = \max(\xi_1, \xi_2)$, and (21) holds. For $m > 2$ the proof holds by induction. In this case $\xi^* = \max(\xi_1, \dots, \xi_m)$. $\square$

On the other hand in terms of downside-risk a well diversified portfolio P should at least satisfy that

$$P\{R_p \leq -x\} \leq P\{R_j \leq -x\} \quad as \quad x \to \infty. \tag{24}$$

For the case of multivariate gaussianity of returns these probabilities satisfy

$$\frac{P\{R_p \leq -x\}}{P\{R_j \leq -x\}} = \frac{1}{\sqrt{m}} \exp^{-\frac{1}{2}(m-1)x^2} [1 + o(1)] \quad as \quad x \to \infty.$$

However for the heavy-tailed portfolio studied here the preceding expression satisfies that

$$\frac{P\{R_p \leq -x\}}{P\{R_j \leq -x\}} = m^{-\frac{1}{\xi_j}} [1 + o(1)] \quad as \quad x \to \infty, \tag{25}$$

with ξ_j the higher tail index in the portfolio. This result is immediate by noting that $P\{S_m \leq -mx\} = P\{R_p \leq -x\}$ and S_m is regularly varying as shown in (21).

The tail of portfolio P is determined by the tail of the asset with heavier tail. The diversification effects of this portfolio are limited. It simply attenuates the downside-risk of the riskier asset by adding independent elements to the portfolio.

If the tail of the riskier return is very heavy tailed (slowly varying distribution: $\xi_j \to \infty$) it satisfies that

$$\frac{P\{R_j \leq -tx\}}{P\{R_j \leq -x\}} = 1 + o(1) \quad for \quad t > 0 \quad as \quad x \to \infty. \tag{26}$$

Thus, with similar arguments to those followed for regularly varying distributions we find that

$$\frac{P\{S_m \leq -tx\}}{P\{S_m \leq -x\}} = 1 + o(1) \quad for \quad t > 0 \quad as \quad x \to \infty, \tag{27}$$

and

$$\frac{P\{R_p \leq -x\}}{P\{R_j \leq -x\}} = 1 + o(1) \quad as \quad x \to \infty, \tag{28}$$

with $t = m$. Portfolios comprising assets with very heavy tailed distributions do not diversify risk by adding independent elements because their shortfall probability is driven by the shortfall probability of the asset with heavier tail.

3.2 Comovement risk

The assumption of multivariate gaussianity implicit in the mean-variance theory has an interesting implication in portfolio theory. The presence of tail dependence diminishes with τ for $\tau \to -\infty$, to the point that is asymptotically zero (see Embrechts, McNeil and Strauman, 1999) and there is no comovement risk. If the multivariate distribution of returns is elliptical but not gaussian there can be tail dependence between returns that does not vanish as $\tau \to -\infty$. Nevertheless the first two moments are sufficient to describe completely the structure of dependence between the variables.

If the assets in the portfolio exhibit comovements and the multivariate distribution is not elliptical investors require further information about the structure of joint dependence. This is particularly challenging in the tails due to the absence of information that hinders nonparametric as well as parametric techniques for modelling joint dependence. By the conditional probability theorem the multivariate distribution function of the vector of returns reads as

$$P\{R_1 \leq \tau, \ldots, R_m \leq \tau\} = P\{R_1 \leq \tau, \ldots, R_m \leq \tau | R_1 \leq \tau_o, \ldots, R_m \leq \tau_o\} P\{R_1 \leq \tau_o, \ldots, R_m \leq \tau_o\} \tag{29}$$

with $(\tau_o, \ldots, \tau_o)$ defining a wider tail region than that determined by the vector of τ's. The joint distribution determined by the vector of τ_o's is well estimated by empirical likelihood methods. Now using Sklar's theorem (1959) the conditional probability can be written in terms of a copula function gauging the dependence structure with margins the conditional probabilities $t_i = P\{R_i \leq \tau | R_1 \leq \tau_o, \ldots, R_m \leq \tau_o\}$ for $i = 1, \ldots, m$. Then

$$P\{R_1 \leq \tau, \ldots, R_m \leq \tau | R_1 \leq \tau_o, \ldots, R_m \leq \tau_o\} = C(t_1, \ldots, t_m). \tag{30}$$

The distribution function C denotes a copula describing the structure of dependence in a $[0, 1]^m$-cube. For a review on copula theory see Joe (1997), Nelsen (1998) or for applications in finance see Cherubini, Luciano and Vechiato (2004).

The use of copulas is criticized by the absence of theory to support ad-hoc choices of copulas often implemented by researchers and by the lack of appropriate goodness of fit tests to validate these choices, see Mikosch (2005) for the former and Chen, Fan and Patton (2004) for the latter.

We instead propose to model tail dependence and comovement risk by using the decomposition (30) and a result from Juri and Wülthrich (2002). These authors find that if returns are polynomially decaying (heavy-tailed) the copula function for the conditional lower tail is approximately described by the Clayton copula (Cl_α) as t goes to zero. More formally, for $t_i = A_i(-\tau)^{-\frac{1}{\xi_i}}$ with A_i some constant and $\tau < 0$,

$$\lim_{t_1, \ldots, t_m \to 0} \quad C(t_1, \ldots, t_m) = Cl_\alpha(t_1, \ldots, t_m) \tag{31}$$

with Cl_α defined by

$$Cl_\alpha(t_1, \ldots, t_m) = (t_1^{-\alpha} + \ldots + t_m^{-\alpha})^{-1/\alpha}. \tag{32}$$

The case $\alpha \to \infty$ describes perfect dependence or comonotonicity. The amount of extreme tail dependence between variables decreases with α. Thus for $\alpha \to 0$, the Clayton copula converges to $Cl_0(t_1, \ldots, t_m) = t_1 \cdots t_m$ that describes asymptotic tail independence. It is easy to see then that comovement risk converges to zero for $\tau \to -\infty$ ($t_i \to 0$, $\forall\ i$).

4 Simulations of portfolios: The Student's-t family

The aim of this simulation experiment is to observe the effect of heavy tails and comovements in an environment where mean-variance agents are correct and see how their decisions worsen as the distribution of the portfolio starts to move away from ideal assumptions given by elliptically distributed returns. In order to do this we simulate four different portfolios consisting of three assets with n=1000 observations each. The multivariate distribution of these portfolios belongs to the Student's-t family. In particular we consider $\nu = 30$, $\nu = 10$, $\nu = 5$ and $\nu = 3$ degrees of freedom, a vector of means $[2 \quad 3 \quad 5]$ and the same following variance-covariance dependence structure for each distribution

$$\begin{pmatrix} 1 & 0.5 & -0.5 \\ 0.5 & 2 & 1 \\ -0.5 & 1 & 3 \end{pmatrix}.$$

It is well known this distribution is elliptical (Patton (2001) or Malevergne et Sornette (2006)). This as described in Embrechts, McNeil and Straumman (1999) implies that mean-variance averse investors will take optimal investment decisions. Nevertheless in contrast to multivariate gaussian distributions the multivariate Student's-t has marginal heavy-tailed distributions and exhibits positive tail dependence that increases as the number of degrees of freedom decreases. This can be observed in figure 4.1.

The tail index of each marginal distribution is well approximated by the inverse of ν (see chapter III in Embrechts, Klüppelberg and Mikosch (1997)). Note then that $\nu = 2$ corresponds to a process with infinite variance. This choice of degrees of freedom implies that *a)* the smaller ν the heavier the marginal tails, and *b)* the tail index of each portfolio is given by the inverse of the common ν describing the tail behaviour of every asset in each portfolio (see Subsection 3.1.)

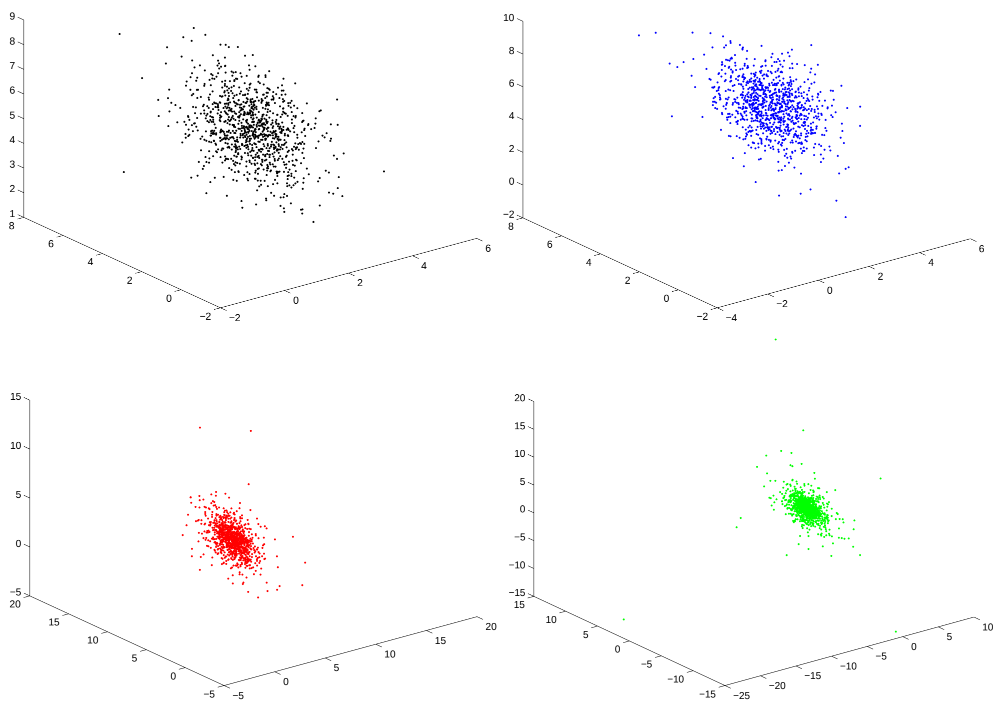

Figure 4.1. *Scatter plot for n=1000 observations of different Student's t distributions. Upper-left panel plots t_{30}; Upper-right t_{10}; Lower-left t_5 and Lower-right t_3.*

The efficient portfolio frontier is consistent across risk measures (see figure 4.2). The most efficient portfolio is $\nu = 30$ for it exhibits the thinner tail and no comovements (it is roughly a multivariate normal distribution). Then t_{10}, t_5 and finally t_3. Mean-variance and downside-risk averse investors agree on their portfolios. It is interesting however that while mean-variance averse investors do not have strong reasons to discard t_3 on the grounds of the variance-covariance structure downside-risk measures are capable of clearly discriminating t_3 from the rest of optimal portfolios.

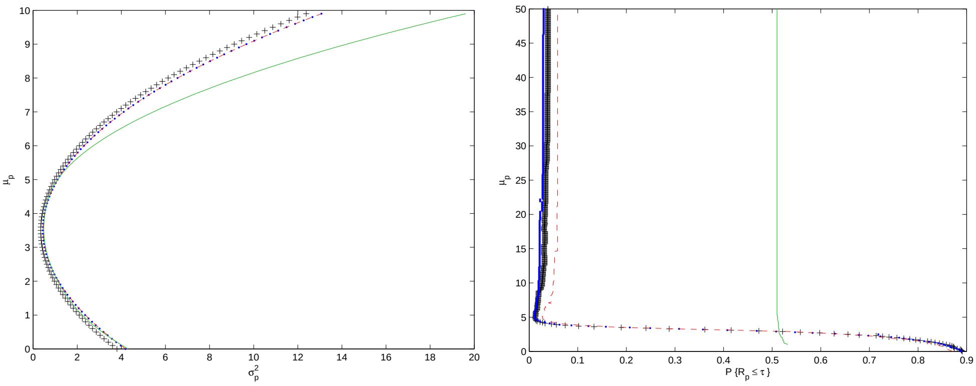

Figure 4.2. *Mean-variance efficient portfolio frontier in the left panel. Shortfall probability efficient portfolio frontier in the right panel.* (+) *in black color describes the curve for* t_{30}; $(\cdot)$ *in blue is used for* t_{10}, $(--)$ *and red color for* t_5 *and* $(-)$ *and green for* t_3.

The efficient portfolio frontiers for LPM_1 (see figure 4.3) support these findings. It is expected that as the returns on the portfolio depart more from the elliptical world downside-risk averse investors take more informed decisions in contrast to mean-variance averse investors. Furthermore, individuals with a higher level of risk aversion as measured by LPM_1 rather than LPM_0 will better discriminate between these portfolios. With this measure we also disregard t_5 in addition to t_3 in the analysis of efficient portfolios. Portfolios t_{30} and t_{10} are still very similar however in terms of downside-risk as it can be seen from figure 4.3.

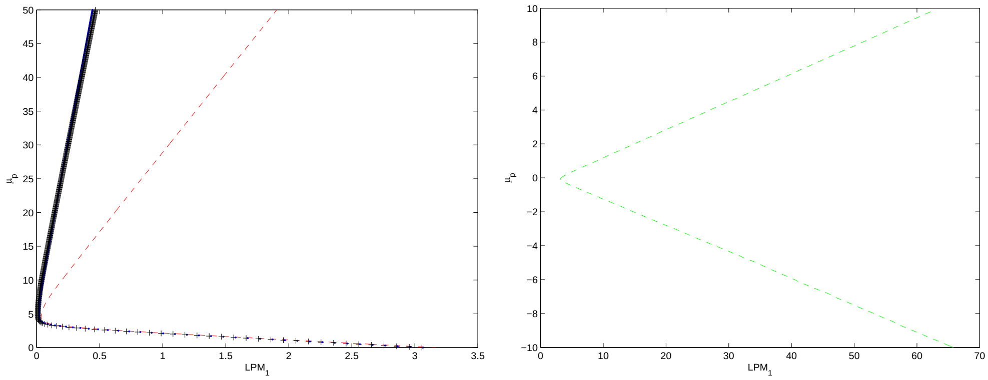

Figure 4.3. *Expected shortfall efficient portfolio frontier for t_{30}, t_{10}, and t_5 in the left panel. Expected shortfall efficient portfolio frontier for t_3 in the right panel.* (+) *in black color describes the curve for t_{30};* ($\cdot$) *in blue is used for t_{10},* ($--$) *and red color for t_5 and* ($-$) *and green for t_3.*

The results of this section are carried out for $\tau = 3$ (roughly the sample mean of an equally-weighted portfolio) but are consistent across targets. Results for other thresholds are available upon request as well as for other non-elliptical distributions (in particular generalized hyperbolic distributions as studied in Mencía and Sentana (2005)) where results are more discordant between mean-variance and downside-risk averse investors due to the impact of heavy tails and comovement risk.

5 A real example of diversification

The aim of this application is to study the impact of heavy tails and comovements, between portfolios of important financial indexes, on constructing well diversified portfolios for mean-variance as well as for downside-risk averse investors. As in Harlow (1991), we work with financial equity and bond indexes. In particular we use data from *US* and *UK*: Dow-Jones Corporate bonds with 2-years maturity Index (*djbc*) describing *US* debt market; Dow-Jones Stock Index (*djsi*) for *US* equity market, and Ftse100 Index (*ftse*) for *UK* equity market. The data spans the period 22/1/2001 - 24/09/2004 and are obtained from *Freelunch.com* website. There are three possible combinations by pairs with these assets: $A = [djbc, djsi]$, $B = [djbc, ftse]$, and $C = [djsi, ftse]$. The scatterplots of log-returns forming these portfolios are in figure 5.1.

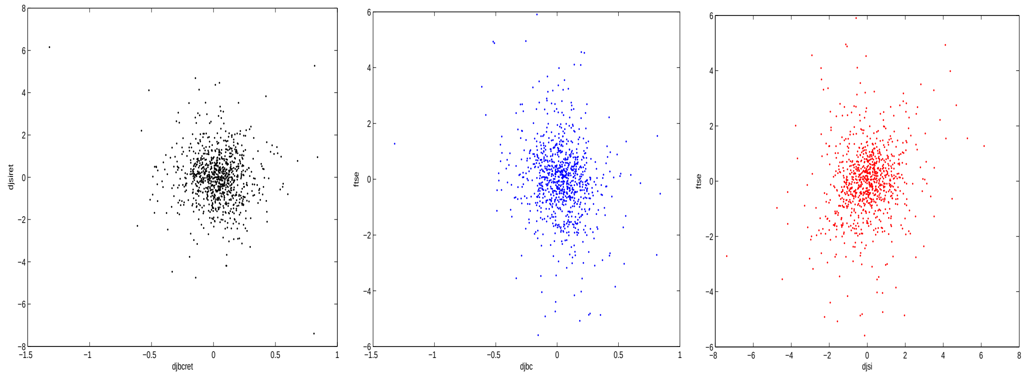

Figure 5.1. *Left panel plots Portfolio A; middle panel Portfolio B; and right panel Portfolio C.*

In contrast to the previous section the variance-covariance matrix is not identical across portfolios. The relevant second order moments are $V(djbc) = 0.0337$, $V(djsi) = 1.579$, $V(ftse) = 1.806$, $Cov(djbc, djsi) = -0.019$, $Cov(djbc, ftse) = -0.046$ and $Cov(djsi, ftse) = 0.3128$. From these values and the plots in figure 5.1 it seems portfolio A has uncorrelated components and exhibits the lower level of tail dependence. Portfolio C on the other hand reports strong positive comovements. The study of the tails reveals that the three assets are heavy-tailed. The tail index takes a value close to 0.3 indicating a significant degree of heaviness. These estimates are obtained by using Hill's estimator (Hill, 1975),

$$\hat{\xi}_n(k) = \frac{1}{k}\sum_{i=1}^{k} ln\, r_{(i)} - ln\, r_{(k+1)},$$

where $r_{(1)} < r_{(2)} < \ldots < r_{(n)}$ denote order statistics corresponding to portfolio returns.[2] Figure 5.2 shows that the estimates of ξ stabilize after inaccurate initial estimates defined by the first order statistics and indexed by k. These results indicate that diversification in this example makes sense and is not driven by the asset with heavier tail.

[2]Note that returns on financial assets are characterized by exhibiting serial dependence on the conditional variance. It is well known that in this framework estimates of ξ are still consistent but no longer efficient unless we filter the dependence in volatility. This is further the intention of our application where we assume returns are serially independent as in the literature in portfolio diversification.

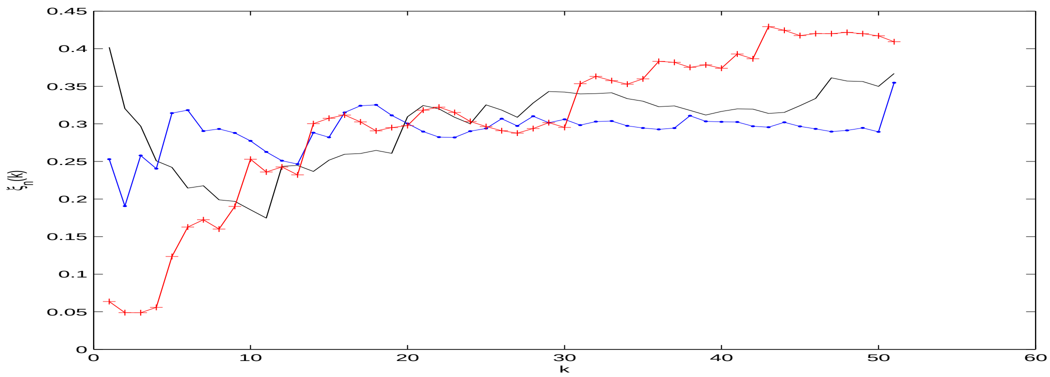

Figure 5.2. *Hill's plot. (-) depicts the path of tail index estimates of DJBC Index, $(\cdot-)$ of DJSI Index and $(+-)$ of Ftse100.*

Figure 5.3 presents the efficient portfolio frontier corresponding to each risk measure for a threshold $\tau = 0$.

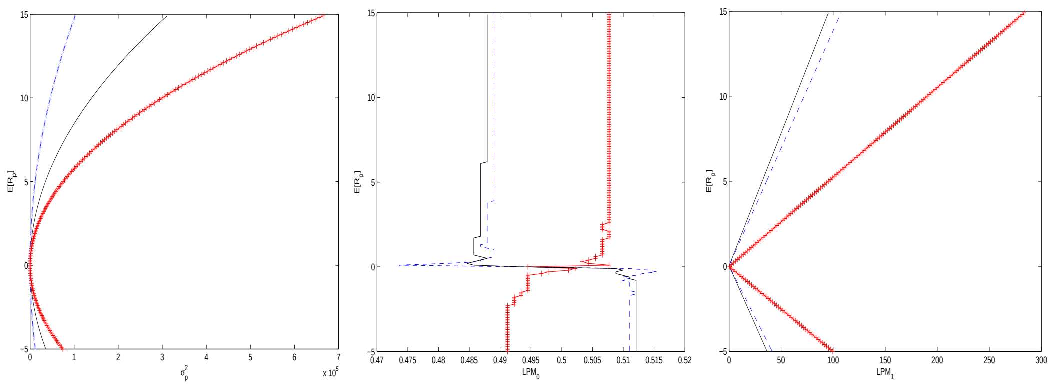

Figure 5.3. *Left panel plots mean-variance efficient portfolio frontier; middle panel mean-LPM_0 efficient curve and right panel mean-LPM_1 both for $\tau = 0$. $(-)$ in black color describes Portfolio A; $(--)$ in blue depicts Portfolio B, and $(+-)$ in red is for Portfolio C.*

In the case of the LPM_1 risk measure the scale of the plot is driven by portfolio C. For this reason, in order to observe properly the distance between the efficient sets for A and B we present figure 5.4.

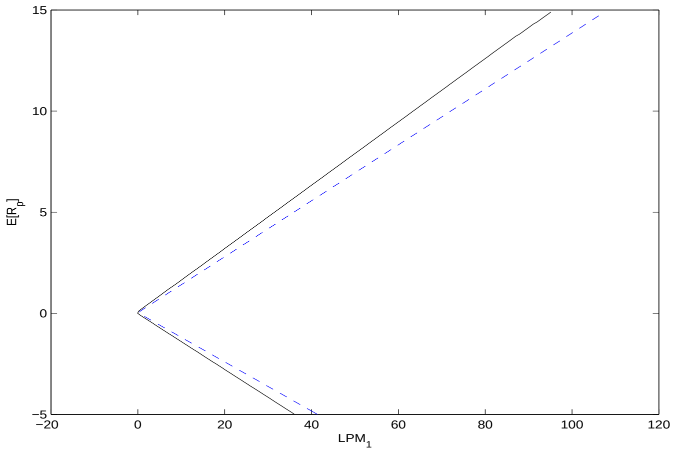

Figure 5.4. *Mean-LPM_1 portfolio frontier for $\tau = 0$. $(-)$ in black color describes Portfolio A and $(--)$ in blue Portfolio B.*

Three conclusions can be obtained from the efficient portfolio frontier. First, it is clear that Portfolio C is ruled out by both mean-variance and downside-risk averse investors. This portfolio has higher positive comovements and higher variance, and the analysis of the efficient portfolio frontier reveals that shorting one of the assets and thereby benefiting from the existence of positive comovements does not lead to portfolios outperforming A and B. This result is in accordance with existing literature in portfolio diversification and flight to quality, where it is commonly agreed that investors prefer to invest in bonds and stocks than solely in stocks in different marketplaces. Second, the ranking of mean-variance averse investors differs from that of those downside-risk averse investors penalizing negative returns on the portfolio. The choice of a threshold $\tau = 0$ is motivated by our willing of studying individuals with high risk aversion profile. These investors are concerned about the occurrence of losses in the portfolio and not just about large negative returns. From the efficient curves in the middle and right panel of figure 5.3 it seems that they prefer portfolio A to B. This outcome is not surprising since this high level of risk aversion and the choice of A over B can be due to country risk, that is, investors overvaluing domestic assets over foreign investments. On the other hand mean-variance averse investors prefer cross-borders diversification. The flight to quality in this case includes fleeing to other international markets. The rationale for this diversification seems to be different from the rationale of downside-risk averse investors. The latter type minimizes losses by exploiting complementarity of domestic financial markets, while the former type smooth investment returns by investing in diverse assets a priori more independent. And third, different downside-risk measures provide the same ranking of portfolios as observed in Danielsson et al. (2006). Note that the efficient sets derived from LPM_0 are not convex. This is, as commented in Section 2.2, because this measure assigns the same weight to each possible negative return below the threshold failing to describe any form of risk aversion.

6 Conclusions

If returns on a portfolio follow an elliptical distribution, mean-variance minimizing agents construct efficient and well diversified portfolios. The empirical evidence however consistently rejects this pointing towards more convoluted multivariate distributions. This phenomenon challenges investors' optimal asset allocation in different ways. Rational investors should not be mean-variance averse but mean-downside-risk averse. The latter type of investors are concerned about lower partial moments of the distribution of returns. These moments depend on the presence of comovements between assets, on the marginal tail behavior of each asset, on an optimal choice of the share invested in each asset, and finally on the tail behaviour and downside variance of the distribution of the optimal portfolio. By uncovering these factors we find that investors only allowed to have long positions construct well diversified portfolios by using asymptotically tail independent assets with tails exponentially decreasing. If they are allowed to hold short positions investment strategies are more involved and could benefit from comovements between assets and from heavy tails.

The case of portfolios with assets exhibiting different tail behaviour is also important. In particular portfolios of assets with very heavy tailed distributions do not diversify risk at all in case of financial distress because the asset with heavier tail drives the return on the overall portfolio down. Adding assets to this portfolio will not diminish risk but add complexity to its management.

Finally from the application to data of *UK* and *US* financial markets we conclude that those portfolios consisting of bonds and stocks achieve higher levels of diversification. This agrees with existing literature on the topic. More importantly, we find that mean-variance and downside-risk averse investors construct different efficient portfolios. Thus, according to downside-risk measures there is also evidence of misleading mean-variance diversification between domestic (*US*) bonds and *UK* stocks given the comovement risk found between *US* and *UK* equity markets.

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