Quanto Implied Correlation in a Multi-Lévy Framework

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Abstract

We propose an integrated model for the joint dynamics of FX rates and asset prices for the pricing of Quanto products; the model is based on the multivariate construction for Lévy processes introduced by Ballotta and Bonfiglioli (2014). The approach gives access to market consistent information on dependence between the relevant financial variables, as it provides an insight into the quanto adjustment showing that it is affected not only by the covariance between FX rates and stock log-returns, but also higher order cumulants of the pure jump part of the systematic risk factor. The model is applied to the USD-denominated Quanto futures on the Nikkei 225 index in the case in of a Variance Gamma framework.

Keywords: FX risk, implied correlation, multivariate Lévy processes, Quanto products, Variance Gamma process.

JEL Classification: G13, G12, C63, D52

1 Introduction

The aim of this paper is to explore the problem of recovering market consistent information on the correlation between financial assets using suitable derivatives contracts. Due to the limited number and trading (usually Over-The-Counter - OTC) of products whose price is related to the existing level of correlation, we focus on the case of Quanto products and specifically Quanto futures, as they offer significant exposure to the correlation between exchange rates and asset prices, and are supported by sufficient liquidity. These are, in fact, financial products with a payoff paid in a different currency from the one in which the underlying asset is traded, allowing investors to participate in the assets profit without facing any exposure to foreign exchange rate risk. Quanto futures are actively traded on the CME (the Nikkei/USD Quanto futures and the USD-denominated Ibovespa Index Futures are two examples of such contracts), the JSE (Energy, Metal and Soft Commodities), the DGCX (Indian Gold and USD/INR), the Euronext Paris such as the Gold Hedged in EUR EasyTRACKER issued by BNP Paribas (see https://easytrackers.bnpparibas.com/) and others.

The interest in market implied metrics of correlation is motivated by the fact that correlation risk is attracting interest for hedging and regulatory purposes. This risk is in fact present in the trading books of a wide range of buy and sell side market participants, such as bank structuring desks and hedge funds for example. Further, the Basel III supervisory regime (Basel, 2010) is focussing in particular on the impact of wrong-way risk effects on the quantification of counterparty credit risk through metrics such as Credit Value Adjustment (CVA), wrong-way risk denoting dependence between specific names in the CVA structure. Capturing correlation risk requires both suitable models for the joint distribution of the relevant variables, and easy-to-implement procedures for the quantification of the parameters controlling the behaviour of the joint distribution of choice. Specifically, regarding the latter issue, we note that possible information sources are either past observed values of the variables in question, or derivatives whose quoted price offers an estimate of the market perception of correlation. The estimation of historical correlation from time series though is significantly affected by the length of the sample, the frequency of observation and the weights assigned to past observations. Further, as historical measures are backward-looking, they do not necessarily reflect market expectations of future joint movements in the financial quantities of interest, which are instead necessary for the assessment of derivatives positions and related capital requirements. Alternatively, over the past few years the CBOE has made available daily quotes of the CBOE S&P 500 Implied Correlation Index (Chicago Board Options Exchange, 2009), which replaces all pairwise correlations with an average one. Although this index in general reflects market capitalization, it might not be suitable for example for pricing and assessing counterparty credit risk, due to the equi-correlation assumption. Hence, our choice to resort to traded multivariate derivative products such as Quanto futures.

As far as modelling is concerned, as reported in the literature, implied correlation - similarly to implied volatility - shows skew patterns (see Da Fonseca et al., 2007; Lucic, 2012; Ballotta and Bonfiglioli, 2014, and references therein, for example) which is not fully consistent with the standard framework based on the Brownian motion, i.e. the Gaussian distribution. A simple but effective way of replacing the Gaussian distribution is the introduction of jumps by adopting Lévy processes, as many analytical formulas established for models based on the Brownian motion can be easily extended to this more general class of processes. With the application involving Quanto futures in mind, we note that the features of asymmetry and excess kurtosis typical of the distributions generated by Lévy processes are consistent with empirical evidence provided for example by Carr and Wu (2007) on the risk neutral conditional distribution of currency returns.

Multivariate constructions for Lévy processes have attracted interest in the literature over the past few years, for example for modelling and pricing in the credit risk and counterparty credit risk area (see Lipton and Sepp, 2009; Ballotta and Fusai, 2015, for example). Although several approaches are available, for a detailed survey of which we refer to Itkin and Lipton (2015) and references therein, in the following we adopt the factor construction of Ballotta and Bonfiglioli (2014), in which the overall risk is split in two components: a systematic one originated by sudden changes affecting the whole market, and an idiosyncratic one capturing instead shocks originated by company specific issues. We note that this assumption of a common source of systematic risk in both stock and foreign exchange returns is consistent with the results of Atanasov and Nitschka (2015). The adopted factor construction also implies that the model shows a flexible correlation structure, a linear dimensional complexity, and readily available characteristic functions, which guarantee a high ease of implementation, and allow us to develop an integrated calibration procedure providing access to information on the dependence structure between the relevant components. We point out that although our framework is based on the model of Ballotta and Bonfiglioli (2014), in which convolution conditions required to recover a known distribution for the margin processes are derived and applied, our model does not need these restrictive conditions, as they are not necessary to retain its mathematical tractability and a limited number of parameters. As observed for example by Eberlein et al. (2008), in fact, the presence of convolution conditions aimed at separating the behaviour of the margin processes from the correlation structure, although intuitive, leads to a biased view of the dependence in place and reduces the flexibility of the factor model as it fails to recognize the different tail behaviour shown by the components of any given multivariate vector.

In light of the discussion above, this paper offers the following contributions. Firstly, we develop a Lévy processes-based multivariate extended FX framework, which also includes additional names to cater for the underlying assets of Quanto products such as Quanto futures and Quanto options.

En route, we show that the part of the framework concerning the multivariate FX model satisfies symmetries with respect to inversion and triangulation. We note that although these properties are important in order to guarantee a fully consistent FX model, it is not trivial to preserve them once we move out of the standard Black-Scholes (BS) framework to include for example stochastic volatility effects; for further details on this matter, we refer for example to De Col et al. (2013). Concerning non-Gaussian framework for Quanto products, we cite amongst others Branger and Muck (2012), who offer an integrated pricing approach for both Quanto and plain-vanilla options on the stock as well as the foreign exchange rate based on Wishart processes for the covariance matrix of log-returns. Secondly, our model gives access to analytical formulae for the correlation coefficient and the indices of tail dependence, which facilitate the recovery of market implied correlation and the assessment of joint movements on the risk position of investors. Thirdly, the proposed model leads to analytical results (up to a Fourier inversion) for the price of both vanilla and Quanto options, which allow for efficient calibration to market quotes. Finally, the application of the proposed model to the pricing of Quanto futures reveals that the quanto adjustment is not only determined by the covariance between asset log-returns (as in the standard Black-Scholes model), but also by higher order cumulants of the jump part of the systematic risk. As these are an explicit function of the parameters of the systematic process, market consistent information on the (in general not observable) common component can be extracted directly from the market, bypassing the need of either imposing unrealistic convolution conditions, or identifying a suitable proxy for this part of the risk. As the same quanto adjustment also enters the pricing formulas of Quanto options, the proposed model allows us to assess the consistency of the information on the existing correlation recovered from Quanto futures and the one extracted from the relevant time series, i.e. the historical correlation commonly used by practitioners in the market. For sake of illustration, in the numerical analysis presented in Section 4, we apply these ideas to the USD-denominated Quanto futures on the Nikkei 225 index to extract the correlation between the USDJPY FX rate and the Nikkei 225 index log-returns.

The rest of this paper is organized as follows. In Section 2, we review the general properties of the adopted factor-based multivariate Lévy processes, with particular attention to the results required for the construction of the multivariate FX model, which is introduced in Section 3, together with its main properties. In Section 3, we also describe the pricing formulas for Quanto futures and Quanto options in our multivariate Lévy FX market. The numerical analysis is offered in Section 4. Section 5 concludes. All the proofs are deferred to the appendices.

2 Preliminaries: Multivariate Lévy processes via linear transformation

Consider a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$. Let $\mathbf{L}(\mathbf{t})$ be a Lévy process in $\mathbb{R}^n$, then its characteristic function is $\phi_{\mathbf{L}}(\mathbf{u};t) = e^{t\varphi(\mathbf{u})}$ with

$$\varphi(\mathbf{u}) = i\langle\gamma, \mathbf{u}\rangle - \frac{1}{2}\langle\mathbf{u}, \Sigma\mathbf{u}\rangle + \int_{\mathbb{R}^d} \left(e^{i\langle\mathbf{u},\mathbf{x}\rangle} - 1 - i\langle\mathbf{u},\mathbf{x}\rangle 1_E(\mathbf{x})\right)\kappa(d\mathbf{x}), \tag{1}$$

where $\gamma \in \mathbb{R}^n$, Σ is a symmetric, non-negative definite $n\times n$ matrix capturing the variance/covariance matrix of the Gaussian component, and κ is a positive measure on $\mathbb{R}^n$ such that

$$\kappa(\{0\}) = 0, \qquad \int_{\mathbb{R}^d} \left(|\mathbf{x}|^2 \wedge 1\right)\kappa(d\mathbf{x}) < \infty$$

for $E = \{\mathbf{x} : |\mathbf{x}| \leq 1\}$. The triplet (γ, Σ, κ) represents the generating triplet of $\mathbf{L}(t)$ and $\varphi(\cdot)$ denotes the characteristic exponent. In this framework, the elements of the variance/covariance matrix of the process $\mathbf{L}(t)$ are of the form

$$\Sigma_{jk} + \int_{\mathbb{R}^d} x_j x_k \kappa(dx_j \times dx_k) \quad j,k = 1,\cdots,n.$$

In order to construct a multivariate Lévy process with dependent components and explicit representation of the characteristic triplet, we use the property that these processes are invariant under linear transformations (see for example Sato, 1999, Proposition 11.10, and Cont and Tankov, 2004, Theorem 4.1). The main result is given in the following (see also Ballotta and Bonfiglioli, 2014).

Proposition 1 *Let $\mathbf{\Lambda}(t) = (Y_1(t),\cdots,Y_n(t),Z(t))^\top$ be a Lévy process in $\mathbb{R}^{n+1}$ with mutually independent components, each with characteristic functions $\phi_{Y_j}(u;t)$, $j=1,\cdots,n$, and $\phi_Z(u;t)$ respectively, and generating triplets (β_j,σ_j,ν_j), $j=1,\cdots,n$, and (β_Z,σ_Z,ν_Z). Then, for $a_j \in \mathbb{R}$, $j=1,...,n$, $\mathbf{L}(t) = (Y_1(t)+a_1Z(t),\cdots,Y_n(t)+a_nZ(t))^\top$ is a multivariate Lévy process in $\mathbb{R}^n$ with characteristic function*

$$\phi_{\mathbf{L}}(\mathbf{u};t) = \phi_Z\left(\sum_{j=1}^n a_j u_j; t\right)\prod_{j=1}^n \phi_{Y_j}(u_j;t),$$

and generating triplet (γ,Σ,κ) such that

- *$\gamma \in \mathbb{R}^n$, $\gamma = (\beta_1 + a_1\beta_Z,\cdots,\beta_n + a_n\beta_Z)^\top + \int_{\mathbb{R}^n}\mathbf{x}\,(1_E(\mathbf{x}) - 1_D(\mathbf{x}))\,\kappa(d\mathbf{x})$, for $E = \{\mathbf{x}\in\mathbb{R}^n : \sum_{j=1}^n x_j^2 \le 1\}$ and $D = \{(y_1+a_1z,\cdots,y_n+a_nz)\in\mathbb{R}^n : \sum_{j=1}^n y_j^2 + z^2 \le 1\}$,*
- *Σ is a $n\times n$ matrix with entries $\Sigma_{jj} = \sigma_j^2 + a_j^2\sigma_Z^2$ and $\Sigma_{jk} = a_j a_k \sigma_Z^2$ for all $j\neq k$,*
- *$\kappa(B) = \sum_{j=1}^n \nu_j(B_j) + \nu_Z(B_a)$,*

for $B\in\mathcal{B}(\mathbb{R}^n)$,

$$B_j = \{y\in\mathbb{R} : (\underbrace{0,\cdots,0}_{j-1\ times}, y, \underbrace{0,\cdots,0}_{n-j\ times}) \in B\},$$

$B_a = \{z : z\in A\}$ and $A = \{z\in\mathbb{R} : (a_1z,\cdots,a_nz)\in B\}$.

Proof. See A.1. ∎

Corollary 2 *Let $\mathbf{L}(t)$ be a $\mathbb{R}^n$- Lévy process as constructed in Proposition 1, with generating triplet (γ,Σ,κ). Then, for $j=1,\cdots,n$, $L_j(t)$ is a Lévy process in $\mathbb{R}$ with triplet $(\gamma_{Lj}, c_j^2, \kappa_j)$ defined as*

- *$\gamma_{Lj} = \gamma_j + \int_{\mathbb{R}^n} x_j\left(1_{x_j^2\le 1} - 1_{\sum_{j=1}^n x_j^2 \le 1}\right)\kappa(d\mathbf{x})$*
- *$c_j^2 = \sigma_j^2 + a_j^2\sigma_Z^2$*
- *$\kappa_j(B) = \kappa(\{\mathbf{x} : x_j = y_j + a_j z \in B\}) = \nu_j(B_j) + \nu_Z(B_{aj})$ for $B\in\mathcal{B}(\mathbb{R})$, $B_j = \{y_j\in\mathbb{R} : y_j\in B\}$, $B_{aj} = \{z\in\mathbb{R} : z\in A_j\}$ and $A_j = \{z\in\mathbb{R} : a_j z\in B\}$.*

Proof. See A.2. ∎

For the case of the proposed construction, the dependence between components of the multivariate Lévy process $\mathbf{L}(t)$ is correctly described (see Embrechts et al., 2002, for example) by the pairwise linear correlation coefficient

$$\rho_{jk}^{\mathbf{L}} = \mathbb{C}orr\left(L_j(t), L_k(t)\right) = \frac{a_j a_k \mathbb{V}ar\left(Z(1)\right)}{\sqrt{\mathbb{V}ar\left(L_j(1)\right)}\sqrt{\mathbb{V}ar\left(L_k(1)\right)}}, \tag{2}$$

which is well defined if all processes have finite moments of all order (specifically the variance). For further details on the dependence structure, we refer to Ballotta and Bonfiglioli (2014).

In terms of tail dependence, the following results apply to the proposed multivariate construction.

Proposition 3 *Consider the multivariate process $\mathbf{L}(t)$ generated by Proposition (1). Then*

a) For $l_j, l_k \downarrow -\infty$ $j \neq k, j = 1, \cdots, n$, $\mathbb{P}(L_j(t) < l_j, L_k(t) < l_k) > 0$ if and only if $\rho^{\mathbf{L}}_{jk} > 0$ for all $t > 0$, and

$$\mathbb{P}(L_j(t) < l_j, L_k(t) < l_k) \simeq \begin{cases} \mathbb{P}\left(Z(t) < \min\left\{\frac{l_j}{a_j}, \frac{l_k}{a_k}\right\}\right) & \text{if } a_j, a_k > 0 \\ \mathbb{P}\left(Z(t) > \max\left\{\left|\frac{l_j}{a_j}\right|, \left|\frac{l_k}{a_k}\right|\right\}\right) & \text{if } a_j, a_k < 0. \end{cases} \tag{3}$$

b) For $l_j, l_k \uparrow \infty$ $j \neq k, j = 1, \cdots, n$, $\mathbb{P}(L_j(t) > l_j, L_k(t) > l_k) > 0$ if and only if $\rho^{\mathbf{L}}_{jk} > 0$ for all $t > 0$, and

$$\mathbb{P}(L_j(t) > l_j, L_k(t) > l_k) \simeq \begin{cases} \mathbb{P}\left(Z(t) > \max\left\{\frac{l_j}{a_j}, \frac{l_k}{a_k}\right\}\right) & \text{if } a_j, a_k > 0 \\ \mathbb{P}\left(Z(t) < \min\left\{-\frac{l_j}{|a_j|}, -\frac{l_k}{|a_k|}\right\}\right) & \text{if } a_j, a_k < 0. \end{cases} \tag{4}$$

c) For $l_j \downarrow -\infty, l_k \uparrow \infty$ $j \neq k, j = 1, \cdots, n$, $\mathbb{P}(L_j(t) < l_j, L_k(t) > l_k) > 0$ if and only if $\rho^{\mathbf{L}}_{jk} < 0$ for all $t > 0$ and

$$\mathbb{P}(L_j(t) < l_j, L_k(t) > l_k) \simeq \begin{cases} \mathbb{P}\left(Z(t) > \max\left\{\left|\frac{l_j}{a_j}\right|, \frac{l_k}{a_k}\right\}\right) & \text{if } a_j < 0 < a_k \\ \mathbb{P}\left(Z(t) < \min\left\{\frac{l_j}{a_j}, -\frac{l_k}{|a_k|}\right\}\right) & \text{if } a_k < 0 < a_j. \end{cases} \tag{5}$$

Proof. See A.3. ■

The above Proposition shows that the tail dependence behaviour is governed by the tail probabilities of the systematic risk process. Further, results $(a) - (b)$ imply that the indices of upper/lower tail dependence are different from zero only when the margin processes are positively correlated, which is consistent with the fact that these coefficients provide a measure of concordance of jumps (see Embrechts et al., 2002). Finally, although the correlation coefficient requires the existence of the processes moments at least up to the second order, the previous Proposition shows that tail dependence is always well defined: the relevant distribution, in fact, can be recovered from the corresponding characteristic function.

We conclude this section by revisiting the results presented in Eberlein et al. (2009) for the multivariate construction given in Proposition 1. In particular, we consider the case of an Esscher probability measure (see Gerber and Shiu, 1994, for example) $\mathbb{P}^{hj}$ with parameter $h_j \in \mathbb{R}$ defined with respect to the j-th component of the vector $\mathbf{L}(t)$. Note that for the purpose of the financial model put forward in the following sections, in the remaining of this paper we assume that the "big" jumps of all the relevant Lévy processes have finite first moment (i.e. $\int_{|x|>1} x\nu(dx) < \infty$), so that we can compensate them to form a martingale. Consequently, the processes have triplets $(\gamma'_{Lj} = \gamma_{Lj} + \int_{|x|>1} x\nu(dx), c_j^2, \kappa_j)$, $(\beta'_j = \beta_j + \int_{|y|>1} y\nu(dy), \sigma_j^2, \nu_j)$ for $j = 1, \cdots, n$ and $(\beta'_Z = \beta_Z + \int_{|z|>1} z\nu(dz), \sigma_Z^2, \nu_Z)$. For sake of simplicity, we suppress the notation $\gamma'_{\cdot}$, $\beta'_{\cdot}$ and write $\gamma_{\cdot}$, $\beta_{\cdot}$ instead. The characteristic exponents now take form

$$
\begin{aligned}
\varphi_{L_j} &= iu\gamma_{Lj} - \frac{c_j^2}{2}u^2 + \int_{\mathbb{R}} \left(e^{iux} - 1 - iux\right)\kappa_j(dx) \quad j = 1, \cdots, n \\
\varphi_{Y_j} &= iu\beta_j - \frac{\sigma_j^2}{2}u^2 + \int_{\mathbb{R}} \left(e^{iuy} - 1 - iuy\right)\nu_j(dy) \quad j = 1, \cdots, n \\
\varphi_Z &= iu\beta_Z - \frac{\sigma_Z^2}{2}u^2 + \int_{\mathbb{R}} \left(e^{iuz} - 1 - iuz\right)\nu_Z(dz).
\end{aligned}
$$

The Esscher change of measure is formalized in the following.

Proposition 4 *Let $\mathbf{\Lambda}(t)$ and $\mathbf{L}(t)$ be multivariate Lévy processes as given in Proposition 1; further, let $\mathbb{P}^{hj}$ be an equivalent probability measure defined by the density process*

$$
\eta(t) = \left.\frac{d\mathbb{P}^{hj}}{d\mathbb{P}}\right|_{\mathcal{F}_t} = e^{-\varphi_{Lj}(-ih_j)t + h_j L_j(t)}, \quad h_j \in \mathbb{R},
$$

for any $j = 1, \cdots, n$. Then, $\mathbf{\Lambda}(t)$ and $\mathbf{L}(t)$ remain Lévy processes under $\mathbb{P}^{hj}$; in particular, the following hold.

a) The components of $\mathbf{\Lambda}(t)$ under $\mathbb{P}^{hj}$ for any $j = 1, \cdots, n$ have triplet

$$
\begin{aligned}
Y_j(t) &: \left(\beta_j + h_j\sigma_j^2 + \int_{\mathbb{R}} y(e^{h_j y} - 1)\nu_j(dy), \sigma_j^2, e^{h_j y}\nu_j\right) \\
Y_k(t) &: \left(\beta_k, \sigma_k^2, \nu_k\right), \quad k \neq j, k = 1, \cdots, n \\
Z(t) &: \left(\beta_Z + h_j a_j \sigma_Z^2 + \int_{\mathbb{R}} z(e^{h_j a_j z} - 1)\nu_Z(dz), \sigma_Z^2, e^{h_j a_j z}\nu_Z\right).
\end{aligned}
$$

b) The components of $\mathbf{L}(t)$ under $\mathbb{P}^{hj}$ for any $j = 1, \cdots, n$ have triplet

$$
\begin{aligned}
L_j(t) &: \left(\gamma_{Lj} + h_j c_j^2 + \int_{\mathbb{R}} x(e^{h_j x} - 1)\kappa_j(dx), c_j^2, e^{h_j x}\kappa_j\right) \\
L_k(t) &: \left(\gamma_{Lk} + h_j a_j a_k \sigma_Z^2 + a_k \int_{\mathbb{R}} z(e^{h_j a_j z} - 1)\nu_Z(dz), c_k^2, \nu_k + e^{h_j a_j z}\nu_Z\right), \quad k \neq j, k = 1, \cdots, n.
\end{aligned}
$$

Proof. See A.4 ■

Proposition 4 implies that Lévy processes are invariant under an Esscher change of measure.

3 A multivariate Lévy (extended) Foreign Exchange market

3.1 The general setting

Consider a frictionless and arbitrage free market in which N currencies are traded. In what follows, we use the convention that the spot FX rate between the l-th and the m-th currency, $X^{m|l}(t)$, is quoted as the amount of currency (l) per unit of currency (m). Further, we assume that interest rates are constant and we let $r_l > 0$, $l = 1, \cdots, N$ denote the continuously compounded interest rate in the l-th currency.

For the purpose of including the pricing of Quanto products, we also consider an asset $S(t)$ traded in the market using the l-th currency. Hence, the total number of assets considered is $n = N + 1$.

We note that for ease of exposition and notation, in the following we consider only the case of one underlying asset; however, the model can be easily generalized to the case of say M assets, so that $n = N + M$.

In order to model the risk dynamics of $S(t)$ and $X^{m|l}(t)$, let $(L_S(t), L_{X_k}(t), k = 1, \ldots, N)$ be a Lévy process in $\mathbb{R}^n$ with dependent components and respecting the construction given in Proposition 1, so that

$$L_j(t) = Y_j(t) + a_j Z(t), \quad j = S, X_k, k = 1, \cdots, N.$$

As shown in Section 2, the full description of $(L_S(t), L_{X_k}(t), k = 1, \cdots, N)$ depends on the idiosyncratic risk processes $(Y_S(t), Y_{X_k}(t), k = 1, \cdots, N)$ and the systematic risk process $Z(t)$; hence for simplicity of notation, we focus only on the properties of these components.

Finally, let $\mathbb{P}^l$ be the risk neutral martingale measure defined by the l-th currency. We note that the proposed market model is incomplete and consequently the risk neutral martingale measure is not unique. Hence we follow standard practice for incomplete markets and fix the risk neutral measure with respect to the chosen currency through the prices of derivative contracts traded in the corresponding market. Under this measure, we assume that all processes have zero-drift, so that the corresponding generating triplets are $(0, \sigma_j^2, \nu_j)$, for $j = S, X_k, k = 1, \cdots, N$ and $(0, \sigma_Z^2, \nu_Z)$ respectively, and the characteristic exponents are therefore

$$\begin{aligned}
\varphi^l_{Y_j}(u) &= -\frac{\sigma_j^2}{2}u^2 + \int_{\mathbb{R}} (e^{iuy} - 1 - iuy)\nu_j(dy) \quad j = S, X_k, k = 1, \cdots, N \\
\varphi^l_Z(u) &= -\frac{\sigma_Z^2}{2}u^2 + \int_{\mathbb{R}} (e^{iuz} - 1 - iuz)\nu_Z(dz).
\end{aligned} \tag{6}$$

In this set-up, the index quoted in the l-th currency, $S(t)$, and the FX spot rate $X^{m|l}(t)$ under the risk neutral measure $\mathbb{P}^l$ are assumed to be of the form

$$\begin{aligned}
S(t) &= S(0)e^{\mu_S t + L_S(t)}, \quad S(0) > 0 \\
X^{m|l}(t) &= X^{m|l}(0)e^{\mu_{X_l} t + L_{X_l}(t)}, \quad X^{m|l}(0) > 0
\end{aligned}$$

with

$$\begin{aligned}
\mu_S &= r_l - \varphi^l_{L_S}(-i) = r_l - \varphi^l_{Y_S}(-i) - \varphi^l_Z(-a_{Y_S} i), \\
\mu_{X_l} &= r_l - r_m - \varphi^l_{L_{X_l}}(-i) = r_l - r_m - \varphi^l_{Y_{X_l}}(-i) - \varphi^l_Z(-a_{X_l} i).
\end{aligned}$$

This choice guarantees that $e^{-r_l t}S(t)$ and $e^{-(r_l - r_m)t}X^{m|l}(t)$ (i.e. the discounted value of one unit of currency (m) invested in the m-denominated currency money market account and converted in the (l) currency) are $\mathbb{P}^l$-martingales.

Up to now, the given market is specified under the risk neutral measure defined by the l-th currency; for practical purposes it is at times convenient to change the measure to any other one based on a numéraire denominated in any other of the N currencies included in the FX market. Without loss of generality, we consider the risk neutral martingale measure defined by the m-th currency. In the given framework by the change-of-numéraire method introduced by Geman et al. (1995), $\mathbb{P}^m \sim \mathbb{P}^l$ is defined by the density process

$$\begin{aligned}
\eta(t) &= \left.\frac{d\mathbb{P}^m}{d\mathbb{P}^l}\right|_{\mathcal{F}_t} = \frac{e^{r_m t}X^{m|l}(t)}{e^{r_l t}X^{m|l}(0)} \\
&= e^{-\varphi^l_{L_{X_l}}(-i)t + L_{X_l}(t)}.
\end{aligned}$$

As $\mathbb{P}^m$ can be considered as an Esscher probability measure with unit parameter, it follows from Proposition 4 that the spot FX rate and the index $S(t)$ remain Lévy processes under the change of measure; further the triplets of the idiosyncratic and systematic processes are

$$\begin{aligned}
Y_{X_l}(t) &: \left(\sigma_{X_l}^2 + \int_{\mathbb{R}} y(e^y - 1)\nu_{X_l}(dy), \sigma_{X_l}^2, e^y \nu_{X_l}\right) \\
Y_j(t) &: \left(0, \sigma_j^2, \nu_j\right), \quad j = S, X_k, k \neq l \\
Z(t) &: \left(a_{X_l}\sigma_Z^2 + \int_{\mathbb{R}} z(e^{a_{X_l} z} - 1)\nu_Z(dz), \sigma_Z^2, e^{a_{X_l} z}\nu_Z\right).
\end{aligned}$$

The corresponding characteristic exponents under the probability measure $\mathbb{P}^m$ are therefore

$$\varphi_{Y_{X_l}}^m(u) = iu\left(\sigma_{X_l}^2 + \int_{\mathbb{R}} y(e^y - 1)\nu_{X_l}(dy)\right) - \frac{\sigma_{X_l}^2}{2}u^2 + \int_{\mathbb{R}}(e^{iuy} - 1 - iuy)e^y \nu_{X_l}(dy) \tag{7}$$

$$\varphi_{Y_j}^m(u) = -\frac{\sigma_j^2}{2}u^2 + \int_{\mathbb{R}}(e^{iuy} - 1 - iuy)\nu_j(dy) \quad j = S, X_k, k \neq l \tag{8}$$

$$\varphi_Z^m(u) = iu\left(a_{X_l}\sigma_Z^2 + \int_{\mathbb{R}} z(e^{a_{X_l} z} - 1)\nu_Z(dz)\right) - \frac{\sigma_Z^2}{2}u^2 + \int_{\mathbb{R}}(e^{iuz} - 1 - iuz)e^{a_{X_l} z}\nu_Z(dz). \tag{9}$$

We note that the proposed multivariate FX model is consistent in terms of symmetries with respect to inversion and triangulation. It follows, in fact, from the invariance of Lévy processes under linear transformation and Esscher change of measure that the "flipped" process $X^{l|m}(t) = 1/X^{m|l}(t)$ follows under the corresponding risk neutral measure $\mathbb{P}^m$ the same type of process as the original $X^{m|l}(t)$ process (symmetry with respect to inversion). Further, the invariance with respect to linear transformation and Esscher change of measure also ensures that the inferred cross rate $X^{m|k}(t) = X^{m|l}(t)/X^{k|l}(t)$ follows under $\mathbb{P}^k$ the same type of process as the original main currency pairs $X^{m|l}(t)$ and $X^{k|l}(t)$ (symmetry with respect to triangulation).

3.2 Quanto futures

As mentioned in Section 1, our main focus is recovering market consistent information on the dependence structure of our model via Quanto products. As Quanto futures are the most frequently traded contracts, we analyse their pricing in the proposed setting as to gain insight into the quanto adjustment.

To this purpose, given that Quanto futures involve only one underlying asset and one FX rate, in the remaining of this paper we consider a reduced version of the multivariate FX market introduced in Section 3.1, with only two currencies: the domestic currency (d), and the foreign (f) currency. Further, for simplicity of notation, we drop the sub-indices from all processes involved, so that under the Foreign Risk Neutral (FRN) martingale measure $\mathbb{P}^f$, the underlying asset price at $t > 0$ is

$$S(t) = S(0)e^{\mu_S^f t + L_S(t)}, \quad S(0) > 0;$$

in accordance with the notation introduced in Section 3.1, the relevant spot FX rate is $X^{d|f}(t)$ defined as

$$X^{d|f}(t) = X^{d|f}(0)e^{\mu_X^f t + L_X(t)}, \quad X^{d|f}(0) > 0,$$

with

$$\begin{aligned}
\mu_S^f &= r_f - \varphi_{Y_S}^f(-i) - \varphi_Z^f(-a_S i), \\
\mu_X^f &= r_f - r_d - \varphi_{Y_X}^f(-i) - \varphi_Z^f(-a_X i).
\end{aligned}$$

It follows by standard no-arbitrage arguments that the price in the foreign economy at time $t \geq 0$ of the futures on S with maturity T equals

$$F^f(t;T) = e^{r_f(T-t)} S(t); \tag{10}$$

similarly, under the assumption that the applied FX rate between the two currencies is set to 1 d/f (see Giese, 2012, for example), the Quanto futures price in the domestic economy (i.e. under the Domestic Risk Neutral - DRN - martingale measure $\mathbb{P}^d$) is given by

$$\begin{aligned} F^d(t;T) &= \mathbb{E}^d[S(T) \mid \mathcal{F}_t] \\ &= e^{q(T-t)} F^f(t;T), \end{aligned} \tag{11}$$

where $q = \varphi_Z^d(-ia_S) - \varphi_Z^f(-ia_S)$ is the "quanto" adjustment. A closer inspection using Equations (6) and (9) shows that

$$\begin{aligned} q &= \varphi_Z^d(-ia_S) - \varphi_Z^f(-ia_S) \\ &= a_S a_X \sigma_Z^2 + \int_{\mathbb{R}} \left(e^{(a_S+a_X)z} - e^{a_X z} - e^{a_S z} + 1 \right) \nu_Z(dz) \\ &= \mathbb{C}ov^f \left(L_S, L_X \right) + \sum_{n=3}^{\infty} qc_Z(n) \end{aligned} \tag{12}$$

for

$$qc_Z(n) = \sum_{k=1}^{n-1} \frac{a_S^{n-k} a_X^k}{k!(n-k)!} \int_{\mathbb{R}} z^n \nu_Z(dz), \tag{13}$$

where the last equality in Equation (12) follows from the Taylor expansion of the exponential function about the origin and the binomial theorem, and $\mathbb{C}ov^f(\cdot)$ denotes the covariance under the initial probability measure $\mathbb{P}^f$. It is clear that in the given framework the quanto adjustment only depends on the parameters of the systematic risk process and the loading factors a_S, a_X.

Further, we note that, in the case in which the driving processes are all Brownian motions (i.e. continuous processes with no jumps), the quanto adjustment reduces to the well known "Black-Scholes type" quanto adjustment

$$q = a_S a_X \sigma_Z^2 = \rho_{SX} \sqrt{\mathbb{V}ar(L_X(1)) \mathbb{V}ar(L_S(1))}, \tag{14}$$

and therefore it only depends on the linear pairwise correlation coefficient between the relevant driving processes. In the more general case, though, Equation (12) shows that the quanto adjustment also depends on higher order cumulants of the pure jump part of the systematic risk process calculated under $\mathbb{P}^f$. In particular, we identify the contributions of the third and fourth order cumulants of Z, which are closely linked to the skewness and the excess kurtosis of the distribution of the process Z

$$qc_Z(3) = \frac{a_S^2 a_X + a_S a_X^2}{2} \int_{\mathbb{R}} z^3 \nu_Z(dz), \tag{15}$$

$$qc_Z(4) = \frac{2a_S^3 a_X + 3a_S^2 a_X^2 + 2a_S a_X^3}{12} \int_{\mathbb{R}} z^4 \nu_Z(dz). \tag{16}$$

It follows from Equations (10)-(11) that the spread between the two futures contracts is to be traced back to the quanto adjustment q. In the case in which the driving processes for S and X are all Brownian motions and volatilities are fixed for example at their at-the-money level (as is common practice in the market), the spread between the futures is only dependent on the correlation parameter ρ_{SX} (see Equation 14); hence, in a market where correlation between the asset S and the FX rate X is positive, the arbitrage free price of the Quanto futures is higher than the futures one, whilst in case of negative correlation, the Quanto futures price is smaller than the futures price, and in the zero correlation case, the two prices are equal. In the more general case of any other Lévy process, though, higher order cumulants of the jump part of the systematic risk process also have an impact on the spread between the two futures contracts. More importantly, as quotes for both futures contracts are readily available from the market, these can be used to calibrate the parameters of the systematic risk factor, and therefore recover information on the "implied" correlation existing between the log-returns of the index and spot FX rate.

3.3 Pricing Quanto options

The arbitrage free price of a (European type) Quanto call option on the (Quanto) futures on the asset S, expressed in units of domestic currency, is given by

$$QC(F^d(T_1;T_2), K, T_1) = e^{-r_d T_1}\mathbb{E}^d[(F^d(T_1;T_2) - K)^+]$$

where $F^d(T_1;T_2)$ is the Quanto futures price at time T_1 with maturity T_2; $T_1 \leq T_2$ is the maturity of the option contract.

It follows from Equation (11) that

$$QC(F^d(T_1;T_2), K, T_1) = e^{-r_d T_1} Q_{adj}\mathbb{E}^d[(S(T_1) - K^*)^+] \tag{17}$$

with

$$\begin{aligned} Q_{adj} &= e^{(r_f+q)(T_2-T_1)}, \\ K^* &= \frac{K}{Q_{adj}}. \end{aligned}$$

A Quanto call option can therefore be seen as a vanilla call on S struck at K^*, rescaled by a constant, Q_{adj}, incorporating the quanto adjustment. As in the market model under consideration relevant characteristic functions are available, the price in Equation (17) can be computed efficiently by means of Fourier inversion based methods, such as the Carr-Madan approach (Carr and Madan, 1999) for example. In this respect, we note that the large majority of options offered on the CME are of American type; the early exercise property can be accommodated in the pricing by adopting either the CONV method of Lord et al. (2008) or the COS method of Fang and Oosterlee (2009) for example. This is left though to future research.

4 Numerical results

In this section, we analyze the performance of our model in terms of calibration, pricing and impact on risk management, using market quotes of the USD-denominated Quanto futures on the Nikkei 225 index. These products are traded on the CME with quarterly maturities (i.e. March, June, September and December) on the second Friday of the contract month; the minimum price change (tick) is 5 index points. Finally, they are characterized by a multiplier of 5 USD for Dollar-denominated CME Nikkei 225 Futures; for more details see e.g. Co et al., 2013.

Calibration is performed using quotes from the Nikkei 225 and USDJPY vanilla option market to anchor the margin processes; further, the fitting of the dependence structure is obtained using the link between the parameters governing the systematic risk process $Z(t)$, the loading coefficients a_S, a_X, and the quanto adjustment extracted from Quanto futures as described in Section 3.2. This calibration procedure is tested against the common market practice of using historical correlation to retrieve information on the dependence in place. For the purposes of the numerical analysis we choose as relevant Lévy process the VG process, originally introduced by Madan and Seneta (1990) and subsequently generalized by Madan and Milne (1991) and Madan et al. (1998). The VG process is a normal tempered stable process obtained by subordinating a Brownian motion with drift by an independent (unbiased) Gamma process. Its characteristic exponent reads

$$\varphi(u) \quad = \quad -\frac{1}{k}\ln\left(1 - iuk\theta + u^2\frac{\sigma^2}{2}k\right), \quad u \in \mathbb{R}, \tag{18}$$

from which it follows the process has mean θt and variance $(\sigma^2 + k\theta^2)t$; the indices of skewness and excess kurtosis are

$$skew\,(t) = \frac{\theta\left(3\sigma^2 k + 2\theta^2 k^2\right)}{\left(\sigma^2 + \theta^2 k\right)^{3/2}\sqrt{t}}, \quad kurt\,(t) = \frac{3k\left(\sigma^4 + 4\sigma^2\theta^2 k + 2\theta^4 k^2\right)}{\left(\sigma^2 + \theta^2 k\right)^2 t}.$$

From the above we observe that the parameter $\theta \in \mathbb{R}$ determines the sign of the skewness of the distribution of the VG process, $\sigma > 0$ controls the overall variance level and $k > 0$ governs the kurtosis or tail heaviness of the distribution. Finally, due to the invariance under an Esscher change of measure of Lévy processes, the VG process remains VG after an appropriate redefinition of the model parameters. The characteristic exponent of the VG process under a h-parameter Esscher probability measure is

$$\varphi^h\,(u) = -\frac{1}{k}\ln\left(1 - iu\theta^h k^h + u^2\frac{\sigma^2}{2}k^h\right), \tag{19}$$

for $\theta^h = \theta + h\sigma^2$, and $k^h = k(1 - h\theta k - h^2\sigma^2/2k)^{-1}$ (see also Hubalek and Sgarra, 2006, for example). Hence, under the assumption that both the systematic risk process and the idiosyncratic risk processes follow a VG process respectively with parameters $(\theta_Z, \sigma_Z, k_Z)$ and $(\theta_{Y_j}, \sigma_{Y_j}, k_{Y_j})$ for $j = S, X$ under $\mathbb{P}^f$, it follows from Equations (11)-(12) that the USD Nikkei 225 index Quanto futures (NKD) can be expressed as

$$F^d(t;T) = F^f(t;T)e^{q(T-t)}, \tag{20}$$

where

$$q = \frac{1}{k_Z}\ln\left(\frac{\left(1 - a_X k_Z\theta_Z - \frac{1}{2}k_Z a_X^2\sigma_Z^2\right)\left(1 - a_S k_Z\theta_Z - \frac{1}{2}k_Z a_S^2\sigma_Z^2\right)}{1 - (a_S + a_X)k_Z\theta_Z - \frac{1}{2}k_Z(a_S + a_X)^2\sigma_Z^2}\right). \tag{21}$$

The price of Quanto options follows from Equation (17); the relevant characteristic function and distribution of the VG process under $\mathbb{P}^d$ follow from Equation (19) for $h = 1$.

Results from the two calibration approaches are presented in Section 4.1. In Section 4.2, we explore the impact of these two alternative calibration procedures on the pricing of Quanto options, and the implied correlation which could be recovered from the market quotes of these contracts. We also investigate in Section 4.3 how the tails of the joint distribution are affected and discuss potential implications on risk management of portfolios of these option contracts.

4.1 Calibration results: Quanto futures implied correlation

We use market data from Bloomberg and the CME free web platform observed on June 13, 2014. Vanilla options on the Nikkei 225 index have maturity of 28 days (July 11, 2014) as these quotes were the most liquid in the market[1]; consequently, we have chosen vanilla options on the USDJPY FX rate with similar maturity, regardless of the fact that the FX market shows high liquidity across other maturities as well. In particular, we consider 9 different strikes for the Nikkei 225 index options and 5 different strikes for the USDJPY exchange rate options. The futures contracts considered have a maturity in 91 days (September 12, 2014); the historical correlation between the log-returns of the Nikkei 225 index and the USDJPY FX rate is estimated using the sample correlation, denoted as ρ^h_{SX}, based on a sample size of 128 days. Data are summarized in Table 1. We just notice that the quotes of the Nikkei 225 index rates are end-of-day quotes, whereas the other quotes were observed at 3 p.m. GMT.

Based on how we incorporate information on correlation, we distinguish between two calibration approaches. In the first one, calibration is performed by minimizing (in least squares sense) simultaneously pricing errors defined as the difference between market prices of both vanilla options and Quanto futures and the corresponding model generated values. The Quanto futures price is computed via the analytical pricing formula given by Equations (20)-(21), whilst option prices are computed using the Carr-Madan method (Carr and Madan, 1999), with the characteristic function generated by our multivariate VG model. Specifically, Quanto futures quotes are used to recover the parameters of the systematic risk process Z and the loading coefficients a_S, a_X through the quanto adjustment – consequently, we denote this procedure as `QF-based calibration´. In the second method, we resort to common market practice of using historical correlation, so that in the minimization procedure we consider vanilla option prices which are computed on the basis of the observed historical correlation ρ^h_{SX} – we denote this procedure as `HC-based calibration´.

Results are summarized in Table 2, where we report the model parameters obtained from the two alternative calibration procedures described above, and Figure 1 in which we show the market volatility smile and the calibrated one originated by our multivariate VG model for both Nikkei 225 and USDJPY vanilla options with parameters from the QF-based calibration (similar results are obtained under the HC-based calibration and are available upon request). In particular, we note that the implied volatilities generated by the calibrated multivariate VG model are always bounded by the corresponding market bid and ask volatilities under both calibration assumptions.

In more details, from Table 2 we observe that, although both procedures are highly accurate as shown by the Root Mean Squared Relative Error (RMSRE, see e.g. Glasserman, 2003), the calibrated parameters generate distributions of the margin processes for the log-returns of the Nikkei 225 index and the USDJPY FX rate which are relatively different under the two calibration assumptions. The assets log-return distributions, in fact, are characterized by very similar volatility (meant as the square root of the process' variance), however the Nikkei 225 index one shows a more pronounced left skew with thicker tails under the HC-based calibration, whilst the USDJPY FX rate distribution presents these features under the QF-based calibration. We also note that the skewness of the distribution of systematic risk process $Z(t)$ changes sign from one calibration procedure to the other.

Finally, Table 2 reports the pairwise linear correlation coefficient between the log-returns of the Nikkei 225 index and the USDJPY spot FX rate computed on the basis of these calibrated parameters and Equation (2). We note the significant difference between the correlation coefficient implied by the QF-based calibration, $\rho^{i,VG}_{SX}$, which returns a value of 81.77%, and the correlation generated by the HC-based calibration, $\rho^{h,VG}_{SX}$, which matches exactly the given 128-day historical correlation value at 28%. For comparison purposes, the historical correlation computed using a sample of 1 year daily data is 37.7%, and 39.72% if a sample of 2 years daily data is considered instead. In both cases, the

[1]Options with maturity of 56 days (August 8, 2014) were also available but with limited liquidity. A set of options with various maturities were quoted by Bloomberg but without trading volume. Prices for these maturities are obtained from Bloomberg models and can therefore not be considered as market prices.

correlation is positive which is consistent with a market quote for the Quanto futures, F^d, greater than the one of the standard futures contract, F^f. We illustrate the idea in Figure 2, in which we plot the spread between market quotes of the Nikkei 225 index (S) and the corresponding futures (F^f) and Quanto futures prices (F^d). This spread is often referred to as correlation trade as it is significantly affected by how information on the correlation is recovered from the market.

To better study the behaviour of the correlation coefficient, we repeat both calibration procedures every day from June 13, 2014 to June 20, 2014. Results are illustrated in Table 3. Specifically, we report the market futures prices, F^f_{mkt}, the market and VG Quanto futures prices, F^d_{mkt} and F^d_{VG}, the at-the-money volatility of both Nikkei 225 index and USDJPY FX rate. The VG prices are obtained using the parameters from both calibration procedures. Further, we report the Quanto futures implied correlation obtained under both the multivariate VG model and the Black-Scholes framework (denoted as $\rho^{i,VG}_{SX}$, $\rho^{i,BS}_{SX}$ respectively) which are compared against the historical correlation; for completeness, we also consider the historical correlation coefficient computed over time periods of different lengths, spanning from 1 month to 2 years. These results show the dynamics over time of the relevant correlation coefficients. In particular we note that the historical correlation is relatively stable, but always fluctuates around a lower level than the implied one. For comparison, we also obtain estimates based on Dynamic Conditional Correlation (DCC) on the basis of multivariate generalized autoregressive conditional heteroskedasticity (GARCH) (see for example Engle 2002), which is shown in Figure 3 for the case of the USDJPY FX rate and the Nikkei 225 index log-returns from January 2000 until February 2015. In particular on June 13th, the DCC estimate is 42.11%, higher than most of the historical estimates but still far away from the one obtained through the Quanto futures implied correlation procedure. The difference between these measures can be compared in a sense to the difference between implied volatility and GARCH type volatilities. Many empirical studies show that implied and historical (including GARCH type) volatilities are quite different, as these measures provide different types of information: the implied volatility is a measure extracted from the market of derivatives and reflects market expectations (and as such it is highly dependent on market news and speculation), whilst the historical volatilities are backward-looking measures. Hence, the results from the calibration exercise show that the market expectation is for much stronger co-movements in the assets of interest (i.e. the Nikkei 225 index and the USDJPY exchange rate) than what experienced in the past.

With hindsight, a possible motivation for the observed differences could be traced back to the unprecedented monetary easing policies implemented by the Japanese government aimed at ending deflation. From this simple analysis it transpires that the market was already anticipating in June 2014 the impact of these monetary policies. Admittedly, 8 months later, in February 2015, the Nikkei Stock Average rose to a 15 years high, whilst the Yen settled around the weakest level against the US Dollar since 2007.

Table 3 also contains the market quanto adjustments and the VG quanto adjustment computed using the parameters obtained from both calibration procedures. Similarly, we report the resulting covariance between the log-returns of the Nikkei 225 index and USDJPY FX rate, which is computed day by day after calibration. Finally, the term corresponding to the cumulants of higher order than two of the jump part of the systematic risk factor in the quanto adjustment (see Equation 12) is presented in the table as well. We note that the largest contribution to the overall quanto adjustment originated by higher order cumulants is due to the skewness of the systematic factor Z, captured by the term $qc_Z(3)$ (see Equation 15), which fluctuates (in absolute value) between 0.22% and 3.51% for the case of the QF-based calibration. The contribution of the excess kurtosis term $qc_Z(4)$ (see Equation 16) counts for up to 0.21% of the overall quanto adjustment, whilst the contribution of the higher order terms ($n > 4$) is negligible in comparison. Similar conclusions hold under the HC-based calibration, except for the fact that the contribution from the skewness term, $qc_Z(3)$, is relatively stable around

1%.

4.2 Implied correlation from Quanto Options

In this section, we aim at testing the consistency of the two calibration procedures introduced in Section 4.1 through the pricing of Quanto options on the Nikkei 225 index. Due to the fact that in our framework these products can be easily priced via analytical formulas (up to a Fourier inversion), these prices can be used to back out the relevant correlation. However, although market quotes for Quanto options are available from the CME platform, we do not have access to them and therefore we base our analysis on model prices obtained using the parameters recovered from both calibration procedures and reported in Table 2. Then, we can recover the value of the correlation coefficient such that the computed Quanto call option prices are matched by the ones obtained in the Black-Scholes model. To this purpose, though, we need first to carefully deal with the volatility smile/skew effect. Common market practice is, in fact, to use the at-the-money implied volatility for both the Nikkei 225 index, and the USDJPY FX spot rate, reported in Table 1. Pricing of multi-asset options, such as Quanto options, though requires consistency with the volatility smile of the corresponding assets (see also Shevchenko, 2006). This is evident when we use Equation (17) with $Q_{adj} = 1$ in the Black-Scholes setting to recover the correlation coefficient value such that the Quanto call prices generated by the multivariate VG model are matched exactly. Results are presented in Figure 4, in which we show the implied correlation coefficient extracted from the Black-Scholes model under the assumption that the volatility of both the Nikkei 225 index and the USDJPY FX rate is set at the corresponding at-the-money value, and under the assumption that the volatility smile of the index is incorporated in the procedure. In details, in the left hand panel of Figure 4, we illustrate the case in which the input Quanto option prices are generated using the parameters from the QF-based calibration; we denote the resulting implied correlation coefficients as $\rho_{SX}^{i,BS}(K; v_1)$ if at-the-money volatilities are used, and $\rho_{SX}^{i,BS}(K; v_2)$ if the whole volatility smile is incorporated instead. Similarly, in the right hand panel of Figure 4 we report the same quantities obtained from input prices generated by the parameters from the HC-based calibration; we denote these coefficients as $\rho_{SX}^{h,BS}(K; v_1)$ and $\rho_{SX}^{h,BS}(K; v_2)$. We note that when input prices are generated with the at-the-money volatilities, implied correlation values are close to their admissible bounds $[-1, 1]$ regardless of the calibration approach adopted; this in turn generates a pronounced mispricing of in-the-money and out-of-the-money options (results available upon request). If instead the volatility smile is used, the resulting implied correlation values show an increasing pattern from 70.91% to 79.54% in the case of input parameters obtained from the QF-based calibration, and 34.74% to 52.76% in the case of input parameters from the HC-based calibration.

Hence, from this simple experiment, we observe that once the volatility smile of the underlying asset is correctly taken into account, information extracted from historical prices generates inconsistent estimates of the correlation value; by using the parameters obtained from the HC-based calibration, in fact, we would expect to recover values of the correlation close to the historical estimate of 28% used in the calibration. This procedure instead generates a discrepancy in the correlation value ranging from 58% to 88%. The parameters obtained from the QF-based calibration, on the other hand, generate values of the correlation relatively close to the one originated by the Quanto futures quotes, as the (percentage) difference ranges from 3% to 13%.

4.3 Tail dependence and risk measures

Proposition 3 shows that in our multivariate Lévy framework, the tail dependence behaviour is governed by the tail probabilities of the systematic risk process Z, and the indices of upper/lower tail dependence are different from zero only when the margin processes are positively correlated, which is the case in our particular example discussed in the previous sections. Hence, we use Equations (3) and (4), derived in Section 2 to compute the indices of upper and lower tail dependence between the log-returns of the Nikkei 225 index and the USDJPY FX rate using the calibrated multivariate VG model; corresponding analytical expressions are recovered following a similar argument as in Barndorff-Nielsen and Shiryaev (2010)[2]. We consider a 1 week horizon. We use the parameters obtained from both calibration procedures described in Section 4.1. Results are shown in Figure 5, from which we note that both calibrated VG models produce a non negligible tail dependence effect. Specifically, in light of the previous discussion, we observe that correlated downwards jumps are more likely according to the prevailing market expectations than what experienced in the past, as the index of lower tail dependence (in the right hand panel of Figure 5) is significantly higher when the parameters of the systematic risk process are recovered from Quanto futures. In other words, information from historical prices leads to underestimating the probability of a joint downward movement; this effect could have an impact on potential capital requirements linked to this measure of risk.

The two alternative correlation assumptions underpinning the calibration approaches discussed in this section also have an influence on univariate contracts. This is evident, for example, in the computation of the Value-at-Risk (VaR) of positions in non linear contracts, as vanilla call options, defined as the potential loss given a prespecified level of probability due to market movements. We illustrate the point by computing the 95% VaR for a short position in one call option on the Nikkei 225 index over both a 1 day and 10 days exposure periods (this example is inspired by Eberlein et al., 1998). Results are presented in Figure 6, which shows that the 95% VaR is higher under the QF-implied calibration procedure, due to the fact that the resulting distribution of the Nikkei 225 index has a relatively heavier right tail, implying relatively more likely upwards movements in the index. In other words, similarly to the case of joint downward risk, information extracted from historical prices could lead to underestimating the risk of losses for sell side market participants, again with cascading effects on potential capital requirements linked to these positions. Similar results can be obtained for alternative levels of confidence.

5 Conclusion

In this paper, we have developed a multivariate Lévy model for the joint dynamics of FX exchange rates and asset prices based on the factor representation of Ballotta and Bonfiglioli (2014). In this setting, we consider the pricing and calibration of Quanto contracts of vanilla nature which are traded over-the-counter in significant size, with the aim of extracting information about the dependence in place. In particular, as in our model Quanto call options can be related to European call options, fast and accurate pricing of these products is achieved via Fourier inversion techniques.

The numerical analysis presented in Section 4 for the CME USD-denominated Quanto futures on the Nikkei 225 index have shown the impact on prices of derivative contracts of different assumptions on the correlation coefficient linking the index to the USDJPY FX rate. In particular, we have shown that the correlation implied by Quanto futures and Quanto options can be quite different (always greater in this empirical example) from the historical correlation and DCC. This has a particular impact on the indices of upper and lower tail dependence and on the computation of risk measures related to portfolios containing these products: information based on historical correlation leads to underestimating both the probability of a joint downward movement in the relevant assets, and the VaR of short positions in the derivative contracts under consideration. Further, as in the proposed construction the quanto adjustment is affected by higher order cumulants of the pure jump part of the systematic risk factor, we are able to estimate the contribution of the skewness, the excess kurtosis

[2]Formulas are available from the Authors upon request.

and the higher order terms. The empirical analysis reveals the predominant role of the skewness of the distribution of the common risk factor, covariance aside.

As the proposed model is based on Lévy processes, i.e. processes with independent and stationary increments, stochastic volatility effects are ignored. For the case of the analysis considered in this paper, this is acceptable due to the very short maturities of the contracts involved. However, stochastic volatility effects can be added in our framework by using an analogous model based on multidimensional versions of time changed Lévy processes; in particular it would be interesting to allow dependence between the stochastic clock and the basis Lévy process: according to e.g. Carr and Wu (2004), this would model more adequately the observed features of stochastically varying return volatilities and correlation. Another aspect not considered in this paper is the simultaneous calibration of FX triangles. These topics are left however to future research.

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A Proofs of results

A.1 Proof of Proposition 1

Let $\mathbf{a} = (a_1, \cdots, a_n)^\top$, and assume that $\boldsymbol{\Lambda}(t)$ has generating triplet (β, Γ, ν). It follows from Sato (1999, E12.10) (see also Cont and Tankov, 2004, Proposition 5.3) that $\beta = (\beta_1, ..., \beta_n, \beta_Z)^\top$, Γ is diagonal and ν is supported by the union of the coordinate axes. Define a $n \times (n+1)$ matrix M as

$$M = \begin{bmatrix} 1 & 0 & ... & 0 & a_1 \\ 0 & 1 & ... & 0 & a_2 \\ \vdots & \vdots & ... & \vdots & \vdots \\ 0 & 0 & ... & 1 & a_n \end{bmatrix}.$$

Then $\mathbf{L}(t) = M\boldsymbol{\Lambda}(t)$; it follows from Sato (1999, Proposition 11.10) (see also Cont and Tankov, 2004, Theorem 4.1) that $\mathbf{L}(t)$ is a Lévy process with drift and diffusion matrix as given. The characteristic function follows from the independence of the components of $\boldsymbol{\Lambda}(t)$. For the construction of the Lévy measure, we note that $\{\mathbf{a}\Delta Z(t) \neq 0, \mathbf{a}\Delta Z(t) \in B\}$ if and only if $\{\Delta Z(t) \neq 0, \Delta Z(t) \in A\}$. As the components of $\boldsymbol{\Lambda}(t)$ are independent, Sato (1999, E12.10) implies that the support of κ is the union of the coordinate axes and the result follows. See Tankov (2004) as well.

A.2 Proof of Corollary 2

Results for γ_{Lj}, c_j^2 follow from Cont and Tankov (2004, Proposition 5.2); the Lévy measure follows from Cont and Tankov (2004, Proposition 5.3) and Sato (1999, E12.10) by recognizing that $L_j(t) = Y_j(t) + a_j Z(t)$ and $Y_j(t)$ is independent of $Z(t)$.

A.3 Proof of Proposition 3

The proof of the above results is based on the fact that the probability of two sums of variables both exceeding some diverging threshold is driven completely by the common component of the sums (see Oh and Patton, 2012, for example).

a) Applying the above, we obtain

$$\begin{aligned}\mathbb{P}\left(L_j(t) < l_j, L_k(t) < l_k\right) &= \mathbb{P}\left(Y_j(t) + a_j Z(t) < l_j, Y_k(t) + a_k Z(t) < l_k\right) \\ &\simeq \mathbb{P}\left(a_j Z(t) < l_j, a_k Z(t) < l_k\right) \quad l_j, l_k \downarrow -\infty.\end{aligned}$$

Hence, if $a_j, a_k > 0$

$$\mathbb{P}\left(L_j(t) < l_j, L_k(t) < l_k\right) \simeq \mathbb{P}\left(Z(t) < \min\left\{\frac{l_j}{a_j}, \frac{l_k}{a_k}\right\}\right) \quad l_j, l_k \downarrow -\infty,$$

whilst, if $a_j, a_k < 0$

$$\mathbb{P}\left(L_j(t) < l_j, L_k(t) < l_k\right) \simeq \mathbb{P}\left(Z(t) > \max\left\{\left|\frac{l_j}{a_j}\right|, \left|\frac{l_k}{a_k}\right|\right\}\right) \quad l_j, l_k \downarrow -\infty.$$

On the other hand, if $\rho^{\mathbf{L}}_{jk} < 0$ for all $t > 0$, we obtain

$$\mathbb{P}\left(L_j(t) < l_j, L_k(t) < l_k\right) \simeq \mathbb{P}\left(Z(t) > \left|\frac{l_j}{a_j}\right|, Z(t) < \frac{l_k}{a_k}\right) \quad l_j, l_k \downarrow -\infty$$

if $a_j < 0 < a_k$, and

$$\mathbb{P}\left(L_j(t) < l_j, L_k(t) < l_k\right) \simeq \mathbb{P}\left(Z(t) < \frac{l_j}{a_j}, Z(t) > \left|\frac{l_k}{a_k}\right|\right) \quad l_j, l_k \downarrow -\infty$$

if $a_k < 0 < a_j$; therefore both probabilities are equal to zero.

b) The result follows by the same argument as above.

c) We note that $\rho^{\mathbf{L}}_{jk} < 0$ if and only if either $a_j < 0 < a_k$ or $a_k < 0 < a_j$. Let us consider first the case in which $a_j < 0 < a_k$; then

$$\mathbb{P}\left(L_j(t) < l_j, L_k(t) > l_k\right) \simeq \mathbb{P}\left(-a_j Z(t) > |l_j|, a_k Z(t) > l_k\right) \quad |l_j|, l_k \uparrow \infty. \tag{A.1}$$

As $a_j < 0$, then

$$\mathbb{P}\left(L_j(t) < l_j, L_k(t) > l_k\right) \simeq \mathbb{P}\left(Z(t) > \max\left\{\left|\frac{l_j}{a_j}\right|, \frac{l_k}{a_k}\right\}\right) \quad |l_j|, l_k \uparrow \infty.$$

On the other hand, if $a_k < 0 < a_j$, by similar argument we obtain

$$\mathbb{P}\left(L_j(t) < l_j, L_k(t) > l_k\right) \simeq \mathbb{P}\left(Z(t) < \min\left\{\frac{l_j}{a_j}, -\frac{l_k}{|a_k|}\right\}\right).$$

If we have positive correlation, i.e. $a_j, a_k > 0$, then Equation (A.1), would read as

$$\mathbb{P}\left(-a_j Z(t) > |l_j|, a_k Z(t) > l_k\right) \simeq \mathbb{P}\left(Z(t) < -\left|\frac{l_j}{a_j}\right|, Z(t) > \frac{l_k}{a_k}\right) \quad |l_j|, l_k \uparrow \infty$$

which is zero, as the event is impossible. Similar argument applies when $a_j, a_k < 0$.

A.4 Proof of Proposition 4

The Girsanov theorem implies that the change of measure is in this case governed by the Esscher parameter $h_j \in \mathbb{R}$, which is constant by construction. Consequently the processes $\mathbf{\Lambda}(t)$ and $\mathbf{L}(t)$ remain Lévy processes under $\mathbb{P}^{hj}$. Therefore, part (*a*) follows directly from the Girsanov theorem. Part (*b*) follows from Proposition 1 and Corollary 2. See also Eberlein et al. (2009).

	Nikkei 225			USDJPY		
Spot	$S(0)$	15097.84	JPY	$X(0)$	102.03	JPY/USD
ATM implied volatility		19.56%			5.42%	
Japan risk free rate of interest	r_f	0.10%				
US risk free rate of interest	r_d	0.25%				
Nikkei 225 futures	$F^f(0,T)$	15030	JPY			
Nikkei 225 Quanto futures	$F^d(0,T)$	15065	USD			
	T	91	days			
Historical correlation	ρ^h_{SX}	28%				

Table 1: Synopsis of market data. Source: Bloomberg, CME free web platform (see http://www.cmegroup.com/). Observation date: June 13, 2014. r_f, r_d: benchmark interest rates of Japan and US, respectively. ρ^h_{SX} : historical correlation between log-returns estimated on a sample size of 128 days.

QF-based calibration						HC-based calibration					
Idiosyncratic process				Systematic process		Idiosyncratic process				Systematic process	
Nikkei 225		USDJPY				Nikkei 225		USDJPY			
θ_{Y_S}	-0.0177	θ_{Y_X}	0.1514	θ_Z	-0.1830	θ_{Y_S}	-0.3825	θ_{Y_X}	-0.1362	θ_Z	0.5978
σ_{Y_S}	0.0150	σ_{Y_X}	0.0070	σ_Z	0.1095	σ_{Y_S}	0.1661	σ_{Y_X}	0.0294	σ_Z	0.0956
κ_{Y_S}	0.0084	κ_{Y_X}	0.0449	κ_Z	0.0522	κ_{Y_S}	0.0750	κ_{Y_X}	0.0456	κ_Z	0.0307
a_S	1.8110	a_X	0.4008			a_S	0.6158	a_X	0.2776		
RMSRE	6.91E-03					RMSRE	6.33E-03				
Margin process				Systematic process		Margin process				Systematic process	
Nikkei 225		USDJPY				Nikkei 225		USDJPY			
$\mathbb{E}L_S(1)$	-0.3491	$\mathbb{E}L_X(1)$	0.0781	$\mathbb{E}Z(1)$	-0.1830	$\mathbb{E}L_S(1)$	-0.0144	$\mathbb{E}L_X(1)$	0.0297	$\mathbb{E}Z(1)$	0.5978
$\sqrt{\mathbb{V}arL_S(1)}$	0.2129	$\sqrt{\mathbb{V}arL_X(1)}$	0.0573	$\sqrt{\mathbb{V}arZ(1)}$	0.1172	$\sqrt{\mathbb{V}arL_S(1)}$	0.2149	$\sqrt{\mathbb{V}arL_X(1)}$	0.0571	$\sqrt{\mathbb{V}arZ(1)}$	0.1418
$s(L_S(1))$	-0.2324	$s(L_X(1))$	-0.0495	$s(Z(1))$	-0.2341	$s(L_S(1))$	-0.2814	$s(L_X(1))$	-0.0393	$s(Z(1))$	0.3173
$\kappa(L_S(1))$	0.1920	$\kappa(L_X(1))$	0.1165	$\kappa(Z(1))$	0.1940	$\kappa(L_S(1))$	0.2379	$\kappa(L_X(1))$	0.1030	$\kappa(Z(1))$	0.1649
$\rho_{SX}^{i,VG}$	81.77%					$\rho_{SX}^{h,VG}$	28.00%				

Table 2: Top panel - Calibrated parameters of the multivariate VG model. QF-based calibration: Z, a_S, a_X calibrated using Quanto futures quotes. HC-based calibration: Z, a_S, a_X calibrated using historical correlation (128 days). Bottom panel - Moments of the resulting margin distribution. $\rho_{SX}^{i,VG}$: pairwise correlation coefficient computed using Equation (2) and the QF-based calibrated parameters. $\rho_{SX}^{h,VG}$: recovered pairwise historical correlation. s and κ denote the indices of skewness and excess kurtosis as in Cont and Tankov (2004). Data: see Table 1.

	Friday 13/06/14	Monday 16/06/14	Tuesday 17/06/14	Wednesday 18/06/14	Thursday 19/06/14	Friday 20/06/14
T(days)	91	88	87	86	85	84
$F^f_{mkt}(0,T)$	15030.00	14950.00	15030.00	15100.00	15365.00	15460.00
$F^d_{mkt}(0,T)$	15065.00	14985.00	15060.00	15130.00	15390.00	15490.00
$F^{d,i}_{VG}(0,T)$	15066.37	14985.41	15059.95	15131.10	15388.83	15488.81
$F^{d,h}_{VG}(0,T)$	15043.15	14960.94	15042.08	15112.85	15374.77	15470.02
Nikkei 225 ATM vol	19.56%	18.14%	18.70%	17.04%	15.63%	18.36%
USDJPY ATM vol	5.42%	5.51%	5.42%	5.55%	4.98%	4.87%
$\rho^{i,BS}_{SX}$	87.90%	97.12%	82.59%	89.07%	89.60%	94.21%
$\rho^{i,VG}_{SX}$	81.77%	91.59%	74.27%	77.21%	77.95%	86.32%
ρ^{128d}_h	28.00%	28.78%	29.29%	30.45%	30.50%	29.81%
ρ^{2y}_h	39.72%	39.76%	39.76%	39.69%	39.63%	39.62%
ρ^{1y}_h	37.70%	39.39%	39.43%	39.58%	38.88%	40.04%
ρ^{6m}_h	29.39%	29.20%	28.51%	28.12%	27.78%	29.99%
ρ^{3m}_h	33.65%	34.35%	35.30%	33.31%	34.34%	29.99%
ρ^{1m}_h	43.15%	49.78%	50.93%	48.29%	43.67%	43.68%
q_{mkt}	9.33E-03	9.70E-03	8.37E-03	8.42E-03	6.98E-03	8.42E-03
q^i_{VG}	9.69E-03	9.81E-03	8.35E-03	8.73E-03	6.65E-03	8.09E-03
$\mathbb{C}ov^i_{VG}(L_S, L_X)$	9.98E-03	9.92E-03	8.63E-03	8.71E-03	6.70E-03	8.19E-03
$qc^i_Z(3)$	-3.03E-04	-1.13E-04	-2.93E-04	1.93E-05	-5.34E-05	-1.05E-04
$qc^i_Z(4)$	2.01E-05	5.52E-06	1.49E-05	4.69E-06	4.80E-06	1.45E-06
residual	-1.27E-06	-8.44E-08	-5.69E-07	1.76E-08	-5.23E-08	-1.86E-08
q^h_{VG}	3.51E-03	3.03E-03	3.37E-03	3.61E-03	2.73E-03	2.81E-03
$\mathbb{C}ov^h_{VG}(L_S, L_X)$	3.44E-03	3.07E-03	3.41E-03	3.57E-03	2.70E-03	2.84E-03
$qc^h_Z(3)$	6.91E-05	-3.26E-05	-4.57E-05	3.56E-05	2.66E-05	-3.06E-05
$qc^h_Z(4)$	1.35E-06	3.40E-07	5.63E-06	4.38E-07	2.52E-07	3.14E-07
residual	2.61E-08	-3.49E-09	-1.03E-07	4.83E-09	2.39E-09	-3.18E-09

Table 3: Time evolution analysis - $F^d_{mkt}(0,T)$: USD-denominated CME Nikkei 225 futures quote (symbol: NKD). $F^f_{mkt}(0,T)$: JPY-denominated CME Nikkei 225 futures quote (symbol: NIY). $F^{d,i}_{VG}(0,T)$: USD-denominated Nikkei 225 futures quote computed with Equation (20)-(21) and QF-based calibration parameters. $F^{d,h}_{VG}(0,T)$: USD-denominated Nikkei 225 futures quote computed with Equation (20)-(21) and HC-based calibration parameters. q_{mkt}: quanto adjustment implied by market data. q^i_{VG} and $\mathbb{C}ov^i_{VG}(L_S, L_X)$: quanto adjustment and covariance computed with QF-based calibration parameters. q^h_{VG} and $\mathbb{C}ov^h_{VG}(L_S, L_X)$: quanto adjustment and covariance computed with HC-based calibration parameters. $qc_Z(3), qc_Z(4)$ as in Equations (15)-(16). Residual: $\sum_{n=5}^{\infty} qc_Z(n)$ - see Equation (13).

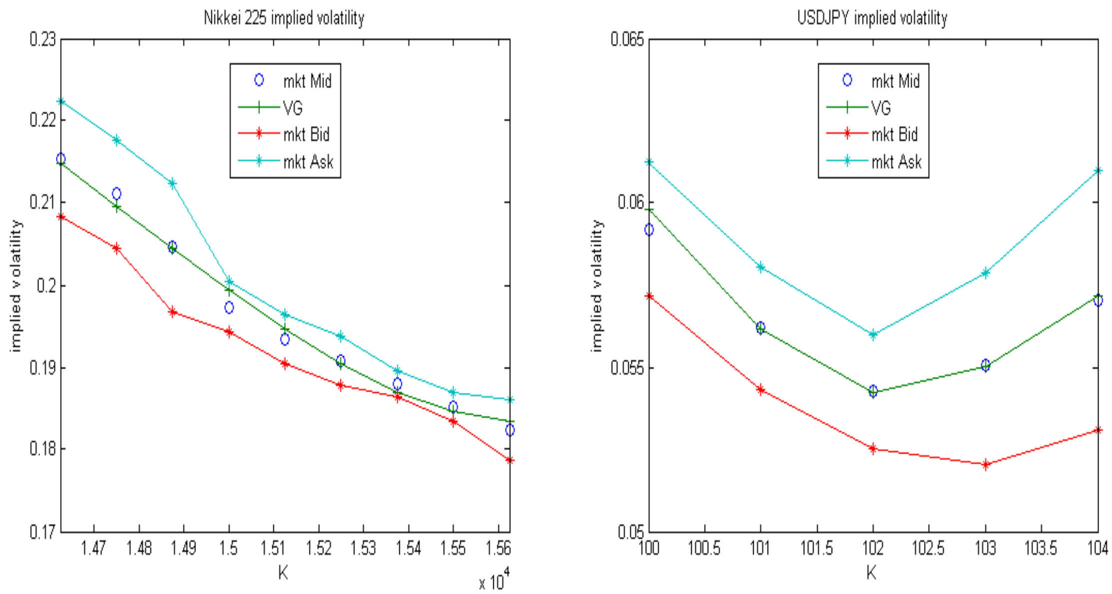

Figure 1: USDJPY and Nikkei 225 implied volatility in function of strike K: market vs calibrated multivariate VG model. Options market data, June 13, 2014: Source: Bloomberg. USDJPY Options maturity: $T = 1$ month. Nikkei 225 Options maturity: $T = 28$ days (July 11, 2014). Market Data: Table 1. Multivariate VG model parameters: Table 2, QF-based calibration.

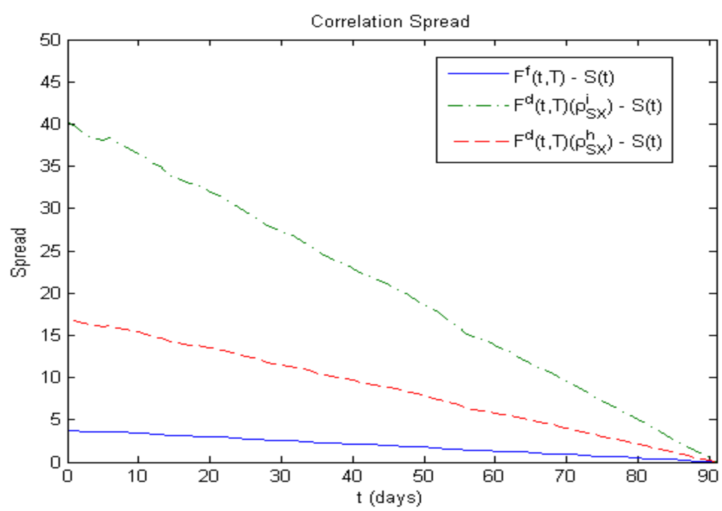

Figure 2: Evolution of the correlation spread in function of time t (in days). Starting date (t=0): June 13, 2014. Maturity T: September 12, 2014. Nikkei 225 index $S(t)$ market data, June 13, 2014 - September 12, 2014: Source: Bloomberg. $F^f(t,T)$: futures prices computed by using Equation (10), $r_f = 0.10\%$. $F^d(t,T)(\rho^i_{SX})$ and $F^d(t,T)(\rho^h_{SX})$: Quanto futures prices computed by using Equation (20) and calibrated parameters of the multivariate VG model from the QF-based calibration and HC-based calibration respectively (see Table 2). Applied FX rate between the two currencies set to 1 USD/JPY.

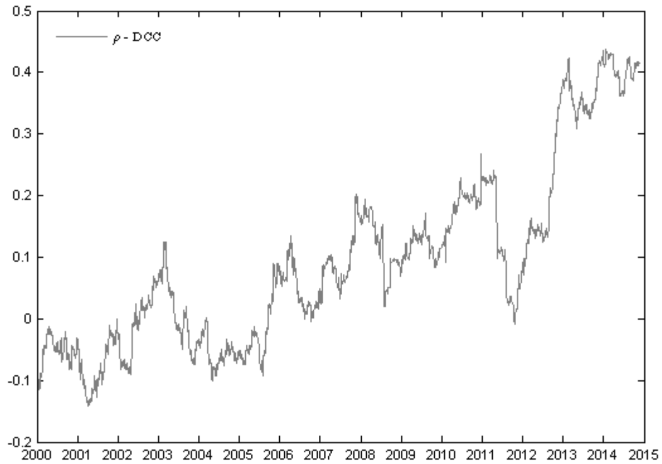

Figure 3: Dynamic Conditional Correlation between USDJPY and Nikkei 225 index log-returns (using daily data from January 2000-February 2015).

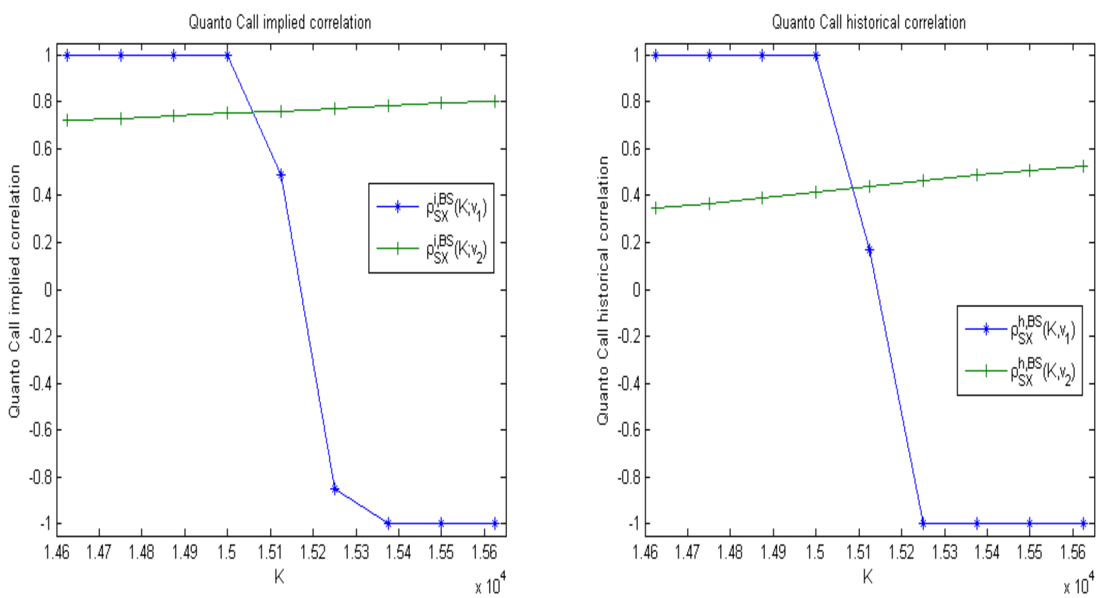

Figure 4: Left hand panel: QF-based calibration. Right hand panel: HC-based calibration. $\rho_{SX}^{\cdot,BS}(K;v_1)$: Quanto call implied correlation in function of strike K, extracted in a BS setting where the Nikkei 225 index and the USDJPY FX rate volatility are set at their at-the-money values (see Table 1). $\rho_{SX}^{\cdot,BS}(K;v_2)$: Quanto call implied correlation in function of strike K, extracted in a BS setting where the strike corresponding Nikkei 225 index implied volatility (Figure 1) is used Market data: see Table 1.

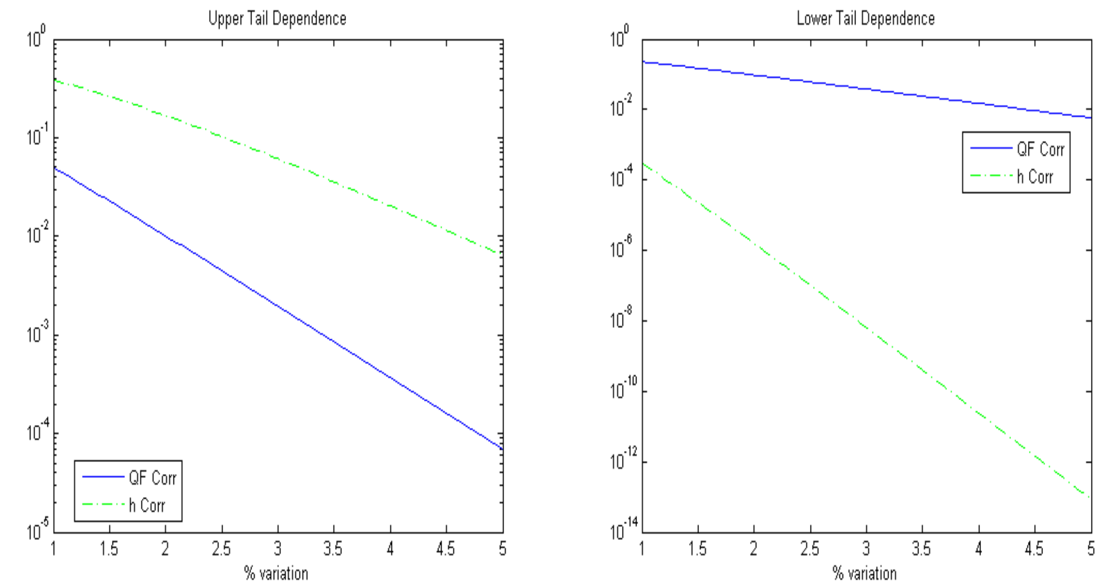

Figure 5: Upper and Lower Tail dependence for given percentage variations of the log-returns (on a log-scale). QF Corr: QF-based calibration. h Corr: HC-based calibration. Parameters: Table 2.

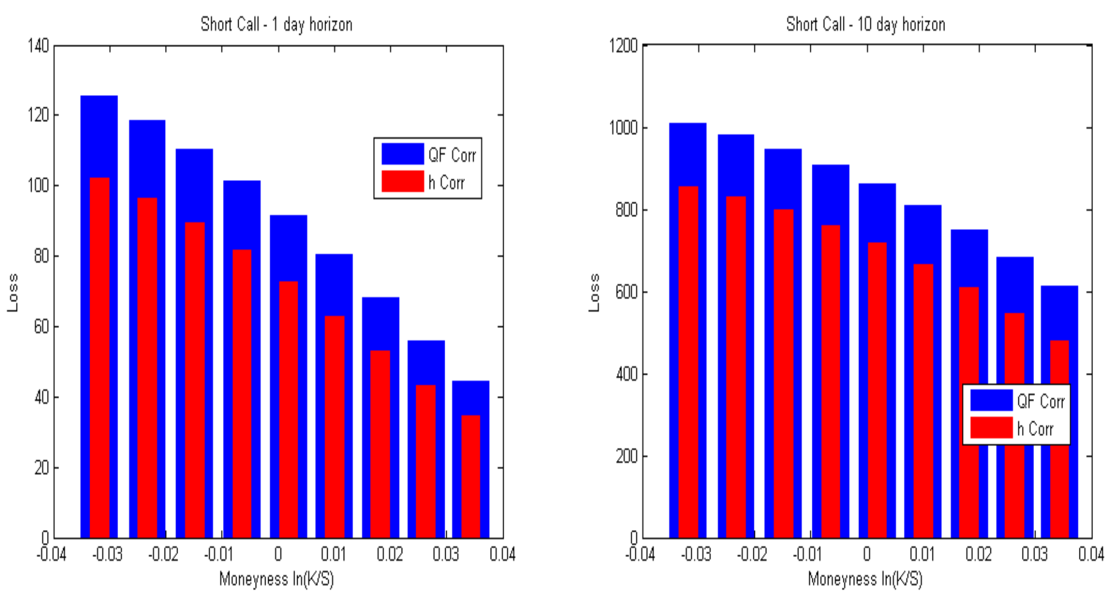

Figure 6: 95% VaR for a non linear short position. Left hand panel: vanilla call option on Nikkei 225 index, 1 day horizon. Right hand panel: vanilla call option on Nikkei 225 index, 10 day horizon. Option prices computed using the Carr-Madan method. QF Corr: QF-based calibration. h Corr: HC-based calibration. Parameters: Table 2.