Risk measures and economic capital for (re)insurers

Andreas Tsanakas
Faculty of Actuarial Science and Insurance
Cass Business School
E-mail: a.tsanakas.1@city.ac.uk

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Abstract

This contribution relates to the use of risk measures for determining (re)insurers' economic capital requirements. Alternative sets of properties of risk measures are discussed. Furthermore, methods for constructing risk measures via indifference arguments, representation results and re-weighting of probability distributions are presented. It is shown how these different approaches relate to popular risk measures, such as VaR, Expected Shortfall, distortion risk measures and the exponential premium principle. The problem of allocating aggregate economic capital to sub-portfolios (e.g. insurers' lines of business) is then considered, with particular emphasis on marginal-cost-type methods. The relationship between insurance pricing and capital allocation is briefly discussed, based on concepts such as the opportunity and frictional costs of capital and the impact of the potential of default on insurance rates.

Keywords: risk measures, economic capital, risk capital, premium principles, choice under risk, solvency, capital allocation, insurance pricing, return on capital.

1 Introduction

A *risk measure* is a function that assigns real numbers to random variables representing uncertain pay-offs, e.g. insurance losses. The interpretation of a risk measure's outcome depends on the context in which it is used. Historically there have been three main areas of application of risk measures:

- As representations of risk aversion in asset pricing models, with a leading paradigm the use of the variance as a risk measure in Markowitz portfolio theory [1].
- As tools for the calculation of the insurance price corresponding to a risk. Under this interpretation, risk measures are called *premium calculation principles* in the classic actuarial literature, e.g. [2].
- As quantifiers of the economic capital that the holder of a particular portfolio or risks should safely invest in e.g. [3].

This contribution is mainly concerned with the latter interpretation of risk measures.

The *economic* or *risk capital* held by a (re)insurer corresponds to the level of safely invested assets used to protect itself against unexpected volatility of its portfolio's outcome. One has to distinguish economic capital from regulatory capital, which is the minimum required economic capital level as set by the regulator. In fact, much of the impetus for the use of risk measures in the quantification of capital requirements comes from the area of regulating financial institutions. Banking supervision [4] and, increasingly, insurance regulation [5] have been promoting the development of companies' *internal models* for modelling risk exposures. In that context, the application of a risk measure (most prominently *Value-at-Risk*) on the modelled aggregate risk profile of the insurance company is required.

Economic capital generally exceeds the minimum set by the regulator. Subject to that constraint, economic capital is determined so as to maximise performance metrics for the insurance company, such as total shareholder return [6]. Such maximisation takes into account two conflicting effects of economic capital [7]:

- An insurance company's holding economic capital incurs costs for its shareholders, which can be opportunity or frictional costs.
- Economic capital reduces the probability of default of the company as well as the severity of such default on its policyholders. This enables the insurance company to obtain a better rating of its financial strength and thereby attract more insurance business at higher prices.

Calculation of the optimal level of economic capital using such arguments is quite complicated and depends on factors that are not always easy to quantify, such as frictional capital costs, and on further constraints, such as the ability of an insurance company to raise capital in a particular economic and regulatory environment.

We could however consider that there is a particular calibration of the (regulatory or other) risk measure, which gives for the insurance company's exposure a level of economic capital that coincides with that actually held by the company. In that sense risk measures can be used to interpret exogenously given economic capital amounts. Such interpretation can be in the context of capital being set to achieve a target rating, often associated with a particular probability of default. Discussion of economic capital in the context of risk measures should therefore be caveated as being *ex-post*.

Finally, we note that the level of economic capital calculated by a risk measure may be a notional amount, as the company will generally not invest all its surplus in risk-free assets. This can be dealt with by absorbing the volatility of asset returns in the risk capital calculation itself.

2 Definition and examples of risk measures

We consider a set of risks $\mathcal{X}$ that the insurance company can be exposed to. The elements $X \in \mathcal{X}$ are random variables, representing losses at a fixed time horizon T. If under a particular state of the world ω the variable $X(\omega) > 0$ we will consider this to be a loss, while negative outcomes will be considered as gains. For convenience it is assumed throughout that the return from risk-free investment is 1 or alternatively that all losses in $\mathcal{X}$

are discounted at the risk-free rate. A risk measure ρ is then defined as a functional

$$\rho : \mathcal{X} \mapsto \mathbb{R}. \tag{1}$$

If X corresponds to the aggregate net risk exposure of an insurance company (i.e. the difference between liabilities and assets, excluding economic capital) and economic capital corresponds to $\rho(X)$, then we assume that the company defaults when $X > \rho(X)$.

In the terminology of [3] (and subject to some simplification), a risky position X is called acceptable if $\rho(X) < 0$, implying that some capital may be released without endangering the security of the holder of X, while $\rho(X) \geq 0$ means that X s a non-acceptable position and that some capital has to be added to it.

Some examples of simple risk measures proposed in the actuarial and financial literature (e.g. [8], [9]) are as follows.

Example 1 (Expected value principle).

$$\rho(X) = \lambda E[X], \;\; \lambda \geq 1 \tag{2}$$

Besides its application in insurance pricing, where it represents a proportional loading, this risk measure in essence underlies simple regulatory minimum requirements, such as the current EU Solvency rules, which determine capital as a proportion of an exposure measure such as premium.

Example 2 (Standard deviation principle).

$$\rho(X) = E[X] + \kappa\sigma[X], \;\; \kappa \geq 0 \tag{3}$$

In this case the loading is risk-sensitive, as it is a proportion of the standard deviation. This risk measure is encountered in reinsurance pricing, while also relating to Markowitz portfolio theory. In the context of economic capital, it is usually derived as an approximation to other risk measures, with this approximation being accurate for the special case of multivariate normal (more generally elliptical) distributions [10].

Example 3 (Exponential Premium Principle).

$$\rho(X) = \frac{1}{a} \ln E[e^{aX}], \; a > 0. \tag{4}$$

The exponential premium principle is a very popular risk measure in the actuarial literature, e.g. [11]. Part of the popularity stems from the fact that, in the classic ruin problem, it gives the required level of premium associated with Kramer-Lundberg bounds for ruin probabilities. We note that this risk measure has been recently considered in the finance literature under the name 'entropic risk measure' [12].

Example 4 (Value-at-Risk).

$$\rho(X) = VaR_p(X) = F_X^{-1}(p), \; p \in (0,1), \tag{5}$$

where F_X is the cumulative probability distribution of X and F_X^{-1} is its (pseudo-)inverse. $VaR_p(X)$ is easily interpreted as the amount of capital that, when added to the risk X, limits the probability of default to $1-p$. Partly because of its intuitive attractiveness Value-at-Risk has become the risk measure of choice for both banking and insurance regulators. For example, the UK regulatory regime for insurers uses $VaR_{0.995}(X)$ [13], while a similar risk measure has been be proposed in the context of the new EU-wide Solvency II regime [5].

Example 5 (Expected Shortfall).

$$\rho(X) = ES_p(X) = \int_p^1 F_X^{-1}(q)dq, \; p \in (0,1). \tag{6}$$

This risk measure, also known as Tail-(or Conditional-)Value-at-Risk, corresponds to the average of all VaR_ps above the threshold p. Hence it reflects both the probability and the severity of a potential default. Expected shortfall has been proposed in the literature as a risk measure correcting some of the theoretical weaknesses of Value-at-Risk [14]. Subject to continuity of F_X at the threshold VaR_p, Expected Shortfall coincides with the *Tail Conditional Expectation*, defined by

$$\rho(X) = E[X|X > F_X^{-1}(p)]. \tag{7}$$

Example 6 (Distortion risk measure).

$$\rho(X) = -\int_{-\infty}^{0} (1 - g(1 - F_X(x)))dx + \int_{0}^{\infty} g(1 - F_X(x))dx, \qquad (8)$$

where $g : [0, 1] \mapsto [0, 1]$ is increasing and concave [15]. This risk measure can be viewed as an expectation under a distortion of the probability distribution effected by the function g. It can be easily shown that Expected Shortfall is a special case obtained by a bilinear distortion [14]. Distortion risk measures can be viewed as Choquet integrals [16], [17], which are extensively used in the economics of uncertainty, e.g. [18]. An equivalent class of risk measures defined in the finance literature are known as *spectral risk measures* [19].

3 Properties of risk measures

The literature is rich in discussions of the properties of alternative risk measures, as well as the desirability of such properties, e.g. [2], [3], [20], [9]. In view of this, the current discussion is invariably selective.

An often required property of risk measures is that of *monotonicity*, stating

$$\text{If } X \leq Y, \text{ then } \rho(X) \leq \rho(Y). \qquad (9)$$

This reflects the obvious requirement that losses that are always higher should also attract a higher capital requirement.

A further appealing property is that of *translation* or *cash invariance*,

$$\rho(X + a) = \rho(X) + a, \text{ for } a \in \mathbb{R}. \qquad (10)$$

This postulates that adding a constant loss amount to a portfolio increases the required risk capital by the same amount. We note that this has the implication that

$$\rho(X - \rho(X)) = \rho(X) - \rho(X) = 0, \qquad (11)$$

which, in conjunction with monotonicity, facilitates the interpretation of $\rho(X)$ as the minimum capital amount that has to be added to X in order to make it acceptable.

Two conceptually linked properties are the ones of *positive homogeneity*,

$$\rho(bX) = b\rho(X), \text{ for } b \geq 0, \tag{12}$$

and *subadditivity*,

$$\rho(X+Y) \leq \rho(X) + \rho(Y), \text{ for all } X, Y \in \mathcal{X}. \tag{13}$$

Positive homogeneity postulates that a linear increase in the risk exposure X also implies linear increase in risk. Subadditivity requires that the merging of risks should always yield a reduction in rick capital due to diversification.

Risk measures satisfying the four properties of monotonicity, translation invariance, positive homogeneity and subadditivity have become widely known as *coherent* [3]. This particular axiomatization, also proposed in an actuarial context [16],[21], has achieved near-canonical status in the world of financial risk management. While Value-at-Risk generally fails the subadditivity property, due to its disregard for the extreme tails of distributions, part of its appeal to regulators and practitioners stems of its use as an approximation to a coherent risk measure.

Nonetheless, coherent risk measures have also attracted criticism because of their insensitivity to the aggregation of large positively dependent risks implied by the latter two properties, e.g. [20]. The weaker property of *convexity* has been proposed in the literature [22], a property already discussed in [23]. Convexity requires that:

$$\rho(\lambda X + (1-\lambda)Y) \leq \lambda\rho(X) + (1-\lambda)\rho(Y), \text{ for all } X, Y \in \mathcal{X} \text{ and } \lambda \in [0,1]. \tag{14}$$

Convexity, while retaining the diversification property, relaxes the requirement that a risk measure must be insensitive to aggregation of large risks. It is noted that subadditivity is obtained by combining convexity with positive homogeneity. Risk measures satisfying convexity and applying increasing penalties for large risks have been proposed in [24].

Risk measures produce an ordering of risks, in the sense that $\rho(X) \leq \rho(Y)$ means that X is considered less risky than Y. One would wish that ordering to conform to standard economic theory, i.e. to be consistent with widely accepted notions of stochastic order such as 1st and 2nd order stochastic dominance and convex order, see [25], [9]. It has been shown that under some relatively mild technical conditions, risk measures that are monotonic and convex produce such a consistent ordering of risks [26].

A further key property relates to the dependence structure between risks under which the risk measure becomes *additive*

$$\rho(X+Y) = \rho(X) + \rho(Y), \tag{15}$$

as this implies a situation where neither diversification credits nor aggregation penalties are assigned. In the context of subadditive risk measures, *comonotonic additivity* is a sensible requirement, as it postulates that no diversification is applied in the case of comonotonicity (the maximal level of dependence between risks, e.g. [27]). On the other hand, one could require a risk measure to be *independent additive*. If such a risk measure is also consistent with the stop-loss or convex order, by the results of [28], it is guaranteed to penalize any positive dependence by being superadditive (i.e. $\rho(X+Y) \geq \rho(X) + \rho(Y)$) and reward any negative dependence by being subadditive.

The risk measures defined above satisfy the following properties:

Expected value principle Monotonic, positive homogenous, additive for all dependence structures.

Standard deviation principle Translation invariant, positive homogenous, subadditive.

Exponential premium principle Monotonic, translation invariant, convex, independent additive.

Value-at-Risk Monotonic, translation invariant, positive homogenous, subadditive for joint-elliptically distributed risks [10], comonotonic additive.

Expected Shortfall Monotonic, translation invariant, positive homogenous, subadditive, comonotonic additive.

Distortion risk measure Monotonic, translation invariant, positive homogenous, subadditive, comonotonic additive.

Finally we note that all risk measures discussed in this contribution are *law invariant*, meaning that $\rho(X)$ only depends on the distribution function of X [21], [29]. This implies that two risks characterised by the same probability distribution would be allocated the same amount of economic capital.

4 Constructions and representations of risk measures

4.1 Indifference arguments

Economic theories of choice under risk seek to model the preferences of economic agents with respect to uncertain pay-offs. They generally have representations in terms of *preference functionals* $V : -\mathcal{X} \mapsto \mathbb{R}$, in the sense that

$$-X \text{ is preferred to } -Y \quad \Leftrightarrow \quad V(-X) \geq V(-Y). \tag{16}$$

(Note that the minus sign is applied because we have defined risk as losses, while preference functionals are typically applied on pay-offs.)

Then a risk measure can be defined by assuming that the addition to initial wealth W of a liability X and the corresponding capital amount $\rho(X)$ does not affect preferences [8]

$$V(W_0 - X + \rho(X)) = V(W_0). \tag{17}$$

Often in this context $W = 0$ is assumed for simplicity.

The leading paradigm of choice under risk is the von Neumann-Morgenstern *expected utility theory* [30], under which

$$V(W) = E[u(W)], \tag{18}$$

where u is an increasing and concave *utility function*. A popular choice of utility function is the *exponential utility*

$$u(w) = \frac{1}{a}\left(1 - e^{-aw}\right), \ a > 0. \tag{19}$$

It can be easily seen that equations (17), (18) and (19) yield the exponential premium principle defined in section 2.

An alternative theory is the *dual theory of choice under risk* [31], under which

$$V(W) = -\int_{-\infty}^{0} (1 - h(1 - F_W(w)))dw + \int_{0}^{\infty} h(1 - F_W(w))dw, \qquad (20)$$

where $h : [0,1] \mapsto [0,1]$ is increasing and convex. It can then be shown that the risk measure obtained from (17) and (20) is a distortion risk measure with $g(s) = 1 - h(1 - s)$. For the function

$$h(s) = 1 - (1 - s)^{\frac{1}{\gamma}},\ \gamma > 1 \qquad (21)$$

the well known *proportional hazards transform* with $g(s) = s^{\frac{1}{\gamma}}$ is obtained [15].

More detailed discussions of risk measures resulting from alternative theories of choice under risk and references to the associated economics literature are given in [24], [32].

It should also be noted that the construction of risk measures from economic theories of choice must not necessarily be via indifference arguments. If a risk measure satisfies the convexity and monotonicity properties, then by setting $U(W) = -\rho(-W)$ we obtain a monotonic concave preference functional. The translation invariance property of the risk measure then makes U also translation invariant. Hence we could consider convex risk measures as the subset of concave preference functionals that satisfy the translation invariance property (subject to a minus sign). Such preference functionals are sometimes called *monetary utility functions*, as their output can be interpreted as being in units of money rather than of an abstract notion of satisfaction.

4.2 Axiomatic characterisations

An alternative approach to deriving risk measures is by fixing a set of properties that risk measures should satisfy and then seeking an explicit functional representation.

For example, coherent (i.e. monotonic, translation invariant, positive homogenous and subadditive) risk measures can be represented by [3]

$$\rho(X) = \sup_{\mathbb{P} \in \mathcal{P}} E_{\mathbb{P}}[X], \tag{22}$$

where $\mathcal{P}$ is a set of probability measures. By adding the comonotonic additivity property one gets the more specific structure of $\mathcal{P} = \{\mathbb{P} : \mathbb{P}(A) \leq v(A)$ for all sets $A\}$, where v is a submodular set function known as a (Choquet) capacity [17]. The additional property of law invariance enables writing $v(A) = g(\mathbb{P}_0(A))$ where $\mathbb{P}_0$ is the objective probability measure and g a concave distortion function [21]. This finally yields a representation of coherent, comonotonic additive, law invariant risk measures as distortion risk measures. An alternative route towards this representation is given by [29].

The probability measures in $\mathcal{P}$ have been termed *generalized scenarios* [3] with respect to which the worst case expected loss is considered. On the other hand, representations such as (22) have been derived in the context of robust statistics [33] and decision theory, known as the multiple-priors model [34].

A related representation result for convex risk measures is derived in [22], while results for independent additive risk measures are given in [35], [36].

4.3 Re-weighting probabilities

An intuitive construction of risk measures is by re-weighting the probability distribution of the underlying risk

$$\rho(X) = E[X\zeta(X)], \tag{23}$$

where ζ is generally assumed to be an increasing function with $E[\zeta(X)] = 1$ and representation (23) could be viewed as an expectation under a change of measure. Representation (23)is particularly convenient when risk measures and related functionals have to be evaluated by Monte-Carlo simulation.

Many well-known risk measures can be obtained in this way. For example, making appropriate assumptions on F_X and g one can easily show that for distortion measures it is

$$\rho(X) = E[Xg'(1 - F_X(X))]. \tag{24}$$

On the other hand the exponential principle can be written as:

$$\rho(X) = E\left[X \int_0^1 \frac{e^{\gamma a X}}{E[e^{\gamma a X}]} d\gamma\right]. \tag{25}$$

The latter representation is sometimes called a 'mixture of Esscher principles' and studied in more generality in [35], [36].

5 Capital allocation

5.1 Problem definition

Often the requirement arises that the risk capital calculated for an insurance portfolio has to be allocated to business units. There may be several reasons for such a capital allocation exercise, the main ones being performance measurement / management and insurance pricing.

Capital allocation is not a trivial exercise, given that in general the risk measure used to set the aggregate capital is not additive. In other words, if one has an aggregate risk Z for the insurance company, breaking down to sub-portfolios $X_1, \ldots, X_n$, such that

$$Z = \sum_{j=1}^{n} X_j, \tag{26}$$

it generally is

$$\rho(Z) \neq \sum_{j=1}^{n} \rho(X_j), \tag{27}$$

due to diversification / aggregation issues.

The capital allocation problem then consists of finding constants $d_1, \ldots, d_n$ such that

$$\sum_{j=1}^{n} d_j = \rho(Z), \tag{28}$$

where the allocated capital amount d_i should in some way reflect the risk of sub-portfolio X_i. Early papers in the actuarial literature that deal with cost allocation problems in insurance are [37], [38], the former taking a risk theoretical view, while the later examining alternative allocation methods from the perspective of cooperative game theory. A specific application of cooperative game theory to risk capital allocation, including a survey of the relevant literature, is [39].

5.2 Marginal cost approaches

Marginal cost approaches associate allocated capital to the impact that changes in the exposure to sub-portfolios have on the aggregate capital. Denote for vector of weights $\mathbf{w} \in [0,1]^n$,

$$Z^{\mathbf{w}} = \sum_{j=1}^{n} w_j X_j. \tag{29}$$

Then the marginal cost of each sub-portfolio is given by

$$MC(X_i; Z) = \frac{\partial \rho(Z^{\mathbf{w}})}{\partial w_i}\bigg|\, \mathbf{w} = \mathbf{1}, \tag{30}$$

subject to appropriate differentiability assumptions. If the risk measure is positive homogenous, then by Euler's theorem we have that

$$\sum_{j=1}^{n} MC(X_j; Z) = \rho(Z) \tag{31}$$

and we can hence use marginal costs directly $d_i = MC(X_i; Z)$ as the capital allocation.

If the risk measure is in addition subadditive then we have that [40]

$$d_i = MC(X_i; Z) \leq \rho(X_i), \tag{32}$$

i.e. the allocated capital amount is always lower than the stand-alone risk capital of the sub-portfolio. This corresponds to the game theoretical concept of the *core*, in that the allocation does not provide an incentive for splitting the aggregate portfolio. This requirement is consistent with the subadditivity property, which postulates that there is always a benefit in pooling risks.

In the case that no such strong assumptions as positive homogeneity (and subadditivity) are made with respect to the risk measure, marginal costs will in general not yield an appropriate allocation, as they will not add up to the aggregate risk. Cooperative game theory then provides an alternative allocation method, based on the *Aumann-Shapley value* [41], which can be viewed as a generalisation of marginal costs

$$AC(X_i, Z) = \int_0^1 MC(X_i; \gamma Z) d\gamma. \tag{33}$$

It can easily be seen that if we set $d_i = AC(X_i, Z)$ then the d_is add up to $\rho(Z)$ and that for positive homogenous risk measures the Aumann-Shapley allocation reduces to marginal costs. Early applications of the Aumann-Shapley value to cost allocation problems are [42], [43].

For the examples of risk measures that were introduced in section 2, the following allocations are obtained from marginal costs / Aumann-Shapley.

Example 7 (Expected value principle).

$$d_i = \lambda E[X_i] \tag{34}$$

Example 8 (Standard deviation principle).

$$d_i = E[X_i] + \kappa \frac{Cov(X_i, Z)}{\sigma[Z]} \tag{35}$$

Example 9 (Exponential Premium Principle).

$$d_i = \int_0^1 \frac{E[X_i \exp(\gamma a Z)]}{E[\exp(\gamma a Z)]} d\gamma \tag{36}$$

Example 10 (Value-at-Risk [44]).

$$d_i = E[X_i | Z = VaR_p(Z)] \tag{37}$$

under suitable assumptions on the joint probability distribution of (X_i, Z).

Example 11 (Expected Shortfall [44]).

$$d_i = E[X_i | Z > VaR_p(Z)], \tag{38}$$

under suitable assumptions on the joint probability distribution of (X_i, Z).

Example 12 (Distortion risk measure [45]).

$$d_i = E[X_i g'(1 - F_Z(Z))] \tag{39}$$

assuming representation (24) is valid.

5.3 Alternative approaches

While marginal cost-based approaches are well-established in the literature, there are a number of alternative approaches to capital allocation. For example, we note that marginal costs generally depend on the joint distribution of the individual sub-portfolio and the aggregate risk. In some cases this dependence may not be desirable, for example when one tries to measure the performance of sub-portfolios to allocate bonuses. In that case, a simple *proportional repartition of costs* [38] may be appropriate:

$$d_i = \rho(X_i) \frac{\rho(Z)}{\sum_{j=1}^n \rho(X_j)}. \tag{40}$$

Different issues emerge when the capital allocation is to be used for managing the performance of the aggregate portfolio, as measured by a particular metric such as return-on-capital. Assume that $\hat{X}_i,\ i = 1, \ldots, n$ correspond to the liabilities from sub-portfolio i minus reserves corresponding to those liabilities, such that $E[\hat{X}_i] = 0$. We then have the breakdown

$$X_i = \hat{X}_i - p_i, \tag{41}$$

where p_i corresponds to the underwriting profit from the insurer's sub-portfolio (e.g. line of business) i, such that $\sum_{j=1}^n \hat{X}_j = \hat{Z}$ and $\sum_{j=1}^n p_j = p$. Then we define the return on capital for the whole insurance portfolio by

$$RoC = \frac{p}{\rho(\hat{Z})}. \tag{42}$$

This is discussed in depth in [44] for the case that ρ is a coherent risk measure. It is then considered whether assessing the performance of sub-portfolios by

$$RoC_i = \frac{p_i}{d_i}, \tag{43}$$

where d_i represents capital allocated to $\hat{X}_i$, provides the right incentives for optimizing performance. It is shown that marginal costs is the unique allocation mechanism that satisfies this requirement as set out in that paper. A closely related argument is that under the marginal cost allocation a portfolio balanced to optimize aggregate return on capital has the property that $RoC = RoC_i$ for all i. While this produces a useful performance yardstick that can be used throughout the company, some care has to be taken when applying marginal cost methodologies. In particular, if the marginal capital allocation to a sub-portfolio is small e.g. for reasons of diversification, the insurer should be careful not to let that fact undermine underwriting standards. A proportional allocation method could also be used for reference, to avoid that danger.

Often one may be interested in calculating capital allocations that are in some sense optimal. For example, in [46] capital allocations are calculated such that a suitably defined distance function between individual sub-portfolios and allocated capital levels is minimized. This methodology reproduces many capital allocation methods found in the literature, while also considering the case that aggregate economic capital is exogenously given rather than calculated via a risk measure. A different optimization approach to capital allocation is presented in [47].

An alternative strand of the literature on capital allocation relates to the pricing of the policyholder deficit due to the insurer's potential default [48]. This is discussed in slightly more detail in section 6.2.

6 Economic capital and insurance pricing

6.1 Cost of capital

A way of associating risk measures and economic capital with insurance prices is via *cost of capital* arguments. It is considered that the shareholders of an insurance company incur an opportunity cost by providing economic capital. Therefore they also require a return on that capital, in excess of the risk free rate. This is typically calculated via equilibrium arguments, with the methodology of *Weighted Average Cost of Capital (WACC)* being the prime example [49]. It is furthermore assumed that the additional return on capital will be earned by including a cost-of-capital adjustment in insurance premiums.

If we denote by CoC the cost of capital associated with the insurance company, then, using the notation of the previous section, the required profit for its insurance portfolio $Z = \hat{Z} - p$ is:

$$p = CoC \cdot \rho(\hat{Z}). \tag{44}$$

It should be apparent from the preceding discussion that cost of capital and return on capital are closely linked concepts; in fact evaluation of the former often leads to a target level for the latter.

A capital allocation $\sum_{j=1}^{n} d_j = \rho(\sum_{j=1}^{n} \hat{X}_j) = \rho(\hat{Z})$ then yields the required profit for each sub-portfolio:

$$p_i = CoC \cdot d_i. \tag{45}$$

6.2 Frictional capital costs and the cost of default

Cost-of-capital approaches to insurance pricing have been criticised in the literature for a number of reasons, including [6]:

- The return on (or cost of) capital considered may be a an inadequate measure of performance, as the total shareholder return is influenced by other factors too.

- The methodology does not explicitly allow for the potential default of the insurer. Hence, by associating a fixed cost with economic capital, the benefits of increased policyholder security are disregarded.

An alternative approach is to break down the insurance price into three parts [7]:

- The economic (market consistent) value of the insurance liability.
- The frictional cost of holding capital.
- The cost of the insurer's default to policyholders.

The economic value of the liability Z, denoted here by $EV(Z)$ where EV is a linear pricing functional derived by a financial valuation method e.g. equilibrium or no-arbitrage arguments. The frictional costs, which may comprise double taxation, agency costs and the costs of financial distress [7], may be written in their simplest form as a fixed percentage fc of aggregate capital, i.e. $fc \cdot \rho(Z)$.

The loss to policyholders caused by the insurer's default or *policyholder deficit* is given by

$$(Z - \rho(Z))_+, \tag{46}$$

that is by the excess of the insurer's liabilities over its assets. It is argued by [48] that the economic value of this cost should be removed from the insurance premium, as it corresponds to a loss to policyholders, given the insurer's shareholders' limited liability; an expression similar to (46) has thus been termed the *limited liability put option* or *default option*.

The total premium for the insurer can then be calculated as

$$EV(Z) + fc \cdot \rho(Z) - EV[(Z - \rho(Z))_+]. \tag{47}$$

Based on expression (48), and assuming ρ is a coherent risk measure, marginal costs give us the allocation of the insurance price for sub-portfolio X_i by

$$EV(X_i) + fc \cdot d_i - EV[(X_i - d_i)\mathbf{I}_{\{Z>\rho(Z)\}}], \tag{48}$$

where $\mathbf{I}_A$ denotes the indicator of set A.

The discussion on allocating capital by the value of the default option is in the main motivated by [48]. That paper, as well as [50] which anticipates it, adopts a slightly different approach, whereby the aggregate capital allocated to sub-portfolios is exogenously given rather than calculated via a risk measure. Moreover, these papers are set in a dynamic framework rather than the simple one-period one adopted in this contribution.

6.3 Incomplete markets and risk measures

As discussed in section 1, risk measures have been traditionally used in the insurance industry to calculate prices. The difference $\rho(X) - E[X]$ then is considered as a *safety loading*. When the risk measure is law invariant, such as those considered here, then the safety loading depends only on the distribution of X and does not reflect market conditions. Moreover, as illustrated in [51], using law-invariant risk measures as pricing functionals, e.g. to reflect market frictions, can also be problematic.

In financial economics securities are priced using no-arbitrage arguments, which in a complete market result in the price of a risk equal to the cost of its replication by traded instruments. Such a market consistent approach to valuation of insurance liabilities is propagated in the context of Solvency II [5]. However, insurance markets are typically incomplete, meaning that no exact replication can be achieved by trading. This motivates approaches where the price equals the cost of replication with some acceptable level of accuracy. Only two examples of incomplete market approaches based on partial replication are mean-variance [52] and quantile hedging [53].

The formulation of such pricing methods often utilizes risk measures to define the quality of replication. Different calibrations of these risk measures, corresponding to different levels of risk aversion, give rise to a range of possible prices. In can thus be argued that there is a continuum between pricing and holding risk capital in incomplete markets, with the main difference being the degree of risk aversion considered in each application. (A rather different approach to incomplete market pricing is based on indifference arguments in a dynamic setting is [54], which is closely related to the risk measures discussed in section 4.1.) Hence one could consider financial pricing of insurance as being a dynamic extension of the traditional actuarial risk measures or premium principles, taking place in a richer economic environment where dynamic trading is possible.

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