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Correction: Exchange Option Under Jump-Diffusion Dynamics

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ABSTRACT *In this note, we provide the correct formula for the price of the European exchange option given in Cheang and Chiarella (2011) in a bi-dimensional jump diffusion model.*

KEY WORDS: exchange option, jump-diffusion.

Theorem 5.1 in Cheang and Chiarella (2011), page 259, gives a formula for the price of a European exchange option under jump diffusion dynamics. The formula is based on a wrong application of the change of numéraire from the risk-neutral to the spot measure. We amend the proof and provide the correct pricing formula for the exchange option.

Theorem 1: *Suppose the asset prices follow the dynamics in formula (38) of Cheang and Chiarella (2011), and the continuous dividend rate for each asset is ξ_i. Then when $S_{1,t} = s_1$ and $S_{2,t} = s_2$, the European exchange option price is*

$$
\begin{aligned}
C_t^E(s_1, s_2) = &\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} e^{-(\tilde{\lambda}_1+\tilde{\lambda}_2+\tilde{\lambda})(T-t)} \frac{(\tilde{\lambda}_1(T-t))^k}{k!}\frac{(\tilde{\lambda}_2(T-t))^m}{m!}\frac{(\tilde{\lambda}(T-t))^n}{n!} \\
&\times \left[s_1 e^{-(\xi_1+\tilde{\lambda}_1\tilde{\kappa}_{Z_1}+\tilde{\lambda}\tilde{\kappa}_1)(T-t)+k\tilde{\alpha}_{11}+\frac{k\delta_{11}^2}{2}+n\tilde{\alpha}_1+\frac{n\delta_1^2}{2}} \Phi(d_{1,t,k,m,n}) \right. \\
&\left. - s_2 e^{-(\xi_2+\tilde{\lambda}_2\tilde{\kappa}_{Z_2}+\tilde{\lambda}\tilde{\kappa}_2)(T-t)+m\tilde{\alpha}_{22}+\frac{m\delta_{22}^2}{2}+n\tilde{\alpha}_2+\frac{n\delta_2^2}{2}} \Phi(d_{2,t,k,m,n}) \right],
\end{aligned} \tag{1}
$$

where

$$
d_{1,t,k,m,n} = \frac{\ln\left(\frac{s_1}{s_2}\right) + (\xi_2 - \xi_1 - \tilde{\lambda}(\tilde{\kappa}_1 - \tilde{\kappa}_2) - \tilde{\lambda}_1\tilde{\kappa}_{Z_1} + \tilde{\lambda}_2\tilde{\kappa}_{Z_2})(T-t) + \mu_{k,m,n} + \frac{\sigma_{k,m,n}^2}{2}(T-t)}{\sigma_{k,m,n}\sqrt{T-t}},
$$

$$
d_{2,t,k,m,n} = d_{1,t,k,m,n} - \sigma_{k,m,n}\sqrt{T-t};
$$

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with

$$\mu_{k,m,n} = k(\tilde{\alpha}_{11} + \delta_{11}^2/2) - m(\tilde{\alpha}_{22} + \delta_{22}^2/2) + n(\tilde{\alpha}_1 - \tilde{\alpha}_2 + \delta_1^2/2 - \delta_2^2/2),$$

and

$$\sigma_{k,m,n}^2 = \sigma^2 + \frac{k\delta_{11}^2}{T-t} + \frac{m\delta_{22}^2}{T-t} + \frac{n(\delta_1^2 + \delta_2^2 - 2\rho_{\mathbf{Y}}\delta_1\delta_2)}{T-t}, \qquad \sigma^2 = \sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2,$$

where Φ *is the standard normal probability distribution function.*

Proof: Without loss of generality, we derive a formula for the exchange option price at time $t = 0$. The option price at $t = 0$ is then

$$C_0^E(S_{1,0}, S_{2,0}) = \mathbb{E}_{\mathbb{Q}}\left[\frac{(S_{1,T} - S_{2,T})^+}{e^{rT}}\right] =$$

$$S_{1,0}e^{-\xi_1 T}\mathbb{E}_{\mathbb{Q}}\left[\exp\left[-\frac{\sigma_1^2}{2}T + \sigma_1\tilde{W}_{1,T} - \tilde{\lambda}\tilde{\kappa}_1 T + \sum_{n=0}^{N_T} Y_{1,n} - \tilde{\lambda}_1\tilde{\kappa}_{Z_1}T + \sum_{i=1}^{N_{1,T}} Z_{1,i}\right]\mathbb{1}_{\{S_{1,T}>S_{2,T}\}}\right]$$
$$-S_{2,0}e^{-\xi_2 T}\mathbb{E}_{\mathbb{Q}}\left[\exp\left[-\frac{\sigma_2^2}{2}T + \sigma_2\tilde{W}_{2,T} - \tilde{\lambda}\tilde{\kappa}_2 T + \sum_{n=1}^{N_T} Y_{2,n} - \tilde{\lambda}_2\tilde{\kappa}_{Z_2}T + \sum_{l=1}^{N_{2,T}} Z_{2,l}\right]\mathbb{1}_{\{S_{1,T}>S_{2,T}\}}\right]. \tag{2}$$

Using twice the change of numéraire from the risk neutral measure $\mathbb{Q}$ to the spot measures $\mathbb{Q}_1$ (stock S_1 is taken as numéraire) and $\mathbb{Q}_2$ (stock S_2 is taken as numéraire), and conditioning on the number of idiosyncratic and common jumps the pricing formula of the exchange option requires the computation of $\mathbb{Q}_1(\mathcal{A}|N_{1,T} = k, N_{2,T} = m, N_T = n)$ and $\mathbb{Q}_2(\mathcal{A}|N_{1,T} = k, N_{2,T} = m, N_T = n)$, where $\mathcal{A}|_{N_{1,T}=k,N_{2,T}=m,N_T=n}$ is the set defined as

$$\left\{\Xi_{k,m,n} > \ln\left(\frac{S_{2,0}}{S_{1,0}}\right) - \left(\xi_2 - \xi_1 - \frac{\sigma_1^2}{2} + \frac{\sigma_2^2}{2} - \tilde{\lambda}(\tilde{\kappa}_1 - \tilde{\kappa}_2) - \tilde{\lambda}_1\tilde{\kappa}_{Z_1} + \tilde{\lambda}_2\tilde{\kappa}_{Z_2}\right)T\right\}$$

and

$$\Xi_{k,m,n} = \sigma_1\tilde{W}_{1,T} - \sigma_2\tilde{W}_{2,T} + \sum_{i=0}^{k} Z_{1,i} - \sum_{l=0}^{m} Z_{2,l} + \sum_{j=0}^{n}(Y_{1,j} - Y_{2,j}).$$

The proof in Cheang and Chiarella (2011) has to be corrected in the specification of the distribution of $\Xi_{k,m,n}$ under $\mathbb{Q}_1$ and $\mathbb{Q}_2$. In particular to compute the distribution of $\mathbf{Y}$ under $\mathbb{Q}_1$ and $\mathbb{Q}_2$, we have to apply Theorem 3.1 of Cheang and Chiarella (2011), according to the following Radon–Nikodým derivatives

$$\left.\frac{d\mathbb{Q}_1}{d\mathbb{Q}}\right|_T = \exp\left[-\frac{\sigma_1^2}{2}T + \sigma_1\tilde{W}_{1,T} - \tilde{\lambda}\tilde{\kappa}_1 T + \sum_{n=1}^{N_T} Y_{1,n} - \tilde{\lambda}_1\tilde{\kappa}_{Z_1}T + \sum_{i=1}^{N_{1,T}} Z_{1,i}\right],$$

and

$$\left.\frac{d\mathbb{Q}_2}{d\mathbb{Q}}\right|_T = \exp\left[-\frac{\sigma_2^2}{2}T + \sigma_2\tilde{W}_{2,T} - \tilde{\lambda}\tilde{\kappa}_2 T + \sum_{n=1}^{N_T} Y_{2,n} - \tilde{\lambda}_2\tilde{\kappa}_{Z_2}T + \sum_{i=1}^{N_{2,T}} Z_{2,i}\right].$$

The parameter $\boldsymbol{\gamma}$ defined in Theorem 3.1 determines the distribution of the jump component $\mathbf{Y}$ through the following relation on the moment-generating function

$$M_{\mathbb{Q}_i,\mathbf{Y}}(\mathbf{u}) = \frac{M_{\mathbb{Q},\mathbf{Y}}(\mathbf{u}+\boldsymbol{\gamma})}{M_{\mathbb{Q},\mathbf{Y}}(\boldsymbol{\gamma})}, \qquad i = 1, 2.$$

Setting $\boldsymbol{\gamma} = [1, 0]^{\intercal}$, Theorem 3.1 implies that the Wiener and the jump components, conditioned on the event $N_{1,T} = k, N_{2,T} = m, N_T = n$, are normally distributed as

$$\Xi_{k,m,n} \sim N((\sigma_1^2 - \rho\sigma_1\sigma_2)T + n(\tilde{\alpha}_1 - \tilde{\alpha}_2 + \delta_1^2 - \rho_{\mathbf{Y}}\delta_1\delta_2) + k(\tilde{\alpha}_{11} + \delta_{11}^2) - m\tilde{\alpha}_{22}, \sigma_{k,m,n}^2 T).$$

The Poisson process N_T has arrival intensity $\hat{\lambda}_1 = \tilde{\lambda}(1 + \tilde{\kappa}_1)$ and the Poisson process $N_{1,T}$ has arrival intensity $\hat{\lambda}_{Z_1} = \tilde{\lambda}_1(1 + \tilde{\kappa}_{Z_1})$ under $\mathbb{Q}_1$, with the intensity of $N_{2,T}$ unchanged.

Similarly setting $\boldsymbol{\gamma} = [0, 1]^{\intercal}$, it follows that the random variable $\Xi_{T,k,m,n}$ is therefore normally distributed as

$$\Xi_{k,m,n} \sim N((\rho\sigma_1\sigma_2 - \sigma_2^2)T + n(\tilde{\alpha}_1 - \tilde{\alpha}_2 + \rho_{\mathbf{Y}}\delta_1\delta_2 - \delta_2^2) + k\tilde{\alpha}_{11} - m(\tilde{\alpha}_{22} + \delta_{22}^2), \sigma_{k,m,n}^2 T).$$

The Poisson process N_T has arrival intensity $\hat{\lambda}_2 = \tilde{\lambda}(1 + \tilde{\kappa}_2)$ and the Poisson process $N_{2,T}$ has arrival intensity $\hat{\lambda}_{Z_2} = \tilde{\lambda}_2(1 + \tilde{\kappa}_{Z_2})$ under $\mathbb{Q}_2$, and the intensity of $N_{1,T}$ unchanged.

Straightforward computations as in Cheang and Chiarella (2011) conclude the proof. □

Table 1 provides numerical results. We consider nine different parameter scenarios (we also set $\xi_1 = \xi_2 = 0$, $r = 0$, $t = 0$ and $T = 1$). Formula (1), row C_t^E, has been computed truncating the triple sum to $n = m = k = 25$. We also provide the Monte Carlo estimate, row MC, obtained with $N^{MC} = 10^7$ random trials, implemented using a control variate method as described in Caldana and Fusai (2013). The row labeled C.I.L. gives the length of the 95% mean-centered Monte Carlo confidence interval. In all cases C_t^E matches the Monte Carlo solution up to the sixth digit.

Scenario	1	2	3	4	5	6	7	8	9
$S_{1,0}$	100.00	100.00	96.00	100.00	100.00	96.00	100.00	100.00	96.00
$S_{2,0}$	96.00	100.00	100.00	96.00	100.00	100.00	96.00	100.00	100.00
σ_1	0.10	0.10	0.10	0.15	0.15	0.15	0.10	0.10	0.10
σ_2	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10
ρ	-0.90	-0.90	-0.90	0.50	0.50	0.50	0.90	0.90	0.90
$\tilde{\lambda}$	0.50	0.50	0.50	0.20	0.20	0.20	0.10	0.10	0.10
$\tilde{\alpha}_1$	0.03	0.03	0.03	0.06	0.06	0.06	0.03	0.03	0.03
$\tilde{\alpha}_2$	0.10	0.10	0.10	0.03	0.03	0.03	0.03	0.03	0.03
δ_1	0.10	0.10	0.10	0.03	0.03	0.03	0.03	0.03	0.03
δ_2	0.03	0.03	0.03	0.09	0.09	0.09	0.03	0.03	0.03
$\rho_{\mathbf{Y}}$	-0.90	-0.90	-0.90	-0.80	-0.80	-0.80	0.90	0.90	0.90
$\tilde{\lambda}_1$	0.50	0.50	0.50	0.20	0.20	0.20	0.10	0.10	0.10
$\tilde{\alpha}_{11}$	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02
δ_{11}	0.01	0.01	0.01	0.06	0.06	0.06	0.01	0.01	0.01
$\tilde{\lambda}_2$	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10
$\tilde{\alpha}_{22}$	0.02	0.02	0.02	-0.07	-0.07	-0.07	0.02	0.02	0.02
δ_{22}	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
C_t^E	10.770907	8.758581	6.694056	7.908547	5.820837	3.949209	4.463981	1.835108	0.463981
MC	10.770907	8.758581	6.694056	7.908547	5.820837	3.949209	4.463981	1.835108	0.463981
C.I.L.	6.048×10^{-7}	6.135×10^{-7}	6.083×10^{-7}	6.034×10^{-7}	6.130×10^{-7}	5.917×10^{-7}	4.785×10^{-7}	6.136×10^{-7}	4.785×10^{-7}

Table 1. Exchange option values are computed for nine different scenarios. C_t^E prices the exchange option according to the analytical formula (1). MC displays the Monte Carlo estimate and C.I.L. gives the length of the 95% mean-centered Monte Carlo confidence interval.

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