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Quantum Integrability of Certain Boundary Conditions [1]

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Abstract

We study the quantum integrability of the $O(N)$ Nonlinear σ (nlσ) model and the $O(N)$ Gross-Neveu (GN) model on the half-line. We show that the nlσ model is integrable with Neumann, Dirichlet and a mixed boundary condition and that the GN model is integrable if $\psi_+^a|_{x=0} = \pm\psi_-^a|_{x=0}$. We also comment on the boundary condition found by Corrigan and Sheng for the $O(3)$ nlσ model.

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1 Introduction

The purpose of this letter is to investigate the quantum integrability of certain boundary conditions of two theories defined on the half-line [1, 2]: the $O(N)$ nonlinear sigma model (nlσ) and the $O(N)$ Gross-Neveu (GN) model. These two models have some very similar properties, such as asymptotic freedom and dynamical mass generation, but are also quite different, the former being bosonic and no bound states, and the latter fermionic and with a very rich spectrum of bound states, for example. Their bulk version has been established to be integrable long ago, at the classical [3, 4] and at the quantum level [5, 6]. The study of these models on the half-line is hindered more difficult, because many of the techniques available on the full-line, such as the Lax pair, cannot be easily extended to the half-line. The structure of this letter is as follows. In the next section we briefly review the two models and exhibit a bulk conserved charge of spin 3 for each model, in section 3 we show that the Neumann, Dirichlet and the "mixed" boundary condition preserve integrability for the nlσ model, and discuss the condition found by Corrigan and Sheng in [7] for the $O(3)$ nlσ model; in section 4 we show that the GN model on the half-line is integrable if $\psi^a_+|_{x=0} = \pm\psi^a_-|_{x=0}$, where $\psi^a_\pm$ are the chiral components of the Majorana fermions. In the final section we present our conclusions and possible extensions of this work.

2 The Models

In this section we briefly review the main properties of the nlσ model and of the GN model. We also discuss the conserved currents of spin 4 that are going to be used later.

2.1 The $O(N)$ Nonlinear σ Model

The $O(N)$ nonlinear σ (nlσ) model is defined by the following Lagrangian

$$\mathcal{L}_{nl\sigma} = \frac{1}{2g_0}\partial_\mu\vec{\mathbf{n}}\cdot\partial^\mu\vec{\mathbf{n}}\ , \tag{2.1}$$

where $\vec{\mathbf{n}}$ is a vector in N-dimensional space, subject to the constraint $\vec{\mathbf{n}}\cdot\vec{\mathbf{n}} = 1$ [1]. We can introduce a Lagrange multiplier ω that takes care of the constraint $\vec{\mathbf{n}}\cdot\vec{\mathbf{n}} = 1$, the Lagrangian being modified to

$$\mathcal{L}'_{nl\sigma} = \frac{1}{2g_0}\partial_\mu\vec{\mathbf{n}}\cdot\partial^\mu\vec{\mathbf{n}} + \omega(\vec{\mathbf{n}}\cdot\vec{\mathbf{n}} - 1)\ . \tag{2.2}$$

[1]Our conventions throughout this paper are: The Minkowski metric is $\eta_{\mu\nu}$ = diag(-1,1), the gamma matrices are $\gamma^0 = i\sigma_2$ and $\gamma^1 = \sigma_1$, where σ_i are the Pauli matrices. The light-cone variables are $x_\pm = (x_0 \pm x_1)/2$.

By using this Lagrangian and the constraint on the length of $\vec{\mathbf{n}}$ it is easy to show that the classical equation of motion in light-cone coordinates is

$$\partial_+\partial_-\vec{\mathbf{n}} + \vec{\mathbf{n}}(\partial_-\vec{\mathbf{n}} \cdot \partial_+\vec{\mathbf{n}}) = 0 \ . \tag{2.3}$$

At the classical level this model is conformally invariant, which implies the vanishing of the off-diagonal components of the energy-momentum tensor

$$T_{+-} = T_{-+} = 0 \ . \tag{2.4}$$

The non-vanishing components are

$$T_{++} = \partial_+\vec{\mathbf{n}} \cdot \partial_+\vec{\mathbf{n}} \qquad \text{and} \qquad T_{--} = \partial_-\vec{\mathbf{n}} \cdot \partial_-\vec{\mathbf{n}} \ . \tag{2.5}$$

Notice that the conservation law

$$\partial_\pm T_{\mp\mp} = 0 \tag{2.6}$$

implies $\partial_\pm (T_{\mp\mp})^n = 0$ for any integer n.

At the quantum level there is dynamical mass generation. This means that this model has an anomaly and so, conserved charges in the classical theory have to be corrected. It is not clear in principle that this model will be still integrable. Nonetheless Polyakov proved the quantum integrability of the nlσ model in [5] (see also [8]).

In [9] Goldschmidt and Witten have analyzed conserved charges of some two-dimensional models in a similar way as Polyakov did for the nlσ model, and showed their quantum integrability. Their argument goes as follows. Since the theory is anomalous, the right hand side of $\partial_\pm (T_{\mp\mp})^2 = 0$ is not zero anymore, and we have to include, in principle, all possible operators of dimension 5 and Lorentz weight ∓ 3. We should then list these operators and check which ones can be written as total derivatives. In the case of the nlσ model they showed that $\partial_\pm (T_{\mp\mp})^2 = 0$ can only pick up anomalous contributions that can be written as total derivatives, namely $\partial_\pm$ of something. So the classical conservation law is inherited to the quantum level.

If we take the spin 3 conservation law, $\partial_\pm (T_{\mp\mp})^2 = 0$, our previous discussion shows that at after quantization it becomes

$$\partial_+(T_{--})^2 = c_1\, \partial_+(\partial_-^2\vec{\mathbf{n}} \cdot \partial_-^2\vec{\mathbf{n}}) + c_2\, \partial_-(\partial_+\vec{\mathbf{n}} \cdot \partial_-\vec{\mathbf{n}}\ \partial_-\vec{\mathbf{n}} \cdot \partial_-\vec{\mathbf{n}}) + c_3\, \partial_-(\partial_-^3\vec{\mathbf{n}} \cdot \partial_+\vec{\mathbf{n}}) \ , \tag{2.7}$$

where the c_i are constants. Of course we have a similar expression for $\partial_-(T_{++})^2$, taking $+ \leftrightarrow -$, with the same coefficients c_i. This result implies the existence of two nontrivial charges in the nlσ model at the quantum level, and therefore its integrability [10].

The boundary version of the nlσ model was first considered by Ghoshal in [11], where it is also conjectured the integrability of the Neumann, ($\partial_1\vec{\mathbf{n}}|_{x=0} = 0$) and Dirichlet

$(\partial_0 \vec{\mathbf{n}}|_{x=0} = 0)$, boundary conditions. In that paper Ghoshal solved the boundary Yang-Baxter equation consistent with this choice (and this is the main argument for its integrability!). The classical integrability of the Neumann condition for the $O(N)$ nlσ model was established by means of a generalization of the Lax pair to the half-line by Corrigan and Sheng in [7].

2.2 The $O(N)$ Gross-Neveu Model

The Gross-Neveu model [12] is a fermionic theory with quartic Fermi coupling defined by the following Lagrangian

$$\mathcal{L}_{gn} = i\bar{\psi}\ \partial\!\!\!/ \psi + \frac{g^2}{4}(\bar{\psi}\psi)^2 \ , \tag{2.8}$$

where ψ is a N component Majorana spinor in the fundamental representation of $O(N)$, with components ψ^a, a from 1 to N. The chiral components of the ψ^a are (ψ^a_+, ψ^a_-). In light-cone coordinates the GN model Lagrangian becomes

$$\mathcal{L}_{gn} = -\psi^a_+ i\partial_- \psi^a_+ - \psi^a_- i\partial_+ \psi^a_- + g^2(\psi^a_+\psi^a_-)^2 \ . \tag{2.9}$$

Notice that $\partial_\pm \to \exp(\pm\theta)\partial_\pm$ and $\psi^a_\pm \to \exp(\pm\theta/2)\psi^a_\pm$, under a Lorentz transformation [2]. This means that $\psi_\pm$ has Lorentz weight $\pm 1/2$, and $\partial_\pm$ has Lorentz weight ± 1. The equations of motion are

$$i\partial_\mp \psi^a_\pm = \pm g^2 \psi^a_\mp (\psi^b_+ \psi^b_-) \ . \tag{2.10}$$

The classical integrability of this model was established by Neveu and Papanicolaou in [4]. The quantum integrability of the GN model was established in [13], where it was proved, in the large N limit, that there is no particle production. The construction of quantum conserved charges for the GN model was done in an analogous way to Polyakov's construction for the nlσ model, in [6].

Following Witten [6], we start by looking at the classical conservation laws due to the conformal invariance of 2.8. The diagonal components of the energy-momentum tensor are $T_{\pm\pm} = \psi^a_\pm \partial_\pm \psi^a_\pm$, the off-diagonal components, T_{+-} and T_{-+} being zero. Let us consider the spin 3 conservation law, $\partial_-(T_{++})^2 = 0$. The left hand side of this equation has dimension 5 and Lorentz weight 3. This implies that the possible anomalies have to be either linear in ∂_- and zeroth order in ψ^a_- or zeroth order in ∂_- and quadratic in ψ^a_-. The operators of the former type can be converted into operators of the latter type by using the equations of motion. Analyzing all local operators with the required properties, Witten showed in [6] that these terms can be written as total derivatives. We list these terms in the appendix. This means that anomalies destroy conformal invariance but do not destroy the conservation law, and the GN model is integrable at the quantum level.

[2] θ is the rapidity variable parameterizing the Lorentz transformation.

3 Integrable Boundary Conditions for GN and nlσ Models

When considering the boundary version of an integrable field theory not all charges will still be conserved. Therefore one should investigate whether some combination of the bulk charges can be preserved after the introduction of a boundary. If we have some spin s conservation law of the form

$$\partial_- J_+^{(s+1)} = \partial_+ R_-^{(s-1)} \qquad \text{and} \qquad \partial_+ J_-^{(s+1)} = \partial_- R_+^{(s-1)} \;, \tag{3.1}$$

then we know that

$$Q_+ = \int_{-\infty}^{+\infty} dx_1 \; (J_+^{(s+1)} - R_-^{(s-1)}) \qquad \text{and} \qquad Q_- = \int_{-\infty}^{+\infty} dx_1 \; (J_-^{(s+1)} - R_+^{(s-1)}) \tag{3.2}$$

are conserved, $\partial_0 Q_\pm = 0$. In proving that these charges are conserved we have to use the fact that we can discard surface terms. When we restrict our model to the half-line we can not do that with one of the surface terms. On the other hand, if the following condition [1] is satisfied

$$J_-^{(s+1)} - J_+^{(s+1)} + R_-^{(s-1)} - R_+^{(s-1)}|_{x=0} = \frac{d}{dt}\Sigma(t) \tag{3.3}$$

for some $\Sigma(t)$, then

$$\widetilde{Q} = \int_{-\infty}^{0} dx_1 \; (J_-^{(s+1)} + J_+^{(s+1)} - R_-^{(s-1)} - R_+^{(s-1)}) - \Sigma(t) \tag{3.4}$$

is a conserved charge. Note that 3.3 depends on the specific boundary action we are considering. In this section we prove the integrability of Neumann ($\partial_1 \vec{\mathbf{n}}|_{x=0} = 0$), Dirichlet ($\vec{\mathbf{n}}|_{x=0} = \vec{\mathbf{n}}_0$ a constant, or equivalently $\partial_0 \vec{\mathbf{n}}|_{x=0} = 0$), and a mixed boundary condition (where some components of $\vec{\mathbf{n}}$ satisfy Neumann and the others Dirichlet). We also analyze the boundary condition proposed by Corrigan and Sheng in [7] for the $O(3)$ nlσ model. For the GN model we show that the spin 4 charge discussed in the previous section, with the boundary condition $\psi_+^a|_{x=0} = \epsilon_a \psi_-^a|_{x=0}$, $\epsilon_a = \pm 1$, provides a conserved charge in the boundary case.

3.1 Nonlinear σ Model

As we explained, we have to look at the combination 3.3 of the spin 4 currents at $x = 0$ and verify that it can be written as a total time derivative. In our case the conservation laws are

$$\begin{aligned}
\partial_+ (T_{--})^2 &= c_1 \; \partial_+ (\partial_-^2 \vec{\mathbf{n}} \cdot \partial_-^2 \vec{\mathbf{n}}) + c_2 \; \partial_- (\partial_+ \vec{\mathbf{n}} \cdot \partial_- \vec{\mathbf{n}} \; \partial_- \vec{\mathbf{n}} \cdot \partial_- \vec{\mathbf{n}}) + c_3 \; \partial_- (\partial_-^3 \vec{\mathbf{n}} \cdot \partial_+ \vec{\mathbf{n}}), \\
\partial_- (T_{++})^2 &= c_1 \; \partial_- (\partial_+^2 \vec{\mathbf{n}} \cdot \partial_+^2 \vec{\mathbf{n}}) + c_2 \; \partial_+ (\partial_- \vec{\mathbf{n}} \cdot \partial_+ \vec{\mathbf{n}} \; \partial_+ \vec{\mathbf{n}} \cdot \partial_+ \vec{\mathbf{n}}) + c_3 \; \partial_+ (\partial_+^3 \vec{\mathbf{n}} \cdot \partial_- \vec{\mathbf{n}}).
\end{aligned} \tag{3.5}$$

The condition we have to analyze is

$$(\partial_-\vec{\mathbf{n}}\cdot\partial_-\vec{\mathbf{n}})^2 - c_1\,\partial_-^2\vec{\mathbf{n}}\cdot\partial_-^2\vec{\mathbf{n}} - (\partial_+\vec{\mathbf{n}}\cdot\partial_+\vec{\mathbf{n}})^2 + c_1\,\partial_+^2\vec{\mathbf{n}}\cdot\partial_+^2\vec{\mathbf{n}} - \\ -c_2\,(\partial_-\vec{\mathbf{n}}\cdot\partial_+\vec{\mathbf{n}}\,\partial_-\vec{\mathbf{n}}\cdot\partial_-\vec{\mathbf{n}}) - c_3\,\partial_-^3\vec{\mathbf{n}}\cdot\partial_+\vec{\mathbf{n}} + \\ +c_2\,(\partial_+\vec{\mathbf{n}}\cdot\partial_-\vec{\mathbf{n}}\,\partial_+\vec{\mathbf{n}}\cdot\partial_+\vec{\mathbf{n}}) + c_3\,\partial_+^3\vec{\mathbf{n}}\cdot\partial_-\vec{\mathbf{n}}|_{x=0} = \frac{d}{dt}\Sigma(t)\ . \tag{3.6}$$

Let us look at the Neumann boundary condition first [3]. Since we have $\partial_1\vec{\mathbf{n}} = 0$ whenever there is a term like $\partial_\pm\vec{\mathbf{n}}$ we can substitute it by $\partial_0\vec{\mathbf{n}}$. The term $\partial_\pm^2\vec{\mathbf{n}}$ becomes $\partial_0^2\vec{\mathbf{n}} \pm \partial_1^2\vec{\mathbf{n}}$. By appropriately combining terms in 3.6, we see immediately that they all add up to zero and we can pick $\Sigma(t) = 0$. This means that the Neumann boundary condition is integrable.

We can now look at the "dual" condition to Neumann, namely the Dirichlet boundary condition, $\vec{\mathbf{n}}|_{x=0} = \vec{\mathbf{n}}_0$, constant, which is equivalent to $\partial_0\vec{\mathbf{n}}|_{x=0} = 0$. Notice that in this case $\partial_0^n\vec{\mathbf{n}} = 0$ for any integer n. The manipulations are very similar as in the Neumann case. Wherever there is $\partial_\pm\vec{\mathbf{n}}$ we should replace by $\pm\partial_1\vec{\mathbf{n}}$, $\partial_\pm^2\vec{\mathbf{n}}$ should be replaced by $\pm 2\partial_0\partial_1\vec{\mathbf{n}} + \partial_1^2\vec{\mathbf{n}}$, and by using the equations of motion and the constraint $\vec{\mathbf{n}}\cdot\vec{\mathbf{n}} = 1$, we see that $\partial_+^3\vec{\mathbf{n}}\cdot\partial_-\vec{\mathbf{n}} - \partial_-^3\vec{\mathbf{n}}\cdot\partial_+\vec{\mathbf{n}} = 0$. So once again, by appropriately collecting terms we see that 3.6 vanishes and we can pick $\Sigma(t) = 0$.

Finally we can look at the more general boundary condition, which is a mixture of Neumann and Dirichlet in the following sense: take $\partial_0 n_i|_{x=0} = 0$ for some collection of indices $\{i\}$, with, say, k elements and $\partial_1 n_j|_{x=0} = 0$ for the remaining $N-k$ indices. Neumann condition is obtained when $k = 0$, and Dirichlet when $k = N$. The analysis is very similar to the preceding cases and we shall skip technical comments. The final conclusion is that this boundary condition too is integrable.

In [7] Corrigan and Sheng showed that (classically) the $O(3)$ nlσ model on the half-line is integrable if

$$\partial_1\vec{\mathbf{n}} = -(\vec{\mathbf{k}}\times\partial_0\vec{\mathbf{n}}) + (\vec{\mathbf{n}}\cdot(\vec{\mathbf{k}}\times\partial_0\vec{\mathbf{n}}))\vec{\mathbf{n}} \qquad \text{and} \qquad \vec{\mathbf{k}}\cdot\partial_0\vec{\mathbf{n}} = 0\ , \tag{3.7}$$

at $x = 0$, $\vec{\mathbf{k}}$ arbitrary. By using the equation of motion plus the constraint $\vec{\mathbf{n}}\cdot\vec{\mathbf{n}} = 1$, this condition is compatible with our spin 4 current, with $\Sigma(t) = 16\,c_3\,\partial_0\vec{\mathbf{n}}\cdot\partial_0\partial_1\vec{\mathbf{n}}$. This indicates that 3.7 is integrable at the quantum level.

3.2 Gross-Neveu Model

Let us consider now the following boundary condition

$$\psi_+^a|_{x=0} = \epsilon_a\psi_-^a|_{x=0}\ , \tag{3.8}$$

[3]Notice that we are always considering fields and their derivatives at $x = 0$ now.

with $a = 1, \ldots, N$ and $\epsilon_a = \pm 1$. Before we continue our analysis, we should add a few remarks about these conditions. The boundary conditions 3.8 can be obtained from the boundary action

$$\mathcal{S}_b = \int_{-\infty}^{+\infty} dx_0 \sum_{a=1}^{N} \frac{i}{2} \epsilon_a \psi_+^a \psi_-^a \ . \tag{3.9}$$

This is the most general form for the boundary potential without introducing new parameters in the theory. If we have N_+ ϵ's equal $+1$ and the remaining $N_- = N - N_+$ ϵ's equal -1, then we are breaking the original $O(N)$ symmetry at the boundary to $O(N_+)$ and $O(N_-)$ symmetric sectors. Therefore there are always two different ways to break the symmetry at the boundary to the same groups (pick N_+ '+' and N_- '−', or N_- '+' and N_+ '−'), which will correspond to different CDD factors in the reflection matrices.

Suppose ϵ_a is different from ± 1 for some a. Then we get that both ψ_+^a and ψ_-^a vanish, which implies that the ath fermion does not propagate, since we have a first order equation of motion.

Let us now return to the main discussion. Condition 3.8 implies that the equations of motion 2.10 give the supplementary condition at the boundary

$$\partial_- \psi_+^a|_{x=0} = \partial_+ \psi_-^a|_{x=0} = 0 \ , \tag{3.10}$$

since for fermion fields $\psi^2 = 0$. The boundary condition 3.8 can be used along with 3.10 to show that

$$\partial_0 \psi_+^a|_{x=0} = \partial_1 \psi_+^a|_{x=0} = \epsilon_a \partial_0 \psi_-^a|_{x=0} = -\epsilon_a \partial_1 \psi_-^a|_{x=0} = 0 \ . \tag{3.11}$$

We should proceed similarly to the nlσ model case, and write down the correspondent condition from 3.3. There are many more terms now and the procedure is a bit tedious, but nonetheless, all appropriately collected terms cancel and we have that we can pick $\Sigma(t) = 0$ again. This shows that the boundary condition 3.8 preserves integrability at the quantum level.

4 Conclusions

We were able to prove the quantum integrability of the Neumann, Dirichlet, and mixed boundary conditions for the nlσ model, and of $\psi_+^a = \pm \psi_-^a$ for the GN model. The reflection matrices for the nlσ model for these conditions were proposed by Ghoshal in [11]. It would be interesting to investigate the boundary Yang-Baxter equation (BYBE) for this model more thoroughly and see if it is possible to find more general solutions [14]. Our results seem to indicate so, since we have a variety of other boundary conditions for the nlσ and GN models.

In [15] Inami, Konno and Zhang studied, via bosonization, some fermionic models on the half-line. In particular, they studied the $O(3)$ GN model and concluded that there were some possible integrable boundary conditions of the same form as proposed here [4]. In particular, in the $O(3)$ GN model it is easy to see that the boundary condition 3.8 either preserves full $O(3)$ invariance or it breaks it to $O(2)$. In each case there are two possibilities, in a similar fashion to the boundary Ising model [1]. The difference between the two correspondent reflection matrices will appear as CDD prefactors.

Another connection we can make to results in the literature is the following. It is possible to relate the $O(2N)$ GN model to the affine Toda field theories (ATFT) with imaginary coupling, using bosonization [16, 17], in a similar fashion to the way Witten used to establish the mapping from the $O(4)$ GN model to two decoupled sine-Gordon models. The ATFT (with real coupling) on the half-line were considered by Bowcock, Corrigan, Dorey and Rietdjik in [18], where they found that there is only a discrete set of integrable boundary conditions. It would be interesting to investigate the relation between their results and our boundary conditions.

One interesting direction to pursue would be to study the most general integrable boundary conditions for these models, compatible with the spin 3 charge that we have analyzed.

As a last remark, it should be interesting to apply our considerations to the local charges in the principal chiral model studied by Evans, Hassan and Mackay in [19].

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[4] In [15] these authors considered a boundary term of the form $g\psi_+\psi_-$ with $g \neq \pm i/2$, which should not be included.

A Appendix

As we mentioned in section 2, here we list the possible anomaly terms that can appear in the right hand side of $\partial_-(T_{++})^2 = 0$ [5]. These operators have to have Lorentz weight 3, dimension 5, and by using equations of motion its easy to show that we can restrict ourselves to operators that are zeroth order in ∂_- and quadratic in ψ_-^a. Witten has shown in [6] that all such operators can be written as $\partial_\pm$ of something. We are actually interested in the *something* structure of these operators. This means that we have to look for operators that have dimension 4 and Lorentz weight 4 (from ∂_- of something) or operators with dimension 4 and Lorentz weight 2 (from ∂_+ of something). The list is as follows

1. *Dimension* 4 *and Lorentz weight* 4

$$\begin{aligned}
&\psi_+^a \partial_+^3 \psi_+^a \ , \\
&\partial_+ \psi_+^a \partial_+^2 \psi_+^a \ , \\
&\psi_+^a \partial_+ \psi_+^a \, \psi_+^b \partial_+ \psi_+^b \ .
\end{aligned}$$

2. *Dimension* 4 *and Lorentz weight* 2

$$\begin{aligned}
&\psi_-^a \partial_+^3 \psi_-^a \ , \\
&\partial_+ \psi_-^a \partial_+^2 \psi_-^a \ , \\
&\psi_-^a \partial_+^2 \psi_+^a \, \psi_+^b \psi_-^b \ , \\
&\psi_-^a \partial_+ \psi_+^a \, \psi_-^b \partial_+ \psi_+^b \ , \\
&\psi_+^a \partial_+ \psi_+^a \, \psi_-^b \partial_+ \psi_-^b \ , \\
&\partial_+ \psi_+^a \partial_+ \psi_-^a \, \psi_+^b \psi_-^b \ , \\
&\psi_+^a \partial_+^2 \psi_+^a \, \psi_+^b \psi_-^b \ , \\
&\psi_+^a \partial_+ \psi_+^a \, (\psi_+^b \psi_-^b)^2 \ .
\end{aligned}$$

[5] There is an analogous analysis for the $\partial_-(T_{++})^2 = 0$ conservation law.

References

[1] S. Ghoshal and A. Zamolodchikov, *Boundary S-Matrix and Boundary State in Two-Dimensional Integrable Quantum Field Theory*, Int. Jour. Mod. Phys. **A9**, (1994) 3841-3886, hep-th/9306002.

[2] A. Fring and R. Köberle *Factorized Scattering in the Presence of Reflecting Boundaries*, Nucl. Phys. **B421**, (1994) 159-172, hep-th/9304141.

[3] K. Pohlmeyer, *Integrable Hamiltonian Systems and Interactions Through Quadratic Constraints*, Comm. Math. Phys. **46**, (1976) 207-221.

[4] A. Neveu and N. Papanicolaou, *Integrability of the Classical* $[\bar{\psi}_i\psi_i]_2^2$ *and* $[\bar{\psi}_i\psi_i]_2^2 - [\bar{\psi}_i\gamma_5\psi_i]_2^2$ *Interactions*, Comm. Math. Phys. **58**, (1978) 31-64.

[5] A. Polyakov, *Hidden Symmetry of the Two-Dimensional Chiral Fields*, Phys. Lett. **B72**, (1977) 224-226.

[6] E. Witten, *Some Properties of the* $(\bar{\psi}\psi)^2$ *Model in Two Dimensions*, Nucl. Phys. **B142**, (1978) 285-300.

[7] E. Corrigan and Z. Sheng, *Classical Integrability of he* $O(N)$ *Nonlinear Sigma Model on a Half Line*, Int. Jour. Mod. Phys. **A12**, (1997) 2825-2834, hep-th/9612150.

[8] A. Zamolodchikov and Al. Zamolodchikov, *Factorized S-Matrices in Two Dimensions as the Exact Solutions of Certain Relativistic Quantum Field Theory Models* Ann. Phys. **120**, (1979) 253-291.

[9] Y. Goldschmidt and E. Witten, *Conservation Laws in Some Two-Dimensional Models*, Phys. Lett. **B91**, (1980) 392-396.

[10] S. Parke, *Absence of Particle Production and Factorization of the S Matrix in* (1+1)-*Dimensional Models*, Nucl. Phys. **B174**, (1980) 166-182.

[11] S. Ghoshal, *Boundary S-Matrix of the* $O(N)$*-Symmetric Non-Linear Sigma Model*, Phys. Lett. **B334**, (1994) 363-368, hep-th/9401008.

[12] D. Gross and A. Neveu, *Dynamical Symmetry Breaking in Asymptotically Free Field Theories*, Phys. Rev. **D10**, (1974) 3235-3253.

[13] A. B. Zamolodchikov and Al. B. Zamolodchikov, *Exact S Matrix of Gross-Neveu "Elementary" Fermions*, Phys. Lett. **B72**, (1978) 481-483.

[14] N. MacKay, *Fusion of SO(N) Reflection Matrices*, Phys. Lett. **B357**, (1995) 89-93, hep-th/9505124.

[15] T. Inami, H. Konno and Y-Z. Zhang, *Integrable Four Fermi Models with a Boundary and Boson-Fermion Duality*, Phys. Lett. **B381**, (1996) 458-464, hep-th/9512085.

[16] R. Shankar, *Some Novel Features of the Gross-Neveu Model*, Phys. Lett. **B92**, (1980) 333-336.

[17] M. Moriconi and A. De Martino, to appear

[18] P. Bowcock, E. Corrigan, P. Dorey and R. Rietdjik, *Classical Integrable Boundary Conditions for Affine Toda Field Theories*, Nucl. Phys. **B344**, (1995) 469-500, hep-th/9501098.

[19] J.M. Evans, M. Hassan and N.J. MacKay, *Local Conserved Charges in Principal Chiral Models*, hep-th/9711140.