AGGREGATION OF RANDOMLY WEIGHTED LARGE RISKS

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Abstract. Asymptotic tail probabilities for linear combinations of randomly weighted order statistics are approximated under various assumptions. One key assumption is the asymptotic independence for all risks. Therefore, it is not surprising that the maxima represents the most influential factor when one investigates the tail behaviour of our considered risk aggregation, which for example, can be found in the reinsurance market. This extreme behaviour confirms the "one big jump" property that has been vastly discussed in the existing literature in various forms whenever the asymptotic independence is present. An illustration of our results together with a specific application are explored under the assumption that the underlying risks follow the multivariate Log-normal distribution.

Keywords and phrases: Davis-Resnick tail property; Extreme value distribution; Max-domain of attraction; Mitra-Resnick model; Risk aggregation.

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1. Introduction

Consider positive dependent random variables (or risks) $X_i, i = 1, \ldots, n$ and let $X_{1,n} \geq \ldots \geq X_{n,n}$ be the corresponding upper order statistics. We investigate the asymptotic tail behaviour of linear combinations of order statistics $L(\mathbf{C}) = \sum_{i=1}^{n} C_i X_{i,n}$ where the C_i's are random deflators/weights and $C_1 > 0$.

Studying the tail probability for such order statistics has multiple financial and insurance applications (for example, see Hashorva, 2007, Ladoucette and Teugels, 2006, Asimit and Jones, 2008 a and b, Jiang and Tang, 2008 and Li and Hashorva, 2013). All of these papers have studied the extreme behaviour of the $L(\mathbf{C})$, where $\mathbf{C}$ was assumed to be a deterministic vector. The tail asymptotics of the total risk $L(\mathbf{1}) = \sum_{i=1}^{n} X_i$ has been investigated in many recent contributions such as Asmussen and Rojas-Nandayapa (2008), Chen and Yuen (2009), Mitra and Resnick (2009), Foss and Richards (2010), Asmussen *et al.* (2011), Kortschak (2012), Hashorva (2013), Embrechts *et al.* (2014), Hashorva *et al.* (2014). It has been seen that the assumption of constant C_i's represents a popular setting considered in the recent past, where the asymptotic tail probabilities of some linear combinations of order statistics have been obtained. Obviously, randomising the C_i's is a more challenging problem to be studied, which is the main purpose of this paper. It is also of interest to recognise situations in which the randomisation represents a problem of interest. This is the case if one is interested in a more accurate risk aggregation, where the time value of the money is introduced in the model. That is, this popular risk evaluation takes into account not only the amount of claim, but also the time when the claim occurs, and therefore $L(\mathbf{C})$ becomes the discounted value of the aggregate risk. Another application that will be detailed in Section 3, is given when the C_i's quantify the random proportions paid by the risk holder in the case of its default in payment. In fact, our results are appealing even when C is deterministic. Take for instance $\mathbf{C} = (2, 1, , \ldots, 1)$, then

$$L(\mathbf{C}) = 2X_{n,n} + \sum_{i=1}^{n-1} X_{i,n} = X_{n,n} + L(\mathbf{1}). \tag{1.1}$$

The maximum risk $X_{n,n}$ and the total risk $L(\mathbf{1})$ are known to behave similarly in terms of large values (tail behaviour) for many tractable models, say for instance the Log-normal one, see e.g., Embrechts *et al.* (2014). However, these two risks are strongly dependent, and therefore, the behaviour of their sum $X_{n,n} + L(\mathbf{1})$ is not intuitively clear. Roughly speaking, our results show that, in many tractable dependence structures, we can substitute $L(\mathbf{1})$ by the largest risk.

We now explain the mathematical framework that will be further assumed in this paper. Under some technical conditions, it is obtained that the most significant contribution to the tail probability of $L(\mathbf{C})$ is given by the largest component, i.e., $C_1 X_{1,n}$. This can be explained by the fact that under asymptotic independence, the "one big jump" property is always present. In other words, as has been observed in the existing literature see e.g., Foss *et al.* (2013), the largest value is the most influential factor in risk aggregation. Our proofs are extremely sensitive to the tail behaviour of the individual risks. Therefore, a characterisation of the tail distribution of a random variable is necessary, which is a classical result of the Extreme Value Theory. A distribution function (df) F is said to belong to the *Maximum Domain of Attraction* (MDA) of a non-degenerate df G, written as $F \in \mathrm{MDA}(G)$, if there are some $a_n > 0$ and $b_n \in \mathbb{R}$ for $n \in \mathbb{N}$ such that for any constant x we have $\lim_{n\to\infty} F^n(a_n x + b_n) = G(x)$, where G is of one of the following three df's:

$$
\begin{array}{lll}
\text{Fréchet:} & \Phi_\alpha(x) = \exp(-x^{-\alpha}), & x > 0, \alpha > 0; \\
\text{Gumbel:} & \Lambda(x) = \exp(-\exp(-x)), & -\infty < x < \infty; \\
\text{Weibull:} & \Psi_\alpha(x) = \exp(-|x|^{\alpha}), & x \le 0, \alpha > 0.
\end{array}
$$

We focus on distributions with unbounded support, i.e., from $\mathrm{MDA}(\Phi_\alpha)$ and $\mathrm{MDA}(\Lambda)$, and therefore only the Fréchet and Gumbel cases will be considered. In Extreme Value Theory, the class of such distributions F is completely characterised. The following section presents our main result. In Section 3, we illustrate our findings with an application, while all the proofs are relegated to the last section.

2. Main Results

We consider first the Fréchet MDA and further include the case where the index α is allowed to be 0, i.e., X_1 may exhibit a slowly regularly varying tail. The mathematical formulation of the tail condition imposed on X_1 is given by

$$
\lim_{t\to\infty} \frac{\mathbb{P}(X_1 > tx)}{\mathbb{P}(X_1 > t)} = x^{-\alpha}, \quad \alpha \ge 0. \tag{2.1}
$$

The index α is crucial, since it determines the existence of the moments of X_1. For instance if $\alpha < 1$, then $\mathbb{E}\{X_1\} = \infty$; clearly such risks cannot be insured. Further, condition (2.1) means that the tail of X_1 can be written simply as $\mathbb{P}(X_1 > x) = x^{-\alpha}\mathcal{L}(x)$ where $\mathcal{L}$ is a slowly varying function (i.e., $\lim_{t\to\infty} \mathcal{L}(tx)/\mathcal{L}(t) = 1, \forall x > 0$). A canonical example for the slowly varying function is $\mathcal{L}(x) = (\log x)^\gamma, \gamma > 0$. Our treatment below is very general, as we do not restrict α or $\mathcal{L}$. Under a moment condition on the scaling random variable C_1, Breiman's Lemma (see Breiman (1965)) shows that the tail of C_1X_1 is up to $\mathbb{E}\{C_1^\alpha\}$ the same as the tail of C_1. For more details on regular variations of random variables and vectors see e.g., Jessen and Mikosch (2006).

Throughout the remainder of the paper $\lambda_1,\ldots,\lambda_n$ are non-negative constants and $\widetilde{\lambda}_n := \sum_{i=1}^n \lambda_i$.

Some standard notation are used, as well as further explanations, in order to provide a precise meaning of our statements. For two positive functions $a(\cdot)$ and $b(\cdot)$, we write $a(\cdot)\sim cb(\cdot)$ to mean asymptotic equivalence, i.e., $\lim a(\cdot)/b(\cdot)=c$ for some positive constant c. We also denote $\limsup a(\cdot)/b(\cdot)\le 1$ by $a(\cdot)\lesssim b(\cdot)$.

Theorem 2.1. *Let $X_1,\ldots,X_n$ be some positive random variables satisfying*

$$\lim_{t\to\infty}\frac{\mathbb{P}(X_i>t)}{\mathbb{P}(X_1>t)}=\lambda_i\in[0,\infty),\ \textit{for all}\ i\in\{1,\ldots,n\}. \tag{2.2}$$

Let $\boldsymbol{C}=(C_1,\ldots,C_n)$ be a random vector such that $C_1>0$ is independent of the maximum $X_{1,n}$. Suppose that (2.1) *holds with $\alpha>0$ and $\mathbb{E}\{C_1^\beta\}<\infty$ for some $\beta\in(\alpha,\infty)$. It is assumed that*

$$\lim_{t\to\infty}\max_{1\le i<j\le n}\frac{\mathbb{P}(\widetilde{C}X_i>t,\widetilde{C}X_j>t)}{\mathbb{P}(X_1>t)}=0, \tag{2.3}$$

with $\widetilde{C}=\max(1,\max_{2\le i\le n}|C_i|)$. If further there exists a positive constant τ such that

$$\mathbb{P}\big(\widetilde{C}X_i>t,\widetilde{C}X_j>t,\widetilde{C}>\tau\big)\ge\kappa_{ij}\mathbb{P}\big(X_i>t/r_{ij},X_j>t/r_{ij}\big) \tag{2.4}$$

holds for all large t and any two indices $i<j$ in $\{1,\ldots,k\}$ with r_{ij} a positive constant, then

$$\mathbb{P}\big(L(\boldsymbol{C})>t\big)\sim\mathbb{P}(C_1X_{1,n}>t)\sim\mathbb{P}(X_1>t)\mathbb{E}\{C_1^\alpha\}\widetilde{\lambda}_n,\quad t\to\infty. \tag{2.5}$$

We now discuss in greater details the conditions imposed in Theorem 2.1. Relation (2.4) is clearly satisfied if $\widetilde{C}$ is independent of $X_i, i\le k$. Another case for which (2.4) still holds is $\widetilde{C}$ has a df with positive lower endpoint $\widetilde{\alpha}$ (take $\tau=r_{ij}=\widetilde{\alpha}$). In numerous applications, risks can be of different nature in terms of their tail behaviour, where both light and heavy-tailed risks can be part of the aggregation process. A classical result in the case of independent risks with non-random weights states that the heaviest tail represents the dominant factor in explaining the extreme events of the aggregate risk. This is also the case for our considered model. Indeed, if X_k has heavier tail than X_1, i.e., (2.2) holds with $\lambda_k=0$, then the result in (2.5) shows that there is no impact of X_k when performing asymptotic evaluations. If $\mathbb{E}\{\widetilde{C}^{\beta}\} < \infty$ for some $\beta \in (\alpha, \infty)$, then by Lemma 4.2 for any $i \neq k, i > 1$

$$\frac{\mathbb{P}(\widetilde{C}X_i > t, \widetilde{C}X_k > t)}{\mathbb{P}(X_1 > t)} \leq \frac{\mathbb{P}(\widetilde{C}X_k > t)}{\mathbb{P}(X_1 > t)} \to 0, \quad t \to \infty$$

and thus condition (2.3) can be relaxed as follows:

$$\lim_{t\to\infty} \max_{(i,j)\in E_+} \frac{\mathbb{P}(\widetilde{C}X_i > t, \widetilde{C}X_j > t)}{\mathbb{P}(X_1 > t)} = 0, \tag{2.6}$$

where $E_+ := \{(i,j) : 1 \leq i \leq j, \lambda_i > 0, \lambda_j > 0\}$. Note in passing that due to the dependence among the C_i's, (2.4) is still needed even if $\lambda_k = 0$ for some $k \leq n$.

It is worth mentioning that (2.4) implies

$$\mathbb{P}\left(X_i > t, X_j > t\right) = o\big(\mathbb{P}(X_1 > t)\big), \quad t \to \infty. \tag{2.7}$$

In non-technical terms, relation (2.3) ensures that C_iX_i's are asymptotically independent, i.e. it is unlikely to observe joint extreme outcomes arising from this set of random variables. The technical assumption from (2.4) does not have an intuitive explanation, but earlier examples showed that it is plausible to fall in one of the given settings.

Next, we discuss the case in which the random scaling are independent of the portfolio risks, and has the advantage of being able to characterise the tail behaviour of $L(\mathbf{C})$ in the presence of slowly variation property of the individual risks.

Theorem 2.2. *Let $X_1, \ldots, X_n$ be some positive random variables satisfying* (2.6)*. Let $\boldsymbol{C} = (C_1, \ldots, C_n)$ be a random vector independent of $X_i, i \leq n$ with $C_1 > 0$. Suppose that* (2.1) *holds with $\alpha \geq 0$ and* (2.3) *is satisfied. If further $\mathbb{E}\{\max_{1\leq i\leq n} |C_i|^{\beta}\} \in (0, \infty)$ for some $\beta \in (\alpha, \infty)$, then* (2.5) *holds and moreover*

$$\mathbb{P}(L(\boldsymbol{C}) > t) \sim \sum_{i=1}^{n} \mathbb{P}(C_iX_{i,n} > t), \quad t \to \infty. \tag{2.8}$$

The Fréchet scenario requires a Pareto-like extreme behaviour for the individual risks, and sometimes leads to an overestimate of the extreme events magnitude. Additionally, this assumption is appropriate in case that not all of the moments of X_1 exist. Therefore, the Gumbel tail assumption represents a valid alternative, which includes from moderately heavy-tailed distributions, such as Log-Normal, to light-tailed distributions with all finite moments, such as Exponential. We further investigate this scenario for which some background is now provided.

It is well-known (see Embrechts et al. 1997) that if $F \in \mathrm{MDA}(\Lambda)$, then there exists a positive, measurable function $a(\cdot)$ such that $\bar{F} := 1 - F$ satisfies

$$\lim_{t\to\infty} \frac{\bar{F}\left(t + a(t)x\right)}{\bar{F}(t)} = \exp(-x) \tag{2.9}$$

for any $x \in \mathbb{R}$. In addition, the latter holds locally uniformly in x (see Resnick, 1987). Recall that the auxiliary function $a(\cdot)$ satisfies $a(t) = o(t)$ and is such that the relation

$$\lim_{t\to\infty} \frac{a\left(t + a(t)x\right)}{a(t)} = 1 \tag{2.10}$$

holds locally uniformly in x.

If X_1 has df F_1 satisfying (2.9), then the asymptotic tail of X_1 can be determined under some weak conditions on the random scaling factor C_1. In the sequel we shall only consider the case in which C_1 is bounded, i.e., its df has some finite upper endpoint $\omega \in (0, \infty)$. Specifically, the following two settings are investigated in this paper:

i) **<u>Model A</u>**: Assume that $\mathbb{P}(C_1 = \omega) = p \in (0, 1]$ and $\mathbb{P}(C_1 \le \eta) = 1 - p$ hold for some $\eta \in (0, \omega)$.

ii) **<u>Model B</u>**: For any $x > 0$ and some $\gamma \in [0, \infty)$, we have

$$\lim_{t\to\infty} \frac{\mathbb{P}\left(C_1 > \omega - x/t\right)}{\mathbb{P}\left(C_1 > \omega - 1/t\right)} = x^{\gamma}. \tag{2.11}$$

Our first model considers the case that the random weight C_1 has a jump at its upper endpoint. This model is of particular interest, since if $p = 1$, one may recover the case in which C_1 is a deterministic constant. As mentioned in the Section 1, our results are new even for such weights. The case in which C_1 has an unbounded upper endpoint, i.e., $\omega = \infty$, is more complex and less tractable. If both C_1 and $X_{1,n}$ have Weibullian tails the exact asymptotic tail behaviour of $C_1 X_{1,n}$ is obtained in Arendarczyk and Dębicki (2011).

The excellent contribution of Mitra and Resnick (2009) derives the asymptotic tail behaviour of the sum of dependent random variables with Gumbel tails, and their sufficient conditions provide the appropriate framework to elaborate our next result, stated as Theorem 2.3. It is worth mentioning that our proof also provides a simplified argumentation of their main result for portfolios consisting of three or more risks. In the following we assume without loss of generality that C_1 has upper endpoint equal to 1.

Theorem 2.3. *Let $X_1, \ldots, X_n$ be some positive random variables and suppose that X_1 has df in the $MDA(\Lambda)$ with infinite right endpoint and an auxiliary function $a(\cdot)$ as defined in (2.9). Let further $\mathbf{C} = (C_1, \ldots, C_n)$ be a given* *random vector such that $C_1>0$ is independent of $X_{1,n}$ and the assumption of Model A or Model B holds with $\omega=1$. If* (2.2) *is satisfied and further*

$$\lim_{t\to\infty}\max_{1\le i\ne j\le n}\frac{\mathbb{P}\left(C^*X_i>t,C^*X_j>a(t)x\right)}{\mathbb{P}(C_1X_1>t)}=0,\quad \text{for all } x>0 \tag{2.12}$$

holds for $C^*:=\max_{1\le i\le n}|C_i|$, then as $t\to\infty$

$$\mathbb{P}(L(\boldsymbol{C})>t)\sim\mathbb{P}(C_1X_{1,n}>t)\sim\widetilde{\lambda}_n\mathbb{P}\left(C_1X_1>t\right),$$

provided that

$$\lim_{t\to\infty}\frac{\mathbb{P}\left(C^*X_i>L_{ij}a(t),C^*X_j>L_{ij}a(t)\right)}{\mathbb{P}(C_1X_1>t)}=0,\ \text{for some } L_{ij}>0 \text{ and all } 1\le i<j\le n. \tag{2.13}$$

Special attention is now given to the conditions imposed in Theorem 2.3. In the above theorem, if C_1 has upper endpoint $\omega\in(0,\infty)$, then $C_1X_{1,n}$ is in $MDA(\Lambda)$ with auxiliary function $a_1(t):=\omega a\big(t/\omega\big)$. Clearly, when $\omega=1$, then $a_1(t)=a(t)$. A simpler case is when $C_1,\dots,C_n$ are independent of $X_1,\dots,X_n$, and this setting is addressed below in greater details under the framework of Log-Normal risks $X_1,\dots,X_n$.

It is worth discussing our model when all $C_i's$ are constant. Consider for simplicity the model in (4.3). For such an instance, Theorem 2.3 states that the total of the maximum risk and the sum of all n risks behaves asymptotically like two times the maximum risk. Additionally, when all risks are asymptotically tail equivalent, i.e., λ_i's are all equal to 1, then the tail probability of $2X_{n,n}$ has the same behaviour as n times the tail probability $\mathbb{P}\left(2X_1>t\right)$ for t large. Therefore, it only remains to determine the tail probability of each risk, which is a great advantage for practical purposes.

The permissible dependence structures for our risks models are sensitive to the random weights. Since risks are assumed to be heavy-tailed in Theorem 2.1, the dependence structure required therein is quite general as compared to what is required in Theorem 2.3. In order to illustrate our point, let us consider the deterministic weights C_i's, as defined in (4.3). The dependence structure assumed in Theorem 2.1 requires the risks to be asymptotically independent, meaning that (2.3) holds with $\widetilde{C}=1$. The asymptotic independence assumption alone is however not enough for the dependence structure assumed in Theorem 2.3. Since the risks in the Gumbel max-domain of attraction can be very light-tailed, the additional conditions imposed in Theorem 2.3 are not restrictive.

Note that our assumptions stated in (2.12) and (2.13) are in fact the conditions from Mitra and Resnick (2009) adapted to our setting. In other words, our assumptions ensure more than the fact that it is unlikely to observe joint extreme outcomes arising from C_iX_i's. In addition, joint extreme outcomes are not likely to occur even if the extreme thresholds are not equal (for asymptotic independence these extreme thresholds are equal to t). The set of random variables with Gumbel tails may behave in a peculiar and uncontrolled manner when the risks are aggregated. Moreover, the classical asymptotic independence assumption has been shown to be insufficient when C's are not random, and therefore, if C's are random, the tail of the aggregated risk is even more expected to exhibit an erratic behaviour.

3. Applications

In this section, we discuss an application of one of our main results, namely an illustration of the approximations found in Theorem 2.3. It is further assumed that the individual risks are multivariate Log-Normal distributed. That is, $X_i = \exp\big(\sigma_i Z_i + \mu_i\big), i \le n$, where $(Z_1, \dots, Z_n)$ is a multivariate Gaussian distributed random vector with Pearson correlation coefficients $\rho_{ij} \in (-1, 1), 1 \le i < j \le n$ and $\sigma_i > 0, \mu_i \in \mathbb{R}$ are some given constants. Further, suppose that $X_i, i \le n$ are independent of C_1, which is a random variable with $[0, 1]$ support. Furthermore, C_1 is Beta distributed with positive parameters α, β. It is well-known that X_i's have distribution functions in the Gumbel MDA with scaling functions $a_i(t) = \sigma_i^2 t / \big(\log(t) - \mu_i\big), i \le n$ (see for example, Embrechts *et al.*, 1997). By using Lemma 4.1, it is not difficult to find that $C_1 X_{1,n}$ has also df in the Gumbel MDA.

The next step is to show that Theorem 2.3 is applicable in the current setting, and therefore we only need to establish that condition (2.13) holds. In view of our assumptions, we have the stochastic representation

$$(X_i, X_j) \stackrel{d}{=} \left(\exp\left(\sigma_i^2 \sin(\Theta) R + \mu_i\right), \exp\left(\sigma_j^2 \left(\rho_{ij} \sin(\Theta) + \sqrt{1 - \rho_{ij}^2} \cos(\Theta)\right) R + \mu_j\right)\right), \quad 1 \le i \ne j < n,$$

where the random angle Θ is uniformly distributed on $(0, 2\pi)$ being independent of the random radius $R > 0$. Note that $\mathbb{P}(R > r) = e^{-r^2/2}, r > 0$. Assume without loss of generality that $\mu_1 > \max_{2 \le i \le n} \mu_i$ and $\sigma_1 \ge \max_{2 \le i \le n} \sigma_i$. Requiring further that C_i's are bounded, say by 1, we have

$$\mathbb{P}\left(C^* X_i > L_{ij} a(t), C^* X_j > L_{ij} a(t)\right) \approx \exp\left(-\frac{(\log t)^2}{2\sigma_1^2 \eta_{ij}^2}\right),$$

where $\approx$ stands for a logarithmic asymptotic equivalence, and

$$\eta_{ij} = \max_{0 \le \theta \le 2\pi} \left(\min\left(\sin(\theta), \rho_{ij} \sin(\theta) + \sqrt{1 - \rho_{ij}^2} \cos(\theta)\right)\right) < 1.$$

Note that $f(t) \approx g(t)$ means that $\log f(t) \sim \log g(t)$ is true. Now, (2.13) holds since $\mathbb{P}\left(C_1 X_1 > t\right) \approx \exp\left(-\frac{(\log t)^2}{2\sigma_1^2}\right)$ is satisfied for all large t. Recall that $\lambda_i = 1$ if $\sigma_i = \sigma_1, \mu_i = \mu_1$ and otherwise $\lambda_i = 0$. Consequently, Theorem 2.3 holds in our setting. The setup formulated in (4.3) leads to

$$\mathbb{P}\left(2X_{n,n} + \sum_{i=1}^{n-1} X_{i,n} > t\right) \sim \sum_{i=1}^{n} \mathbb{P}\left(2X_i > t\right) \sim \sum_{i=1}^{n} \mathbb{P}\left(2\sigma_i Z > \log(t) - \mu_i\right) \tag{3.1}$$

as $t \to \infty$, where Z is an $N(0,1)$ random variable. Hence, as it can be easily seen above, determining σ_i's and μ_i's is crucial in the Log-normal model. Several extensions of this model can be still covered by our main results. Clearly, the technical assumption that C_1 is beta distributed can be weakened to C_1 has finite upper endpoint with a regularly varying tail. Additionally, as in Embrechts et al. (2014) we can easily include in our application the log-normal model with stochastic volatilities. For instance, we can assume that σ_i's are random with finite upper endpoint satisfying the assumptions of Model A or Model B. Note that the case of Model A has been considered in the aforementioned paper for C_i's being all equal. Another interesting and feasible extension is to consider the more general framework of Log-elliptical risks. For such a model, our results generalise the recent findings of Kortschak and Hashorva (2013). The Log-normal assumption is widely accepted by practitioners as a distribution for modelling individual risks. Choosing an appropriate dependence to model the association among various risks is a difficult tasks, and therefore, the Gaussian dependence is an acceptable choice due to many convenient features, such as the availability of relatively simple estimation and simulation methods. Moreover, as mentioned above, our results allow for dependent risks with Log-elliptical dependence structure and stochastic volatility, which are of interest in particular when the Log-normality is not tenable.

Let us now discuss the parametric model considered above. Consider the situation in which the holder of this portfolio, named *insurer*, prefers to transfer the first k largest claim amounts, with $1 \le k < n$, to a different insurance player, namely *reinsurer*. This risk transfer contract is also known as the *Large Claims Reinsurance (LCR)* (see for example, Ladoucette and Teugels, 2006) and the reinsurer is liable to pay $\sum_{i=1}^{k} X_{i,n}$, which might not always be paid in full due the possibility of default in payment. Therefore, the insurer expects to pay an additional amount (as a result of the default event) of $L_k(\mathbf{C}) := \sum_{i=1}^{k} C_i X_{i,n}$, where $0 \le C_i = 1 - RecR_i \le 1$ are some random weights with $RecR_i$ being the so-called *recovery rate* corresponding to the i^{th} largest claim. Our assumptions require that $RecR_1$ is Beta distributed with parameters β and α. Assume next for simplicity that $\sigma_1 > \sigma_j, j \ge 1$. Theorem 2.3 shows that $\mathbb{P}\left(L_k(\mathbf{C}) > t\right) \sim \mathbb{P}\big(C_1 X_1 > t\big)$, and consequently, the insurer may easily understand the severity of the extreme events associated with the reinsurer default. Specifically, the asymptotic result can be used in approximating tail risk measures such as *Value-at-Risk (VaR)* and *Expected Shortfall (ES)*:

$$ES_p\left(L_k(\mathbf{C})\right) \sim VaR_p\left(L_k(\mathbf{C})\right) \sim VaR_p\big(C_1 X_1\big) \ \text{ as } p \uparrow 1, \tag{3.2}$$

since $C_1 X_{1,n}$ is in the Gumbel MDA (for more details, see Asimit and Badescu, 2010). Recall that for a generic random variable Z, $VaR_p(Z)$ represents the p^{th} quantile and $ES_p(Z) := \mathbb{E}\{Z|Z > VaR_p(Z)\}$. It can be easily seen that evaluating the extreme events associated with $L_k(\mathbf{C})$ has been drastically reduced via our findings from Theorem 2.3. Moreover, the claim in (3.2) holds for general Log-elliptical risks with underlying random radius in the Gumbel MDA.

4. Further Results and Proofs

We display next some lemmas which are of some independent interests and then proceed with the proofs of the main results.

Lemma 4.1. *Let C and X be two independent positive random variables. Suppose that C has upper endpoint $\omega \in (0,\infty)$ and X has df in $MDA(\Lambda)$ with scaling function $a(\cdot)$.*

i) *If C obeys Model A, then for a function $R(t)$ with $R(t) = o\left(\left(\frac{a(t)}{t}\right)^{\xi}\right)$, for every $\xi > 0$*

$$\mathbb{P}\left(CX > t\omega\right) \sim p\mathbb{P}\left(X > t\right)\left(1 + R(t)\right).$$

ii) *If C satisfies the assumption* (2.11) *of Model B, then*

$$\mathbb{P}\left(CX > t\omega\right) \sim \Gamma(\gamma+1)\mathbb{P}\left(C > \omega - \frac{\omega a(t)}{t}\right)\mathbb{P}\left(X > t\right),$$

where $\Gamma(\cdot)$ is the Euler gamma function.

Proof of Lemma 4.1 i) The crucial asymptotic result for establishing the claim is the so-called Davis-Resnick tail property of distributions in the $MDA(\Lambda)$. Namely, by Proposition 1.1 in Davis and Resnick (1988)

$$\lim_{t\to\infty}\left(\frac{a(t)}{t}\right)^{\mu}\frac{\mathbb{P}\left(X > Kt\right)}{\mathbb{P}\left(X > t\right)} \quad = \quad 0$$

holds for any $\mu \in \mathbb{R}, K > 1$. The latter and the fact that $\eta \in (0,\omega)$ implies the proof. $\square$

ii) Since $C^* := C/\omega$ has df with upper endpoint 1 and is regularly varying at 1 with index γ, the claim follows immediately from Theorem A.2 in Hashorva (2015), and thus the proof is now complete. $\square$

Lemma 4.2. *Let C, X and Y be three positive random variables such that C is independent from X, Y and X satisfies* (2.1) *with some constant $\alpha \geq 0$. Suppose that there exists a positive function $h(\cdot)$ such that $\lim_{t\to\infty} h(t) = \lim_{t\to\infty} t/h(t) = \infty$ and*

$$\lim_{t\to\infty} \frac{\mathbb{P}(C > h(t))}{\mathbb{P}(X > t)} = 0. \tag{4.1}$$

If further

$$\lim_{t\to\infty} \frac{\mathbb{P}(Y > t)}{\mathbb{P}(X > t)} = 0, \tag{4.2}$$

then we have

$$\lim_{t\to\infty} \frac{\mathbb{P}(CY > t)}{\mathbb{P}(X > t)} = 0. \tag{4.3}$$

Furthermore, if $\mathbb{E}\{C^\beta\} < \infty$ for some $\beta > \alpha$, then (4.1) *is valid.*

Proof of Lemma 4.2 By the assumptions

$$\begin{aligned}
\frac{\mathbb{P}(CY > t)}{\mathbb{P}(X > t)} &= \frac{\mathbb{P}(CY > t, C \leq h(t)) + \mathbb{P}(CY > t, C > h(t))}{\mathbb{P}(X > t)} \\
&\leq \frac{\mathbb{P}(CY > t, C \leq h(t)) + \mathbb{P}(C > h(t))}{\mathbb{P}(X > t)}
\end{aligned}$$

holds for all large t. Denote by F the df of C and set $h^*(t) = t/h(t)$. Since $\lim_{t\to\infty} h^*(t) = \lim_{t\to\infty} h(t) = \infty$, then for any large M we can find $n(M)$ so that for all $t > n(M)$ we have $h^*(t) > M$ and $h(t) > M$. Further, by (4.2) for any $\varepsilon > 0$ and for some M' (take for simplicity $M' = M$), we have $\frac{\mathbb{P}(Y>t)}{\mathbb{P}(X>t)} \leq \varepsilon, \forall t > M$. Consequently, for any $c \in (0,1)$ we have $h^*(t)/c > M/c > M$ implying $\mathbb{P}(Y > h^*(t)/c)/\mathbb{P}(X > h^*(t)/c) \leq \varepsilon, \forall t > n(M)$. The independence of C and Y together with equation (4.2) yield $\Big(\text{set } G(c) := F\big(h(t)c\big)\Big)$

$$\begin{aligned}
\frac{\mathbb{P}(CY > t, C \leq h(t))}{\mathbb{P}(CX > t)} &= \int_0^{h(t)} \frac{\mathbb{P}(Y > t/c)}{\mathbb{P}(CX > t)}\, dF(c) \\
&= \frac{1}{\mathbb{P}(CX > t)} \int_0^1 \frac{\mathbb{P}(Y > h^*(t)/c)}{\mathbb{P}(X > h^*(t)/c)} \mathbb{P}(X > h^*(t)/c)\, dG(c) \\
&\leq \frac{\varepsilon}{\mathbb{P}(CX > t)} \int_0^1 \mathbb{P}(X > h^*(t)/c)\, dG(c) \\
&= \varepsilon \frac{\mathbb{P}\big(CX > t, C \leq h(t)\big)}{\mathbb{P}(CX > t)} \leq \varepsilon
\end{aligned}$$

for any $t > n(M)$, and thus (4.3) follows. Next, if $\mathbb{E}\{C^\beta\} < \infty$ for some $\beta > \alpha$, then the random variable X has heavier tail than C, i.e., $\mathbb{E}\{X^{\alpha+\varepsilon}\} = \infty$ and $\mathbb{E}\{(C^{1+\varepsilon'})^{\alpha+\varepsilon''}\} < \infty$ for some $\varepsilon, \varepsilon', \varepsilon''$ positive with $\varepsilon'' > \varepsilon$ implying thus (4.1) holds since further by the assumption $\lim_{t\to\infty} h(t)/t = 0$, hence the proof is complete. □

In the following we shall need the concept of vague convergence. Let $\{\mu_n, n \geq 1\}$ be a sequence of measures on a locally compact Hausdorff space $\mathbb{B}$ with countable base. Then μ_n converges vaguely to some measure μ, written as $\mu_n \xrightarrow{v} \mu$, if for all continuous functions f with compact support we have

$$\lim_{n\to\infty} \int_{\mathbb{B}} f \,\mathrm{d}\mu_n = \int_{\mathbb{B}} f \,\mathrm{d}\mu.$$

A thorough background on vague convergence is given in Resnick (1987).

Lemma 4.3. *If the assumptions of Theorem 2.1 are satisfied with $\alpha > 0$ and $C_i \geq 0$ almost surely for all $2 \leq i \leq n$, then the following vague convergence*

$$\frac{\mathbb{P}\big((C_1X_{1,n}/t, \ldots, C_nX_{n,n}/t) \in \cdot\big)}{\mathbb{P}(C_1X_{1,n} > t)} \xrightarrow{v} \mu(\cdot), \quad t \to \infty \tag{4.4}$$

holds on $[0,\infty] \times [M,\infty] \times \cdots \times [M,\infty] \setminus \{(0,0,\ldots,0)\}$ for any $M < 0$ where the limit measure μ is given by

$$\mu(dx_1, dx_2, \ldots, dx_n) := \alpha x_1^{-\alpha-1} dx_1 \epsilon_0(dx_2) \cdots \epsilon_0(dx_n),$$

where $\epsilon_0(\cdot)$ denotes the Dirac measure.

Proof of Lemma 4.3 First note that by Bonferroni's inequality for any real t we have

$$\sum_{i=1}^n \mathbb{P}(X_i > t) - \sum_{1\leq i<j\leq n} \mathbb{P}(X_i > t, X_j > t) \leq \mathbb{P}(X_{1,n} > t) \leq \sum_{i=1}^n \mathbb{P}(X_i > t). \tag{4.5}$$

Clearly, equation (2.7) suggests that $\sum_{1\leq i<j\leq n} \mathbb{P}(X_i > t, X_j > t) = o\big(\mathbb{P}\left(X_1 > t\right)\big)$ as $t \to \infty$. Hence, equation (2.2) implies that $\lim_{t\to\infty} \frac{\mathbb{P}(X_{1,n}>t)}{\mathbb{P}(X_1>t)} = \widetilde{\lambda}_n$, which in turn by Breiman's Lemma (see Breiman, 1965) yields

$$\lim_{t\to\infty} \frac{\mathbb{P}(C_1X_{1,n} > t)}{\mathbb{P}(X_1 > t)} = \mathbb{E}\{C_1^\alpha\}\widetilde{\lambda}_n. \tag{4.6}$$

Next, we show the vague convergence only for the first two largest order statistics, since the high dimensional case follows easily by using further the fact that $X_{1,n} \geq X_{2,n} \geq \cdots \geq X_{n,n}$ almost surely. The above-mentioned convergence of measures holds if the convergence is valid over the following relative compact sets:

i) $(x,\infty] \times (y,\infty]$, where $x > 0$, $y \geq M$ and $y \neq 0$;

ii) $[0,\infty] \times (y,\infty]$, where $y > 0$.

Part i) is now investigated for which $\mathbb{P}(C_1 X_{1,n} > tx, C_2 X_{2,n} > ty) = \mathbb{P}(C_1 X_{1,n} > tx)$ holds for all $y < 0$ due to the positivity assumption of the X_i's. Consequently, equation (4.6) yields

$$\lim_{t\to\infty} \frac{\mathbb{P}\big((C_1 X_{1,n}/t, C_2 X_{2,n}/t) \in (x,\infty] \times (y,\infty]\big)}{\mathbb{P}(C_1 X_{1,n} > t)} = x^{-\alpha} = \mu\left((x,\infty] \times (y,\infty]\right).$$

For any $y > 0$, the following is true

$$\mathbb{P}(X_{2,n} > y) \le \sum_{1\le i<j\le n} \mathbb{P}(X_i > y, X_j > y). \tag{4.7}$$

Now,

$$\begin{aligned}
&\frac{\mathbb{P}\big((C_1 X_{1,n}/t, C_2 X_{2,n}/t) \in (x,\infty] \times (y,\infty]\big)}{\mathbb{P}(C_1 X_{1,n} > t)} \\
&\quad \le \quad \frac{\mathbb{P}(C_2 X_{2,n} > ty)}{\mathbb{P}\left(C_1 X_{1,n} > t\right)} \\
&\quad \le \quad \frac{\mathbb{P}(X_1 > ty)}{\mathbb{P}(X_1 > t)} \frac{\mathbb{P}(X_1 > t)}{\mathbb{P}(C_1 X_{1,n} > t)} \sum_{1\le i<j\le n} \frac{\mathbb{P}(C_2 X_i > ty, C_2 X_j > ty)}{\mathbb{P}(X_1 > ty)} \\
&\quad \sim \quad y^{-\alpha} \left(\mathbb{E}\{C_1^\alpha\}\widetilde{\lambda}_n\right)^{-1} \sum_{1\le i<j\le n} \frac{\mathbb{P}(C_2 X_i > ty, C_2 X_j > ty)}{\mathbb{P}(X_1 > ty)} \\
&\quad \to \quad 0 = \mu\left((x,\infty] \times (y,\infty]\right), \quad t \to \infty,
\end{aligned} \tag{4.8}$$

where the third implication is due to equations (2.1) and (4.6), while the fourth implication is a consequence of (2.3). Thus, part i) is fully justified. Finally, part ii) can be shown in the same manner as displayed in (4.8). □

Lemma 4.4. *Let us assume that the assumptions of Theorem 2.3 are satisfied such that $C_i \ge 0$ almost surely for all $2 \le i \le n$. Then as $t \to \infty$*

$$\frac{\mathbb{P}\Big(\big((C_1 X_{1,n} - t)/a(t), C_2 X_{2,n}/a(t), \ldots, C_n X_{n,n}/a(t)\big) \in \cdot\Big)}{\mathbb{P}(C_1 X_{1,n} > t)} \xrightarrow{v} \nu(\cdot) \tag{4.9}$$

holds on $[M,\infty] \times [-\infty,\infty] \times \cdots \times [-\infty,\infty]$ for any $M < 0$ with limiting measure ν given by

$$\nu(dx_1, dx_2, \ldots, dx_n) := \exp(-x_1)dx_1 \epsilon_0(dx_2) \cdots \epsilon_0(dx_n).$$

Proof of Lemma 4.4 The proof is similar to that of Lemma 4.3 and it is sufficient to verify the convergence only over the following compact sets:

i) $(x_1,\infty] \times (x_2,\infty] \times \ldots \times (x_n,\infty]$, where $x_1 > M$ and $x_i < 0$ for all $i \ge 2$;

ii) $(x_1,\infty]\times(x_2,\infty]\times\ldots\times(x_n,\infty]$, where $x_1>M$ and $x_i>0$ for some $i\geq 2$.

For any set from part i)

$$\mathbb{P}\big(C_1X_{1,n}>t+a(t)x_1, C_iX_{i,n}>a(t)x_i,\ \text{for all } i\geq 2\big)=\mathbb{P}\big(C_1X_{1,n}>t+a(t)x_1\big).$$

By Lemma 4.1, under the assumptions of Model A or Model B the random variable C_1X_1 is in the $MDA(\Lambda)$ with auxiliary function $a(\cdot)$. Note that when $\omega\neq 1$, then the auxiliary function is not $a(\cdot)$ but $\omega a(t/\omega)$. Moreover, $C_1X_{1,n}$ is also in the $MDA(\Lambda)$ with auxiliary function $a(\cdot)$ under the assumptions of Model A or Model B, provided that $X_{1,n}$ is in $MDA(\Lambda)$ with auxiliary function $a(\cdot)$, which we show next. We shall show that

$$\lim_{t\to\infty}\frac{\mathbb{P}(C_1X_{1,n}>t)}{\mathbb{P}(C_1X_1>t)}=\widetilde{\lambda}_n \tag{4.10}$$

holds, and thus both $X_{1,n}$ and $C_1X_{1,n}$ have df in the Gumbel MDA. The fact that $C_1>0$, equations (2.2) and (4.5), and the main result of Lemma 4.1 suggest that (4.10) is satisfied as long as

$$\mathbb{P}(C_1X_i>t, C_1X_j>t)=o\big(\mathbb{P}(C_1X_1>t)\big),\quad t\to\infty,\ \text{for all } 1\leq i<j\leq n. \tag{4.11}$$

Recall that $a(t)=o(t)$ as $t\to\infty$. The latter and equation (2.12) yield that for large t we have

$$\begin{aligned}\mathbb{P}(C_1X_i>t, C_1X_j>t) &\leq \mathbb{P}(C^*X_i>t, C^*X_j>t)\\ &\leq \mathbb{P}\big(C^*X_i>t, C^*X_j>a(t)\big)=o\big(\mathbb{P}(C_1X_1>t)\big),\quad t\to\infty,\end{aligned}$$

which justifies (4.11). Consequently,

$$\begin{aligned}&\lim_{t\to\infty}\frac{\mathbb{P}\Bigg(\bigg(\frac{C_1X_{1,n}-t}{a(t)},\frac{C_2X_{2,n}}{a(t)},\ldots,\frac{C_nX_{n,n}}{a(t)}\bigg)\in(x_1,\infty]\times(x_2,\infty]\times\ldots\times(x_n,\infty]\Bigg)}{\mathbb{P}(C_1X_{1,n}>t)}\\ &=\lim_{t\to\infty}\frac{\mathbb{P}\big(C_1X_{1,n}>t+a(t)x_1\big)}{\mathbb{P}(C_1X_{1,n}>t)}\\ &=e^{-x_1}=\nu\big((x_1,\infty]\times(x_2,\infty]\times\ldots\times(x_n,\infty]\big).\end{aligned}$$

For the second part, without loss of generality $x_2 > 0$ is further assumed. Now,

$$\begin{aligned}
&\frac{\mathbb{P}\Big(\Big(\frac{C_1X_{1,n}-t}{a(t)}, \frac{C_2X_{2,n}}{a(t)}, \ldots, \frac{C_nX_{n,n}}{a(t)}\Big) \in (x_1,\infty] \times (x_2,\infty] \times \ldots \times (x_n,\infty]\Big)}{\mathbb{P}(C_1X_{1,n} > t)} \\
&\leq \quad \frac{\mathbb{P}\Big(\Big(\frac{C_1X_{1,n}-t}{a(t)}, \frac{C_2X_{2,n}}{a(t)}\Big) \in (x_1,\infty] \times (x_2,\infty]\Big)}{\mathbb{P}\left(C_1X_{1,n} > t\right)} \\
&\leq \quad \frac{\mathbb{P}\left(C_1X_1 > t\right)}{\mathbb{P}\left(C_1X_{1,n} > t\right)} \frac{\mathbb{P}\big(C_1X_{1,n} > t + a(t)x_1, C_2X_{2,n} > a(t)x_2\big)}{\mathbb{P}\left(C_1X_1 > t\right)} \\
&\leq \quad \frac{\mathbb{P}\big(C^*X_{1,n} > t + a(t)x_1, C^*X_{2,n} > a(t)x_2\big)}{\mathbb{P}\left(C_1X_1 > t\right)} \\
&\leq \quad \sum_{1\leq i\neq j\leq n} \frac{\mathbb{P}\big(C^*X_i > t + a(t)x_1, C^*X_j > a(t)x_2\big)}{\mathbb{P}\left(C_1X_1 > t\right)} \\
&\to \quad 0 = \nu\big((x_1,\infty] \times (x_2,\infty] \times \ldots \times (x_n,\infty]\big), \quad t \to \infty
\end{aligned}$$

follows from (2.12), (4.7) and the fact that

$$\max_{1\leq i\neq j\leq n} \mathbb{P}\big(C^*X_i > t + a(t)x_1, C^*X_j > a(t)x_2\big) = o\big(\mathbb{P}(C_1X_1 > t)\big).$$

The latter is justified in few steps. Equation (2.10) yields that $(1-\varepsilon)a(t) \leq a\left(t + a(t)x_1\right) \leq (1+\varepsilon)a(t)$ for any arbitrarily fixed $0 < \varepsilon < 1$ and all large t. Recall that C_1X_1 is in the $MDA(\Lambda)$ with scaling function $a(\cdot)$. Consequently,

$$\begin{aligned}
\frac{\mathbb{P}\left(C^*X_i > t + a(t)x_1, C^*X_j > a(t)x_2\right)}{\mathbb{P}(C_1X_1 > t)} \quad &\leq \quad \frac{\mathbb{P}\left(C^*X_i > t + a(t)x_1, C^*X_j > a\left(t + a(t)x_1\right)\frac{x_2}{1+\varepsilon}\right)}{\mathbb{P}(C_1X_1 > t)} \\
&\sim \quad \frac{\mathbb{P}\left(C^*X_i > t + a(t)x_1, C^*X_j > a\left(t + a(t)x_1\right)\frac{x_2}{1+\varepsilon}\right)}{\mathbb{P}(C_1X_1 > t + a(t)x_1)} e^{-x_1} \\
&\to \quad 0, \quad t \to \infty
\end{aligned}$$

holds for any $x_1 \in \mathbb{R}$ and $x_2 > 0$, which is a consequence of relations (2.9) and (2.12), and thus the claim follows. $\Box$

Proof of Theorem 2.1 In the first instance, we assume that $C_i \geq 0$ for all $i \geq 2$. Clearly,

$$\mathbb{P}\left(\sum_{i=1}^{n} C_iX_{i,n} > t\right) = \mathbb{P}\left(\sum_{i=1}^{n} C_iX_{i,n} > t, C_1X_{1,n} > 0\right). \tag{4.12}$$

In addition,

$$\lim_{t\to\infty} \frac{\mathbb{P}\left(\sum_{i=1}^{n} C_iX_{i,n} > t, C_1X_{1,n} > 0\right)}{\mathbb{P}(C_1X_{1,n} > t)} = \lim_{t\to\infty} \frac{\mathbb{P}\Big(\big(C_1X_{1,n}/t, C_2X_{2,n}/t \ldots, C_nX_{n,n}/t\big) \in A_1\Big)}{\mathbb{P}(C_1X_{1,n} > t)} = \mu(A_1) = 1, \tag{4.13}$$

where $A_1 := \{\underline{x} : \sum_{i=1}^n x_i > 1, x_1 > 0, x_i > M \text{ for all } i \geq 2\}$ and M is a negative constant. Note that the second step is due to the fact that Proposition A2.12 of Embrechts et al. (1997, p. 563) can be applied in (4.4) since A_1 does not put any mass on its boundary. In addition, the whole mass over the set A_1 is concentrated on the line $(1, \infty] \times \{0\}^{n-1}$. Combining (4.12) and (4.13) we have

$$\mathbb{P}\left(\sum_{i=1}^n C_i X_{i,n} > t\right) \sim \mathbb{P}\left(C_1 X_{1,n} > t\right), \quad t \to \infty. \tag{4.14}$$

Similarly,

$$\lim_{t\to\infty} \frac{\mathbb{P}\left(C_1 X_{1,n} - \sum_{i=2}^n C_i X_{i,n} > t, C_1 X_{1,n} > 0\right)}{\mathbb{P}(C_1 X_{1,n} > t)} \quad = \quad \mu(A_2) = 1,$$

where $A_2 := \{\underline{x} : x_1 - \sum_{i=2}^n x_i > 1, x_1 > 0, x_i > M \text{ for all } i \geq 2\}$. Once again, the entire mass over the set A_2 is concentrated on the line $(1, \infty] \times \{0\}^{n-1}$. Thus,

$$\mathbb{P}\left(C_1 X_{1,n} - \sum_{i=2}^n C_i X_{i,n} > t\right) \sim \mathbb{P}(C_1 X_{1,n} > t), \quad t \to \infty. \tag{4.15}$$

We may now drop the non-negativity assumption for the $C_i, i \geq 2$ since

$$\mathbb{P}\left(C_1 X_{1,n} - \sum_{i=2}^n C_i^- X_{i,n} > t\right) \leq \mathbb{P}\left(\sum_{i=1}^n C_i X_{i,n} > t\right) \leq \mathbb{P}\left(\sum_{i=1}^n C_i^+ X_{i,n} > t\right), \tag{4.16}$$

where $C_i^+ = \max\{C_i, 0\}$ and $C_i^- = \max\{-C_i, 0\}$. The latter, together with (4.6), (4.14) and (4.15) completes the proof for this case. $\square$

Proof of Theorem 2.2 We show first that for any index i such that $2 \leq i \leq n$

$$\frac{C_i^+ X_{i,n}}{t}\Big|(C_1 X_{1,n} > t) \xrightarrow{p} 0$$

is valid where $\xrightarrow{p}$ and $\xrightarrow{d}$ stand for convergence in probability and in distribution, respectively letting the argument $t \to \infty$. Indeed, by (2.3) for any $y > 0$ we obtain applying Breiman's Lemma (set $\overline{C} := \max_{2 \le i \le n} C_i^+$)

$$
\begin{aligned}
&\mathbb{P}\left(\frac{C_i^+ X_{i,n}}{t} > y \big| C_1 X_{1,n} > t\right) \\
&\quad = \quad \frac{\mathbb{P}\big(C_i^+ X_{i,n} > ty, C_1 X_{1,n} > t\big)}{\mathbb{P}\left(C_1 X_{1,n} > t\right)} \\
&\quad \le \quad \frac{\mathbb{P}\big(\overline{C} X_{2,n} > ty\big)}{\mathbb{P}\left(C_1 X_{1,n} > t\right)} \\
&\quad \le \quad \frac{\mathbb{P}(X_1 > ty)}{\mathbb{P}(X_1 > t)} \frac{\mathbb{P}(X_1 > t)}{\mathbb{P}(C_1 X_{1,n} > t)} \sum_{1 \le i < j \le n} \frac{\mathbb{P}\big(\overline{C} X_i > ty, \overline{C} X_j > ty\big)}{\mathbb{P}\left(X_1 > ty\right)} \\
&\quad \to \quad 0 \quad \text{as } t \to \infty
\end{aligned}
$$

since by the assumption on C_i's

$$
\lim_{t \to 0} \frac{\mathbb{P}\big(\overline{C} X_i > ty, \overline{C} X_j > ty\big)}{\mathbb{P}\left(X_1 > ty\right)} \le \lim_{t \to 0} \frac{\mathbb{P}\big(\widetilde{C} X_i > ty, \widetilde{C} X_j > ty\big)}{\mathbb{P}\left(X_1 > ty\right)} = 0.
$$

Therefore, $\lim_{t\to\infty} \mathbb{P}\left(\frac{C_i^+ X_{i,n}}{t} > y \big| C_1 X_{1,n} > t\right) = 0$. Further, equation (4.6) implies that $\frac{C_1 X_{1,n}}{t} \Big| (C_1 X_{1,n} > t) \xrightarrow{d} W$, where the random variable $W \ge 1$ has survival function $x^{-\alpha}, x \ge 1$. Thus,

$$
\left(\frac{C_1 X_{1,n}}{t}, \frac{C_2^+ X_{2,n}}{t}, , \dots, \frac{C_n^+ X_{n,n}}{t}\right) \Big| (C_1 X_{1,n} > t) \xrightarrow{d} (W, 0, \dots, 0) \tag{4.17}
$$

implying

$$
\left(\frac{C_1 X_{1,n}}{t} + \sum_{i=2}^{n} \frac{C_i^+ X_{i,n}}{t}\right) \Big| (C_1 X_{1,n} > t) \xrightarrow{d} W. \tag{4.18}
$$

When $\alpha = 0$, then $W = 1$ and hence the convergence holds in probability. Similarly, we obtain

$$
\left(\frac{C_1 X_{1,n}}{t} + \sum_{i=2}^{n} \frac{C_i^- X_{i,n}}{t}\right) \Big| (C_1 X_{1,n} > t) \xrightarrow{d} W.
$$

Consequently,

$$
\begin{aligned}
\lim_{t \to \infty} \frac{\mathbb{P}\left(C_1 X_{1,n} - \sum_{i=2}^{n} C_i^- X_{i,n} > t\right)}{\mathbb{P}\left(C_1 X_{1,n} > t\right)} &= \lim_{t \to \infty} \frac{\mathbb{P}\left(C_1 X_{1,n} - \sum_{i=2}^{n} C_i^- X_{i,n} > t, C_1 X_{1,n} > t\right)}{\mathbb{P}\left(C_1 X_{1,n} > t\right)} \\
&= \lim_{t \to \infty} \mathbb{P}\left(C_1 X_{1,n} - \sum_{i=2}^{n} C_i^- X_{i,n} > t \big| C_1 X_{1,n} > t\right) = \mathbb{P}\left(W > 1\right) = 1.
\end{aligned}
$$

If $\overline{C}=0$ almost surely the proof follows, therefore let us assume that $\overline{C}>0$. Suppose for notational simplicity that $C_i>0, i\le n$. For any $\varepsilon>0$ we have

$$\begin{aligned}\mathbb{P}\left(\sum_{i=1}^n C_iX_{i,n}>t\right) &\le \mathbb{P}\left(\sum_{i=1}^n C_iX_{i,n}>t, C_1X_{1,n}>t(1-\varepsilon)\right)+\mathbb{P}\left(\sum_{i=2}^n C_iX_{i,n}>\varepsilon t\right)\\ &\le \mathbb{P}\left(\sum_{i=1}^n C_iX_{i,n}>t, C_1X_{1,n}>t(1-\varepsilon)\right)+\mathbb{P}\left(n\overline{C}X_{2,n}>\varepsilon t\right).\end{aligned}$$

By (4.8) we have $\lim_{t\to\infty}\frac{\mathbb{P}\left(n\overline{C}X_{2,n}>\varepsilon t\right)}{\mathbb{P}(C_1X_{1,n}>t)}=0$. Thus, in view of (4.17)

$$\lim_{t\to\infty}\frac{\mathbb{P}\left(\sum_{i=1}^n C_iX_{i,n}>t, C_1X_{1,n}>t(1-\varepsilon)\right)}{\mathbb{P}\left(C_1X_{1,n}>t(1-\varepsilon)\right)}=\lim_{t\to\infty}\mathbb{P}\left(\sum_{i=1}^n C_iX_{i,n}>t/(1-\varepsilon)|C_1X_{1,n}>t\right)=\mathbb{P}\big(W>1/(1-\varepsilon)\big),$$

and hence the proof follows from equation (4.16) and letting $\varepsilon\downarrow 0$. $\square$

Proof of Theorem 2.3 The proof is based on Lemma 4.4 and similar arguments as provided for Theorem 2.1. We first assume that $C_i\ge 0$ for all $2\le i\le n$ and let $M<0$ such that $-M>(n-1)L^*$, where $L^*=\max_{1\le i<j\le n}L_{ij}$. Obviously,

$$\mathbb{P}\left(\sum_{i=1}^n C_iX_{i,n}>t\right)=\mathbb{P}\left(\sum_{i=1}^n C_iX_{i,n}>t, C_1X_{1,n}\le t+Ma(t)\right)+\mathbb{P}\left(\sum_{i=1}^n C_iX_{i,n}>t, C_1X_{1,n}>t+Ma(t)\right). \tag{4.19}$$

Further, we have

$$\begin{aligned}\mathbb{P}\left(\sum_{i=1}^n C_iX_{i,n}>t, C_1X_{1,n}\le t+Ma(t)\right) &\le \mathbb{P}\left(\sum_{i=2}^n C_iX_{i,n}>-Ma(t)\right)\\ &\le \mathbb{P}\left(C^*X_{2,n}>-\frac{M}{n-1}a(t)\right)\\ &\le \sum_{1\le i<j\le n}\mathbb{P}\left(C^*X_i>-\frac{M}{n-1}a(t), C^*X_j>-\frac{M}{n-1}a(t)\right)\\ &= o\big(\mathbb{P}(C_1X_1>t)\big),\quad t\to\infty,\end{aligned} \tag{4.20}$$

where the last implication is due to (2.13) and the fact that $-M>(n-1)L_{ij}$ for all $1\le i<j\le n$. Next, the second term from (4.19) is investigated via (4.9), i.e.,

$$\begin{aligned}&\lim_{t\to\infty}\frac{\mathbb{P}\left(\sum_{i=1}^n C_iX_{i,n}>t, C_1X_{1,n}>t+Ma(t)\right)}{\mathbb{P}(C_1X_{1,n}>t)}\\ &= \frac{\mathbb{P}\Big(\big((C_1X_{1,n}-t)/a(t), C_2X_{2,n}/a(t),\dots,C_nX_{n,n}/a(t)\big)\in B_1\Big)}{\mathbb{P}(C_1X_{1,n}>t)}=\nu(B_1)=1,\end{aligned} \tag{4.21}$$

where $B_1 := \{\underline{x} : \sum_{i=1}^n x_i > 0, x_1 > M\}$. Now, the second step is due to the fact that Proposition A2.12 of Embrechts et al. (1997, p. 563) can be applied in (4.9) since B_1 does not put any mass on its boundary. In addition, the whole mass over the set B_1 is concentrated on the line $(0,\infty] \times \{0\}^{n-1}$. Combining equations (4.19), (4.20) and (4.21), we get

$$\mathbb{P}\left(\sum_{i=1}^n C_i X_{i,n} > t\right) \sim \mathbb{P}(C_1 X_{1,n} > t), \quad t \to \infty. \tag{4.22}$$

Similarly,

$$\begin{aligned}
&\lim_{t\to\infty} \frac{\mathbb{P}\left(C_1 X_{1,n} - \sum_{i=2}^n C_i X_{i,n} > t, C_1 X_{1,n} > t + Ma(t)\right)}{\mathbb{P}(C_1 X_{1,n} > t)} \qquad (4.23)\\
&= \frac{\mathbb{P}\Big(\big((C_1 X_{1,n} - t)/a(t), C_2 X_{2,n}/a(t), \ldots, C_n X_{n,n}/a(t)\big) \in B_2\Big)}{\mathbb{P}(C_1 X_{1,n} > t)} = \nu(B_2) = 1,
\end{aligned}$$

where $B_2 := \{\underline{x} : x_1 - \sum_{i=2}^n x_i > 0, x_i > M\}$. Once again, the entire mass over the set B_2 is concentrated on the line $(0,\infty] \times \{0\}^{n-1}$. Note that

$$\mathbb{P}\left(C_1 X_{1,n} - \sum_{i=2}^n C_i X_{i,n} > t, C_1 X_{1,n} \le t + Ma(t)\right) = 0$$

due to the non-negativity assumption of the C_i's. The latter and (4.23) yield that

$$\mathbb{P}\left(C_1 X_{1,n} - \sum_{i=2}^n C_i X_{i,n} > t\right) \sim \mathbb{P}(C_1 X_{1,n} > t), \quad t \to \infty. \tag{4.24}$$

Therefore, equations (4.22), (4.24) and (4.16) help in dropping the non-negativity assumption for the C_i for all $2 \le i \le n$, and the proof follows utilising further (4.10). □

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