General closed-form basket option pricing bounds

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This article presents lower and upper bounds on the prices of basket options for a general class of continuous-time financial models. The techniques we propose are applicable whenever the joint characteristic function of the vector of log-returns is known. Moreover, the basket value is not required to be positive. We test our new price approximations on different multivariate models, allowing for jumps and stochastic volatility. Numerical examples are discussed and benchmarked against Monte Carlo simulations. All bounds are general and do not require any additional assumption on the characteristic function, so our methods may be employed also to non-affine models. All bounds involve the computation of one-dimensional Fourier transforms, hence they do not suffer from the curse of dimensionality and can be applied also to high dimensional problems where most existing methods fail. In particular we study two kinds of price approximations: an accurate lower bound based on an approximating set and a fast bounded approximation based on the arithmetic-geometric mean inequality. We also show how to improve Monte Carlo accuracy by using one of our bounds as a control variate.

Keywords: Basket option, Option pricing, Fourier inversion, Control variate

JEL Classification: C63, G13

Basket options are popular derivative contracts which are becoming increasingly widespread in many financial markets, for example equity, FX and commodity markets. Given a vector of weights \( \mathbf{w} = (w_1, \ldots, w_n) \in \mathbb{R}^n \), the basket is defined as the weighted arithmetic average of the \( n \) stock prices \( S_1(t), \ldots, S_n(t) \) at time \( T \):

\[
A_n(T) = \sum_{k=1}^{n} w_k S_k(T).
\]

We assume, without loss of generality, that \( \sum_{k=1}^{n} w_k = 1 \). A basket call option gives the holder the right, but not the obligation, to purchase the portfolio of assets at a fixed price \( K \), known as the option’s strike price. We consider European-style options, where the buyer has the right to exercise the option only at maturity \( T \). Hence, the basket option payoff at time \( T \) is \((A_n(T) - K)^+\). Another important example is the spread option, where the payoff involves the difference of two or more underlyings, see e.g. Carmona and Durrelman (2003) and Caldana and Fusai (2013).
time \( t \) no-arbitrage fair price of the basket option is

\[ C_K(t) = e^{-r(T-t)}E_t[(A_n(T) - K)^+] , \]

where the \( t \)-conditional expectation is computed with respect to a risk-neutral measure and \( r \) is a constant riskless interest rate.

A basket option is similar to an Asian claim, where the payoff is determined by the average underlying price over some predetermined period of time. In most contributions from the literature on the valuation of such products the underlying asset prices are assumed to follow lognormal processes. However, the celebrated Black and Scholes (1973) formula cannot be easily extended to the basket option case, since the lognormal distribution is not closed under summation. Several approaches have been proposed to solve the problem, including Monte Carlo simulations, tree-based methods, partial differential equations, and analytical approximations. The last category is the most appealing because most other methods are computationally expensive due to the large dimension of the problem. In addition, it is not easy to extend such methods to a non-Gaussian setting.

Under the assumption that the dynamics of the underlying follows a multivariate geometric Brownian motion, several accurate analytical approximations are available. Curran (1994) introduces the idea of a conditioning variable and conditional moment matching. In particular, he proposes a method based on conditioning on the geometric mean. Assuming \( \Lambda \) is a random variable correlated with \( A_n \) and satisfying \( A_n \geq K \), whenever \( \Lambda \geq \kappa \) for some constant \( \kappa \), the option price is decomposed into two parts:

\[ E_t[(A_n(T) - K)^+] = E_t[(A_n(T) - K)I(\Lambda > \kappa)] + E_t[(A_n(T) - K)^+I(\Lambda < \kappa)] , \]

where \( I(\cdot) \) is the indicator function, taking unit value whenever the argument is true and zero otherwise. By choosing \( \Lambda \) to be the geometric average, the first part can be calculated exactly. The second part can be computed approximately by means of the conditional moment matching method. A similar conditioning argument has been used by Rogers and Shi (1995), where lower and upper bounds for Asian options are derived. Since the approach for Asian options can be easily adapted to basket options and vice-versa, Thompson (1999) and Beisser (2001) extend to basket options the idea of Rogers and Shi (1995) and examine the bound

\[ E_t[(A_n(T) - K)^+] \geq E_t[(E[A_n(T)|\Lambda] - K)^+] . \]

The approximation in formula (2) can be computed in closed-form in the lognormal framework. It is a lower bound but it turns out to be very close to the true option value in many practical situations. Rogers and Shi (1995) also give an upper bound to the true value, which was later improved by Nielsen and Sandmann (2003) as

\[ E_t[(A_n(T) - K)^+] \leq E_t[(E[A_n(T)|\Lambda] - K)^+] + \frac{1}{2}E_t[\text{var}(A_n(T)|\Lambda)I(\Lambda < \kappa)]^{1/2}E_t[I(\Lambda < \kappa)]^{1/2} . \]

Other bounds proposed in the literature exploit comonotonicity. In this case the central idea consists in replacing the original basket by another one, with a simpler dependence structure. The newly introduced basket involves the components of the comonotonic version of the original random vector, see for example Dhaene et al. (2002b) and Dhaene et al. (2002a). Vyncke et al. (2004) propose a two-moment matching approximation with a convex combination of the comonotonic lower and upper bounds for Asian options while Vanmaele et al. (2004) suggest a similar approximation for basket options. Deelstra et al. (2004) develop a general framework for pricing basket and Asian options via conditioning and derive lower and upper bounds based on comonotonic risks. The case of Asian basket option is discussed in Deelstra et al. (2008). All mentioned bounds are derived in
the lognormal framework.

Other authors tried to approximate the basket by using the so-called moment matching method. The idea is to approximate \( A_n(T) \) via \( \hat{A}_n(T) \), where \( \hat{A}_n(T) \) is a random variable with a suitable distribution, chosen to be “close” to the distribution of \( A_n(T) \). For example, Gentle (1993) approximates the arithmetic average in the basket payoff by a geometric average. The fact that a geometric average of lognormal random variables is again lognormally distributed allows the application of a Black–Scholes-type valuation formula for pricing the approximating payoff. Vorst (1992) uses the arithmetic-geometric mean inequality to produce lower and upper bounds to the option price and proposes an approximation lying between bounds. Levy (1992) approximates the distribution of the basket by a lognormal distribution such that its first two moments coincide with those of the original distribution of the weighted sum of the stock prices. Huynh (1993) applies the Edgeworth expansion method to basket option valuation for Asian options. Milevsky and Posner (1998a) use the reciprocal Gamma distribution as an approximation for the distribution of the basket. The motivation is the fact that the distribution of correlated lognormally distributed random variables converges to a reciprocal Gamma distribution as the dimension of the basket increases, under special assumptions about the covariance structure. Milevsky and Posner (1998b) use distributions from the Johnson (1949) family as state–price densities to match the higher moments of the arithmetic mean distribution. Ju (2002) considers a Taylor expansion of the ratio of two characteristic functions: the one of the arithmetic average and the one of an approximating lognormal random variable. Such Taylor expansion is computed around zero volatility. Zhou and Wang (2008) approximate the basket distribution by a log-extended-skew-Normal distribution. Further extensions and applications are discussed by Lord (2006).

Many of the methods listed above have limited validity or scope. They may require a basket with positive weights or they may not identify the sensitivities with respect to each basket component. In this regard, Alexander and Venkatramanan (2012) derive a general analytic approximation for pricing basket options expressing each option’s price as a sum of the prices of various compound exchange options. They derive an analytic approximation for the price of the compound exchange option, first under the assumption that the underlying assets of these options follow correlated lognormal processes, and then under more general assumptions for the asset price processes. The case of a basket where not all assets have positive weights \( (w_k < 0 \text{ for some } k) \) is discussed by Borovkova et al. (2007), Li et al. (2010) and Deelstra et al. (2010) in a lognormal setting. Borovkova et al. (2007) approximate the basket distribution by using a generalized family of lognormal distributions. Li et al. (2010) provide an extended Kirk approximation and a second-order boundary approximation for pricing spread options on a basket. Deelstra et al. (2010) develop approximations based on comonotonicity theory and moment matching methods for spread options, basket spread options, and Asian basket spread options.

Few results are available in the non-Gaussian setting. Flamouris and Giamouridis (2007) propose the use of a simplified jump process, namely, a Bernoulli jump process, and obtain approximate basket option valuation formulas. Xu and Zheng (2009) show that a lower bound similar to that of Rogers and Shi (1995) can also be calculated exactly in a special jump diffusion model with constant volatility and two types of Poisson jumps. An asymptotic expansion with a variance approximation and a lower bound to basket option values for local volatility jump diffusion models are studied by Xu and Zheng (2010, 2014), respectively.

In practice, it is sometimes useful to have model free pricing methods. This part of the literature considers the set of all models consistent with observed prices of vanilla options and recovers distribution free upper and lower bounds to the basket option price. The seminal paper is Bertsimas and Popescu (2002). Then, in a series of papers, Laurence and Wang (2004, 2005), Hobson et al. (2005b,a) and D’Aspremont and El Ghaoui (2006) derived distribution free bounds in the case of basket options with positive weights. Model free upper and lower bounds to the basket spread option are investigated in Laurence and Wang (2008). Lower and upper bounds based on comonotonicity theory are theoretically applicable to general dynamics, but research of such methods outside the lognormal setting is still in its early stage.
Another approach assumes the knowledge of the model characteristic function. In such a framework, Hurd and Zhou (2010) propose a general pricing method for a two-dimensional spread option and describe how to generalize it to a multidimensional payoff. Their pricing method is exact and based on an explicit formula for the Fourier transform of the spread option payoff in terms of the Gamma function. Lord et al. (2008) and Jackson et al. (2008) proposed a general fast Fourier transform (FFT) pricing framework for multi-asset options. All these methods require some particular assumptions on the characteristic function specification, ruling out important models such as mean-reverting models. The work of Jackson et al. (2008) has been later generalized by Jaimungal and Surkov (2011) to a cross-commodity modeling framework, allowing pricing for mean-reverting assets. The main drawback of all these methods is that they need an $n$-dimensional FFT to price an $n$-dimensional basket option. To this end, Leentvaar and Oosterlee (2008) propose a parallel partitioning approach to tackle the so-called curse of dimensionality when the number of underlying assets becomes large. However, they did not provide results for baskets with dimension greater than seven in their paper.

Readers interested in other basket option pricing methods, based on partial differential equations, Monte Carlo simulations, binomial trees and lattice techniques, are referred to the list of references given in Zhou and Wang (2008).

In conclusion, the existing literature on basket option approximation methods has three weak points:

(i) Many methods have limited applicability because they require the positivity of the basket weights, so they cannot deal with the basket spread option case.

(ii) Most studies are limited to the lognormal case. The study of general pricing methods is still limited.

(iii) Analytical formulas are available in the non-Gaussian case but they involve an $n$-dimensional FFT and, in practice, they are of little help for applications involving a large number of assets.

This article presents lower and upper bounds for the basket option price, assuming very general dynamics for the $n$ underlyings. The only quantity we need to know explicitly is the joint characteristic function of the log-returns of the assets. All bounds are general and do not require any additional assumption on the characteristic function specification. In particular, we do not assume that the characteristic function is exponential affine with respect to the initial state of the log asset price vector. Our procedure allows the computation for a very large class of stochastic dynamics like mean reverting and non-affine models. Moreover, the basket weights are not required to be positive. Our bounds involve the computation of a univariate Fourier inversion, hence they do not suffer from the curse of dimensionality. This makes our methodologies particularly appealing for higher dimensional problems. To our knowledge, no other general method is successfully applicable to the basket option pricing problem when the basket dimension is large. In general all existing methods face unaffordable computational cost. The only feasible alternative to our approximations is Monte Carlo simulation. However, by using one of our bounds as a control variate, we can also significantly improve the accuracy of the Monte Carlo method itself. In particular we study two kinds of price approximations: an accurate lower bound based on an approximating set, and a fast bounded approximation based on the arithmetic-geometric mean inequality. We test the bounds on different models, including non-Gaussian ones. Numerical examples are discussed and benchmarked against Monte Carlo simulations. The wide range of contexts in which basket option pricing problems arise means that the relevance of our result falls also beyond exotic option valuation. For example, the probability distribution of a basket is required in portfolio allocation problems as well. For such problems, a weight optimization is often required, thus a fast procedure to compute the portfolio distribution is needed.

The article is outlined as follows: Section 1 discusses an accurate lower bound based on an approximating set. Section 2 put forward a fast bounded approximation obtained through the arithmetic-geometric mean inequality. The geometric Brownian motion case is discussed in Section 4.
1. An accurate lower bound through an approximating set

Lower bounds to spread and basket option price can be obtained by approximating the option exercise region via an event set defined through a suitable random variable. Examples in the lognormal framework are Rogers and Shi (1995), Thompson (1999), Carmona and Durrelman (2003) and Bjerksund and Stensland (2014). Extensions to some jump diffusion models are given in Xu and Zheng (2009, 2014). The contribution of this section is the original extension of this popular category of lower bounds to a characteristic function framework. Caldana and Fusai (2013) provide a similar extension, limiting their analysis to options written on the spread between two assets.

Given the set
\[ A = \{ \omega \in \Omega : A_n(T) > K \}, \]
the value of the basket option price is
\[ C_K(t) = e^{-r(T-t)} \mathbb{E}_t [(A_n(T) - K)^+] = e^{-r(T-t)} \mathbb{E}_t [(A_n(T) - K) I(A)] . \] (3)

For any event set \( \mathcal{G} \subset \Omega \)
\[ \mathbb{E}_t [(A_n(T) - K) I(\mathcal{G})] \leq \mathbb{E}_t [(A_n(T) - K)^+ I(\mathcal{G})] \leq \mathbb{E}_t [(A_n(T) - K)^+] . \]

Applying the positive part and discounting, it follows that
\[ C_G^K(t) = e^{-r(T-t)} \mathbb{E}_t [(A_n(T) - K)^+ I(\mathcal{G})] \leq C_K(t) , \] (4)
Depending on the set \( \mathcal{G} \), the value of \( C_G^K(t) \) is a lower bound to the basket option price \( C_K(t) \). We define the set \( \mathcal{G} \) using the geometric average \( G_n(T) \) of the underlying prices,
\[ G_n(T) = \prod_{k=1}^{n} S_k(T)^{w_k} . \]

Being \( Y_n(T) = \ln G_n(T) \), we set \( \mathcal{G} = \{ \omega : Y_n(T) > \kappa \} \). This choice, which is intuitive and technically convenient, also turns out to be very accurate. We address the choice of the parameter \( \kappa \) shortly. Let \( X_k(T) \) be the log-return over the period \([t, T] \):
\[ X_k(T) = \ln \left( \frac{S_k(T)}{S_k(t)} \right) . \]

We assume that the risk-neutral joint characteristic function of the \( n \) stock returns is known:
\[ \varphi_T(\gamma) = \mathbb{E}_t \left[ e^{i \sum_{k=1}^{n} \gamma_k X_k(T)} \right] , \] (5)

3 and some non-Gaussian models are shown in Section 4. Finally, Section 5 presents numerical experiments.
where $\gamma = [\gamma_1, \gamma_2, \ldots, \gamma_n]$. Simple algebra shows that

$$Y_n(T) = \sum_{k=1}^{n} w_k \ln S_k(T)$$

$$= \sum_{k=1}^{n} w_k X_k(T) + \sum_{k=1}^{n} w_k \ln S_k(t)$$

$$= \sum_{k=1}^{n} w_k X_k(T) + Y_n(t).$$

so the joint characteristic function of the log-returns and the log-geometric average is

$$\Phi_T(\gamma_0, \gamma, w, Y_n(t)) = E_t \left[ e^{i \sum_{k=1}^{n} \gamma_k X_k(T) + i \gamma_0 Y_n(T)} \right]$$

$$= E_t \left[ e^{i \sum_{k=1}^{n} (\gamma_k + w_k \gamma_0) X_k(T) + i \gamma_0 Y_n(T)} \right]$$

$$= e^{i \gamma_0 Y_n(t)} \phi_T(\gamma + \gamma_0 w)$$

and $\gamma + \gamma_0 w$ is the vector with components $\gamma_k + \gamma_0 w_k$. In particular, the characteristic function of the log-geometric average is given by $\Phi_T(\gamma_0, 0, w, Y_n(t))$. Following Lee (2004), we denote by $A_X$ the interior of the set

$$\left\{ \nu \in \mathbb{R}^n \mid E_t \left[ e^{i \sum_{k=1}^{n} \nu_k X_k(T)} \right] < \infty \right\}.$$ 

The explicit computation of the lower bound in (4) is given in the following proposition.

**Proposition 1.1** Let $\delta > 0$ and assume that $\{e_k, \delta w + e_k\} \in A_X$, $\forall k = 1, \ldots, n$, for $e_k$ denoting the $k$-th element of the canonical basis in $\mathbb{R}^n$. A lower bound to the basket option price is given by the following formula

$$C^G_K(t) = \max_{\kappa \in \mathbb{R}} C^G_K(t, \kappa),$$

where

$$C^G_K(t, \kappa) = \left( e^{-\delta \kappa - r(T-t)} \frac{1}{\pi} \int_0^{+\infty} e^{-i\gamma \kappa} \Psi_T(\gamma; \delta) d\gamma \right)^+,$$

$$\Psi_T(\gamma; \delta) = \frac{1}{i\gamma + \delta} \left[ \sum_{k=1}^{n} w_k S_k(t) \Phi_T(\gamma - i\delta, -ie_k, w, Y_n(t)) - K \Phi_T(\gamma - i\delta, 0, w, Y_n(t)) \right].$$

**Proof:** See Appendix A. □

Some remarks are in order about the above formula. First, the computation of the lower bound requires an univariate Fourier transform inversion and an optimization with respect to the parameter $\kappa$. The damping factor $\exp(-\delta \kappa)$, for $\delta > 0$, is introduced in (8) to ensure the existence of the Fourier transform, as Carr and Madan (2000) do. However the numerical inversion is not restricted to this approach and can be performed by using alternative representations, as in Lewis (2000) and Lee (2004). In particular, by following Lee (2004), it is possible to write the lower bound for any
Second, if the characteristic function $\Phi_T$ is explicitly known, then the Fourier transform of the lower bound can be expressed in closed form as well in terms of the complex function $\Psi_T$. The integral in (8) can be easily computed using standard numerical quadratures (e.g. NIntegrate in Mathematica or quadgk in Matlab) or via an FFT algorithm.

The third remark is relative to the characteristic function. The only requirement we set on it is its availability. In particular, we do not require the characteristic function to be exponential affine with respect to the initial value of the state variables. In contrast to this, existing Fourier-based methods for basket options are limited to affine models. In addition, no assumption on the sign of basket weights is introduced in our case.

The fourth remark is about the optimal value of $\kappa = \kappa^*$. Figure 1 shows a typical shape for $C^g_K(t, \kappa)$, as a function of the parameter $\kappa$.

Our lower bound requires the maximization of $C^g_K(t, \kappa)$. In practice, the optimization can be accelerated by using a one-dimensional FFT to bound the optimization interval and to guess the starting optimization value $\kappa^{start}$. Therefore we adopt a two-step strategy which results in a significant time saving:

**Step 1 – Bounding the search domain** We compute formula (8) via FFT and we obtain $C^g_K(t, \kappa)$ on an equally spaced grid $\{\kappa_1, \ldots, \kappa_M\}$. Then we perform a grid search to find $\kappa_m$ such that

$$\kappa_m = \arg \max_{\kappa \in \{\kappa_1, \ldots, \kappa_M\}} C^g_K(t, \kappa),$$

i.e. an estimate of the lower bound on such a grid. Since $C^g_K(t, \kappa_m)$ is the best approximation we can get via FFT, we select $\kappa_m$ as the starting point of the optimization routine in the second step. Extensive numerical tests show that the target function is unimodal, so the maximum of $C^g_K(t, \kappa)$ should lie in the interval $[\kappa_{m-1}, \kappa_{m+1}]$. If the maximum is not unique (i.e. it is achieved on two different points of the grid), we restrict the optimization to the interval delimited by these two values and we use their average as starting point.

**Step 2 – Constrained optimization** We perform an optimization for the integral in (8) to all

![Figure 1. Lower bound $C^g_K(t, \kappa)$ as a function of the parameter $\kappa$ for a mean reverting jump diffusion model. The basket is composed by four assets. Parameter values are as in Table 4 and strike price $K = 30$.](image)
Figure 2. Optimization procedure for a mean reverting jump diffusion model. The basket is composed by four assets. Parameter values are as in Table 4 and strike price $K = 30$. The blue line indicates the values $C^G_K(t, \kappa)$ as a function of the parameter $\kappa$. The red markers refer to values obtained via FFT. The black marker indicates the optimized lower bound $C^G_K = 7.5059$. The search domain is restricted to the red segment. The true price estimated via Monte Carlo simulation is $7.5768$.

$\kappa$ in the range $[\kappa_{m-1}, \kappa_{m+1}]$. We assume that the numerical quadrature is performed using a grid with $N$ integration points. Given the integration grid and a maturity $T$, we notice that all evaluations of the function $\Psi_T$ in (9) do not depend on the variable $\kappa$ over which the optimization is performed. Hence it is possible to evaluate and store all instances of this function computed on the quadrature nodes and then use the stored values in the optimization. Figure 2 shows the two-step procedure with reference to a mean reverting jump diffusion model.

The lower bound can also be used for the computations of Greeks. The envelope theorem guarantees that changes in the optimizer of the objective do not contribute to the change in the objective function, see Takayama (1974) page 160. Therefore, assuming that interchange of differentiation and integration is allowed, the first-order sensitivity of the basket option price to a change in the spot price of a generic asset is given by

$$
\frac{\partial C^G_K(t, \kappa^*)}{\partial S_k} = \mathcal{I} \left( \int_0^{+\infty} e^{-i\gamma\kappa^*} \Psi_T(\gamma; \delta) d\gamma \geq 0 \right) e^{-\delta \kappa^* - r(T-t)} \frac{1}{\pi} \int_0^{+\infty} e^{-i\gamma\kappa^*} \frac{\partial \Psi_T(\gamma; \delta)}{\partial S_k} d\gamma,
$$

Similar formula can be computed for other Greeks.

Finally, the main point concerning formula (7) is that the approximated option price is always obtained through the optimization of a univariate Fourier inversion. The computational cost of the method is $O(n^2 N + M \log(M))$, and increases quadratically with the number of assets $n$ composing the basket. Therefore our technique does not suffer from the curse of dimensionality as it happens for many other Fourier inversion methods proposed in the literature (Hurd and Zhou (2010), Lord et al. (2008), Jackson et al. (2008) and Jaimungal and Surkov (2011)). These methods provide an exact solution but, requiring a multivariate FFT, they have a cost that is of order $O(nN^n \log(N))$. Due to their computational cost, they are not applicable to the basket option problem when the basket dimension is high. Indeed, the largest dimension of the basket we found in the literature seven and the result is obtained by means of a parallel partitioning approach, see Leentvaar and
Oosterlee (2008).
Looking at the remaining literature, the model free bounds of Hobson et al. (2005b) and Laurence and Wang (2008) are the only bounds so general to cover all the models and the basket sizes we are interested in. Such model free bounds require one to compute prices of European call and put on each underlying and solve an optimization problem. However numerical experiments, available upon request, show that their performance in terms of computational speed and accuracy is very poor with respect to the pricing problem under examination.

Our method guarantees faster approximations. Basket options can be easily priced also for high dimensions, and a broader class of problems can be considered. Up to our knowledge, the only feasible alternative to our approximations is Monte Carlo simulation. However, using the bound \( C_{G}^{K} \) as a control variate\(^1\), we can also improve the accuracy of a Monte Carlo method. Indeed, we rewrite Eq. (1) as

\[
C_{K}(t) = C_{G}^{K}(t) + e^{-r(T-t)}E_t \left[ (A_n(T) - K)^+ \right] - e^{-r(T-t)}E_t \left[ ((A_n(T) - K)I(G))^+ \right].
\]

We calculate \( C_{G}^{K}(t) \) via formula (7) on the optimal approximating set \( G \) and we use Monte Carlo simulation to compute the two expected values, which are highly correlated. In this way the simulation error is considerably reduced. Our formula provides a ready-to-use control variate estimate that allows us to improve the accuracy of Monte Carlo simulations. The accuracy of our lower bound as well as our control variate are proved via extensive numerical tests on a battery of different models in section 5.

2. A fast bounded approximation through the arithmetic-geometric mean inequality

We discuss here new upper and lower bounds to the basket option price and we propose a price approximation lying between such bounds, exploiting the so-called geometric-arithmetic mean inequality. This consists in a generalization of the Vorst (1992) approach to a characteristic function framework, allowing the basket weights to be negative.

Denoting \( J^{pos} \) and \( J^{neg} \) the sets of the indices corresponding to the positive and negative weights respectively, the basket can be rewritten as

\[
A_n(T) = \sum_{k \in J^{pos}} w_k S_k(T) - \sum_{k \in J^{neg}} |w_k| S_k(T) = c^{pos} A^{pos}_n(T) - c^{neg} A^{neg}_n(T),
\]

where

\[
A^{pos}_n(T) = \frac{\sum_{k \in J^{pos}} w_k S_k(T)}{\sum_{k \in J^{pos}} w_k}, \quad A^{neg}_n(T) = \frac{\sum_{k \in J^{neg}} |w_k| S_k(T)}{\sum_{k \in J^{neg}} |w_k|},
\]

and

\[
c^{pos} = \sum_{k \in J^{pos}} w_k, \quad c^{neg} = \sum_{k \in J^{neg}} |w_k|.
\]

We define \( w^{pos} \) the vector having as \( k \)-th component \( w^{pos}_k = w_k / \sum_{k \in J^{pos}} w_k \), when \( k \in J^{pos} \) and 0 when \( k \in J^{neg} \). Similarly, we define \( w^{neg} \) the vector having \( w^{neg}_k = |w_k| / \sum_{k \in J^{neg}} |w_k| \) in the \( k \)th

\(^1\)See e.g. Glasserman (2003).
position when $k \in J^{neg}$ and 0 when $k \in J^{pos}$. We also define

$$G^{pos}_n(T) = \prod_{k \in J^{pos}} S_k(T)w^{'pos}_k, \quad G^{neg}_n(T) = \prod_{k \in J^{neg}} S_k(T)w^{'neg}_k,$$

$Y^{pos}_n(T) = \ln G^{pos}_n(T)$ and $Y^{neg}_n(T) = \ln c^{neg}_n(T)$.

Assuming $K > 0$, we can now provide upper and lower bounds to the basket option price. We also obtain an approximation lying between such bounds, in this way generalizing Vorst (1992).

**Proposition 2.1** A lower bound $L^AG_K(t)$, an upper bound $U^AG_K(t)$ and an approximation $C^AG_K(t)$ to the basket option value (1), such that $L^AG_K(t) \leq C^AG_K(t) \approx C^AG_K(t) \leq U^AG_K(t)$, are obtained as

$$L^AG_K(t) = e^{-r(T-t)}E_t[c^{pos}A^{pos}_n(T) - c^{neg}A^{neg}_n(T) - K^+],$$

$$U^AG_K(t) = e^{-r(T-t)}E_t[c^{pos}A^{pos}_n(T) - c^{neg}A^{neg}_n(T) - K^+],$$

$$C^AG_K(t) = e^{-r(T-t)}E_t[c^{pos}A^{pos}_n(T) - c^{neg}A^{neg}_n(T) - K^+] +$$

$$e^{neg}e^{-r(T-t)}\{E_t[c^{neg}G^{neg}_n(T) - E_t[A^{neg}_n(T)]\}, \quad \text{ (13)}$$

$$U^AG_K(t) = e^{-r(T-t)}E_t[c^{pos}G^{pos}_n(T) - c^{neg}G^{neg}_n(T) - K^+] +$$

$$e^{pos}e^{-r(T-t)}\{E_t[A^{pos}_n(T)] - E_t[G^{pos}_n(T)]\}, \quad \text{ (14)}$$

$$C^AG_K(t) = e^{-r(T-t)}E_t[c^{pos}G^{pos}_n(T) - c^{neg}G^{neg}_n(T) - K^+] +$$

$$e^{neg}e^{-r(T-t)}\{E_t[c^{neg}G^{neg}_n(T)]\}, \quad \text{ (15)}$$

where

$$K^* = K - E_t[c^{pos}A^{pos}_n(T)] + E_t[c^{pos}G^{pos}_n(T)] + E_t[c^{neg}A^{neg}_n(T)] - E_t[c^{neg}G^{neg}_n(T)]. \quad \text{ (16)}$$

**Proof:** See Appendix B.

In the spirit of Vorst (1992), approximation $C^AG_K(t)$ replaces the random variable $c^{pos}A^{pos}_n(T) - c^{neg}A^{neg}_n(T) - K$ with $c^{pos}G^{pos}_n(T) - c^{neg}G^{neg}_n(T) - K^*$ in the basket spread option payoff. Then the strike price $K^*$ is corrected as in formula (16), so that the unbiasedness condition on the first moment is guaranteed.

We observe that pricing formulae (13), (14) and (15) depend on the value of a call option written on the difference between $c^{pos}G^{pos}_n(T)$ and $c^{neg}G^{neg}_n(T)$, that is

$$e^{-r(T-t)}E_t[(c^{pos}G^{pos}_n(T) - c^{neg}G^{neg}_n(T) - K^+].$$

Therefore, we need the pricing of a spread option, that can be easily performed via the Hurd and Zhou (2010) method or through the approximation in Caldana and Fusai (2013), that we recall in Appendix C. These methods require the joint characteristic function of $[\ln(c^{pos}G^{pos}_n(T)), \ln(c^{neg}G^{neg}_n(T))]'$, that we state here for the sake of completeness:

$$E_t[e^{i\gamma_1\ln(c^{pos}G^{pos}_n(T))+i\gamma_2\ln(c^{neg}G^{neg}_n(T))}] = e^{i\gamma_1\ln(c^{pos}G^{pos}_n(T))+i\gamma_2\ln(c^{neg}G^{neg}_n(T))}[\varphi_T(\gamma_1w^{pos} + \gamma_2w^{neg})].$$

When the basket is strictly positive, $G_n(T) = G^{pos}_n(T)$ and $c^{pos} = 1$ by assumption. We have then the following corollary:

**Corollary 2.2** When the basket is strictly positive, a lower bound $L^AG_K(t)$, an upper bound $U^AG_K(t)$ and an approximation $C^AG_K(t)$ for the basket option value (1), such that $L^AG_K(t) \leq C^AG_K(t) \approx C^AG_K(t) \leq U^AG_K(t)$, are obtained as

1The strike price $K \leq 0$ leads to $E_t[(A_n(T) - K)^+] = E_t[(A_n(T) - K)$ for a positive basket.
where

\[ K^* = K - \mathbb{E}[A_n(T)] + \mathbb{E}[G_n(T)]. \]  

Proof: We consider Proposition (2.1) with \( G_n^{pos}(T) = G_n(T), \) \( c^{pos} = 1 \) and \( c^{neg} = 0. \) When weights are positive, pricing formulae (17),(18) and (19) require the pricing of a call option written on \( G_n(T), \) rather than pricing a spread option. The call price is easily computed via Fourier inversion as

\[ e^{-r(T-t)}\mathbb{E}[(G_n(T) - K)^+] = \frac{e^{-\delta \ln K - r(T-t)}}{\pi} \int_0^\infty e^{-i\gamma \ln K} \Psi_T^G(\gamma; \delta) d\gamma, \]  

where the characteristic functions \( \Psi_T^G \) of \( \ln G_n(T) \) is

\[ \Psi_T^G(\gamma; \delta) = \Phi_T(\gamma - i(\delta + 1), 0, w, Y_n(t)), \]  

and the parameter \( \delta \) tunes the damping factor.

Lower and upper bounds \( L^G_K(t) \) and \( U^G_K(t) \) in formulae (13), (14), (17) and (18) provide an interval for the approximation error of \( C^G_K(t). \) Depending on the expected differences between the arithmetic and the geometric average, such an interval may be small or wide.

The main advantage of pricing based on the arithmetic-geometric mean inequality is its computational speed. In the positive basket case, the computation of the bounded approximation requires one-dimensional integrations. The basket spread option case requires two-dimensional FFTs using the Hurd and Zhou (2010) method, or one-dimensional integrations using the approximation in Caldana and Fusai (2013). The computation of \( L^G_K(t), C^G_K(t) \) and \( U^G_K(t) \) is very fast, regardless of the basket dimension. Its computational cost is linearly increasing in the number of assets, rather than quadratically as for the lower bound of section 1.

Proposition 2.1 and Corollary 2.2 provide very general bounded approximations, that can be computed when the model characteristic function is known. Numerical experiments in Section 5 show that the approximation \( C^G_K(t), \) based on the arithmetic-geometric mean inequality, is in general less accurate than the lower bound \( C_K(t) \) discussed in Section 1. However the former is much faster than the latter and computing \( L^G_K(t), C^G_K(t) \) and \( U^G_K(t) \) may be very useful in all applications involving a large number of underlyings.

3. The geometric Brownian motion case

This section discusses in greater detail the geometric Brownian motion case and the explicit computation of the previously introduced price approximations.

We consider a multivariate Black–Scholes model. The dynamics are given by

\[ dS(t) = \text{Diag}(S(t)) \left( (r1 - q)dt + \sqrt{\Sigma}dW(t) \right), \]  

November 20, 2014 Quantitative Finance basket˙option˙qf
where \( r \) is the risk-free rate, \( \mathbf{q} \) is the vector of dividend yields for each asset, \( \mathbf{1} \) is a vector whose entries are all equal to one, \( \Sigma \) is the covariance matrix, and \( \mathbf{W} \) is an \( n \)-dimensional Brownian motion.

The risk-neutral joint characteristic function of the \( n \) stock returns in the geometric Brownian motion case is

\[
\varphi_T(\gamma) = e^{\gamma^\top \mathbf{m}(T-t) - \frac{1}{2} \gamma^\top \Sigma \gamma(T-t)},
\]

(24)

where

\[
\mathbf{m} = r \mathbf{1} - \mathbf{q} - \frac{1}{2} \text{Vec}(\Sigma_{kk})
\]

(25)

and \( \text{Vec}(\cdot) \) is the vectorization operator. From (6), the joint characteristic function of the log-returns and the log-geometric average is

\[
\Phi_T(\gamma_0, \gamma, \mathbf{w}, Y_n(t)) = e^{\gamma_0 Y_n(t) + i(\gamma^\top \gamma_0 \mathbf{w}^\top) \mathbf{m}(T-t) - \frac{1}{2} (\gamma^\top + \gamma_0 \mathbf{w}^\top) \Sigma(\gamma + \gamma_0 \mathbf{w})(T-t)}.
\]

(26)

Expression (26) can be used to compute the proposed lower and upper bounds; however, in the geometric Brownian motion setting, all formulas can be explicitly computed, see details in Appendix D. The lower bound \( C^D_K(t) \) given in formula (8) becomes

\[
C^D_K(t) = \max_{\kappa} e^{-r(T-t)} \left( \sum_{k=1}^{n} w_k S_k(t) e^{(r-q_k)(T-t)} \mathcal{N}(a_k \sqrt{T-t} - d) - K \mathcal{N}(-d) \right)^+.
\]

(27)

where

\[
d = \frac{\kappa - \mathbf{w}^\top (\ln(S(t)) + \mathbf{m}(T-t))}{\sigma^* \sqrt{T-t}}, \quad a_k = \frac{\sum_{j=1}^{n} w_j \Sigma_{kj}}{\sigma^*}, \quad \sigma^* = \sqrt{\mathbf{w}^\top \Sigma \mathbf{w}}
\]

and we indicate with \( \mathcal{N}(\cdot) \) the standard Normal distribution function. The following value for \( \kappa \) is a good starting point to implement the maximization in (27)

\[
\kappa^\text{start} = \sigma^* - \frac{K - \sum_{k=1}^{n} w_k S_k(t) e^{(r-q_k)(T-t)}}{\sum_{k=1}^{n} w_k a_k S_k(t) e^{(r-q_k)(T-t)} + \sum_{k=1}^{n} w_k \left( \ln S_k(t) + \left( r - q_k - \frac{\Sigma_{kk}}{2} \right)(T-t) \right)}.
\]

The expectations \( \mathbb{E}_t[A_n(T)] \) and \( \mathbb{E}_t[G_n(T)] \) are

\[
\mathbb{E}_t[A_n(T)] = \sum_{k=1}^{n} w_k S_k(t) e^{(r-q_k)(T-t)} \quad \text{and} \quad \mathbb{E}_t[G_n(T)] = G(t) e^{\left( \mathbf{w}^\top \mathbf{m} + \frac{\sigma^*}{2} \right)(T-t)}.
\]

In the positive basket case, call option values involved in bounds through the arithmetic-geometric mean inequality can be obtained using a Black–Scholes formula. Indeed, each asset price \( S_k(T) \) is lognormally distributed. Since \( G(T) \) is a product of lognormally distributed variables, it is also lognormally distributed

\[
G(T) \sim \mathcal{L}\mathcal{N}(\ln(G(t)) + \mathbf{w}^\top \mathbf{m}(T-t), \sigma^*^2(T-t))
\]

and clearly also \( e^{X_k(T)} \) is lognormally distributed

\[
e^{X_k(T)} \sim \mathcal{L}\mathcal{N}(m_k(T-t), \Sigma_{kk}(T-t)).
\]
Then, given a lognormal distributed random variable $Z \sim \mathcal{LN}(\mu, \sigma^2)$,

$$
\mathbb{E}[(Z - K)^+] = e^{\mu+\sigma^2/2}N\left(\frac{\mu - \ln(K) + \sigma^2}{\sigma}\right) - K N\left(\frac{\mu - \ln(K)}{\sigma}\right).
$$

Trivial modifications occur to compute the bounded approximation for basket spread options, see Caldana and Fusai (2013).

4. Non-Gaussian price models

This section presents several price models on which we analyze the performance of our novel bounds. For each model, we give a brief description and we provide the risk-neutral joint characteristic function of asset log-returns $\varphi_T(\gamma)$. The joint characteristic function of the log-returns and the log-geometric average $\Phi_T(\gamma, \omega, Y_0(t))$ used in the bounds computation is then immediately obtained via formula (6).

4.1. A jump diffusion stock price model

Let us consider a generalization to an $n$-dimensional case of the two-dimensional jump diffusion process with asymmetric Laplace distributed jump size discussed in Huang and Kou (2006), with reference to the pricing of two dimensional barrier options in equity markets.

The components of the stock price vector, for $k = 1, \ldots, n$, have the form

$$
S_k(t) = S_k(0) \exp\left(\left(\rho - \frac{\sigma^2}{2} - \lambda \sigma - \lambda k_{Z_k} - \lambda \sigma Z_k\right) t + \sigma_k W_k(t) + \sum_{m = 1}^{N_k(t)} Z_{k}^{(m)} + \sum_{m = 1}^{N(t)} Y_{k}^{(m)}\right),
$$

where $\sigma_k > 0$, for $k = 1, \ldots, n$, and $W_k, W_j$ are risk-neutral Brownian motions with instantaneous correlation $\rho_{kj}, |\rho| < 1$, for $k, j = 1, \ldots, n$. In addition, $\sum_{m = 1}^{N_k(t)} Z_{k}^{(m)}$, for $k = 1, \ldots, n$, are $n$ univariate compound Poisson processes driven by the Poisson processes $N_k$ with intensity rate $\lambda_k$. This jump component is unique to each stock and describes the idiosyncratic shocks for that particular asset only. The idiosyncratic jump sizes $Z_k$ are independently and identically distributed according to an asymmetric Laplace distribution $\mathcal{AL}(\alpha_{kk}, \xi_{kk})$. The model also allows for macroeconomic shocks described by

$$
\sum_{m = 1}^{N(t)} Y_{k}^{(m)} = \sum_{m = 1}^{N(t)} Y_{1}^{(m)}(m_Y), \ldots, \sum_{m = 1}^{N(t)} Y_{n}^{(m)}(m_Y),
$$

which is a $n$-dimensional compound Poisson process with intensity rate $\lambda$. Under the risk-neutral measure $\mathbb{Q}$ the jump sizes $Y$ are assumed to be independently and identically distributed according to a multivariate asymmetric Laplace distribution $\mathcal{MAL}(\alpha, \Sigma_Y)$, where $\alpha = (\alpha_1, \ldots, \alpha_n)^T$ and $\Sigma_Y$ is an $n \times n$ matrix whose elements are defined as

$$(\Sigma_Y)_{k,j} = \xi_k \xi_j \rho_{kj}, \quad k, j = 1, \ldots, n.$$

Finally, the quantities $\sigma_k$ and $k_{Z_k}, k = 1, \ldots, n$, in (28) are, respectively,

$$
\sigma_k = \int_{\mathbb{R}^2} [e^{y_k} - 1] m_Q(dy) = \int_{\mathbb{R}} [e^{y_k} - 1] m_Q(dy_k) = \frac{1}{1 - \alpha_k - \xi_k^2/2} - 1,
$$
Proposition 4.1 The joint characteristic function of the log-returns for the asymmetric Laplace jump diffusion model is

\[ \varphi_T(\gamma) = \exp \left[ (T - t) \left( i \gamma^\top \eta - \frac{1}{2} \gamma^\top \Sigma \gamma + \frac{\lambda}{1 - i \gamma^\top \alpha + \gamma^\top \Sigma \gamma/2} - \lambda + \sum_{k=1}^n \left( \frac{\lambda_k}{1 - i \gamma_k \alpha_{kk} + \gamma_k^2 \Sigma_{kk}/2} - \lambda_k \right) \right) \right], \quad (29) \]

where \((\Sigma)_{k,j} = \sigma_k \sigma_j \rho_{k,j}\) and \(\eta_k := r - q_k - \sigma_k^2/2 - \lambda \kappa_k - \lambda_k \kappa Z_k, k = 1, \ldots, n.\)

Proof: Straightforward generalization of Huang and Kou (2006) to the \(n\)-dimensional case. \(\Box\)

4.2. Mean-reverting jump diffusion model

The third model is a mean-reverting jump diffusion that generalizes the model proposed by Hambly et al. (2009) to describe the electricity spot price in energy markets. For \(k = 1, \ldots, n\), the spot price process \(S_k(t)\) is defined as the exponential of the sum of three components: a deterministic function \(f_k(t)\), a Gaussian Ornstein–Uhlenbeck process \(X_k(t)\), and a mean-reverting process with a jump component \(Y_k(t)\):

\[
\begin{align*}
S_k(t) &= \exp \left( f_k(t) + X_k(t) + Y_k(t) \right), \\
\frac{dX_k}{dt} &= -\alpha_k X_k(t) dt + \sigma_k dW_k, \\
\frac{dY_k}{dt} &= -\alpha_k Y_k(t-) dt + J_k^+ dN_k^+ - J_k^- dN_k^- .
\end{align*}
\]

The parameter \(\sigma_k\) is strictly positive and \(W_k\) is a risk-neutral Brownian motion. We assume a speed of mean reversion \(\alpha_k > 0\) for both the diffusion process \(X_k(t)\) and the jump process \(Y_k(t)\).

The Brownian motions \(W_k\) and \(W_j\) have instantaneous correlation \(\rho_{k,j}, |\rho_{k,j}| < 1\) for \(k \neq j\) and equal to 1 for \(k = j\). We denote with \(N_k^+\) and \(N_k^-\) Poisson processes with intensity \(\lambda_k^+\) and \(\lambda_k^-\), respectively, and describe the positive and negative jump arrivals separately. The terms \(J_k^+\) and \(J_k^-\) are independent identically distributed random variables representing the jump size and we assume they are exponentially distributed with parameters \(0 < \mu_k^+ < 1\) and \(\mu_k^- > 0\), respectively. We denote with \(\eta(T)\) the vector having elements

\[
\eta_k(T) = (X_k(t) + Y_k(t))(e^{-\alpha_k(T-t)} - 1) + f_k(T) - f_k(t)
\]

and \(\Sigma(T)\) the matrix having elements

\[
\Sigma_{kj}(T) = \rho_{k,j} \frac{\sigma_k \sigma_j}{\alpha_k + \alpha_j} \left( 1 - e^{-(\alpha_k + \alpha_j)(T-t)} \right).
\]

Assuming independence between the jump processes we get the following result:

Proposition 4.2 The joint characteristic function of the log-returns for the mean reverting jump
The resulting dynamics of $S^{i,j} = (S^{i,j}(t))_{t \geq 0}$ under the $\mathbb{Q}$ risk neutral measure is shown to be

$$d\langle Z_k, W_h \rangle(t) = \rho_{kh} dt, \quad k, h = 1, \ldots, d,$$

(32)

together with $d\langle Z_k, Z_k \rangle(t) = \delta_{kk} dt$ and $d\langle W_k, W_h \rangle(t) = \delta_{kh} dt$.

The philosophy behind this approach is that each exchange rate is driven by several independent noises $Z_k$ ($k = 1, \ldots, d$), each with an independent stochastic variance factor $V_k$, to which $Z_k$ is partially correlated via $\rho_k$. The vectors $a^i$ ($i = 1, \ldots, n$) describe how much each of the different volatilities contributes to the dynamics of $S^{0,i}$.

Following De Col et al. (2013), we now turn our attention to the exchange rate $S^{i,j}$ between two different currencies, say $i$ and $j$. We set by definition $S^{i,j} = S^{0,j}/S^{0,i}$.

The resulting dynamics of $S^{i,j} = (S^{i,j}(t))_{t \geq 0}$ under the $\mathbb{Q}$ risk neutral measure is shown to be
equal to
\[ dS^{i,j}(t) = S^{i,j}(t) \left( (r^i - r^j)dt + (a^i - a^j)^T \sqrt{\text{Diag}(V(t))} d\mathbf{Z}^{i,j}(t) \right). \] (33)

The valuation of vanilla FX options can be performed using standard Fourier based techniques, see De Col et al. (2013). In the sequel, we provide the characteristic function of the log-returns in order to approximate FX basket options.

**Proposition 4.3** The joint characteristic function of the log-returns in the model of De Col et al. (2013) is given by
\[ \varphi_T(\gamma) = e^{A(\tau) + \sum_{k=1}^d B_k(\tau) V_k(t)}, \] (34)

where, for \( \tau = T - t \) we have

\[ A(\tau) = \sum_{j=1, j \neq i}^n (r^i - r^j) i \gamma_j \tau + \sum_{k=1}^d \frac{\zeta_k \theta_k}{\xi_k^2} \left[ (Q_k - d_k) \tau - 2 \log \frac{1 - c_k e^{-d_k \tau}}{1 - c_k} \right], \]
\[ B_k(\tau) = \frac{Q_k - d_k}{\xi_k^2} \left( 1 - e^{-d_k \tau} \right), \]
\[ d_k = \sqrt{Q_k^2 - 4R_k P_k}, \]
\[ c_k = \frac{Q_k - d_k}{Q_k + d_k}, \]
\[ P_k = \frac{1}{2} \sum_{j,l=1, j,l \neq i}^n i \gamma^j \gamma^l \left( a^i_k - a^j_k \right) \left( a^l_k - a^i_k \right) - \frac{1}{2} \sum_{j=1, j \neq i}^n (a^i_k - a^j_k)^2 i \gamma^j, \]
\[ Q_k = \zeta_k - \sum_{j=1, j \neq i}^n i \gamma^j B_k(\tau) \left( a^i_k - a^j_k \right) \rho_k \xi_k, \]
\[ R_k = \frac{1}{2} \xi_k^2. \]

**Proof.** See De Col et al. (2013).

---

**4.4. The WASC model**

The Wishart Affine Stochastic Correlation (WASC) model, introduced by Da Fonseca et al. (2007), is a model which is applicable to different asset classes whenever a realistic and analytically tractable description of instantaneous correlations among state variables is required see e.g. Escobar et al. (2012). It describes an \( n \)-dimensional vector of assets \((S_1(t), \ldots, S_n(t))^{\top}, t \geq 0\) according to the following dynamics:
\[ d\mathbf{S}(t) = \text{Diag}(\mathbf{S}(t)) \left( \mathbf{r} dt + \sqrt{\Sigma(t)} d\mathbf{Z}(t) \right), \] (35)
where $Z(t) \in \mathbb{R}^n$ is a vector Brownian motion and the returns’ covariance matrix $\Sigma(t)$ evolves according to the following matrix SDE:

$$d\Sigma(t) = \left( \alpha Q^\top Q + M \Sigma(t) + \Sigma(t) M^\top \right) dt + \sqrt{\Sigma(t)} dW(t) Q + Q^\top dW(t) \sqrt{\Sigma(t)},$$  \hfill (36)

$$\Sigma(0) \in S_n^+, \hfill (37)$$

where $S_n^+$ denotes the cones of positive semidefinite matrices endowed with the scalar product given by the trace operator applied to the matrix product. In the dynamics above, $M, Q \in GL(n)$ and we assume that $M$ has negative eigenvalues, so as to ensure the stationarity of the process. Furthermore, we assume $\alpha \geq n - 1$, see Cuchiero et al. (2011).

The asymmetric correlation effects are modeled by introducing the following correlation structure among Brownian motions:

$$dZ(t) = \sqrt{1 - \rho^\top \rho} dB(t) + dW(t) \rho$$  \hfill (38)

where $\rho \in \mathbb{R}^n$, with $\rho \in [-1, 1]^n$ and $\rho^\top \rho \leq 1$. The model belongs to the class of multidimensional stochastic volatility models. In the following, we follow the approach of Grasselli and Tebaldi (2008) and Da Fonseca et al. (2007) and report their result on the joint Fourier transform of assets’ returns, that we adapt to our setting.

**Proposition 4.4** Let $\tau := T - t$. Given a real vector $\gamma \in \mathbb{R}^n$, the characteristic function of the WASC model is given by

$$\varphi_T(\gamma) = \exp \{ A(\tau) + Tr [B(\tau) \Sigma(t)] \}$$

where

$$B(\tau) = B_{22}(\tau)^{-1} B_{21}(\tau),$$

$$\begin{pmatrix} B_{11}(\tau) & B_{12}(\tau) \\ B_{21}(\tau) & B_{22}(\tau) \end{pmatrix} = \exp \left( \begin{pmatrix} M + i Q^\top \rho (\gamma^\top) & -2 Q^\top Q \\ \Lambda & - (M + i Q^\top \rho (\gamma^\top)) \end{pmatrix} \right),$$

$$\Lambda = -\frac{1}{2} \left( (\gamma) (\gamma^\top) + i \sum_{j=1}^n (\gamma)_j e_{jj} \right),$$

$$A(\tau) = -\frac{\alpha}{2} \left[ \log(B_{22}(\tau)) + \tau \left( M + i Q^\top \rho (\gamma^\top) \right) \right] + i \gamma^\top 1 \tau,$$

where $e_{jj}$ denotes a matrix with a unique non zero entry along the main diagonal equal to one on the position $jj$.

**Proof.** See Da Fonseca et al. (2007). \hfill $\square$

### 5. Numerical results

This section discusses numerical results with reference to the models we introduced in the previous section. Numerical experiments were coded and implemented in Matlab version 7.14.0 on an Intel Core i5 2.40 GHz machine running under Mac OS X with 4 GB physical memory. We compute the fair value of basket option contracts, spanning different strike prices, for the geometric Brownian
motion case and for each non-Gaussian model presented in Section 4. Numerical results are reported in Tables (2–11). Results for the positive basket case are given in Tables (2–6). Results for the basket spread option are in Tables (7–11).

At first we have to select a benchmark. To this aim we consider Monte Carlo simulation, using the lower bound \( C^G_K(t) \) as a control variate, as described in Section 1. In this way the simulation error is considerably reduced. The Monte Carlo simulation for the geometric Brownian motion has been implemented by sampling from the lognormal distribution. For the jump diffusion model we sample from the Gaussian noise and the jump components. The other three models have been simulated through the Euler-Maruyama discretization scheme. The number of simulations is chosen depending on the model, as indicated in each table caption. Columns labeled with C.I. length give the length of the 95% mean-centered Monte Carlo confidence interval.

Table 1 compares the control variate (MC) and the crude (MC\textsuperscript{cr}) Monte Carlo for each model. Model parameters are set as in Tables 2–6 and the option strike price \( K \) is chosen to be close to at the money. The confidence interval length of each Monte Carlo simulation is also provided. Using the lower bound \( C^G_K(t) \) as a control variate in the simulation, the standard error and therefore the confidence interval of the crude Monte Carlo estimate are significantly reduced. For example the length of the confidence interval in the geometric Brownian motion model is reduced from the confidence interval of the crude Monte Carlo estimate are significantly reduced. For example, in the FX stochastic volatility model the confidence interval length is reduced from the lower bound \( C^G_K(t) \) as a control variate in the simulation, the standard error and therefore the confidence interval of the crude Monte Carlo estimate are significantly reduced. For example, the length of the confidence interval in the geometric Brownian motion model is reduced from 8.2416 \( \times 10^{-2} \) to 3.3707 \( \times 10^{-3} \). A substantial reduction is also obtained for all other models. For example, in the FX stochastic volatility model the confidence interval length is reduced from 1.2348 \( \times 10^{-1} \) to 8.8989 \( \times 10^{-4} \).

The bounds for the geometric Brownian motion in Table 2 and 7 are computed by exploiting the explicit formulas of section 3. For non-Gaussian models, integrals involved in all lower and upper bound computations are evaluated by means of a Gauss–Kronrod quadrature rule, using Matlab’s built-in function \texttt{quadgk}. The optimization involved in the computation of \( C^G_K(t) \) is performed via the Matlab function \texttt{fminunc} for the geometric Brownian motion and using \texttt{fminbnd} in remaining cases. Spread options in formulae (13), (14) and (15) have been evaluated using formula (C2) in Appendix C. For all computations involving a Fourier inversion, we used a damping parameter \( \delta = 0.75 \). The bottom line of each table shows the average CPU time of a single option price evaluation. The quantity is measured in seconds and depends on the model and the pricing method.

The best performances in terms of accuracy are generally obtained by the lower bound \( C^G_K(t) \), that outperforms the approximation \( C^{AG}_K(t) \) in most cases. The accuracy of the approximations depends on the basket distribution and is affected by the basket size and the presence of negative weights. In particular, the lower bound \( C^G_K(t) \) is usually more accurate when the basket is small and weights are positive. Indeed, when the basket weights are strictly positive (Tables 2–6), the bound based on the approximating set argument \( C^G_K(t) \) always outperforms the arithmetic-geometric mean inequality approximation \( C^{AG}_K(t) \). In the basket spread case (Tables 7–11), i.e. when weights can assume negative values, the lower bound \( C^G_K(t) \) still provides good results but it is sometimes outperformed by the bounded approximation \( C^{AG}_K(t) \). In particular, \( C^{AG}_K(t) \) performs better than \( C^G_K(t) \) for out of the money options in Tables 7 and 6 and for in the money options in Table 8. In the pure spread option case in Table 10, the bounded approximation becomes exact and it is always more accurate than \( C^G_K(t) \).

Depending on expected differences between the arithmetic and the geometric average, the interval between lower and upper bounds \( [U^{AG}_K(t) - L^{AG}_K(t)] \) can be very small or very wide. For example, it is null in the spread option case of Table 10, providing an exact result. It is very large and practically useless for basket spread options in Table 8. In general, the true price is closer to \( U^{AG}_K(t) \) when the option is in the money, while it is closer to \( L^{AG}_K(t) \) when the option is out of the money.

We consider now the GBM model and we compare Table 2 with the results obtained for different approximation methods in Krekel et al. (2004) (see tables reported in that paper). Using the same parameter setting, the lower bound \( C^G_K(t) \) is as accurate as the best methods for a positive basket (Beisser (2001) and Ju (2002)). Concerning the basket spread option case, we compare Table 7 with the results obtained for different approximation methods in Deelstra et al. (2010) (see tables reported in that paper). On the same parameter setting, the lower bound \( C^G_K(t) \) is less accurate
than the best methods they considered for a basket spread option (Borovkova et al. (2007) and Deelstra et al. (2010)).

The MRJD model, presented in Tables (4) and (6), is a nice example of the generality of our bounds. Other methods such as Hurd and Zhou (2010), Lord et al. (2008) and Jackson et al. (2008) cannot cope with this model, because they require assumptions on the model characteristic function that rule out mean reverting models.

As an example of application, we tested two models calibrated to current market data. Results in Tables (5) and (10) refer to the stochastic volatility model for currencies of Section 4.3. We take the parameters from Table 1 in De Col et al. (2013), which features the result of a calibration performed on market data as of July 23rd 2010. Taking the perspective of a Japanese investor who seeks protections against fluctuations of both EUR-JPY and USD-JPY, we evaluate the payoff

\[
(w_1 S_{USD,JPY}^T + w_2 S_{EUR,JPY}^T - K)^+.
\]

Results in Table (6) refers to a two-asset basket option on FTSE and Eurostoxx 50 modeled with the WASC process of Section 4.4. The parameters refer to a calibration performed on August 20th 2008, as in Da Fonseca and Grasselli (2011).

The computational cost of our price approximation methods varies depending on the performed tests. Computations are extremely fast for the GBM, where no numerical integration is required and in addition we are able to choose a good starting point for the optimization problem. As the complexity of the model characteristic function increases, the computational cost increases as well. In Tables 3 and 8 we tested our methods by considering large baskets consisting of ten and twenty assets. The price approximation is always obtained at a reasonable time cost. Both methodologies involve the computation of one dimensional integrals, and hence they do not suffer from the curse of dimensionality, as opposed to the approach of Hurd and Zhou (2010), Lord et al. (2008), Jackson et al. (2008) and Jaimungal and Surkov (2011). All these latter require an n-dimensional quadrature, and cannot be used in the practice when the basket dimension is high.

Table 12 investigates the relation between CPU time and basket size, as we increase the number of assets up to 100. We select a basket with positive weights under the jump diffusion model of Section 4.1. We consider the two price approximations \( C_G(t) \) and \( C_{AG}(t) \) for an increasing basket size \( n \). Column CPU\(^G\) provides the CPU time for the lower bound in formula (7). CPU times for the approximation based on the arithmetic-geometric mean inequality of formula (19) are given in column CPU\(^{AG}\). The computation of the approximation \( C_{AG}(t) \) is much faster than the lower bound \( C_G(t) \) because its complexity is linear in the basket dimension and it does not require any optimization. This approximation can be very useful for pricing basket options written on an high number of underlying assets, in particular when the model characteristic function valuation is computationally expensive. On the other hand, the lower bound \( C_G(t) \) is generally more accurate and it is also reasonably fast because its CPU time grows quadratically in the basket dimension. In our experiments, the price for a basket on 100 underlyings is computed in only 15.53 seconds. Considering basket spread options does not change the above considerations. Monte Carlo simulation in such a case turns out to be very slow and, given the same time budget of competing procedures, quite inaccurate. Numerical results are plotted in Figure 3.

Finally, Figure 4 shows that our approximate methods can be used to compute option Greeks. We consider the computation of Deltas for a basket spread option with strike price \( K = 100 \) and model parameters as in Table 11. We compare our approximate sensitivities with those computed via Monte Carlo simulation and a finite difference scheme. Both approximate methods provide accurate estimates of the Delta. With respect to the Monte Carlo benchmark, the relative error is smaller than 2%.
6. Conclusions

Most methodologies in the basket option pricing literature are either restricted to a simple lognormal setting or to a model-dependent framework. In this paper we introduced novel methods which allow for a fast and reliable approximation of basket options, via lower and upper bounds. Such bounds rely on a rather weak assumption, i.e. the characteristic function of the vector of log-prices is known. This assumption is very general as most models which are commonly found in the literature allow for an explicit expression of this quantity either by means of the Lévy-Khintchine formula for Lévy processes or systems of (generalized) Riccati equations for affine processes. However, our approach is not restricted to those classes of models, provided the multivariate characteristic function is known. We study the case of strictly positive basket weights as well as the negative one, i.e. the so-called basket spread option. We test our methodologies on different models: a Gaussian model, a jump diffusion model, a mean reverting jump diffusion model, a multi-factor stochastic volatility model with common factors among the underlyings and finally a stochastic correlation model. In particular we study two kinds of price approximations: an accurate lower bound based on an approximating set and a fast bounded approximation based on the arithmetic-geometric mean inequality. Both approximations are particularly appealing for higher dimensional problems, versus most existing methods in the literature that cannot be applied when the basket dimension is large, due to the significant computational cost. Moreover, by using one of our bounds as a control variate, we can largely improve the accuracy of the Monte Carlo estimate.

We believe that our solutions opens up the possibility of performing further investigations, for example in the study of the relationship between the implied volatility of a basket and its constituents, such as the S&P 500 index, and in portfolio selection applications.

References


Huang, Z. and Kou, S., First passage times and analytical solutions for options on two assets with jump risk [online], 2006. (accessed ????).


Appendix A: Proof of Proposition 1.1

We denote by \( f(X_k, Y_n) \) the joint bivariate probability density of \( X_k(T) \) and the log-geometric average \( Y_n(T) \). We consider the lower bound to the basket option payoff in \( T \), as in formula (4):

\[
E_t \left[ (A_n(T) - K) I(\mathcal{G}) \right],
\]

where \( \mathcal{G} = \{ \omega : Y_n(T) > \kappa \} \). We introduce the damping factor \( \exp(\delta \kappa) \) according to Carr and Madan (2000) and compute the Fourier transform with respect to \( \kappa \). We obtain

\[
\Psi_T(\gamma; \delta) = \int_{\mathbb{R}} e^{i \gamma \kappa + \delta \kappa} E_t \left[ (A_n(T) - K) I(\mathcal{G}) \right] d\kappa
\]

\[
= \int_{\mathbb{R}} e^{i \gamma \kappa + \delta \kappa} E_t \left[ \left( \sum_{k=1}^{n} w_k S_k(T) - K \right) I(\mathcal{G}) \right] d\kappa
\]

\[
= \int_{\mathbb{R}} e^{i \gamma \kappa + \delta \kappa} \sum_{k=1}^{n} w_k E_t \left[ S_k(T) I(\mathcal{G}) \right] d\kappa - \int_{\mathbb{R}} e^{i \gamma \kappa + \delta \kappa} E_t \left[ K I(\mathcal{G}) \right] d\kappa
\]

\[
= \Psi_T^1(\gamma; \delta) - \Psi_T^2(\gamma; \delta).
\]
So the first part becomes:

\[
\Psi_1^T(\gamma; \delta) = \int e^{i(\gamma + \delta x)} \left[ \sum_{k=1}^{n} w_k \mathbb{E}_t [S_k(T)I(G)] \right] d\kappa
\]

\[
\begin{align*}
&= \int e^{i\gamma x + \delta \kappa} \left[ \sum_{k=1}^{n} w_k \int_{-\infty}^{+\infty} S_k(t) e^{X_k(T)} f(X_k, Y_n) dX_k dY_n \right] d\kappa \\
&= \frac{1}{i\gamma + \delta} \sum_{k=1}^{n} w_k \int_{-\infty}^{+\infty} e^{i(\gamma - i\delta)Y_n(T)} S_k(t) e^{X_k(T)} f(X_k, Y_n) dX_k dY_n \\
&= \frac{1}{i\gamma + \delta} \sum_{k=1}^{n} w_k S_k(t) \mathbb{E}_t \left[ e^{i(\gamma - i\delta)Y_n(T) + X_k(T)} \right]
\end{align*}
\]

\[
\begin{align*}
&= \frac{1}{i\gamma + \delta} \sum_{k=1}^{n} w_k S_k(t) \Phi_T \left( \gamma - i\delta - iE_k, w, Y_n(t) \right).
\end{align*}
\]

The second part becomes:

\[
\Psi_2^T(\gamma; \delta) = \int e^{i\gamma x + \delta \kappa} \mathbb{E}_t [KI(G)] d\kappa
\]

\[
\begin{align*}
&= \int e^{i\gamma x + \delta \kappa} \left[ \int_{-\infty}^{+\infty} K f(X_k, Y_n) dX_k dY_n \right] d\kappa \\
&= \int e^{i\gamma x + \delta \kappa} \left[ \int_{-\infty}^{+\infty} K f(X_k, Y_n) dX_k dY_n \right] d\kappa \\
&= \frac{1}{i\gamma + \delta} \int e^{i(\gamma - i\delta)Y_n(T)} K f(X_k, Y_n) dX_k dY_n \\
&= \frac{1}{i\gamma + \delta} K \mathbb{E}_t \left[ e^{i(\gamma - i\delta)Y_n(T)} \right]
\end{align*}
\]

\[
\begin{align*}
&= \frac{1}{i\gamma + \delta} K \Phi_T \left( \gamma - i\delta, 0, w, Y_n(t) \right).
\end{align*}
\]

Remembering the damping factor, we read the Fourier inversion as

\[
\frac{e^{-\delta \kappa}}{\pi} \int_{0}^{+\infty} e^{-i\gamma \kappa} \Psi(\gamma; \delta) d\gamma.
\]

Formula (8) is obtained by discounting, taking the positive part and maximizing with respect to \( \kappa \). The moment condition \( \{ e_k, \delta w + e_k \} \in A_X, \forall k = 1, \ldots, n \), can be easily deduced from (Lee 2004, Theorem 5.1)
Appendix B: Proof of Proposition 2.1

Due to the arithmetic-geometric mean inequality, $A_n^{\text{pos}}(T) \geq G_n^{\text{pos}}(T)$, and $A_n^{\text{neg}}(T) \geq G_n^{\text{neg}}(T)$. Through the put-call parity we have

$$ (c^{\text{pos}} A_n^{\text{pos}}(T) - c^{\text{neg}} A_n^{\text{neg}}(T) - K)^+ \geq (c^{\text{pos}} G_n^{\text{pos}}(T) - c^{\text{neg}} A_n^{\text{neg}}(T) - K)^+ =$$

$$ (c^{\text{neg}} A_n^{\text{neg}}(T) + K - c^{\text{pos}} G_n^{\text{pos}}(T))^+ + c^{\text{pos}} G_n^{\text{pos}}(T) - c^{\text{neg}} A_n^{\text{neg}}(T) - K \geq$$

$$ (c^{\text{neg}} G_n^{\text{neg}}(T) + K - c^{\text{pos}} G_n^{\text{pos}}(T))^+ + c^{\text{pos}} G_n^{\text{pos}}(T) - c^{\text{neg}} A_n^{\text{neg}}(T) - K =$$

$$(c^{\text{pos}} G_n^{\text{pos}}(T) - c^{\text{neg}} G_n^{\text{neg}}(T) - K)^+ + c^{\text{neg}} G_n^{\text{neg}}(T) - c^{\text{neg}} A_n^{\text{neg}}(T),$$

and

$$(c^{\text{pos}} A_n^{\text{pos}}(T) - c^{\text{neg}} A_n^{\text{neg}}(T) - K)^+ \leq (c^{\text{pos}} A_n^{\text{pos}}(T) - c^{\text{neg}} G_n^{\text{neg}}(T) - K)^+ =$$

$$(c^{\text{neg}} G_n^{\text{neg}}(T) + K - c^{\text{pos}} A_n^{\text{pos}}(T))^+ + c^{\text{pos}} A_n^{\text{pos}}(T) - c^{\text{neg}} G_n^{\text{neg}}(T) - K \leq$$

$$(c^{\text{neg}} G_n^{\text{neg}}(T) + K - c^{\text{pos}} A_n^{\text{pos}}(T))^+ + c^{\text{pos}} A_n^{\text{pos}}(T) - c^{\text{neg}} G_n^{\text{neg}}(T) - K =$$

$$(c^{\text{pos}} G_n^{\text{pos}}(T) - c^{\text{neg}} G_n^{\text{neg}}(T) - K)^+ + c^{\text{pos}} A_n^{\text{pos}}(T) - c^{\text{pos}} G_n^{\text{pos}}(T).$$

The proof ends taking the expectation of above inequalities and discounting.

Appendix C: Spread option approximation

We recall here the spread option pricing formula proposed in Caldana and Fusai (2013). Let $S_1(t)$ and $S_2(t)$ be two stock price processes. The time 0 no-arbitrage fair price of a spread option is

$$ \text{Spread}_K(0) = e^{-rT}E \left[(S_1(T) - S_2(T) - K)^+ \right], \quad (C1) $$

Let $u = (u_1, u_2)^T \in \mathbb{R}^2$ and $X(t) = (\ln S_1(t), \ln S_2(t))^T$ and consider the joint characteristic function

$$ \Phi_T(u) = \Phi_T(u_1, u_2) = E \left[e^{iu_1 \ln S_1(T) + iu_2 \ln S_2(T)} \right] = E \left[e^{iu^T X(T)} \right]. $$

**Proposition C.1** The approximate spread option value $C_{K,a}(0)$ is given in terms of a Fourier inversion formula as

$$ \text{Spread}_{K,a}^a(0) = \left( \frac{e^{-\delta k - rT}}{\pi} \int_0^{+\infty} e^{-i\gamma k} \Psi_T(\gamma; \delta, \alpha) d\gamma \right)^+, \quad (C2) $$

where

$$ \Psi_T(\gamma; \delta, \alpha) = \frac{e^{i(\gamma - i\delta) \ln(\Phi_T(0, -i\alpha))}}{i(\gamma - i\delta)} \left[ \Phi_T(\gamma - i\delta - i, -\alpha(\gamma - i\delta) - i) - K \Phi_T(\gamma - i\delta, -\alpha(\gamma - i\delta)) \right] \quad (C3) $$
and

\[
\alpha = \frac{F_2(0, T)}{F_2(0, T) + K}, \quad (C4)
\]

\[
k = \ln \left( F_2(0, T) + K \right). \quad (C5)
\]

The quantity \( F_2(0, T) = \mathbb{E}[S_2(T)] \) in formulas (C4) and (C5) is the forward price of the second asset at time 0 for delivery at future date \( T \). Using the characteristic function properties, we can write \( F_2(0, T) = \Phi_T(0, -i) \). The parameter \( \delta \) tunes an exponentially decaying term introduced to allow the integration in the Fourier space.

**Appendix D: Proofs for the geometric Brownian motion case**

Let us introduce the notation

\[
\ln(S(t)) = \begin{pmatrix}
\log S_1(t) \\
\vdots \\
\log S_n(t)
\end{pmatrix}.
\]

We consider the set

\[
\mathcal{G} = \{ \omega : Y_n(T) > \kappa \} = \left\{ \omega : w^\top \left( \ln(S(t)) + \left( r1 - q - \frac{1}{2} \text{vec}(\Sigma_{kk}) \right) (T-t) + \sqrt{\Sigma} W(T-t) \right) > \kappa \right\}.
\]

We see that \( w^\top \sqrt{\Sigma} W(T-t) \) has the same distribution as a univariate Brownian motion \( \sigma^* W^* (T-t) \), where \( \sigma^* = \sqrt{w^\top \Sigma w} \). Considering \( m \) as in formula (25), we can write the set \( \mathcal{G} \) as

\[
\mathcal{G} = \left\{ \omega : Z > d = \frac{\kappa - w^\top (\ln(S(t)) + m(T-t))}{\sigma^* \sqrt{T-t}} \right\},
\]

where \( Z \) is a standard Normal random variable. We can write the expectation in (4) as

\[
\mathbb{E}_t \left[ (A_n(T) - K) I(\mathcal{G}) \right] = \mathbb{E}_t \left[ \mathbb{E}_t \left[ A_n(T) - K | \mathcal{G} \right] I(\mathcal{G}) \right]^+
\]

\[
= \mathbb{E}_t \left[ \mathbb{E}_t \left[ A_n(T) - K | Z \right] I(Z > d) \right]^+.
\]

Conditionally to the random variable \( Z \), the vector \( \sqrt{\Sigma} W(T-t) \) is distributed like a multivariate Normal with mean \( \mu \) and variance \( V \), with their elements defined for \( k, j = 1, \ldots, n \) as

\[
\mu_k = Z a_k \sqrt{T-t}, \quad V_{kj} = (T-t)(\Sigma_{kj} - a_k a_j), \quad a_k = \sum_{j=1}^n w_j \Sigma_{kj} / \sigma^*.
\]

and we indicate with \( \Sigma_{kj} \) the element of \( \Sigma \) in position \((k,j)\). Due to this fact, \( S(T)|Z \) follows a multivariate lognormal \( \mathcal{MLN}(\hat{\mu}, \hat{V}) \), where, for \( k, j = 1, \ldots, n \),

\[
\hat{\mu}_k = \ln S_k(t) + (r - q_k - \Sigma_{kk}/2)(T-t) + Z a_k \sqrt{T-t},
\]

\[
\hat{V}_{kj} = V_{kj}.
\]
We can now compute the inner expectation of the payoff, using the lognormal distribution properties

\[
E_t \left[ E \left[ A_n(T) - K | Z \right] 1_{(Z \geq d)} \right] +
\]

\[
= E_t \left[ \left( \sum_{k=1}^{n} w_k e^{\ln S_k(t) + (r - q_k - a_k^2/2)(T-t) + Z a_k \sqrt{T-t} - K} \right) I (Z \geq d) \right] +
\]

We solve the above expectation by using the partial expectation property of the lognormal distribution. Discounting and maximizing with respect to \( \kappa \), we obtain the lower bound

\[
C_{G K}(t) = \max_{\kappa} e^{-r(T-t)} \left( \sum_{k=1}^{n} w_k S_k(t)e^{(r - q_k)(T-t)}N(a_k \sqrt{T-t} - d) - K N(-d) \right)^+. \tag{D1}
\]

We indicate with \( N(\cdot) \) the standard Normal distribution function. The formula above still depends on maximizing with respect to \( \kappa \), involved in the definition of \( d \). Maximization must be carried out by a numerical search, equating to zero the first derivative with respect to \( \kappa \).

We need to solve the equation

\[
\sum_{k=1}^{n} w_k S_k(t)e^{(r - q_k)(T-t)} \phi(a_k \sqrt{T-t} - d) - K \phi(-d) = 0, \tag{D2}
\]

where we indicate with \( \phi(\cdot) \) the standard Normal density function. Using a linearization argument, we can provide the starting point \( \kappa^{start} \) of the numerical search. We approximate the term \( \phi(a_k \sqrt{T-t} - d) \) in formula (D2) with a first-order Taylor expansion centered at \(-d\),

\[
\phi(a_k \sqrt{T-t} - d) \approx \phi(-d) + a_k \sqrt{T-t} \phi'(-d) = \phi(-d) + da_k \sqrt{T-t} \phi(-d),
\]

obtaining

\[
\sum_{k=1}^{n} w_k S_k(t)e^{(r - q_k)(T-t)} \left( 1 + da_k \sqrt{T-t} \right) - K = 0.
\]

Substituting the definition of \( d \) and rearranging terms, it is easy to obtain the following approximation for the value of \( \kappa \) in which the option price is maximum:

\[
\kappa^{start} = \sigma \sqrt{T-t} \left( \frac{K}{\sum_{k=1}^{n} w_k S_k(t)e^{(r - q_k)(T-t)}} + \sum_{k=1}^{n} w_k \left( \ln S_k(t) + \left( r - q_k - \frac{\Sigma_{kk}}{2} \right) (T-t) \right) \right).
\]
Figure 3. The CPU time (seconds) for the jump diffusion model of Section 4.1 as a function of the basket dimension. Numerical values are given in Table 12.

Figure 4. Deltas for a three-asset basket spread option with strike price \( K = 100 \) and model parameters as in Table 11. We consider sensitivities computed via the approximating set lower bound of Section 1 and via the price approximation of Section 2. A benchmark is obtained through Monte Carlo simulation and a finite difference scheme.

Table 1. The control variate (MC) and the crude (MC\(^{cr} \)) Monte Carlo are compared for each model. Simulation settings and model parameters are set as in Tables 2–6. The option strike price \( K \) is chosen to be close to at the money. The confidence interval length of the Monte Carlo simulation is also provided.

<table>
<thead>
<tr>
<th>Model</th>
<th>( K )</th>
<th>Parameters</th>
<th>MC</th>
<th>C.I. length</th>
<th>MC(^{cr} )</th>
<th>C.I. length(^{cr} )</th>
</tr>
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<td>3.3707 × 10(^{-5} )</td>
<td>28.0447</td>
<td>8.2416 × 10(^{-2} )</td>
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<tr>
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<td>18.3813</td>
<td>1.0106 × 10(^{-2} )</td>
<td>18.2822</td>
<td>2.0322 × 10(^{-1} )</td>
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<tr>
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<td>Table 4</td>
<td>10.0060</td>
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<td>9.9641</td>
<td>2.0639 × 10(^{-1} )</td>
</tr>
<tr>
<td>FX-SV</td>
<td>100</td>
<td>Table 5</td>
<td>7.4990</td>
<td>8.8989 × 10(^{-4} )</td>
<td>7.4717</td>
<td>1.2348 × 10(^{-1} )</td>
</tr>
<tr>
<td>WASC</td>
<td>5000</td>
<td>Table 6</td>
<td>1024.27</td>
<td>2.4257 × 10(^{-1} )</td>
<td>1032.2709</td>
<td>9.6196 × 10(^{1} )</td>
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</tbody>
</table>
Table 2. Prices of basket options computed for different strikes $K$ in the geometric Brownian motion model of Section 3. The basket weights are $w = [0.25; 0.25; 0.25; 0.25]$. The parameter values are $T - t = 5$, $r = 0$, $S_k(t) = 100$, $q_k = 0$, $\sigma_k = 40\%$ and $\rho_{kj} = 0.5$, for $k = 1, \ldots, 4$ and $k \neq j$. Column $C^G_k(t)$ contains the lower bound in formula (4). Column $L^G_k(t)$, $C^G_k(t)$ and $U^G_k(t)$ contain the lower bound, the upper bound and the approximation based on the arithmetic-geometric mean inequality as in formulae (17), (19) and (18), respectively.

Columns MC and C.I. length contain the Monte Carlo prices and confidence intervals for $10^7$ random trials respectively.

<table>
<thead>
<tr>
<th>$K$</th>
<th>$C^G_k(t)$</th>
<th>$L^G_k(t)$</th>
<th>$U^G_k(t)$</th>
<th>$C^G_k(t)$</th>
<th>$C^G_k(t)$</th>
<th>MC</th>
<th>C.I. length</th>
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<tr>
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<td>11.5319</td>
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<td>15.1638</td>
<td>4.3800 $\times 10^{-3}$</td>
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**Time** 0.0451  0.0085  9.4026

Table 3. Prices of basket options computed for different strikes $K$ in the jump diffusion model of Section 4.1. The basket weights are $w = \frac{1}{20}1_{20}$. The parameter values are $T - t = 1$, $r = 1\%$, $S_k(t) = 100$, $\sigma_k = 40\%$, $\xi_k = 0.5$, $\xi_{kk} = 0.3$, $\alpha_k = \alpha_{kk} = -0.05$, $\lambda = 1$, $\lambda_k = 0.5$, $\rho_{kj} = 0.5$ and $\rho_{kk} = 0.5$ for $k = 1, \ldots, 20$ and $k \neq j$. The Monte Carlo price is obtained with $10^6$ simulations. Column labels are as in Table 2.

<table>
<thead>
<tr>
<th>$K$</th>
<th>$C^G_k(t)$</th>
<th>$L^G_k(t)$</th>
<th>$U^G_k(t)$</th>
<th>$MC$</th>
<th>C.I. length</th>
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<td>25.2013</td>
<td>33.6724</td>
<td>36.9109</td>
<td>34.9339</td>
</tr>
<tr>
<td>80</td>
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<td>19.4182</td>
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<td>28.2421</td>
</tr>
<tr>
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<td>18.0726</td>
<td>11.4574</td>
<td>15.6289</td>
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<td>18.3813</td>
</tr>
<tr>
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<td>14.6576</td>
<td>8.9072</td>
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</tr>
<tr>
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<td>10.3153</td>
</tr>
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<td>4.5925</td>
<td>5.8579</td>
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<td>8.7358</td>
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<td>3.8030</td>
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<td>7.4913</td>
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</tbody>
</table>

**Time** 2.0381  0.0430  10.5049
Table 4. Prices of basket options computed for different strikes $K$ in the mean reverting jump diffusion model of Section 4.2. The basket weights are $\mathbf{w} = [0.25; 0.25; 0.25; 0.25]$. The parameter values are $T-t=1$, $r=0$, $f_s(T) = \log(25)$, $\alpha = [0.1; 0.2; 0.1; 0.3]$, $f_x(t) = X_k(t) = Y_k(t) = 0$ for $k = 1, \ldots, 4$. Jump parameters are $\lambda^+ = \lambda^- = [0.1; 0.2; 0.3; 0.2]$, $\mu^+ = \mu^- = [0.1; 0.1; 0.3; 0.3]$. The covariance matrix of the Brownian motion is $[0.5, 0.35, 0.35, 0.25; 0.35, 0.5, 0.475, 0.15; 0.35, 0.475, 0.5, 0.15; 0.25, 0.15, 0.15, 0.5]$. The Monte Carlo price is obtained with $10^5$ random trials and 100 time steps. Column labels are as in Table 2.

<table>
<thead>
<tr>
<th>$K$</th>
<th>$C_G^0(t)$</th>
<th>$L_{G}^0(t)$</th>
<th>$C_{G}^0(t)$</th>
<th>$U_{G}^0(t)$</th>
<th>MC</th>
<th>C.I. length</th>
</tr>
</thead>
<tbody>
<tr>
<td>15</td>
<td>17.0447</td>
<td>14.6763</td>
<td>16.8630</td>
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<td>17.0666</td>
<td>3.2606 × 10^{-3}</td>
</tr>
<tr>
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<td>13.1066</td>
<td>10.9662</td>
<td>12.7435</td>
<td>13.5023</td>
<td>13.1485</td>
<td>5.0113 × 10^{-3}</td>
</tr>
<tr>
<td>30</td>
<td>7.5059</td>
<td>5.9190</td>
<td>6.9338</td>
<td>8.4551</td>
<td>7.5768</td>
<td>8.0097 × 10^{-3}</td>
</tr>
<tr>
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<td>4.3327</td>
<td>5.0748</td>
<td>6.8688</td>
<td>5.7404</td>
<td>1.0593 × 10^{-2}</td>
</tr>
<tr>
<td>40</td>
<td>4.2798</td>
<td>3.1793</td>
<td>3.7181</td>
<td>5.7154</td>
<td>4.3615</td>
<td>1.1832 × 10^{-2}</td>
</tr>
<tr>
<td>45</td>
<td>3.2497</td>
<td>2.3431</td>
<td>2.7337</td>
<td>4.8792</td>
<td>3.3314</td>
<td>1.2047 × 10^{-2}</td>
</tr>
<tr>
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<td>2.4809</td>
<td>1.7364</td>
<td>2.0200</td>
<td>4.2724</td>
<td>2.5603</td>
<td>1.2114 × 10^{-2}</td>
</tr>
<tr>
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<td>1.2945</td>
<td>1.5013</td>
<td>3.8306</td>
<td>1.9791</td>
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</tr>
<tr>
<td>Time</td>
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<td>0.0461</td>
<td>6.9586</td>
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</table>

Table 5. Prices of basket options computed for different strikes $K$ in the stochastic volatility model of Section 4.3. The basket weights are $\mathbf{w} = [0.4; 0.6]$. The parameter values are $T-t=1$, $S_{JPY,USD} = 86.90$, $S_{JPY,EUR} = 112.29$, $V_1 = 0.0137$, $V_2 = 0.0391$, $a_{USD}^{USD} = 0.6650$, $a_{EUR}^{USD} = 1.0985$, $a_{EUR}^{USD} = 1.6177$, $a_{EUR}^{EUR} = 1.3588$, $a_{EUR}^{EUR} = 0.2995$, $a_{EUR}^{EUR} = 1.6214$, $\lambda_1 = 0.9418$, $\lambda_2 = 1.7909$, $\theta_1 = 0.0370$, $\theta_2 = 0.0909$, $\xi_1 = 0.4912$, $\xi_2 = 1$, $\rho_1 = 0.5231$ and $\rho_2 = -0.3980$. The Monte Carlo price is obtained with $10^5$ random trials and 100 time steps. Column labels are as in Table 2.

<table>
<thead>
<tr>
<th>$K$</th>
<th>$C_G^0(t)$</th>
<th>$L_{G}^0(t)$</th>
<th>$C_{G}^0(t)$</th>
<th>$U_{G}^0(t)$</th>
<th>MC</th>
<th>C.I. length</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>52.2555</td>
<td>51.1796</td>
<td>52.2871</td>
<td>52.2988</td>
<td>52.2560</td>
<td>3.3805 × 10^{-4}</td>
</tr>
<tr>
<td>60</td>
<td>42.3929</td>
<td>41.3311</td>
<td>42.4279</td>
<td>42.4503</td>
<td>42.3939</td>
<td>4.6861 × 10^{-4}</td>
</tr>
<tr>
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<td>32.6765</td>
<td>31.6344</td>
<td>32.7076</td>
<td>32.7536</td>
<td>32.6782</td>
<td>6.5341 × 10^{-4}</td>
</tr>
<tr>
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<td>23.2842</td>
<td>22.2763</td>
<td>23.2967</td>
<td>23.3955</td>
<td>23.2873</td>
<td>7.7204 × 10^{-4}</td>
</tr>
<tr>
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<td>7.4935</td>
<td>6.7392</td>
<td>7.3923</td>
<td>7.8584</td>
<td>7.4990</td>
<td>8.8989 × 10^{-4}</td>
</tr>
<tr>
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<td>3.0988</td>
<td>2.6440</td>
<td>2.9503</td>
<td>3.7631</td>
<td>3.1043</td>
<td>9.6854 × 10^{-4}</td>
</tr>
<tr>
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<td>1.2843</td>
<td>1.0498</td>
<td>1.1551</td>
<td>2.1690</td>
<td>1.2833</td>
<td>8.6991 × 10^{-4}</td>
</tr>
<tr>
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<td>0.6279</td>
<td>0.4926</td>
<td>0.5318</td>
<td>1.6117</td>
<td>0.6305</td>
<td>7.0428 × 10^{-4}</td>
</tr>
<tr>
<td>140</td>
<td>0.3561</td>
<td>0.2672</td>
<td>0.2845</td>
<td>1.3863</td>
<td>0.3581</td>
<td>7.4411 × 10^{-4}</td>
</tr>
<tr>
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<td>0.2237</td>
<td>0.1601</td>
<td>0.1489</td>
<td>1.2793</td>
<td>0.2250</td>
<td>6.7663 × 10^{-4}</td>
</tr>
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<td>0.0260</td>
<td>2.9647</td>
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</tr>
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</table>
Table 6. Prices of basket options computed for different strikes $K$ in the WASC model of Section 4.4. The basket weights are $w = [0.5; 0.5]$. The parameter values are $T - t = 1$, $S(t) = [5371.80; 3295.28]$, $Q = [0.3296, 0.2866; 0.3446, 0.3524]$, $M = [-0.9886, -0.3631; -0.4464, -0.7599]$, $\rho = [-0.2675; -0.5496]$, $\beta = 10.8247$ and $r = 0$. The Monte Carlo price is obtained with $10^4$ random trials and 100 time steps. Column labels are as in Table 2.

<table>
<thead>
<tr>
<th>$K$</th>
<th>$C_{K}^{M}(t)$</th>
<th>$L_{K}^{M}(t)$</th>
<th>$C_{K}^{AG}(t)$</th>
<th>$U_{K}^{AG}(t)$</th>
<th>MC</th>
<th>C.I. length</th>
</tr>
</thead>
<tbody>
<tr>
<td>4000</td>
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<td>1291.48</td>
<td>1348.57</td>
<td>1428.48</td>
<td>1389.78</td>
<td>$1.9805 \times 10^{-1}$</td>
</tr>
<tr>
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<td>1102.68</td>
<td>1151.49</td>
<td>1239.68</td>
<td>1193.17</td>
<td>$2.5758 \times 10^{-1}$</td>
</tr>
<tr>
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<td>1023.87</td>
<td>941.50</td>
<td>983.14</td>
<td>1078.50</td>
<td>1024.27</td>
<td>$2.4257 \times 10^{-1}$</td>
</tr>
<tr>
<td>5500</td>
<td>879.12</td>
<td>804.14</td>
<td>839.61</td>
<td>941.14</td>
<td>879.52</td>
<td>$2.8705 \times 10^{-1}$</td>
</tr>
<tr>
<td>6000</td>
<td>755.22</td>
<td>687.22</td>
<td>717.40</td>
<td>824.22</td>
<td>755.60</td>
<td>$2.9166 \times 10^{-1}$</td>
</tr>
<tr>
<td>6500</td>
<td>649.24</td>
<td>587.76</td>
<td>613.43</td>
<td>724.76</td>
<td>649.55</td>
<td>$2.4128 \times 10^{-1}$</td>
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<tr>
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<td>503.16</td>
<td>525.00</td>
<td>640.16</td>
<td>558.91</td>
<td>$2.5190 \times 10^{-1}$</td>
</tr>
<tr>
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<td>481.09</td>
<td>431.20</td>
<td>449.78</td>
<td>568.20</td>
<td>481.42</td>
<td>$3.0524 \times 10^{-1}$</td>
</tr>
<tr>
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<td>369.97</td>
<td>385.78</td>
<td>506.97</td>
<td>415.18</td>
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<tr>
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<td>331.30</td>
<td>454.83</td>
<td>358.50</td>
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<td>284.88</td>
<td>410.41</td>
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<td></td>
<td></td>
<td></td>
<td>85.9311</td>
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</table>

Table 7. Prices of basket spread options computed for different strikes $K$ in the geometric Brownian motion model of Section 3. The basket weights are $w = [1; -1; -1]$. The parameter values are $T - t = 1$, $r = 5\%$ $S(t) = [100; 63; 12]$, $q = [0; 0; 0]$, $\sigma = [0.21; 0.34; 0.63]$, $\rho_{12} = 0.87$, $\rho_{13} = 0.3$, $\rho_{23} = 0.43$. Column $C_{K}^{M}(t)$ contains the lower bound in formula (4). Column $L_{K}^{M}(t)$, $C_{K}^{AG}(t)$ and $U_{K}^{AG}(t)$ contains the lower bound, the upper bound and the approximation based on the arithmetic-geometric mean inequality as in formulas (13), (15) and (14), respectively. Columns MC and C.I. length contain the Monte Carlo prices and confidence intervals for $10^5$ random trials.

<table>
<thead>
<tr>
<th>$K$</th>
<th>$C_{K}^{M}(t)$</th>
<th>$L_{K}^{M}(t)$</th>
<th>$C_{K}^{AG}(t)$</th>
<th>$U_{K}^{AG}(t)$</th>
<th>MC</th>
<th>C.I. length</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>25.6962</td>
<td>23.6509</td>
<td>24.6434</td>
<td>27.8943</td>
<td>26.7762</td>
<td>$4.6745 \times 10^{-3}$</td>
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<td>19.6852</td>
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<tr>
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<td>18.0691</td>
<td>16.7042</td>
<td>$9.4315 \times 10^{-3}$</td>
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<td>13.2363</td>
<td>11.8478</td>
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<td></td>
<td></td>
<td>2.8915</td>
</tr>
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</table>
The basket weights are \( w = [1; 1; 1; 1; 1; 1; 1; 1; 0.1; 0.1; -0.9; 0.1; 0.1; 0.09; -0.9; -0.9; -0.9; -0.9; -0.9; -0.9; -0.9]. \)

The parameter values are \( T - t = 1, r = 1\%\), \( S_0(t) = 100\), \( \sigma_k = 20\%\), \( \xi_k = 0.25\), \( \xi_{kk} = 0.15\), \( \alpha_k = \alpha_{kk} = -0.05\). \( \lambda = 1, \lambda_k = 0.1\), \( \rho_{kj} = 0.75\) and \( \rho_{kk} = 0.75\) for \( k = 1, \ldots, 20\) and \( k \neq j\). The Monte Carlo price is obtained with \( 10^6 \) simulations. Column labels are as in Table 7.

<table>
<thead>
<tr>
<th>( K )</th>
<th>( C^G_{K}(t) )</th>
<th>( L_{K}^{AG}(t) )</th>
<th>( C_{K}^{AG}(t) )</th>
<th>( U_{K}^{AG}(t) )</th>
<th>MC</th>
<th>C.I. length</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
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<td>47.9137</td>
<td>59.9191</td>
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<td>52.4154</td>
<td>40.5551</td>
<td>52.4967</td>
<td>63.8976</td>
<td>52.8422</td>
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<td>45.6139</td>
<td>57.0587</td>
<td>45.9051</td>
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<td>39.3213</td>
<td>50.8690</td>
<td>39.7300</td>
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<td>28.6166</td>
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</tr>
<tr>
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<td>20.3964</td>
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<td>20.8955</td>
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<td>29.0577</td>
<td>17.6438</td>
<td>1.1814 \times 10^{-2}</td>
</tr>
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<td>3.0185</td>
<td>14.3852</td>
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<td>14.8924</td>
<td>1.2630 \times 10^{-2}</td>
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<td>24.1002</td>
<td>12.5764</td>
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<tr>
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<td>6.3182</td>
<td></td>
<td></td>
<td></td>
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</table>
Table 10. Prices of basket spread options computed for different strikes $K$ in the FX stochastic volatility model of Section 4.3. The basket weights are $w = [2; -1]$. The parameter values are $T - t = 1$, $S^{JPY,USD} = 86.90$, $S^{JPY, EUR} = 112.29$, $V_1 = 0.0137$, $V_2 = 0.0391$, $a_1^{USD} = 0.6650$, $a_1^{EUR} = 1.0985$, $a_2^{EUR} = 1.6177$, $a_2^{EUR} = 1.3588$, $a_1^{JPY} = 0.2995$, $a_2^{JPY} = 1.6214$, $\kappa_1 = 0.9418$, $\kappa_2 = 1.7909$, $\theta_1 = 0.0370$, $\theta_2 = 0.0990$, $\xi_1 = 0.4912$, $\xi_2 = 1$, $\rho_1 = 0.5231$ and $\rho_2 = -0.3980$. The Monte Carlo price is obtained with $10^5$ random trials and 100 time steps.

<table>
<thead>
<tr>
<th>$K$</th>
<th>$C_{AG}^K(t)$</th>
<th>$L_{AG}^K(t)$</th>
<th>$C_{AG}^K(t)$</th>
<th>$U_{AG}^K(t)$</th>
<th>MC</th>
<th>C.I. length</th>
</tr>
</thead>
<tbody>
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<td>56.7048</td>
<td>56.7049</td>
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<tr>
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<td>51.5996</td>
<td>51.7630</td>
<td>51.7630</td>
<td>51.7631</td>
<td>1.8136 \times 10^{-4}</td>
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<td>15</td>
<td>46.6555</td>
<td>46.8368</td>
<td>46.8368</td>
<td>46.8371</td>
<td>2.4466 \times 10^{-4}</td>
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<td>20</td>
<td>41.7420</td>
<td>41.9396</td>
<td>41.9396</td>
<td>41.9402</td>
<td>4.8875 \times 10^{-4}</td>
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<tr>
<td>30</td>
<td>32.0764</td>
<td>32.3139</td>
<td>32.3139</td>
<td>32.3158</td>
<td>7.9035 \times 10^{-4}</td>
<td></td>
</tr>
<tr>
<td>40</td>
<td>22.8299</td>
<td>23.0913</td>
<td>23.0913</td>
<td>23.0947</td>
<td>1.0828 \times 10^{-3}</td>
<td></td>
</tr>
<tr>
<td>60</td>
<td>7.8445</td>
<td>8.0167</td>
<td>8.0167</td>
<td>8.0300</td>
<td>2.3239 \times 10^{-3}</td>
<td></td>
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<tr>
<td>70</td>
<td>3.7239</td>
<td>3.8079</td>
<td>3.8079</td>
<td>3.8255</td>
<td>3.2946 \times 10^{-3}</td>
<td></td>
</tr>
<tr>
<td>80</td>
<td>2.5116</td>
<td>2.5589</td>
<td>2.5589</td>
<td>2.5768</td>
<td>3.5316 \times 10^{-3}</td>
<td></td>
</tr>
<tr>
<td>80</td>
<td>1.6886</td>
<td>1.7073</td>
<td>1.7073</td>
<td>1.7287</td>
<td>4.4332 \times 10^{-3}</td>
<td></td>
</tr>
</tbody>
</table>

| Time  | 0.6483       | 0.2401       | 2.4542       |

Table 11. Prices of basket spread options computed for different strikes $K$ in the WASC model of Section 4.4. The basket weights are $w = [1; 1; -1]$. The parameter values are $T - t = 1$, $S(t) = 100 \cdot 1_3$, $Q = 0.25 \cdot I_3$, $M = -0.5 \cdot I_3$, $\rho = -0.3 \cdot 1_3$, $\beta = 10.8247$ and $\tau = 0$. The Monte Carlo price is obtained with $10^4$ random trials and 100 time steps.

<table>
<thead>
<tr>
<th>$K$</th>
<th>$C_{AG}^K(t)$</th>
<th>$L_{AG}^K(t)$</th>
<th>$C_{AG}^K(t)$</th>
<th>$U_{AG}^K(t)$</th>
<th>MC</th>
<th>C.I. length</th>
</tr>
</thead>
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<tr>
<td>50</td>
<td>64.3911</td>
<td>65.0700</td>
<td>64.0552</td>
<td>67.6858</td>
<td>65.2016</td>
<td>2.1340 \times 10^{-1}</td>
</tr>
<tr>
<td>60</td>
<td>57.4289</td>
<td>48.4728</td>
<td>56.8742</td>
<td>61.0886</td>
<td>58.3278</td>
<td>2.3614 \times 10^{-1}</td>
</tr>
<tr>
<td>70</td>
<td>50.9008</td>
<td>42.3712</td>
<td>50.1516</td>
<td>54.9870</td>
<td>51.7403</td>
<td>2.2825 \times 10^{-1}</td>
</tr>
<tr>
<td>80</td>
<td>44.8258</td>
<td>36.7837</td>
<td>43.9182</td>
<td>49.3995</td>
<td>45.6697</td>
<td>2.3646 \times 10^{-1}</td>
</tr>
<tr>
<td>90</td>
<td>39.2168</td>
<td>31.7180</td>
<td>38.1950</td>
<td>44.3338</td>
<td>40.0216</td>
<td>2.2475 \times 10^{-1}</td>
</tr>
<tr>
<td>100</td>
<td>34.0799</td>
<td>27.1708</td>
<td>32.9926</td>
<td>39.7866</td>
<td>34.9146</td>
<td>2.2828 \times 10^{-1}</td>
</tr>
<tr>
<td>110</td>
<td>29.4146</td>
<td>23.1290</td>
<td>28.3108</td>
<td>35.7448</td>
<td>30.2865</td>
<td>2.3272 \times 10^{-1}</td>
</tr>
<tr>
<td>120</td>
<td>25.2135</td>
<td>19.5703</td>
<td>24.1386</td>
<td>32.1861</td>
<td>26.0794</td>
<td>2.3005 \times 10^{-1}</td>
</tr>
<tr>
<td>130</td>
<td>21.4630</td>
<td>16.4654</td>
<td>20.4561</td>
<td>29.0812</td>
<td>22.4785</td>
<td>2.5191 \times 10^{-1}</td>
</tr>
<tr>
<td>140</td>
<td>18.1441</td>
<td>13.7799</td>
<td>17.2356</td>
<td>26.3957</td>
<td>19.3800</td>
<td>3.0061 \times 10^{-1}</td>
</tr>
<tr>
<td>150</td>
<td>15.2327</td>
<td>11.4761</td>
<td>14.4440</td>
<td>24.0919</td>
<td>16.4885</td>
<td>3.0263 \times 10^{-1}</td>
</tr>
</tbody>
</table>

| Time  | 4.8373       | 1.1070       | 98.7995      |
Table 12. The CPU time (seconds) for the jump diffusion model of Section 4.1 is given. We consider the two classes of price approximations for an increasing basket dimension $n$. The basket weights are $w = \frac{1}{n} 1_n$. The parameter values are $T - t = 1$, $r = 1\%$, $S_k(t) = 100$, $\sigma_k = 40\%$, $\xi_k = 0.5$, $\xi_{kk} = 0.3$, $\alpha_k = \alpha_{kk} = -0.5$, $\lambda = 1$, $\lambda_k = 0.5$, $\rho_{kj} = 0.5$ and $\rho_{kj}^Y = 0.5$ for $k = 1, \ldots, n$ and $k \neq j$. The strike price is $K = 100$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\text{CPU}^B$</th>
<th>$\text{CPU}^A$</th>
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<td>2</td>
<td>0.299</td>
<td>0.067</td>
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<tr>
<td>5</td>
<td>0.644</td>
<td>0.071</td>
</tr>
<tr>
<td>10</td>
<td>1.029</td>
<td>0.073</td>
</tr>
<tr>
<td>20</td>
<td>1.874</td>
<td>0.075</td>
</tr>
<tr>
<td>30</td>
<td>2.800</td>
<td>0.075</td>
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<tr>
<td>40</td>
<td>3.788</td>
<td>0.077</td>
</tr>
<tr>
<td>50</td>
<td>4.905</td>
<td>0.081</td>
</tr>
<tr>
<td>60</td>
<td>6.347</td>
<td>0.081</td>
</tr>
<tr>
<td>70</td>
<td>9.273</td>
<td>0.128</td>
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<tr>
<td>80</td>
<td>10.564</td>
<td>0.115</td>
</tr>
<tr>
<td>90</td>
<td>13.004</td>
<td>0.122</td>
</tr>
<tr>
<td>100</td>
<td>14.152</td>
<td>0.123</td>
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