# Optimal insurance with counterparty default risk*

Enrico Biffis[†] Pietro Millossovich[‡]

This version: April 15, 2012.

**Abstract**

We study the design of optimal insurance contracts when the insurer can default on its obligations. In our model default arises endogenously from the interaction of the insurance premium, the indemnity schedule and the insurer's assets. This allows us to understand the joint effect of insolvency risk and background risk on optimal contracts. The results may shed light on the aggregate risk retention schedules observed in catastrophe reinsurance markets, and can assist in the design of (re)insurance programs and guarantee funds.

*Keywords*: insurance demand, default risk, catastrophe risk, limited liability, incomplete markets.

*JEL classification*: D52, D81, G22.

*We benefited from suggestions by seminar participants and discussants at Tel Aviv University, LMU Munich, Imperial College London, Cass Business School, Chinese University of Hong Kong, University of New South Wales, King's College London, ISFA Lyon, University of Bonn, University of Hanover, Queen's University Belfast, BCAM Bilbao, MAF Ravello, IME Toronto, IWAP Madrid, WRIEC/EGRIE Singapore, Risk Theory Society Meeting 2011, EEA-ESEM Oslo. Any errors are our own responsibility.

[†]Biffis (`E.Biffis@imperial.ac.uk`) is at Imperial College Business School, Imperial College London, South Kensington Campus, SW7 2AZ United Kingdom.

[‡]Millossovich (`Pietro.Millossovich.1@city.ac.uk`) is at Cass Business School, City University, London, EC1 Y8TZ United Kingdom.

# 1 Introduction

Global insurers are exposed to substantial default risk, given the sheer size of losses that may materialize in the event of financial crises and major catastrophes, such as earthquakes, hurricanes, and terrorist attacks. Commonly used reinsurance counterparty risk indices[1] suggest that default risk is both sizable and time varying, due to occasional catastrophic losses and wider market conditions affecting insurers' funding costs. For example, in the last financial crisis the deterioration of credit markets generated solvency concerns for a number of insurers that had to mark down their assets or step up liability provisions.[2] Although counterparty risk can be reduced by spreading insurance purchases across a large number of insurers, diversification fails in the presence of systematic risk, as represented by market shocks and catastrophe losses cutting across different business lines and insurers, and government guarantees are still essential in a number of situations.[3]

In this work we study the effect of counterparty default risk on optimal insurance contracts. The importance of insolvency risk in insurance markets has been emphasized by several authors (e.g., Cummins and Danzon, 1997; Cummins, Doherty and Lo, 2002), but fewer studies have considered its impact on the design of optimal risk transfers. Notable exceptions are represented by Tapiero, Kahane and Jacque (1986), Schlesinger and Schulenburg (1987), Doherty and Schlesinger (1990), Cummins and Mahul (2003) and Richter (2003), whose contributions are reviewed more in detail below. As suggested by Schlesinger and Schulenburg (1987) and Doherty and Schlesinger (1990), default is a form of contract nonperformance,[4] under which the insurance buyer is hit twice: first,

[1]See, for example, the TRX P&C Reinsurance Index V2, `http://communities.thomsonreuters.com/ILS`.

[2]See Thompson (2010) for an analysis of the effects of counterparty risk in credit default swaps.

[3]Recent examples are the Terrorism Risk and Insurance Act (TRIA) following the 9/11 attacks (see Kunreuther and Michel-Kerjan, 2006) and the intervention following Hurricane Katrina (see Kunreuther and Pauly, 2006).

[4]Other reasons why insurance contracts may fail to perform are ex-post underwriting and litigation (claims contested in court, delays in claims payments), or contract conditions introducing delay or exclusions in claim payments (e.g, probationary periods).

through the exposure to realized losses, as the indemnity may not be paid in full; second, through the loss of premium dollars, which introduces positive dependence between the amount of insurance purchased and the magnitude of default losses. Here we consider one more channel affecting insurance demand, namely the fact that default may occur in states of the world associated with low realizations of the insurance buyer's wealth. Hence our focus on measures of dependence between the agents' holdings and the insurable loss.

In our model, insolvency arises endogenously from the interaction of the insurance premium, the indemnity schedule, and the insurer's assets. This represents a departure from the extant literature, where default is typically triggered exogenously and can therefore be seen as a special form of background risk. Default endogeneity allows us to better understand the trade-offs that shape optimal insurance contracts, bringing into the picture the value of the insurer's default option, which is affected in a nonlinear way by the insurer's holdings, the promised coverage level, and the premium size. The main result of the paper is an explicit characterization of these trade-offs for different forms of dependence between the risks involved. We find that in several cases optimal contracts involve retention schedules that are tent-shaped, meaning that smaller and larger risks are retained, whereas medium sized risks are largely insured. This is consistent with the empirical evidence on catastrophe reinsurance markets (Froot, 2001; Cummins and Mahul, 2003) and corporate insurance purchases (Doherty and Smith, 1993).

A more detailed summary of our findings is as follows. We prove that, in the presence of bankruptcy costs, optimal contracts entail no insurance for some realizations of the insurable loss. Whether this is in the form of a deductible or an upper limit on coverage depends on the way the insurance buyer's wealth is affected by the insurable loss. When policyholder's wealth is negatively affected by realized losses, optimal contracts contain deductibles. The intuition is that the dollar premiums saved by retaining small losses can be used as a buffer in states of the world where default risk is higher and the policyholder more vulnerable. Hence, aversion to bankruptcy costs alone may explain deductibles without requiring the presence of administrative costs (see Raviv, 1979) or background risk (see Dana and Scarsini, 2007). When the policyholder's wealth is positively dependent on insurable losses, perhaps because of hedging instruments or government guarantees, the optimal indemnity gives rise to an upper limit on coverage. The intuition is that limited liability and bankruptcy costs make insurance demand vanish for loss realizations associated with higher default probabilities. The introduction of bankruptcy costs in a setting where insurance demand is already weak may therefore explain upper limits, without requiring regulatory constraints (see Raviv, 1979; Jouini, Schachermayer and Touzi, 2008) or policyholders' limited liability (see Huberman, Mayers and Smith, 1983). Irrespective of the sign and degree of dependence between the insurance buyer's wealth and the insurable loss, we find that, whenever optimal policies involve coinsurance, the marginal demand for coverage is shaped in a nontrivial way by terms accounting for the conditional default probability and the expected loss given default. This is the case even when background risk is independent of the insurable loss. The results shed light on some counterintuitive, nonmonotonic relationships between insurance demand and default probabilities documented in Doherty and Schlesinger (1990). When we abstract from bankruptcy costs and focus on the role of limited liability, we find that insurance demand is almost entirely driven by background risk. The reason is that in this case there is no loss of premium dollars upon default, and the policyholder can fully benefit from her claim on the insurer's residual assets, as the default option is fairly priced in equilibrium.

The results have both positive and normative implications. On the positive side, the analysis may shed light on the patterns of (re)insurance purchases documented in the literature. For example, there is evidence that corporations and primary insurers tend to retain very large exposures to catastrophic events (see figure 1). Froot (2001) offers several possible explanations based on different types of frictions (including default risk) affecting both the demand and the supply side. Cummins and Mahul (2003) give an explanation based on the insured and the insurer having divergent assessments of (exogenous) default probabilities. Our findings may provide a rational explanation of low insurance demand for high layers of exposure based on counterparty default risk in an otherwise standard framework. In particular, we show how the indemnity schedules used in the analysis of Froot (2001) can be endogenized by allowing for bankruptcy risk. On the normative side, the analysis offers insights into how government intervention schemes and guarantee funds can be optimally designed to account for the credit quality of private insurance providers (e.g., Jaffee and Russell, 1997; Kunreuther and Pauly, 2006). Finally, some of our analytical results may help assist the development of (re)insurance programs in the presence of counterparty default risk (e.g., CEIOPS, 2009).

**Related literature.** The article is related to the literature on optimal risk sharing originated with Borch (1962), Arrow (1963, 1974), and Raviv (1979). Although the traditional complete market framework has been extended to different forms of market incompleteness (see Schlesinger, 2000, for an overview), insurance demand under insolvency risk has been studied in a limited number of articles.[5] Tapiero, Kahane and Jacque (1986) examine insolvency risk in the context of mutual insurance, and consider the delicate relationship between premium and default, but do not study optimal indemnity schedules. Schlesinger and Schulenburg (1987) use a three-state model to examine the effects of relative risk aversion and total default on insurance demand. Doherty and Schlesinger (1990) use a similar setting to show that full insurance is never optimal when default is total. For more general situations, they find that the relationship between risk retention levels and default probabilities can be nonmonotonic. Cummins and Mahul (2003) consider a setting where the insurer and the insurance buyer disagree on the default probability. They provide conditions under which the risk retention schedule is nondecreasing in the exposure, but their contract reduces to full insurance above a de-

[5] In a related contribution, Huberman, Mayers and Smith (1983) consider policyholders with limited liability (e.g., government intervention). Our analysis encompasses their setup, although in reduced form; see section 3.

ductible as soon as disagreement is assumed away. Richter (2003) examines the trade-off between default risk and basis risk for a primary insurer deciding between reinsurance and hedging instruments. As opposed to the above contributions, we consider default as being endogenously determined by the optimal contract and the performance of the the insurer's assets. We further allow for randomness in the insurance buyer's wealth, as its dependence on default risk plays an important role in shaping optimal insurance decisions. The work on background risk by Dana and Scarsini (2007) is therefore very relevant for our study. They do not study limited liability and bankruptcy costs, but use concepts of positive dependence that prove to be essential for our characterization of optimal contracts under counterparty default risk.

The rest of the paper is organized as follows. The next section introduces the model setup, whereas section 3 recaps three useful concepts of positive dependence between random variables: association, stochastic increasingness, affiliation. In section 4 we study optimal indemnity schedules for given premium, distinguishing between when bankruptcy costs are present and absent. Section 5 extends the analysis to optimal insurance contracts, optimizing over both the premium level and the indemnity schedule. We provide conditions that are both necessary and sufficient for optimality under default risk, and discuss the existence of optimal contracts. Section 6 offers some concluding remarks. All proofs are collected in the appendix, together with additional details and results.

## 2 Model setup

We consider a one-period model with two agents, a risk-averse insurance buyer and a risk-neutral insurer. The insurance buyer has utility function $u$ and end-of-period wealth $W - X$, with $X$ representing random losses that can be insured and $W \geq 0$ capturing other variations in wealth. We assume that $u$ is twice continuously differentiable, with

$u' > 0$ and $u'' < 0$, and that $X$ takes values in the compact interval $[0, \overline{x}]$. The exposure $X$ is insurable, in the sense that an indemnity schedule can be purchased from the insurer by payment of a premium $P \geq 0$. By indemnity we mean a function $I(\cdot) : [0, \overline{x}] \to \mathbb{R}_+$ mapping the coverage level associated with each loss realization. We require the indemnity to cover at most the realized loss, i.e., to satisfy $I \leq Id$, with $Id$ the identity function. The insurer has terminal wealth $A + P$, the random variable $A \geq 0$ representing the insurer's capital at the end of the period.[6] To obtain neater result, we do not allow for discounting[7] and require the triple $(A, W, X)$ to have a density[8] positive on $\mathbb{R}^2_{++} \times (0, \overline{x}]$ and null elsewhere. We further require the joint density to be continuous on $\mathbb{R} \times \mathbb{R}_{++} \times [0, \overline{x}]$. We denote its marginals by $f_A, f_W$ and $f_X$, and the conditional density of $(A, W)$, given $X = x$, by $f(\cdot|x)$. We will assume these densities to be differentiable whenever the analysis requires it.

The insurer defaults on its obligations if total assets, $A + P$, are insufficient to cover the promised indemnity, $I(X)$. We assume that on the event $\{A + P < I(X)\}$ only a given fraction $\gamma$ (with $0 \leq \gamma \leq 1$) of the insurer's assets can be recovered, making the insurance buyer's terminal wealth drop from the amount $\widetilde{W} := W - P - X + I(X)$ to the level $\widetilde{W}(\gamma) := W - P - X + (A + P)\gamma \leq \widetilde{W}$. The parameter $\gamma$ captures deadweight bankruptcy costs; the extension to non-fractional recovery rules, covered in the appendix, yields similar results.

An insurance contract is identified by a pair $(P, I)$. By limited liability, it yields the insurer an expected profit equal to $V(P, I) := E(\max\{\widetilde{A}, 0\})$, with $\widetilde{A} := A + P - I(X)$.

[6] Alternatively, we may allow $A$ to be $\mathbb{R}$-valued, so that $A$ would represent the insurer's net assets and our risk-sharing problem would cover the sale of one extra policy to a risk-averse insurance buyer.

[7] An alternative formulation would have the insurer hold initial capital $a > 0$, which is invested with the premium $P$ to earn a random return $R$, yielding terminal wealth of amount $(a + P)(1 + R)$. The insurance buyer's wealth could be modeled along the same lines. Our formulation yields equivalent results.

[8] The point is to avoid using nonsmooth analysis techniques. The essential assumption is that the conditional law of $A$, given $X = x$, is continuous (and if its support is bounded above, that the upper bound is greater than $x$). Also, the requirement that $W$ admits a density is for notational simplicity only (for example, one can verify that the results for independent $W$ and $(A, X)$ appearing in the next sections also cover the case of deterministic $W$). Similarly, we may assume $X$ to have a mass at zero at the expense of more involved notation.

On the other hand, the insurance buyer derives expected utility

$$U(P, I) := E\left(u\left(1_{\widetilde{A}\geq 0}\widetilde{W} + 1_{\widetilde{A}<0}\widetilde{W}(\gamma)\right)\right). \tag{2.1}$$

With no bankruptcy costs, the policyholder recovers the entire amount $A + P$ upon default, and the above simplifies to

$$U(P, I) = E\left(u(W - X - P + \min\{I(X), A + P\})\right), \tag{2.2}$$

which is clearly monotone in both $P$ and $I$, a property that is not satisfied by (2.1) in general.

The default event is endogenous in this setting, as solvency of the insurer not only depends on the pair $(A, X)$, but also on the insurance contract $(P, I)$ itself. The contract may fail to perform and hit the policyholder in three different ways: first, through a reduction in the indemnity payment, which may drop to $(A + P)\gamma \leq I(X)$; second, through the loss of premium dollars in the default state; third, because default may occur in states of the world when the policyholder is more vulnerable (low realizations of $W$).

A contract $(P, I)$ strictly dominates another contract $(\hat{P}, \hat{I})$ if $U(P, I) \geq U(\hat{P}, \hat{I})$ and $V(P, I) \geq V(\hat{P}, \hat{I})$, with at least one inequality strict. A contract is Pareto optimal if it is not strictly dominated. The condition $V(P, I) \geq V(0, 0) = E(A) > 0$ is necessary for the insurer to offer $(P, I)$, whereas $U(P, I) \geq U(0, 0) = E(u(W - X))$ is necessary for the insurance buyer to consider entering the contract. In what follows, we focus on Pareto optimal contracts ensuring that these participation constraints are satisfied.

Optimal contracts can be characterized by examining the solutions of the following problem for different levels of the insurer's reservation profit $\underline{v}$:

$$\begin{cases} \sup_{(P,I)\in\ \mathbb{R}_+\times\mathcal{A}} & U(P,I) \\ \text{subject to:} & V(P,I) \geq \underline{v} \\ & 0 \leq I(x) \leq x \text{ for all } x \in [0,\overline{x}], \end{cases} \tag{2.3}$$

where $\mathcal{A}$ denotes the set measurable functions defined on $[0,\overline{x}]$. We assume that the pair $(A,W)$ is integrable, so that the problem is well posed. It is immediate to see that condition $\underline{v} \geq E(A)$ ensures that the contract is acceptable for the insurer. The insurance buyer's participation constraint is less straightforward and will be examined in section 5. Although Pareto optimal contracts are certainly solutions to problem (2.3), it is in general not obvious whether a solution to (2.3) is also Pareto optimal. In case of no bankruptcy costs ($\gamma = 1$), the equivalence is immediate, due to the monotone behavior of the objective functional (2.2). The general case of $\gamma \in [0,1]$ is more challenging, and will be addressed in Proposition 5.3 below.

To understand the role of default endogeneity, it is useful to rewrite the insurer's participation constraint as

$$P \geq E(I(X)) + (\underline{v} - E(A)) - E(\max\{I(X) - (A+P), 0\}). \tag{2.4}$$

The right-hand side includes the actuarial value of the indemnity, $E(I(X))$, the insurer's required excess return on capital, $\underline{v} - E(A)$, and the cost (to the insurance buyer) of the insurer's option to default, $E(\min\{A+P-I(X),0\})$. Note that an increase in the coverage level has the effect of increasing both the actuarial value of the promised indemnity and the value (to the insurer) of the default option; hence limited liability introduces a complex interaction between the optimal premium and the optimal indemnity schedule. As bankruptcy costs do not explicitly appear on the right-hand side of (2.4), it may be tempting to conclude that, for some insurance coverage to be optimal, the insurance buyer would have to be sufficiently risk averse to enter the contract and bear both the expected bankruptcy costs and the extra loading $\underline{v} - E(A)$. We will prove in section 5, however, that zero coverage (no insurance) is optimal only in degenerate cases, and that constraint (2.4) is always binding. The reason is that the insurer's participation constraint can fully internalize the expected bankruptcy costs via the optimal indemnity schedule.

We develop the analysis of optimal insurance contracts in two steps. We first consider agents that are price takers, and determine optimal indemnity schedules for fixed premium level $P \geq 0$ (section 4). We then undertake the more difficult task of jointly optimizing with respect to the indemnity schedule and the premium level (section 5). The issue of existence of optimizers is addressed separately for the two cases. Before proceeding, we introduce tools to compare different risky prospects.

## 3 Concepts of dependence

The results derived in the next sections apply to a large class of risks and decision makers. To limit parametric assumptions on the utility and distribution functions, we rely on the following notions of dependence, which impose increasingly strong restrictions on the dependence structure of a random vector (e.g., Müller and Stoyan, 2002). In the following, a real valued function defined on $\mathbb{R}^n$ is said to be monotone if it is monotone in each of its arguments. Inequalities between random variables are in the almost sure sense.

**Definition 3.1.** *A random vector $Y$ is* **associated**, *if* $\mathrm{Cov}(g(Y), k(Y)) \geq 0$ *for all pair of nondecreasing functions $g$ and $k$ such that the covariance exists. A random vector $Y$ is* **conditionally associated**, *given a $\sigma$-field $\mathcal{G}$, if* $\mathrm{Cov}(g(Y), k(Y)|\mathcal{G}) \geq 0$ *for all pairs of nondecreasing functions $g$ and $k$.*

We say that a random vector $Y$ is stochastically increasing in another vector $Z$ if the conditional distribution of $Y$ becomes larger (in the sense of first order stochastic dominance) when conditioning on higher values of $Z$.

**Definition 3.2.** *A random vector $Y$ is* **stochastically increasing** *(strictly stochastically increasing) in another random vector $Z$, denoted $Y \uparrow_{st} Z$ ($Y \uparrow_{sst} Z$), if the map $z \to E\left(g(Y)|Z = z\right)$ is nondecreasing (increasing) for every nondecreasing (increasing) function $g$ such that the expectation of $g(Y)$ exists.*

We write $Y \downarrow_{st} Z$ and $Y \downarrow_{sst} Z$ for the stochastic increasingness relations $Y \uparrow_{st} -Z$ and $Y \uparrow_{sst} -Z$.

A further characterization of the idea that high values of some of the components of a random vector make the other components more likely to be high than small is given by the concept of affiliation (or monotone likelihood ratio property; see Milgrom and Weber, 1982).

**Definition 3.3.** *A random vector $Y = (Y_1, \ldots, Y_n)$ with joint density $f$ is* **affiliated** *if $f$ is log-supermodular, i.e., if it satisfies*

$$f(y \wedge z)f(y \vee z) \geq f(y)f(z) \quad \text{for all } y, z \in \mathbb{R}^n, \tag{3.1}$$

*where $y \wedge z$ and $y \vee z$ denote the componentwise minimum and maximum of $y$ and $z$.*

The intuition behind (3.1) is that compound returns in one variable are nondecreasing in each other variable (e.g., Topkis, 1998; Athey, 2002). When $f$ is positive and twice continuously differentiable, an equivalent characterization of (3.1) can be given in terms of second order derivatives:

$$\frac{\partial^2}{\partial y_i \partial y_j} \ln f(y) \geq 0 \quad \text{for all } y \in \mathbb{R}^n, 1 \leq i \neq j \leq n.$$

Affiliation implies stochastic increasingness (of each component with respect to the others), which in turn implies association. An important implication of affiliation is that the semielasticity of the conditional density $f(y_{-i}|y_i)$ with respect to any component $y_i$ is nondecreasing in each other component of $y$, denoted by $y_{-i}$; see the appendix. Defined as $\ln_{y_i} f(y_{-i}|y_i) := \partial/\partial y_i \ln f(y_{-i}|y_i)$, the semielasticity measures the approximate percentage change in $f(\cdot|y_i)$, given a unit increase in the $i$-th component of $y$.

For the risk sharing problem at hand, the following situations are of particular interest (see Dana and Scarsini, 2007, for several other examples):

- *$A$ and $W$ are conditionally independent, given $X$* (denoted $A \perp_X W$). This means that once the insurable loss is taken into account, the residual randomness affecting the agents' holdings is idiosyncratic. An example is represented by catastrophic risks, which may impact the overall economy and introduce dependence between otherwise uncorrelated endowments.

- *$A \uparrow_{st} X$ and $W \uparrow_{st} X$.* High losses are likely to make both the insurance buyer and the insurer better off. This may be the case when the insurer is invested in hedging instruments (e.g., industry loss warranties) or has access to post-loss financing tools (e.g., contingent surplus notes, loss equity puts). On the policyholder side, this may be the case when the insurance buyer holds (say) credit default swaps, or when the insured event is catastrophic enough to trigger government intervention (e.g., Huberman *et al.*, 1983).

- *$A \downarrow_{st} X$ and $W \downarrow_{st} X$.* This represents situations where large losses are more likely to negatively affect both agents' holdings, as for catastrophic risks.

- *$W - X \downarrow_{sst} X$ or $W - X \uparrow_{st} X$.* The first case may capture contagion between the insurable loss and any uninsurable exposure included in $W$. The second case may reflect a situation where the policyholder is overinvested in hedging instruments or can rely on government guarantees.

- *$(A, W, -X)$ is affiliated.* This implies that both $(A, -X)$ and $(W, -X)$ are affiliated, as well as the stochastic decreasingness relations $A \downarrow_{st} X$, $W \downarrow_{st} X$. A further implication is that $(A, W)$ is affiliated, reflecting the common dependence of agents' holdings on (say) macroeconomic conditions.

## 4 Optimal insurance policy schedules for given premium

Following Raviv (1979), it is convenient to see (2.3) as an optimal control problem where the indemnity schedule is the control and the insurer's expected return is the state variable. For fixed premium level $P \geq 0$, we can use the law of iterated expectations to write problem (2.3) as

$$\begin{cases} \sup_{I \in \mathcal{A}} & \int_0^{\overline{x}} E^x \left( 1_{\widetilde{A} \geq 0} u(\widetilde{W}) + 1_{\widetilde{A} < 0} u(\widetilde{W}(\gamma)) \right) f_X(x) \mathrm{d}x \\ \text{subject to} & \int_0^{\overline{x}} E^x(\max\{\widetilde{A}, 0\}) f_X(x) \mathrm{d}x \geq \underline{v} \\ & 0 \leq I(x) \leq x \text{ for all } x \in [0, \overline{x}], \end{cases} \tag{4.1}$$

where $E^x(\cdot)$ denotes the conditional expectation operator $E(\cdot|X = x)$; we write $\mathbb{P}^x(\cdot)$ for the conditional probability $\mathbb{P}(\cdot|X = x)$. The problem is feasible for reservation utility levels satisfying $E(A) \leq \underline{v} \leq E(A) + P$. Although the usual concavity requirements are not satisfied by (4.1), the problem always admits a solution (see Neustadt, 1963; Cesari, 1983). We write the Lagrangian as

$$L(x, I(x), \lambda_0, \lambda_1(x), \lambda_2(x); P) := H(x, I(x), \lambda_0; P) + \lambda_1(x)I(x) + \lambda_2(x)(x - I(x)),$$

where the function $H$ (the Hamiltonian) is defined as

$$H(x, I(x), \lambda_0; P) := E^x \left( 1_{\widetilde{A} \geq 0} u(\widetilde{W}) + 1_{\widetilde{A} < 0} u(\widetilde{W}(\gamma)) \right) f_X(x) + \lambda_0 E^x(\max\{\widetilde{A}, 0\}) f_X(x),$$

with the multipliers $\lambda_0, \lambda_1, \lambda_2 \geq 0$ satisfying the complementarity conditions $\lambda_0(V(P, I^*) - \underline{v}) = 0$, $\lambda_1(x)I(x) = 0$ and $\lambda_2(x)(x - I(x)) = 0$ for all $x \in [0, \overline{x}]$. If $I^*$ is a solution of problem (4.1), $I^*(x)$ maximizes $H(x,\cdot,\lambda_0;P)$ over $[0,x]$ for every $x\in[0,\overline{x}]$.

Default risk means that the policyholder may not receive the promised indemnity schedule, $I(X)$, and may recover only a fraction of the insurer's residual assets, $(A+P)\gamma$. The utilities derived in the 'no-default state' and 'default state' differ by $\Delta u(\gamma,I):=u(\widetilde{W})-u(\widetilde{W}(\gamma))$. For each loss realization $x$, the expected utility loss incurred at the default boundary when default occurs can be written as[9]

$$E^x(\Delta u(\gamma,I)|\widetilde{A}=0)=\lim_{a\uparrow 0}E^x(\Delta u(\gamma,I)|\widetilde{A}=a)$$
$$=\int_{\mathbb{R}_+}[u(w-P-x+I(x))-u(w-P-x+\gamma I(x))]\,f_W(w|A=I(x)-P,X=x)\mathrm{d}w,$$

is nonincreasing in $\gamma$ and bounded below by $E^x(\Delta u(1,I)|\widetilde{A}=0)=0$ for all admissible indemnity schedules. The first order necessary conditions can then be characterized by the following functions:

$$\overline{J}(x)\;:=\;E^x(u'(W-P-x)-\lambda_0), \tag{4.2}$$

$$\begin{aligned}\underline{J}(x)\;:=\;&\mathbb{P}^x(A\geq x-P)E^x\left(u'(W-P)-\lambda_0\big|A\geq x-P\right)\\&-f_A(x-P|x)E^x\left(\Delta u(\gamma,Id)\big|A=x-P\right),\end{aligned} \tag{4.3}$$

$$\begin{aligned}K(x)\;:=\;&\mathbb{P}^x(A^*\geq 0)E^x\left(u'(W^*)-\lambda_0\big|A^*\geq 0\right)\\&-f_A(I^*(x)-P|x)E^x\left(\Delta u(\gamma,I^*)\big|A^*=0\right),\end{aligned} \tag{4.4}$$

where the terms $W^*,W^*(\gamma),A^*$ denote the random variables $\widetilde{W},\widetilde{W}(\gamma)$ and $\widetilde{A}$ computed at the optimal schedule $I^*$. The first two functions quantify the variational trade-off between insurance supply and demand when the constraints on the indemnity schedule are binding ($I^*(x)=0$ or $I^*(x)=x$), the last function when the optimal indemnity is interior. The functions $\overline{J},\underline{J}$ are continuous and coincide with $K$ in zero. Partitioning the

[9] Note that the expression is not defined on $\{I(x)<P\}$. In the following, we often consider expectations conditional on $\{\widetilde{A}=0\}$, but they are always pre-multiplied by $f_A(I(x)-P|x)$, which clearly vanishes on $\{I(x)<P\}$.

interval $(0,\overline{x}]$ into the subsets $\mathcal{X}_1 := \{x \in (0,\overline{x}] \ : 0 < I^*(x) < x\}$, $\mathcal{X}_2 := \{x \in (0,\overline{x}] \ : I^*(x) = x\}$, $\mathcal{X}_3 := \{x \in (0,\overline{x}] \ : \ I^*(x) = 0\}$, we can summarize the necessary conditions as follows:

**Proposition 4.1.** *If $I^*$ is a solution of Problem* (4.1)*, then the following implications are satisfied for all $x \in (0,\overline{x}]$: if $x \in \mathcal{X}_1$, then $K(x) = 0$; if $x \in \mathcal{X}_2$, then $\underline{J}(x) \geq 0$; if $x \in \mathcal{X}_3$, then $\overline{J}(x) \leq 0$.*

The conditions of Proposition 4.1 are only necessary for maximality. They would be sufficient if, for example, the Hamiltonian were concave in $I(x)$ for all $x$, but this is not the case in general, even when $\gamma = 1$. On the other hand, the maximized Hamiltonian does not depend on the state variable (an Arrow-type sufficient condition; see Seierstad and Sydsaeter, 1987, p. 107) and hence a schedule satisfying the necessary conditions will indeed be optimal. The situation is more complicated when the premium is no longer a fixed parameter; see Theorem 5.8 below.

In the traditional case of no background risk (deterministic $A$ and $W$) and no default risk, both $\overline{J}$ and $\underline{J}$ are monotone and may cross zero at most once; see Raviv (1979). When the pair $(A, W)$ is random, but no default is considered, the behavior of $\overline{J}, \underline{J}$ can be understood by imposing restrictions on the dependence structure of the triple $(A, W, X)$; see Dana and Scarsini (2007). In our setting, $\overline{J}$ and $\underline{J}$ are in general nonmonotone, may cross zero several times, and may be difficult to order. In particular, the implications in Lemma 4.1 cannot be inverted unless additional assumptions are made.[10]

Before considering separately the cases of $\gamma = 1$ and $0 \leq \gamma < 1$, we recall some definitions of common indemnity schedules.

**Definition 4.2.** *For each $\hat{x} \in (0,\overline{x}]$, if there exists $\delta > 0$ such that, for all $\varepsilon \in (0,\delta)$,*

*i) $\hat{x} \in \mathcal{X}_3$ and $\hat{x} + \varepsilon \in \mathcal{X}_1$, then $\hat{x}$ is a* **generalized deductible** *(which is called* **standard deductible** *if $I^*(\hat{x} + \varepsilon) = \varepsilon$);*

[10] Proposition A.3 in the appendix offers partial results in this direction.

*ii)* $\hat{x} \in \mathcal{X}_3$ *and* $\hat{x}+\varepsilon \in \mathcal{X}_1$*, and* $Id - I^*$ *is nonincreasing on* $[\hat{x}, \overline{x}]$*, then* $\hat{x}$ *is a* **disappearing deductible***;*

*iii)* $\hat{x}-\varepsilon \in \mathcal{X}_1$ *and* $\hat{x} \in \mathcal{X}_3$*, then* $\hat{x}$ *represents an* **upper limit on coverage***;*

*iv)* $\hat{x} \in \mathcal{X}_2$ *and* $\hat{x}+\varepsilon \in \mathcal{X}_1$*, then* $\hat{x}$ *represents an* **upper limit on full coverage***.*

We use the terms 'deductible' and 'upper limit' locally, rather than for a contract as a whole, due to the diverse indemnity schedules our setup may give rise to. Figure 2 depicts an example with both a disappearing deductible and an upper limit on full coverage.

## 4.1 No bankruptcy costs

Setting $\gamma = 1$, in this section we abstract from bankruptcy costs and emphasize the role of limited liability. We first characterize interior policy schedules under a differentiability assumption.

**Proposition 4.3.** *Let* $\gamma = 1$ *and consider a schedule* $I^*$ *differentiable on* $\mathcal{X}_1$*. If* $I^*$ *is optimal, it satisfies the differential equation*

$$I^{*\prime}(x) = 1 - \frac{\delta_1(x) - h(x)\delta_2(x)}{\delta_0(x) - h(x)\delta_2(x)}, \qquad x \in \mathcal{X}_1, \tag{4.5}$$

*where the function* $h(x)$ *is defined in* (4.6) *below, and the terms* $\delta_0, \delta_1, \delta_2$ *are given by*[11]

$$\delta_0(x) := E^x(u''(W^*)|A^* \geq 0)$$

$$\delta_1(x) := \mathrm{Cov}^x(u'(W^*), \ln_x f(A, W|x)\big|A^* \geq 0)$$

$$\delta_2(x) := E^x(u'(W^*)|A^* = 0) - E^x(u'(W^*)|A^* \geq 0),$$

*with* $\ln_x f(\cdot|x)$ *denoting the semielasticity of* $f(\cdot|x)$ *with respect to* $x$*.*

[11]Given the random variables $X, Y, Z$, and the event $H$, we use the notation $\mathrm{Cov}^x(Y, Z|H)$ for the conditional covariance of $Y$ and $Z$, given $X$ and $H$, computed at $X = x$.

The result shows that the marginal demand for coverage is shaped by terms accounting for the dependence structure of $(A, W, X)$ and the insurance buyer's risk preferences, exactly as in the case of background risk (e.g., Gollier, 1996; Dana and Scarsini, 2007). However, all these terms are conditional on no default occurring, and there is a specific component, $h(x)\delta_2(x)$, stemming from limited liability. We have that $\delta_2$ quantifies the expected loss in marginal utility incurred by crossing the default boundary, whereas the function $h$ is essentially a hazard rate, defined as

$$h(x) := \frac{f_A(I^*(x) - P|x)}{\mathbb{P}^x(A^* \geq 0)} = \lim_{\epsilon \downarrow 0} \frac{\mathbb{P}^x(A^* \leq \epsilon | A^* \geq 0)}{\epsilon}. \tag{4.6}$$

Under the conditional independence assumption $A \perp_X W$, the term $\delta_2$ is null, and both $\delta_0$ and $\delta_1$ no longer depend on the event $\{A^* \geq 0\}$. In particular, we have $\delta_0(x) = E^x(u''(W^*))$ and $\delta_1(x) = \mathrm{Cov}^x(u'(W^*), \ln_x f_W(W|x))$. This suggests that limited liability may affect indemnity schedules the most when the dependence between $A$ and $W$ is not channeled by the insurable loss alone. On the other hand, conditional independence allows us to provide a number of sharper results, as we now show. First, we note that the lack of concavity of the Hamiltonian means that condition (4.5) is only necessary for local extrema. To focus on local maxima, we need to impose the second order condition $(\partial^2 H/\partial I(x)^2)|_{I=I^*} = \delta_0(x) - h(x)\delta_2(x) \leq 0$, which is always satisfied (strictly) under the assumption $A \perp_X W$. Second, we can invert the implications of Proposition 4.1, and thus obtain necessary and sufficient conditions for optimal policies directly based on comparison of the functions $\underline{J}$ and $\overline{J}$, exactly as in Raviv (1979).

**Theorem 4.4.** *Assume that $A \perp_X W$ and $\gamma = 1$. If $I^*$ is an optimal solution to problem* (4.1) *then, for all $x \in (0, \overline{x}]$, one of the following exhaustive and mutually exclusive conditions is satisfied: $x \in \mathcal{X}_1$ if and only if $\underline{J}(x) < 0 < \overline{J}(x)$; $x \in \mathcal{X}_2$ if and only if $\underline{J}(x) \geq 0$; $x \in \mathcal{X}_3$ if and only if $\overline{J}(x) \leq 0$.*

Theorem 4.4 can be used to characterize optimal contracts as follows. Denote by

$x_i$ (with $i = 1, 2, \ldots$) the $i$-th positive zero of either $\overline{J}$ or $\underline{J}$ (in the two cases we write $x_i = \overline{x}_j$ or $x_i = \underline{x}_k$, with $j, k \leq i$), with the convention that $x_i = +\infty$ means that there are fewer than $i$ zeros. Each $x_i$ gives then rise to changes in the indemnity schedule that can be classified according to Definition 4.2. An illustration is given in Figure 2, which depicts the behavior of the functions $\underline{J}$ and $\overline{J}$, as well as the optimal policy schedule $I^*$. The optimal contract involves a deductible given by the first zero of $\overline{J}$, $x_1 = \overline{x}_1$, which is followed by coinsurance until the first zero of $\underline{J}$, $x_2 = \underline{x}_1$. Full insurance then follows, but the second zero of $\underline{J}$, $x_3 = \underline{x}_2$, gives rise to an upper limit on full coverage. (If we had $x_3 \geq \overline{x}$, the contract would be a disappearing deductible.) In this example, the fraction of exposure that is insured, $I^*(x)/x$, is tent-shaped: small losses are retained, medium sized losses are substantially insured, whereas a significant fraction of large losses is retained (the retention schedule $Id - I^*$ is increasing from $x_3$ onwards).

## 4.2 Bankruptcy costs

The introduction of bankruptcy costs adds a range of interesting features to optimal indemnity schedules. We can see it immediately by examining interior differentiable solutions.

**Proposition 4.5.** *For fixed* $0 \leq \gamma < 1$*, consider a schedule* $I^*$ *differentiable on* $\mathcal{X}_1$*. If* $I^*$ *is optimal, it satisfies the differential equation*

$$I^{*\prime}(x) = \frac{\delta_0(x) - \delta_1(x) - h(x)(\delta_5(x,\gamma) - \delta_6(x,\gamma) + \delta_7(x,\gamma))}{\delta_0(x) - h(x)\left(\delta_2(x) + \delta_3(x,\gamma) + \delta_4(x,\gamma)\right)}, \quad x \in \mathcal{X}_1, \tag{4.7}$$

*with the terms* $\delta_3, \delta_4, \delta_5, \delta_6, \delta_7$ *admitting the explicit expressions* (B.2)-(B.6) *reported in the appendix.*

Apart from the case of $x \in \mathcal{X}_1 \cap [0, P]$, which yields $I^{*\prime}(x) = 1 - \delta_1(x)/\delta_0(x)$, the marginal coverage level is shaped by six more terms than when bankruptcy costs are absent (compare with Proposition 4.3), demonstrating the complexity of trade-offs solved by the optimal contract. This provides a rational explanation for some of the apparently counterintuitive results documented in Doherty and Schlesinger (1990), who could not identify a monotone relationship between insurance demand and the probability of default.

As in the case with no bankruptcy costs, the differential equation (4.7) only identifies local extrema for the Hamiltonian. Writing the second order condition, we obtain that local maxima must in addition satisfy the following inequality on $\mathcal{X}_1$:

$$\delta_0(x) - f_A(I^*(x) - P|x)\delta_3(x,\gamma) - h(x)\left(\delta_2(x) + \delta_4(x,\gamma) + h(x)\right) \leq 0. \tag{4.8}$$

The stronger condition that $A \perp_X W$ and $\ln_a f(I^*(x) - P|x) \geq -h(x)$ for all $x \in \mathcal{X}_1$, a property used in the next section for the sufficiency of the first-order conditions, ensures that the second-order condition (4.8) is automatically satisfied.

From expression (4.7), we can understand the salient features of interior indemnity schedules by elaborating on some special cases.

**Proposition 4.6.** *Let $I^*$ be a solution of problem* (4.1)*, differentiable on $\mathcal{X}_1$. For all $x \in \mathcal{X}_1$ we have:*

*i) If $A, W, X$ are independent, then*

$$I^{*\prime}(x) = \frac{\delta_0(x) - h(x)\delta_5(x,\gamma)}{\delta_0(x) - h(x)(\delta_3(x,\gamma) + \delta_4(x,\gamma))}. \tag{4.9}$$

*ii) If $\gamma = 0$, $W \perp (A, X)$, and $(A, -X)$ is affiliated, then $Id - I^*$ is nondecreasing.*

Although it is in general difficult to characterize the optimal risk retention schedule, statement *(ii)* shows that it is nondecreasing whenever there is total default and the insurer's holdings are negatively dependent on the insurable loss (in the sense of affiliation). Point *(i)* demonstrates that bankruptcy costs are incompatible with straight deductibles and constant coinsurance rates, even when background risk is independent of the insurable loss; compare with Proposition 4.3. Shutting off the background risk channel is important to isolate the impact of default endogeneity on coinsurance rates: the result should be contrasted with Cummins and Mahul (2003, Corollary 2), who obtain optimality of full insurance above a deductible when default is exogenous and there is no recovery.

Note that with bankruptcy costs the characterization of optimal indemnities provided by Theorem 4.4 no longer applies, and the optimality conditions do not admit an immediate interpretation in terms of the functions $\underline{J}$ and $\overline{J}$. The main issue is to compare the functions $K$ and $\underline{J}$ when $\overline{J} > 0$: the behavior of $f_A(I^*(x) - P|x)E^x(\Delta u(\gamma, I^*)|A^* = 0)$ is difficult to characterize, unless explicit assumptions are made on the utility function and the conditional law of $A$. In the following section, we will nonetheless derive a number of results without imposing further restrictions beyond conditional independence.

# 5 Pareto optimal policies

In this section we derive contracts that are optimal with respect to both the premium level and the indemnity schedule. Although the standard concavity assumptions ensuring existence of optimizers do not apply to this case, one can show that $U^x(P, I)$ belongs to a class of integrands satisfying a Cesari-type growth condition (see Zaslavski, 2001). We can then invoke Theorem 1.2 in Zaslavski (2001), and claim that in this class there exists a dense subset of integrands for which our problem has a unique solution, and this solution is stable under small perturbations of the integrand. In the following, we assume the integrand $U^x(P, I)$ to belong to this subset.

As is customary in control problems depending on a parameter, it is convenient to see the premium as an additional state variable. This means that the previous analysis still applies, but one more optimality condition needs to be considered: an optimal contract

$(P^*, I^*)$ must also satisfy the restriction (e.g., Seierstad and Sydsaeter, 1987)

$$\int_0^{\overline{x}} \frac{\partial}{\partial P} L(x, I^*(x), \lambda_0, \lambda_1(x), \lambda_2(x); P)\Big|_{P=P^*} \mathrm{d}x \leq 0. \tag{5.1}$$

The inequality reduces to an equality when $P^* > 0$. A calculation developed in the appendix shows that in the latter case we have

$$\int_{\mathcal{X}_2} \underline{J}(x) f_X(x)\mathrm{d}x + (1-\gamma)E\left(u'(W^*(\gamma))1_{A^*<0}\right) = \int_{\mathcal{X}_3} (-\overline{J}(x)) f_X(x)\mathrm{d}x, \tag{5.2}$$

where $(1-\gamma)E\left(u'(W^*(\gamma))1_{A^*<0}\right)$ is nonnegative, and equal to zero if and only if $\gamma = 1$ or $I^*(x) \leq P^*$ for all $x$. Note that in this case the optimal policy does not satisfy the insurance buyer's participation constraint, as one can show that $U(P, I) < U(0, 0)$ for all insurance contracts with coverage bounded by the (positive) premium.

Recalling that $\underline{J}(x) \geq 0$ on $\mathcal{X}_2$, and $\overline{J}(x) \leq 0$ on $\mathcal{X}_3$, from (5.2) we see that the no insurance region is used to balance the provision of full insurance and the expected loss of marginal utility arising from bankruptcy costs. This allows us to make the following general statement.

**Proposition 5.1.** *When $0 \leq \gamma < 1$, any optimal contract must provide no insurance on a set of positive measure.*

Hence bankruptcy costs are incompatible with full insurance, with coinsurance, and with indemnity schedules that offer a mixture of full and partial coverage against loss realizations. In other words, insurance contracts must provide deductibles and/or upper limits on coverage to be optimal.

In the absence of bankruptcy costs, the impact of limited liability is less dramatic and the joint provision of full insurance and no insurance is essentially driven by background risk. As the following proposition shows, the dependence between the insurance buyer's wealth and the insurable loss is particularly relevant in this context.

**Proposition 5.2.** *Let $\gamma = 1$ and assume that either $W \downarrow_{st} X$ or $W - X \uparrow_{sst} X$ ($W \downarrow_{sst} X$ or $W \uparrow_{sst} X$) holds. Then any optimal contract offering no insurance (full insurance) on a set of positive measure must offer full insurance (no insurance) on a nonnegligible set.*

The previous results allow us to finally establish the Pareto optimality of solutions to problem (2.3). As the following proposition shows, the insurer's participation constraint is binding at an optimum and hence the compelling premium representation (2.4) applies.

**Proposition 5.3.** *For fixed $\underline{v} \geq E(A)$, let $(P^*, I^*)$ solve problem* (2.3). *Then, the optimal premium $P^*$ is related to the optimal schedule $I^*$ by*

$$P^* = E(I^*(X)) + (\underline{v} - E(A)) - E(\max\{I^*(X) - (A + P^*), 0\}), \tag{5.3}$$

*and the pair $(P^*, I^*)$ is Pareto optimal.*

As was discussed in section 2, the fact that bankruptcy costs do not explicitly appear in expression (5.3) may suggest that the insurance buyer may not have an incentive to trade if risk aversion is not large enough to offset both the expected bankruptcy costs and the insurer's required return on capital. It turns out, however, that the insurer's participation constraint fully internalizes both aspects, although indirectly, via the optimal indemnity schedule, and no insurance can only be optimal in degenerate cases. The next proposition shows that there is always insurance, for example, when $W \downarrow_{st} X$ (which encompasses $W \perp X$) or $W - X \uparrow_{sst} X$ (which implies $W \uparrow_{sst} X$).

**Proposition 5.4.** *If no insurance is optimal, then $E^x(u'(W - x))$ is constant.*

To obtain sharper results on Pareto optimal contracts, it is convenient to work under the conditional independence assumption $A \perp_X W$, exactly as we did in section 4. The structure of efficient policies crucially depends on the way the insurance buyer's wealth is affected by the insurable loss. We focus first on the case of negative dependence and no bankruptcy costs.

**Theorem 5.5.** *Assume $A \perp_X W$ and $W \downarrow_{st} X$. If $\gamma = 1$, any optimal insurance contract is either full insurance or admits a generalized deductible followed by first coinsurance and then full insurance. We further have:*

*i) If $(W, -X)$ is affiliated, optimal contracts are disappearing deductibles.*

*ii) If $W \perp X$, full insurance is optimal.*

We note that, when there are no bankruptcy costs, background risk is the main factor shaping optimal contracts, and optimal policies are remarkably similar to those arising in the absence of limited liability (e.g., Gollier, 1996; Dana and Scarsini, 2007). The intuition is that the contract gives the policyholder a claim on the insurer's residual assets, thus allowing the risk averse agent to extend her balance sheet in case of default. As the insurer's default option is fairly priced via (5.3), and there are no premium losses upon default, insurance demand is essentially unaffected by default risk.

The situation changes when introducing bankruptcy costs. As the insurer's participation constraint does not explicitly take into account the expected bankruptcy costs, the insurance buyer needs to adjust the demand for coverage to internalize expected default costs. An immediate consequence is that insurance demand becomes weaker for large loss realizations associated with higher default risk. More importantly, there is always a positive deductible, even in the independence case. The intuition is that the insurance buyer reduces the potential loss in premium dollars by giving up coverage for low loss realizations. As a result, we have that bankruptcy costs may explain deductibles without requiring the introduction of administrative costs or even background risk. Moreover, optimal risk retention schedules can be nondecreasing, giving rise to a tent-shaped pattern in the fraction $I^*(x)/x$ of exposure insured. We prove this for the case of total default, $\gamma = 0$, which is easier to analyze.

**Theorem 5.6.** *Assume $A \perp_X W$ and $W \downarrow_{st} X$. If $0 \leq \gamma < 1$, any optimal contract involves a positive deductible. In particular, we have:*

*i) If $W \perp X$, the deductible is followed by coinsurance.*

*ii) If $W \perp X$, $\gamma = 0$, $(A, -X)$ is affiliated, and $I^*$ is differentiable on $\mathcal{X}_1$, then the risk retention schedule $Id - I^*$ is nondecreasing on $\mathcal{X}_1$.*

Finally, we consider the case when the policyholder's wealth is positively dependent on the insurable loss, which may capture (in reduced form) the limited liability constraints studied by Huberman, Mayers and Smith (1983), or may reflect the fact that the buyer is partially hedged against the insurable loss. In this case we obtain the natural result that limited liability and bankruptcy costs make insurance demand vanish for loss realizations associated with higher default probabilities. As a result, bankruptcy costs may explain upper limits in situations where insurance demand is naturally weak.

**Theorem 5.7.** *Assume $A \perp_X W$ and $W - X \uparrow_{sst} X$. Then any optimal contract involves full insurance followed by coinsurance and no insurance.*

Some numerical examples for the optimal schedule $I^*(x)$ and insured fraction $I^*(x)/x$ are presented in figures 3-4. We use exponential utility and work under the assumption of independence, with lognormal $A, W$, and $X$ distributed according to a truncated Gamma. Although optimal indemnity schedules do not present an upper limit on coverage, the indemnity flattens out for high loss realizations, tending towards a 'stop-loss contract'. Taking into account the presence of deductibles, we have therefore an example of how the insurance layers considered in Froot (2001) can be endogenized by allowing for bankruptcy costs.

We conclude this section by addressing the issue of sufficiency of the optimality conditions discussed so far. As opposed to section 4, the problem is now more challenging, as the premium level is an additional state variable and hence the optimized Hamiltonian does depend on the state. The following theorem, however, shows that an Arrow-type sufficiency result can be obtained under mild conditions.

**Theorem 5.8.** *Let $(\hat{P}, \hat{I})$ be an admissible contract, with $\hat{\lambda}_0 \geq 0$ the corresponding adjoint constant, and suppose that $H(x, \hat{I}(x), \hat{\lambda}_0; \hat{P}) = \max_{z\in[0,x]} H(x, z, \hat{\lambda}_0; \hat{P})$ for all $x \in [0, \overline{x}]$ and condition (5.1) is satisfied. Moreover, assume $A \perp_X W$ and that either of the following conditions hold:*

*i) there are no bankruptcy costs, $\gamma = 1$;*

*ii) there is total default, $\gamma = 0$, and $\ln_a f_A(\hat{I}(x) - \hat{P}|x) \geq -h(x)$ for all $x \in [0, \overline{x}]$.*

*Then, $(\hat{P}, \hat{I})$ is optimal.*

A simple calculation shows that the condition $\ln_a f_A(\hat{I}(x) - \hat{P}|x) \geq -h(x)$ is satisfied if, for all $x \in [0, \overline{x}]$, the hazard rate $f_A(a|x)/\mathbb{P}^x(A \geq a)$ is nondecreasing on the interval $[0, \max(0, \hat{I}(x) - \hat{P})]$. In other words, sufficiency can be enforced by restricting the set of admissible contracts to pairs $(P, I)$ offering coverage no higher than the premium level whenever the insurer's hazard rate is decreasing. Note that condition *(ii)* automatically ensures that the second-order condition (4.8) is satisfied.

# 6 Conclusion

In this work we have characterized optimal insurance contracts when the insurer can default on its obligations and the insurance buyer is risk-averse. We have shown how bankruptcy costs can explain the presence of deductibles and upper limits in optimal contracts, thus endogenizing the risk-sharing arrangements often considered in the literature studying catastrophe (re)insurance markets. We have also demonstrated how default risk may give rise to nonmonotonic relationships between the marginal demand for insurance and conditional default probabilities, consistently with what was pointed out in Doherty and Schlesinger (1990). The analysis is based on a static model with symmetric information. Our results are therefore just a first step towards understanding how couterparty default risk may jointly affect optimal insurance premiums and indemnity schedules in the presence of additional frictions. Extensions to include information asymmetries, for example by allowing the insurer to have superior information on the quality of her assets, seem of particular interest, but are clearly quite challenging.

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# Appendix

# A Additional discussion and results

## A.1 Further results on dependence

The proofs of the main results (appendix B) rely on some consequences of the dependence concepts introduced in section 3. We begin by elaborating on the relation between association and stochastic increasingness.

**Proposition A.1.** *Let $X$, $Y$, $Z$ be univariate random variables. Then, the following properties hold:*

*i)* *$X$ is associated conditionally on $Y$, and $(X, Y)$ is associated conditionally on $(Y, Z)$;*

*ii)* *if $X \uparrow_{st} (Y, k(Z))$ for strictly monotone function $k$, then $(X, Y)$ is associated conditionally on $Z$;*

*iii)* *If $Z \uparrow_{st} (k(X), Y)$ for strictly monotone function $k$, then for every nondecreasing function $f$ we have $E^x(f(Y, Z)|Y \geq 0) \geq E^x(f(Y, Z)|Y = 0)$ for all $x$ such that $\mathbb{P}^x(Y \geq 0) > 0$.*

*Proof.* Part *i)* follows directly from the fact that any univariate random variable is associated. To prove part *ii)*, consider nondecreasing functions $f$ and $g$. We have

$$\begin{aligned}\mathrm{Cov}(f(X, Y), g(X, Y)|Z) =& E(\mathrm{Cov}(f(X, Y), g(X, Y)|Y, Z)|Z) \\ &+ \mathrm{Cov}(E(f(X, Y)|Y, Z), E(g(X, Y)|Y, Z)|Z).\end{aligned}$$

The first term is nonnegative by *i)*. With regard to the second term, we have that $\widetilde{f}(y, z) := E(f(X, Y)|Y = y, Z = z)$ is nondecreasing in $y$, and likewise $\widetilde{g}(y, z) := E(g(X, Y)|Y = y, Z = z)$ is nondecreasing in $y$. It follows again by *i)* that

$$\mathrm{Cov}(E(f(X, Y)|Y, Z), E(g(X, Y)|Y, Z)|Z) = \mathrm{Cov}(\widetilde{f}(Y, Z), \widetilde{g}(Y, Z)|Z) \geq 0.$$

Part *iii)* follows from observing that

$$E^x(f(Y, Z)|Y \geq 0) = \frac{1}{\mathbb{P}^x(Y \geq 0)} \int_0^\infty E^x(f(y, Z)|Y = y) F_Y(\mathrm{d}y|x),$$

where $F_Y(\cdot|x)$ is the conditional law of $Y$, given $X = x$, and from the fact that the function $y \mapsto E^x(f(y, Z)|Y = y)$ is nondecreasing in $y$. □

The next proposition shows why we need to resort to the stronger notion of affiliation to study the marginal demand for insurance when it involves semielasticities.

**Proposition A.2.** *Let $X$, $Y$, $Z$ be random variables with joint density $h$, and let $h(\cdot|z)$ denote the conditional density of $(X, Y)$ given $Z = z$. Then, the following properties hold:*

*i)* *If $(X, Y, Z)$ is affiliated, the conditional density function $h(x, y|z)$ is log-supermodular.*

*ii)* *If $X$ and $Y$ are independent conditionally on $Z$, then $(X, Y, Z)$ is affiliated if and only if $(X, Z)$ and $(Y, Z)$ are affiliated.*

*iii)* *If $(X, Y, Z)$ $((X, Y, -Z))$ is affiliated, the semielasticity $\ln_z h(x, y|z)$ is nondecreasing (nonincreasing) in $x, y$.*

*Proof.* Part *i)* is obvious. Part *ii)* follows from the fact that by conditional independence we have $h(x,y,z) = h_X(x|z)h_Y(y|z)h_Z(z)$, where $h_X(\cdot|z)$ and $h_Y(\cdot|z)$ are the conditional densities of $X$ and $Y$ given $Z = z$, and $h_Z$ is the density of $Z$. Part *iii)* follows from the observation that, for example, $\frac{\partial}{\partial x} \ln_z h(x,y|z) = \frac{\partial}{\partial x \partial z} \ln h(x,y,z) \geq 0$. □

## A.2 Characterizing the behavior of $\underline{J}, \overline{J}, K$.

In section 4 we have shown that the presence of bankruptcy costs prevents us from fully characterizing optimal indemnities in terms of the zeros of the the functions $\underline{J}, \overline{J}, K$, even when working under the conditional independence assumption $A \perp_X W$. The next proposition collects a number of results that can be obtained in different cases.

**Proposition A.3.** *For $x \in (0, \bar{x}]$, let $I^*$ be a solution of Problem* (4.1)*. Then, the following properties hold:*

1. *If $\underline{J}(x) < 0 < \overline{J}(x)$, then $x \in \mathcal{X}_1$.*
2. *When $A \perp_X W$, we have: (i) if $\underline{J}(x) \geq 0$, then $\overline{J}(x) > 0$; (ii) if $\overline{J}(x) \leq 0$, then $K(x) < 0$, $\underline{J}(x) < 0$ and $x \in \mathcal{X}_3$; (iii) if $x \in \mathcal{X}_1$, then $\overline{J}(x) > 0$.*
3. *When $A \perp_X W$ and $\gamma = 1$, we have: (j) $x \in \mathcal{X}_1$ if and only if $\underline{J}(x) < 0 < \overline{J}(x)$; (jj) $x \in \mathcal{X}_2$ if and only if $\underline{J}(x) \geq 0$; (jjj) $x \in \mathcal{X}_3$ if and only if $\overline{J}(x) \leq 0$.*
4. *If $x \leq P$ then (3.j), (3.jj), and (3.jjj) always hold.*

*Proof.* Part *(1)* follows directly from Theorem 4.1. To prove part *(2)*, note first that under $A \perp_X W$ we have

$$\underline{J}(x) = \mathbb{P}^x(A \geq x - P)E^x(u'(W-P) - \lambda_0) - f_A(x - P|x)E^x(\Delta u(\gamma, Id))$$
$$K(x) = \mathbb{P}^x(A^* \geq 0)E^x(u'(W^*) - \lambda_0) - f_A(I^*(x) - P|x)E^x(\Delta u(\gamma, I^*)).$$

We also note that

$$\begin{aligned} E^x(u'(W-P) - \lambda_0) &\leq E^x(u'(W - P - x + I^*(x)) - \lambda_0) \\ &\leq E^x(u'(W-P-x) - \lambda_0) = \overline{J}(x), \end{aligned} \tag{A.1}$$

where the first inequality is strict if $I^*(x) < x$, the second one if $I^*(x) > 0$. Finally, we have $\mathbb{P}^x(A \geq z) > 0$ for all $z$. Hence *(i)*, *(ii)* and *(iii)* immediately follow. To prove part *(3.jj)*, let $\gamma = 1$ and assume $\underline{J}(x) \geq 0$. If $I^*(x) = 0$, then $\overline{J}(x) \leq 0$ but (A.1) gives $\overline{J}(x) > 0$. If $0 < I^*(x) < x$, we then have $K(x) = 0$, but (A.1) implies $\underline{J}(x) < 0$. It then follows that $I^*(x) = x$ is the only admissible case and *(jj)* is proved. Points *(j)* and *(jjj)* can be proved in a similar way.

Finally, part *(4)* follows from the fact that for $x \leq P$ we have $\underline{J}(x) \leq K(x) \leq \overline{J}(x)$, the first inequality being strict if $I^*(x) < x$, the second one if $I^*(x) > 0$. $\square$

## A.3 Beyond fractional recovery rules

The assumption of fractional recovery can be relaxed without affecting the main conclusions of the paper. Let $\Gamma$ be a smooth, nonnegative function defined on $\mathbb{R}_+$ and satisfying $\Gamma \leq Id$. We assume that on the default event $\{A + P < I(X)\}$ the insurance buyer recovers $\Gamma(A + P)$. The relevant random variables depending on the recoverable assets are redefined accordingly; for example, $W(\gamma)$ is replaced by $W(\Gamma) := W - P - X + \Gamma(A+P)$. Then all the results on optimal indemnity schedules for given premium appearing in section 4 still apply, provided the condition $\gamma = 1$ (no bankruptcy costs) is replaced with $\Gamma = Id$, and the condition $0 \leq \gamma < 1$ (bankruptcy costs) with $\Gamma \neq Id$. As for Pareto optimal contracts, condition (5.2) now takes the form

$$\int_{\mathcal{X}_2} \underline{J}(x) f_X(x)\mathrm{d}x + E\left((1 - \Gamma'(A + P^*))u'(W^*(\Gamma))1_{A^*<0}\right) = \int_{\mathcal{X}_3} (-\overline{J}(x)) f_X(x)\mathrm{d}x.$$

Finally, all the results given in section 5 still apply, provided the additional condition $\Gamma' \leq 1$ is satisfied.

# B Proofs

## B.1 Proofs for section 4

**Proposition 4.1**

*Proof.* For fixed $x \in [0, \overline{x}]$ and $P \geq 0$, the Hamiltonian can be written as

$$H(x, I(x), \lambda_0; P) = \left[U^x(P, I) + \lambda_0 V^x(P, I)\right] f_X(x)$$

where $V^x(P, I) := E^x(\max\{\widetilde{A}, 0\})$ and $U^x(P, I)$ is similarly defined. The multiplier $\lambda_0 \geq 0$ is constant, as the Hamiltonian is not state-dependent, and the participation constraint $\int_0^{\overline{x}} V^x(P, I) f_X(x)\mathrm{d}x \geq \underline{v}$ induces the transversality condition $\lambda_0(V(P, I) - \underline{v}) = 0$. The Lagrangian is then given by

$$L(x, I(x), \lambda_0, \lambda_1(x), \lambda_2(x); P) = H(x, I(x), \lambda_0; P) + \lambda_1(x)I(x) + \lambda_2(x)(x - I(x)),$$

for multipliers $\lambda_1, \lambda_2 \geq 0$ satisfying $\lambda_1(x)I(x) = 0$, $\lambda_2(x)(x - I(x)) = 0$ for all $x \in [0, \overline{x}]$. As the conditional distribution of $A$, given $X$, is continuous, we can differentiate the Lagrangian with respect to $I(x)$, to obtain

$$\begin{aligned}\frac{\partial}{\partial I}L(x, I(x), \lambda_0, \lambda_1(x), \lambda_2(x); P) =& E^x(1_{\widetilde{A}\geq 0}(u'(\widetilde{W}) - \lambda_0))f_X(x) \\ &- f_A(I(x) - P|x)E^x(\Delta u(\gamma, I)|\widetilde{A} = 0)f_X(x) + \lambda_1(x) - \lambda_2(x).\end{aligned} \tag{B.1}$$

Fix now $x \in (0, \overline{x}]$. Assume that $0 < I^*(x) < x$, so that $\lambda_1(x) = \lambda_2(x) = 0$ and the first order condition reads as $K(x) = 0$, with $K$ defined in (4.4). Next consider the case $I^*(x) = 0$, which implies $\lambda_2(x) = 0$. As $\lambda_1(x) \geq 0$ and $f_X > 0$ on $[0, \overline{x}]$, the first order condition now reads $\overline{J}(x) = -\frac{\lambda_1(x)}{f_X(x)} \leq 0$, with $\overline{J}(x)$ defined in (4.2). Finally, consider the case $I^*(x) = x$, so that $\lambda_1(x) = 0$. As $\lambda_2(x) \geq 0$, and $f_X > 0$ on $[0, \overline{x}]$, we can write the first order condition as $\underline{J}(x) = \frac{\lambda_2(x)}{f_X(x)} \geq 0$, with $\overline{J}(x)$ defined in (4.3). $\square$

**Proposition 4.1**

*Proof.* If $\underline{v} = E(A) + P$ then no insurance is the only feasible contract. Conversely, if $I^* \equiv 0$ is optimal then $V(P, I^*) = E(A) + P \geq v$. If $E(A) + P > \underline{v}$ then $\lambda_0 = 0$ and $\overline{J}(x) = E^x(u'(W - P - x)) > 0$, thus contradicting Proposition 4.1. $\square$

**Proposition 4.3**

*Proof.* When $\gamma = 1$, the function $K$ simplifies to

$$K(x) = E^x\left((u'(W^*) - \lambda_0)1_{A^*\geq 0}\right).$$

Noting that $K(x) = 0$ for all $x \in \mathcal{X}_1$, by total differentiation we get

$$I^{*\prime}(x) = \frac{f_A(I^*(x) - P|x)E^x(u'(W^*) - \lambda_0|A^* = 0) - \mathbb{P}^x(A^* \geq 0)E^x(u''(W^*)|A^* \geq 0)}{E^x((u'(W^*) - \lambda_0)\ln_x f(A, W|x)1_{A^*\geq 0}) - \mathbb{P}^x(A^* \geq 0)E^x(u''(W^*)|A^* \geq 0)},$$

where we have used the fact that $I^*$ is differentiable on $\mathcal{X}_1$. Replacing $\lambda_0$ recovered again from $K(x) = 0$, and rearranging terms, we finally obtain (4.5). $\square$

**Theorem 4.4**

*Proof.* See Proposition A.3. $\square$

**Proposition 4.5**

*Proof.* Proceeding as in Proposition 4.3, lengthy calculations yield (4.7), with the terms $\delta_3, \delta_4, \delta_5, \delta_6, \delta_7$ admitting the following explicit expressions:

$$\delta_3(x,\gamma) := \frac{E^x(\Delta u(\gamma, I^*)|A^* = 0)}{\mathbb{P}^x(A^* \geq 0)}, \tag{B.2}$$

$$\delta_4(x,\gamma) := \frac{\partial}{\partial I(x)} E^x(\Delta u(\gamma, I)|A^* = 0)\Big|_{I=I^*} \tag{B.3}$$

$$\delta_5(x,\gamma) := E^x(u'(W^*) - u'(W^*(\gamma))|A^* = 0) \tag{B.4}$$

$$\delta_6(x,\gamma) := \mathrm{Cov}^x(\Delta u(\gamma, I^*), \ln_x f(I^*(x) - P, W|x)\big|A^* = 0) \tag{B.5}$$

$$\delta_7(x,\gamma) := E^x(\Delta u(\gamma, I^*)|A^* = 0)\left[E^x(\ln_x f(A, W|x)\big|A^* \geq 0) - E^x(\ln_x f(A, W|x)|A^* = 0)\right]. \tag{B.6}$$

□

**Proposition 4.6**

*Proof.* *(i).* If $A, W, X$ are independent, we have $\delta_1 = \delta_2 = \delta_6 = \delta_7 = 0$. The result then follows by (4.7). *(ii).* We first note that by $W \perp (A, X)$ we have $\delta_1 = \delta_2 = \delta_6 = 0$. Rearranging terms in (4.7), we obtain

$$I^{*\prime}(x) = 1 + \frac{h(x)(\delta_5(x,\gamma) - \delta_4(x,\gamma) - \delta_3(x,\gamma) + \delta_7(x,\gamma))}{|\delta_0(x)| + h(x)(\delta_3(x,\gamma) + \delta_4(x,\gamma))}.$$

As $\gamma = 0$, we also have $\delta_4(x, 0) = E^x(u'(W^*)) > 0$, and $\delta_5(x, 0) - \delta_4(x, 0) < 0$. We also have $\delta_3 > 0$ and $\delta_7 \leq 0$. To show the latter, note that affiliation of $(A, -X)$ implies that $\ln_x f_A(\cdot|x)$ is nonincreasing (see Proposition A.2), and hence by Proposition A.1 we obtain

$$\delta_7(x, 0) = E^x(\Delta u(0, I^*))\big(E^x(\ln_x f_A(A|x)|A^* \geq 0) - E^x(\ln_x f_A(A|x)|A^* = 0)\big) \leq 0.$$

Putting everything together, we finally obtain $I^{*\prime}(x) \leq 1$ for all $x \in \mathcal{X}_1$, and hence $Id - I^*$ is nondecreasing on $\mathcal{X}_1$.

□

## B.2 Proofs for section 5

**Derivation of expression** (5.2)

*Proof.* For $x \in \mathcal{X}_1$, we have

$$\frac{\partial}{\partial P} L(x, I^*(x), \lambda_0, \lambda_1(x), \lambda_2(x); P) = \big[f_A(I^*(x) - P|x)E^x(\Delta u(\gamma, I^*)|A^* = 0) \\ - E^x(u'(W^*)1_{A^* \geq 0}) - (1-\gamma)E^x(u'(W^*(\gamma))1_{A^*<0}) + \lambda_0 \mathbb{P}^x(A^* \geq 0)\big] f_X(x) \\ = - K(x) f_X(x) - (1-\gamma)E^x(u'(W^*(\gamma))1_{A^*<0}) f_X(x).$$

Similarly, for $x \in \mathcal{X}_2$ we can write

$$\frac{\partial}{\partial P} L(x, x, \lambda_0, \lambda_1(x), \lambda_2(x); P) = -\underline{J}(x) f_X(x) - (1-\gamma)E^x(u'(\widetilde{W}(\gamma))1_{A<x-P}) f_X(x),$$

whereas for $x \in \mathcal{X}_3$ we have $(\partial/\partial P)L(x, 0, \lambda_0, \lambda_1(x), \lambda_2(x); P) = -\overline{J}(x) f_X(x)$. Noting that for $I = I^*$ the function $K$ is null on $\mathcal{X}_1$, we can then write the necessary condition for an optimum as

$$\int_0^{\overline{x}} \frac{\partial}{\partial P} L(x, I^*(x), \lambda_0, \lambda_1(x), \lambda_2(x); P)\Big|_{P=P^*} \mathrm{d}x = \int_{\mathcal{X}_2} (-\underline{J}(x)) f_X(x) \mathrm{d}x + \int_{\mathcal{X}_3} (-\overline{J}(x)) f_X(x) \mathrm{d}x \\ - (1-\gamma)E(u'(W^*(\gamma))1_{A^*<0}) \leq 0,$$

where the inequality holds with equality if $P^* > 0$. □

**Proposition 5.1**

*Proof.* If $0 \leq \gamma < 1$, the term $(1-\gamma)E(u'(W^*(\gamma))1_{A^*<0})$ is always positive, unless $I^*(x) \leq P^*$ for all $x \in [0, \overline{x}]$. If that were the case, however, there would be no insurance, as it can be easily shown that $U(0,0) > U(P, I)$ for all contracts $(P, I)$ such that $I(x) \leq P$ for all $x$ and $P > 0$. If there is insurance, the left hand side of (5.2) is therefore positive. But then we must have $\int_{\mathcal{X}_3} (-\overline{J}(x)) f_X(x) \mathrm{d}x > 0$. □

**Proposition 5.2**

*Proof.* If $\gamma = 1$, the term $(1-\gamma)E(u'(W^*(\gamma))1_{A^*<0})$ in condition (5.2) is always null. Consider first the case either $W \downarrow_{st} X$ or $W - X \uparrow_{sst} X$ hold. If $(P^*, I^*)$ provides no insurance on a nonnegligible set, then the right hand side of (5.2) is positive, as now $\overline{J}$ is increasing (if $W \downarrow_{st} X$) or decreasing (if $W - X \uparrow_{sst} X$) and nonpositive on $\mathcal{X}_3$. For equality to hold in (5.2), we then need $\int_{\mathcal{X}_2} \underline{J}(x) f_X(x) \mathrm{d}x > 0$. The case in which either $W \downarrow_{st} X$ or $W - X \uparrow_{sst} X$ hold is proved similarly, by noting that $\underline{J}$ is increasing (if $W \downarrow_{sst} X$) or decreasing (if $W \uparrow_{sst} X$) and nonnegative on $\mathcal{X}_2$. □

**Proposition 5.3**

*Proof.* Assume that $(P^*, I^*)$ solves (2.3) and the constraint (2.4) is not binding, that is $V(P^*, I^*) > \underline{v} \geq E(A)$, so that $P^* > 0$ and $\lambda_0 = 0$. When $\gamma = 1$, this is clearly a contradiction, as we could increase the insured's expected utility by lowering the premium. When $0 \leq \gamma < 1$, we know by Proposition 5.1 that the contract must provide no insurance for some loss realization, but this contradicts Proposition 4.1, as $\overline{J}(x) = E^x(u'(W - P^* - x)) > 0$ for all $x$ when $\lambda_0 = 0$. Now, assume that $(P^*, I^*)$ is not Pareto optimal. This means there is a contract $(\widetilde{P}, \widetilde{I})$ which dominates $(P^*, I^*)$. By optimality and inefficiency of $(P^*, I^*)$ we must then have $U(\widetilde{P}, \widetilde{I}) = U(P^*, I^*)$ and $V(\widetilde{P}, \widetilde{I}) > V(P^*, I^*) \geq \underline{v}$. This means that $(\widetilde{P}, \widetilde{I})$ is also optimal. Slackness of the state constraint, however, would give a contradiction, as discussed in the first part of the proof. □

**Proposition 5.4**

*Proof.* Let $(P = 0, I = 0)$ be Pareto optimal. Then it is optimal for $\underline{v} = E(A)$. We then have $\mathcal{X}_3 = (0, \overline{x}]$, and hence

$$\overline{J}(x) = E^x(u'(W - x)) - \lambda_0 \leq 0 \text{ for all } x \in [0, \overline{x}]. \tag{B.7}$$

On the other hand, condition (5.2) implies

$$\int_{\mathcal{X}_3} (-\overline{J}(x)) f_X(x) \mathrm{d}x - (1 - \gamma) E(u'(W - X) 1_{A<0}) = \lambda_0 - E(u'(W - X)) \leq 0. \tag{B.8}$$

From (B.7)-(B.8) we obtain $|E(\overline{J}(X))| \leq 0$, and the result follows.
In particular, if $W \downarrow_{st} X$ (if $W - X \uparrow_{sst} X$) then $E^x(u'(W - x))$ is increasing (decreasing). In both cases the contract $(P = 0, I = 0)$ cannot be Pareto optimal.

□

**Theorem 5.5**

*Proof.* Under the assumptions $A \perp_X W$ and $\gamma = 1$ we can use Theorem 4.4 to characterize the optima in terms of the zeros of $\underline{J}, \overline{J}$. By Proposition 5.4 $W \downarrow_{st} X$ implies that no insurance is never Pareto optimal. Also, $\overline{J}$ is increasing and $\underline{J}(x)$ is given by $E^x(u'(W - P^*) - \lambda_0) \mathbb{P}^x(A > x - P^*)$, with $E^x(u'(W - P^*) - \lambda_0)$ nondecreasing. If $J(0) > 0$, then $\underline{J} > 0$ and $\mathcal{X}_2 = (0, \bar{x}]$, thus contradicting (5.2). If $J(0) = 0$, then $\underline{J} \geq 0$, and there is full insurance. If $J(0) < 0$, then $\overline{J}$ must cross zero and there is therefore a deductible, followed by first coinsurance and then full insurance (by Proposition 5.2).
*i)* The assumption that $(W, -X)$ is affiliated implies $W \downarrow_{st} X$ (see section 3), and the previous analysis applies. By Propositions A.1-A.2 we also have that $Id - I^*$ is nonincreasing on $\mathcal{X}_1$. By Definition 4.2,

optimal contracts are disappearing deductibles.

*ii)* As we work under the assumption $A \perp_X W$, if $W \perp X$ we have that $E^x(u'(W-P^*)-\lambda_0 | A \geq x-P^*) = E(u'(W-P^*)-\lambda_0)$, and hence the sign of $\underline{J}$ is constant. If $J(0) < 0$, then $\underline{J} < 0$, whereas $\overline{J}$ is initially negative and crosses zero once. Hence a deductible can be followed by coinsurance only, contradicting Proposition 5.2. The only remaining possibility is therefore $J(0) = 0$, in which case $\overline{J}$ is nonnegative and $\underline{J}$ is null on $[0, \overline{x}]$; there is therefore full insurance. □

**Theorem 5.6**

*Proof.* By Proposition 5.4 the assumption $W \downarrow_{st} X$ implies that no insurance cannot be optimal, i.e., $\mathcal{X}_3 \neq (0, \bar{x}]$. On the other hand, by Proposition 5.1, we know that no insurance must be offered on a set of positive measure and hence $\mathcal{X}_3 \neq \emptyset$.

The assumption $W \downarrow_{st} X$ implies $W - X \downarrow_{sst} X$ and hence $\overline{J}$ is increasing; the case $J(0) \geq 0$ can therefore be excluded, as it contradicts Proposition 5.1. Let us assume $J(0) < 0$. $\overline{J}$ must then cross zero once, for otherwise the contract would be no insurance by Proposition A.3. Letting $x_1$ denote the zero of $\overline{J}$, we have $\mathcal{X}_3 = (0, x_1]$. If $\underline{J} < 0$ for all $x \in [0, \overline{x}]$, then $\mathcal{X}_1 = (x_1, \bar{x}]$; otherwise we have that the contract is coinsurance (if $\underline{J} < 0$) and full insurance (only if $\underline{J} \geq 0$).

*(i)* The assumption $W \perp X$ implies that $\underline{J} < 0$, because $J(0) < 0$ and hence $\underline{J}(x) = J(0)\mathbb{P}^x(A > x - P^*) - f_A(x - P^*|x)E(\Delta u(\gamma, Id)) < 0$. We have therefore coinsurance on the set $(x_1, \bar{x}]$.

*(ii)* If $W \perp X$, $\gamma = 0$, and $(A, -X)$ affiliated, we can use Propositions 4.6 to obtain the result. □

**Theorem 5.7**

*Proof.* By Proposition 5.4, $W - X \uparrow_{sst} X$ implies that no insurance cannot be Pareto optimal, i.e. $\mathcal{X}_3 \neq (0, \bar{x}]$, and $P^* > 0$. If $A \perp_X W$ and $W - X \uparrow_{sst} X$, then $\overline{J}$ and $E^x(u'(W-P^*)-\lambda_0)$ are decreasing. If $J(0) \leq 0$ then $\overline{J} < 0$, but we would then have $\mathcal{X}_3 = (0, \bar{x}]$ by Proposition A.3. Hence $J(0) > 0$ and the contract is initially full insurance, again by Proposition A.3, as $P^* > 0$. If $\gamma = 1$, it follows by Proposition 5.2 that no insurance must be offered on a nonnegligible set. The same conclusion holds if $0 \leq \gamma < 1$, by Proposition 5.1. Therefore in both cases we have $\mathcal{X}_3 \neq \emptyset$. But then $\overline{J}$ must cross zero once at $\overline{x}_1 \in (0, \overline{x})$ and $\mathcal{X}_3 = (\overline{x}_1, \overline{x}]$, by Proposition A.3. Focusing then on the set $(0, \bar{x}_1)$, when $\gamma = 1$ we have $\mathcal{X}_2 = (0, \underline{x}_1]$ and $\mathcal{X}_1 = (\underline{x}_1, \overline{x}_1)$, with $\underline{x}_1$ the first zero of $\underline{J}$. When $0 \leq \gamma < 1$, on every subinterval of $(0, \bar{x}_1)$ where $\underline{J} < 0$, and in particular on an interval $(\underline{x}_{\text{last}}, x_1)$, where $\underline{x}_{\text{last}}$ is the last zero of $\underline{J}$, the contract offers coinsurance, and on the remaining sets the contract may offer full insurance. □

**Theorem 5.8**

*Proof.* The Hamiltonian for problem (2.3) is still given by $H(x, I(x), \lambda_0; P)$ (see section 4), but with the premium now acting as the initial/terminal value of a fictitious, constant state variable. We define the maximized Hamiltonian as

$$\widehat{H}(x, P, \lambda_0) = \max_{z \in [0,x]} H(x, z, \lambda_0; P).$$

If $H(x, z, \lambda_0; P)$ is concave in $P$, then $\widehat{H}(x, \cdot, \lambda_0)$ is concave (e.g. Seierstad and Sydsaeter, 1987, p. 164). This means that the Arrow condition applies and the necessary conditions for optimality given by the Maximum principle are also sufficient (e.g. Seierstad and Sydsaeter, 1987, p. 107). In particular, letting $(\hat{P}, \hat{I})$ denote an admissible insurance contract with associated adjoint constant $\hat{\lambda}_0 \geq 0$, we have that $(\hat{P}, \hat{I})$ is optimal if and only if all the conditions of the Maximum Principle are satisfied. To prove concavity of $H(x, z, \hat{\lambda}_0; P)$ in $P$, we compute the second order derivative $(\partial^2/\partial P^2) H(x, I(x), \hat{\lambda}_0; P)$ to obtain, after some calculations, the explicit expression

$$\frac{\partial^2}{\partial P^2} H(x, I(x), \hat{\lambda}_0; P) = E^x \left( u''(\widetilde{W}) 1_{\widetilde{A} \geq 0} \right) - f_A(I(x) - P|x) \Big\{ E^x \left( u'(\widetilde{W}) \big| \widetilde{A} = 0 \right) - E^x \left( u'(\widetilde{W}) \big| \widetilde{A} \geq 0 \right)$$
$$+ E^x \left( u'(\widetilde{W}) - \gamma u'(\widetilde{W}(\gamma)) \big| \widetilde{A} = 0 \right) + E^x \left( \Delta u(\gamma, I(x)) \left( \ln_a f_A(I(x) - P, W|x) + h(x) \right) \big| \widetilde{A} = 0 \right) \Big\}.$$

Under $A \perp_X W$, the above is clearly strictly negative for $\gamma = 1$. When $\gamma = 0$, imposing the additional condition $\ln_a f_A(I(x) - \hat{P}|x) \geq -h(x)$ is sufficient for the last term to be nonnegative. The result then follows. $\square$

# C Figures and tables

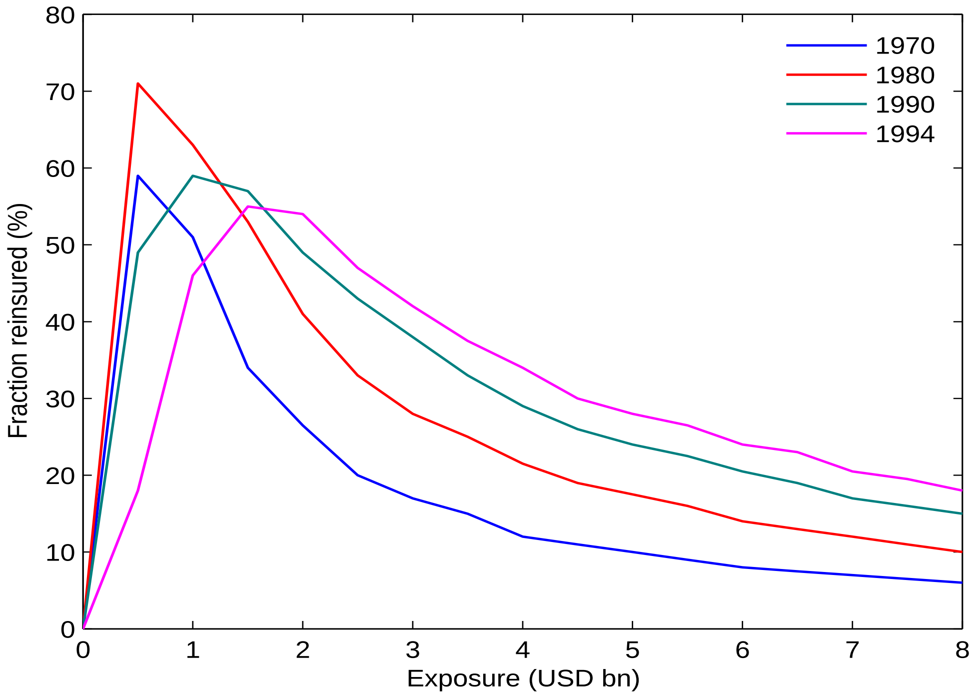


Figure 1: Source: Guy Carpenter & Co., see Froot (2001)

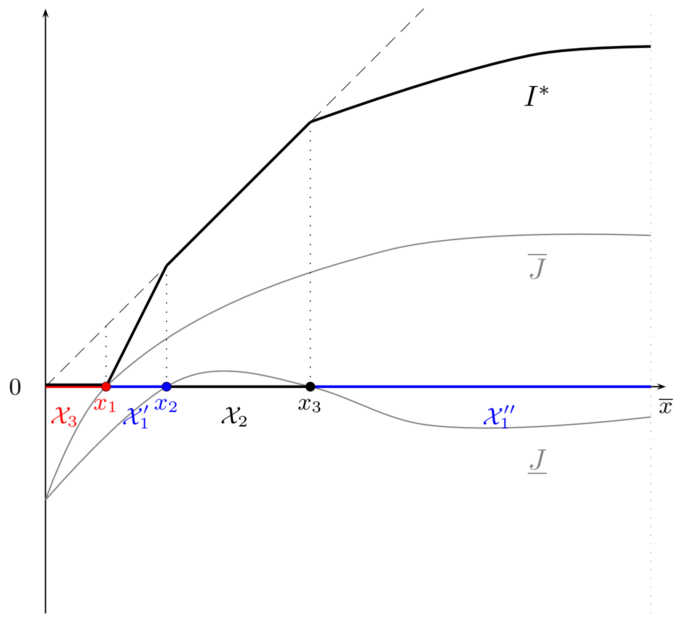


Figure 2: The coinsurance set is given by $\mathcal{X}_1 = \mathcal{X}_1' \cup \mathcal{X}_2''$, the full insurance set by $\mathcal{X}_2$, and the no insurance set by $\mathcal{X}_3$. There is a deductible, $x_1 = \overline{x}_1$, followed by coinsurance until $x_2 = \underline{x}_1$, and then full insurance. There is also an upper limit to full coverage at $x_3 = \underline{x}_2$.

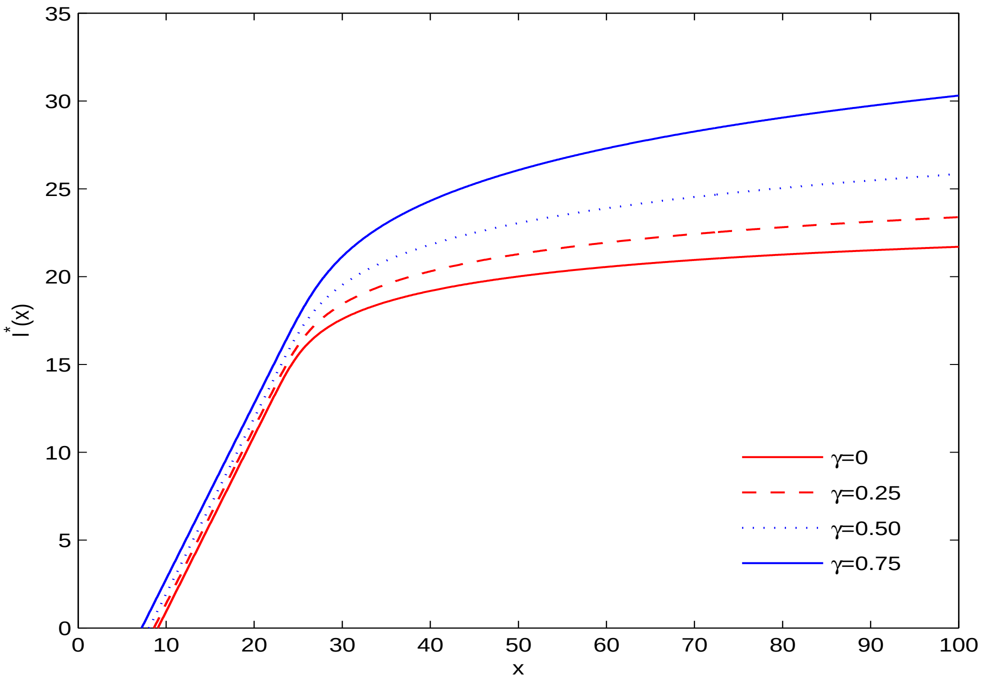


Figure 3: Insurance coverage ($I^*(x)$) for different levels of fractional recovery. Assumptions: $u(w) = -\mathrm{e}^{-\alpha w}$, with $\alpha = 0.05$ (note that in this case the efficient contract does not depend on the insurance buyer's wealth distribution); $A, W, X$ independent; $A$ log-normally distributed, with $A \sim \ln \mathcal{N}(30, 15)$; $X$ has Gamma distribution truncated at $\bar{x} = 100$ with mean 20 and standard deviation 10; $\underline{v} = 30$.

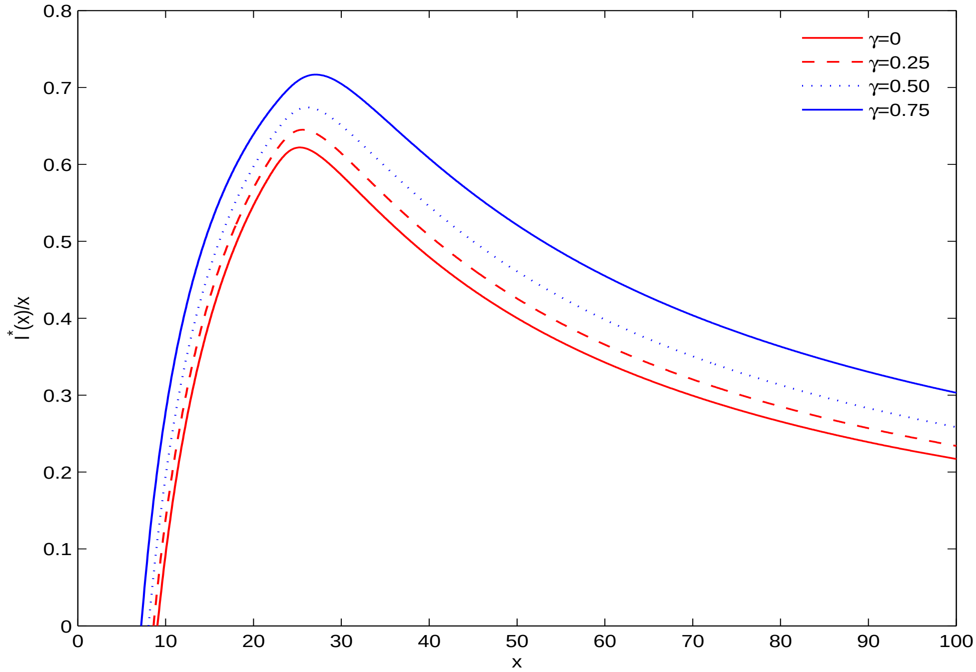


Figure 4: Insurance coverage as a fraction of the exposure ($I^*(x)/x$) for different levels of fractional recovery.