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# Squeezed coherent states for noncommutative spaces with minimal length uncertainty relations

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ABSTRACT: We provide an explicit construction for Gazeau-Klauder coherent states related to non-Hermitian Hamiltonians with discrete bounded below and nondegenerate eigenspectrum. The underlying spacetime structure is taken to be of a noncommutative type with associated uncertainty relations implying minimal lengths. The uncertainty relations for the constructed states are shown to be saturated in a Hermitian as well as a non-Hermitian setting for a perturbed harmonic oscillator. The computed value of the Mandel parameter dictates that the coherent wavepackets are assembled according to sub-Poissonian statistics. Fractional revival times, indicating the superposition of classical-like sub-wave packets are clearly identified.

## 1. Introduction

Noncommutative spacetime structures are suggested by gravitational stability [1] in almost all promising approaches to quantum gravity, such as string theory [2, 3, 4] or loop quantum gravity [5, 6] as well as black hole physics [7]. Besides the numerous possible structures we focus here on a particularly interesting one giving rise to minimal measurable distance beyond which the entire concept of length becomes meaningless. Such type of cutoff in our possible knowledge of space results from generalized versions of Heisenberg's uncertainty relations, i.e. from modifying standard spacetime to certain noncommutative versions in a specific way. These type of spaces have attracted considerable attention in recent years in the mathematical and physical literature [8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21].

So far most of the attention has been paid to the algebras, but fairly little, in comparison to standard systems, to the actual nature and properties of the states it acts on. Here we focus on the explicit construction of coherent states respecting these modified uncertainty relations. Generic expressions for coherent states were propose by Gazeau and Klauder (GZ) [22, 23]. By construction the states are expected to be stable when evolved in time, in the sense that they remain coherent during the evolution process. For a one dimensional

harmonic oscillator on a noncommutative space we present here a computation of these states in first order perturbation theory. When dealing with noncommutative spacetime structures of the aforementioned type one also encounters an addition complication due to the fact that the associated canonical variables are in general no longer Hermitian with regard to standard inner products [10]. Consequently Hamiltonians formulated in terms of these variables are also no longer Hermitian. We adopt here a recent approach to non-Hermitian systems [24, 25, 26, 27, 28] which renders them self-consistent and physically meaningful.

A striking feature of the coherent states presented here is the well known fact that in a certain parameter regime the original wave packet can be fully reconstructed after a specific time and can be interpreted as a superposition of classical-like sub-wave packets. In the system considered here the existence of these structures is a signature of the spacetime deformation and disappears in a standard setting. This indicates the interesting possibility that the structures of noncommutative spaces could actually be probed experimentally.

Many of the computations presented here have been attempted previously by Ghosh and Roy in [29], but almost all our results disagree with their findings and conclusions which we believe to be conceptually and compositionally incorrect as we will point in the course of our presentation.

Our manuscript is organized as follows: In section 2 we assemble various generalities on GZ-coherent states and show how the construction needs to be altered for a non-Hermitian setting. We also set up our notation for a standard perturbative treatment. In section 3 we construct GZ-states and evaluates various expectation values for a Hermitian as well as non-Hermitian setting, which we use to test the Ehrenfest theorem and the equivalence principle. In section 4 we present our analysis for the revival times structure. Our conclusions are stated in section 5.

## 2. Perturbative GZ-coherent states for non-Hermitian Hamiltonians

We commence by establishing our notations by collecting some well known facts about GZ-coherent states and indicate the necessary modifications needed for a non-Hermitian setting. The GZ-coherent states [22, 23] for a Hermitian Hamiltonian $h$ with discrete bounded below and nondegenerate eigenspectrum are defined as a two parameter set

$$|J, \gamma, \phi\rangle = \frac{1}{\mathcal{N}(J)} \sum_{n=0}^{\infty} \frac{J^{n/2} \exp(-i\gamma e_n)}{\sqrt{\rho_n}} |\phi_n\rangle, \qquad J \in \mathbb{R}_0^+, \gamma \in \mathbb{R}. \qquad (2.1)$$

The states $|\phi_n\rangle$ are the orthonormal eigenstates of $h$, that is $h |\phi_n\rangle = \hbar \omega e_n |\phi_n\rangle$. The probability distribution and normalization constant are

$$\rho_n := \prod_{k=1}^{n} e_k \qquad \text{and} \qquad \mathcal{N}^2(J) := \sum_{k=0}^{\infty} \frac{J^k}{\rho_k}, \qquad (2.2)$$

respectively, where the latter results from the requirement $\langle J, \gamma, \phi | J, \gamma, \phi \rangle = 1$.

Here we will consider Hamiltonians $H$ on noncommutative spaces which are non-Hermitian with regard to the standard inner product. The construction for the coherent

states is then easily adoptable when we assume that the Hamiltonian $H$ is pseudo/quasi Hermitian, i.e. the non-Hermitian Hamiltonian $H$ and the Hermitian Hamiltonian $h$ are related by a similarity transformation $h = \eta H \eta^{-1}$, with $\eta^\dagger \eta$ being a positive definite operator playing the role of the metric. The corresponding eigenstates $|\Phi\rangle$ and $|\phi\rangle$ of $H$ and $h$, respectively, are then simply related as $|\Phi\rangle = \eta^{-1} |\phi\rangle$. As observables are expected to be Hermitian, we need to change the metric when computing expectation values for operators associated to the non-Hermitian system [24, 25, 26, 27, 28]. The same reasoning has to be adopted for the evaluation of expectation values with regard to the coherent states. Therefore the expectation value for a non-Hermitian operator $\mathcal{O}$ related to a Hermitian operator $o$ by a similarity transformation $o = \eta \mathcal{O} \eta^{-1}$ is computed as

$$\langle J, \gamma, \Phi| \, \mathcal{O} \, |J, \gamma, \Phi\rangle_\eta := \langle J, \gamma, \Phi| \, \eta^\dagger \eta \mathcal{O} \, |J, \gamma, \Phi\rangle = \langle J, \gamma, \phi| \, o \, |J, \gamma, \phi\rangle . \tag{2.3}$$

Our notation is to be understood in the sense that in the state $|J, \gamma, \phi\rangle$ and $|J, \gamma, \Phi\rangle$ we sum over the eigenstates of the Hermitian Hamiltonian $h$ and non-Hermitian Hamiltonian $H$, respectively. These states are continuous in the two variables $(J, \gamma)$, provide a resolution of the identity, are temporarily stable, in the sense that they remain coherent states under time evolution, and satisfy the action identity

$$\langle J, \gamma, \Phi| \, H \, |J, \gamma, \Phi\rangle_\eta = \langle J, \gamma, \phi| \, h \, |J, \gamma, \phi\rangle = \hbar \omega J. \tag{2.4}$$

This identity ensures that $(J, \gamma)$ are action angle variables [22, 23].

The main purpose is here to consider a model on a noncommutative space with nontrivial commutation relations for their canonical variables giving rise to minimal uncertainties. It is then interesting to investigate how close the GK-states approach the minimum uncertainty product and eventually might even become squeezed states. Thus for a simultaneous measurement of two observables $A$ and $B$ in this system we need to evaluate the left and right hand side of the generalized version of Heisenberg's uncertainty relation

$$\Delta A \Delta B \geq \frac{1}{2} \left| \langle J, \gamma, \Phi| \, [A, B] \, |J, \gamma, \Phi\rangle_\eta \right| . \tag{2.5}$$

The uncertainties are computed as $\Delta A = \langle J, \gamma, \Phi| \, A^2 \, |J, \gamma, \Phi\rangle_\eta - \langle J, \gamma, \Phi| \, A \, |J, \gamma, \Phi\rangle_\eta^2$ and analogously for $\Delta B$. In order to test the quality of the coherent states, i.e. to see how closely they resemble classical mechanics, we may also test Ehrenfest's theorem for an operator $A$

$$i\hbar \frac{d}{dt} \langle J, \gamma + t\omega, \Phi| \, A \, |J, \gamma + t\omega, \Phi\rangle_\eta = \langle J, \gamma + t\omega, \Phi| \, [A, H] \, |J, \gamma + t\omega, \Phi\rangle_\eta . \tag{2.6}$$

We used in (2.6) the fact that the time evolution for the states $|J, \gamma, \Phi\rangle$ is simply implemented as $\exp(-iHt/\hbar) |J, \gamma, \Phi\rangle = |J, \gamma + t\omega, \Phi\rangle$, see [22, 23]. Specifying the operators $A$ and $B$ we will also test below the correspondence principle.

Here we present a perturbative treatment of the above considerations around $h_0$ for a Hamiltonian decomposable as $h = h_0 + h_1$, with $h_0 |n\rangle = e_n^{(0)} |n\rangle$. According to standard

Rayleigh-Schrödinger perturbation theory the first order expansions of the eigenenergies and the eigenstates are

$$e_n = e_n^{(0)} + \langle n| h_1 |n\rangle + \mathcal{O}(\tau^2) \quad \text{and} \quad |\phi_n\rangle = |n\rangle + \sum_{k \neq n} \frac{\langle k| h_1 |n\rangle}{e_n^{(0)} - e_k^{(0)}} |k\rangle + \mathcal{O}(\tau^2), \quad (2.7)$$

respectively. Wherever appropriate, we then simply use these expressions in (2.1) for our computations.

## 3. GK-coherent states for the noncommutative harmonic oscillator

We will now construct the GK-coherent states and various expectation values for the one dimensional harmonic oscillator [8, 10, 13]

$$H = \frac{P^2}{2m} + \frac{m\omega^2}{2} X^2 - \hbar\omega \left( \frac{1}{2} + \frac{\tau}{4} \right) \quad (3.1)$$

defined on the noncommutative space satisfying

$$[X, P] = i\hbar \left( 1 + \check{\tau} P^2 \right), \qquad X = (1 + \check{\tau} p^2) x, \qquad P = p. \quad (3.2)$$

Here $\check{\tau} := \tau/(m\omega\hbar)$ has the dimension of an inverse squared momentum with $\tau$ being dimensionless. We also provided in (3.2) a representation for the noncommutative variables in terms of the standard canonical variables $x$, $p$ satisfying $[x, p] = i\hbar$. The ground state energy is conveniently shifted to allow for a factorization of the energy. The Hamiltonian in (3.1) in terms of $x$, $p$ differs from the one treated recently in [29] as we take a different representation for $X$ and $P$, which we believe to be incorrect in [29] even up to $\mathcal{O}(\tau)$. The so-called Dyson map $\eta$, whose adjoint action relates the non-Hermitian Hamiltonian in (3.1) to its isospectral Hermitian counterpart $h$, is easily found to be $\eta = (1 + \check{\tau} p^2)^{-1/2}$. With the help of this expression we evaluate

$$h = \eta H \eta^{-1} = \frac{p^2}{2m} + \frac{m\omega^2}{2} x^2 + \frac{\omega\tau}{4\hbar}(p^2 x^2 + x^2 p^2 + 2xp^2 x) - \hbar\omega \left( \frac{1}{2} + \frac{\tau}{4} \right) + \mathcal{O}(\tau^2). \quad (3.3)$$

Taking now $h_0$ to be the standard harmonic oscillator, the energy eigenvalues for $H$ and $h$ were computed to lowest order in perturbation theory [8, 13] to

$$E_n = \hbar\omega e_n = \hbar\omega n \left[ 1 + \frac{\tau}{2}(1 + n) \right] + \mathcal{O}(\tau^2). \quad (3.4)$$

According to (2.7) we now calculate the first order expression for the wavefunctions to

$$|\phi_n\rangle = |n\rangle - \frac{\tau}{16} \sqrt{(n-3)_4} \, |n-4\rangle + \frac{\tau}{16} \sqrt{(n+1)_4} \, |n+4\rangle + \mathcal{O}(\tau^2). \quad (3.5)$$

where $(x)_n := \Gamma(x + n)/\Gamma(x)$ denotes the Pochhammer symbol. We stress that it is vital to include the second and third term in $|\phi_n\rangle$ in order to achieve an accuracy of order $\tau$ for expectation values. In [29], where a similar computation was attempted, these terms were incorrectly ignored. The expression for $E_n$ coincides with the one found in [29] for $\tau \to 2\lambda$,

as this computation only involves $\langle k| h_1 |n\rangle$. Given $e_n$ as defined by by the relation (3.4), we compute the probability density and the expansions of its inverse

$$\rho_n = \frac{1}{2^n}\tau^n n! \left(2 + \frac{2}{\tau}\right)_n \qquad \text{and} \qquad \frac{1}{\rho_n} = \frac{1}{n!} - \frac{3+n}{4(n-1)!}\tau + \mathcal{O}(\tau^2), \qquad (3.6)$$

We use the latter expression to evaluate the normalization constant in (2.2)

$$\mathcal{N}^2(J) = e^J \left(1 - \tau J - \frac{\tau}{4}J^2\right) + \mathcal{O}(\tau^2). \qquad (3.7)$$

We have now assembled all the necessary quantities to define the GZ-coherent states $|J, \gamma, \phi\rangle$ in (2.1) and are in the position to verify the validity of some of the crucial requirements on them, test their behaviour and compute expectation values.

As is well known [24, 25, 26, 27, 28], in a non-Hermitian setting the observables are not dictated by the Hamiltonian and it becomes a matter of choice to select them or equivalently the metric [24]. In fact, this is also true for a Hermitian system, where, however, the choice of the standard metric seems to be the most natural one. Here we are mainly interested in the Hamiltonian $H$ of (3.1) with $X$ and $P$ as observables, but it will also be instructive to consider first the Hermitian system described by $h$ with $x$ and $p$ being the observables of choice.

## 3.1 Obervables in the Hermitian system

At first we consider the Hamiltonian $h$ in (3.3) as fundamental and treat the variables $x$ and $p$ as observables in that system. Expectation values are then most easily computed by taking the states $|n\rangle$ to be the normalized standard Fock space eigenstates of the harmonic oscillator with usual properties $a^\dagger |n\rangle = \sqrt{n+1} |n+1\rangle$ and $a |n\rangle = \sqrt{n} |n-1\rangle$. To first order in $\tau$, we then compute the expectation values of the creation and annihilation operators

$$\langle J, \gamma, \phi| a |J, \gamma, \phi\rangle = \sqrt{J}e^{-i\gamma}\left[1 - \frac{\tau}{4}\left(2 + J + 4i\gamma(1+J)\right)\right] + \frac{\tau}{4}J^{3/2}e^{3i\gamma} + \mathcal{O}(\tau^2), \quad (3.8)$$

$$\langle J, \gamma, \phi| a^\dagger |J, \gamma, \phi\rangle = \sqrt{J}e^{i\gamma}\left[1 - \frac{\tau}{4}\left(2 + J - 4i\gamma(1+J)\right)\right] + \frac{\tau}{4}J^{3/2}e^{-3i\gamma} + \mathcal{O}(\tau^2). \quad (3.9)$$

For the details of the computation we refer the reader to the appendix. In what follows we will often drop the explicit mentioning of the order in $\tau$, understanding that all our computations are carried out to first order. Using the fact that $x = \sqrt{\hbar/(2m\omega)}(a^\dagger + a)$ and $p = i\sqrt{\hbar m\omega/2}(a^\dagger - a)$, the expectation values

$$\langle J, \gamma, \phi| x |J, \gamma, \phi\rangle = \sqrt{\frac{2J\hbar}{m\omega}}\left[\cos\gamma - \tau\left[\gamma\sin\gamma + \frac{\cos\gamma}{2} + J\sin\gamma\left(\gamma + \frac{\sin 2\gamma}{2}\right)\right]\right], \quad (3.10)$$

$$\langle J, \gamma, \phi| p |J, \gamma, \phi\rangle = -\sqrt{2Jm\omega\hbar}\left[\sin\gamma + \tau\left[\gamma\cos\gamma - \frac{\sin\gamma}{2} + J\cos\gamma\left(\gamma - \frac{\sin 2\gamma}{2}\right)\right]\right],$$

then follow trivially from (3.8) and (3.9). Expanding $x^2$ and $p^2$ in terms of $a^\dagger$ and $a$, a similar, albeit more lengthy, computation yields

$$\langle J, \gamma, \phi |\, x^2 \,| J, \gamma, \phi \rangle = \frac{\hbar}{2m\omega} \left[ 1 + 4J \cos^2 \gamma - \tau J \left( 6\gamma \sin 2\gamma + \cos 2\gamma + 2 \right) \right. \quad (3.11)$$
$$\left. - \tau J^2 (4\gamma \sin 2\gamma - \cos 4\gamma + 1) \right],$$

$$\langle J, \gamma, \phi |\, p^2 \,| J, \gamma, \phi \rangle = \frac{\hbar m\omega}{2} \left[ 1 + 4J \sin^2 \gamma + \tau J \left( 6\gamma \sin 2\gamma + \cos 2\gamma - 2 \right) \right. \quad (3.12)$$
$$\left. + \tau J^2 (4\gamma \sin 2\gamma + \cos 4\gamma - 1) \right].$$

These two expressions may be used to compute the expectation value of $h$, as defined in (3.3), with regard to the GZ-coherent states. The remaining term in $h$ only needs to be computed to zeroth order to achieve an overall accuracy of order $\tau$. We therefore calculate

$$\langle J, \gamma, \phi |\, p^2 x^2 + x^2 p^2 + 2xp^2 x \,| J, \gamma, \phi \rangle = \hbar^2 (1 + 4J + 2J^2 - 2J^2 \cos 4\gamma) + \mathcal{O}(\tau). \quad (3.13)$$

Summing the contributions from (3.10), (3.11) and (3.12), together with the required prefactors to make up the Hamiltonian $h$, yields the action identity (2.4) as expected. We remark that this crucial identity was violated in [29].

Employing the above quantities we can also investigate how close the coherent states approach the minimum uncertainty product. Assembling the required expectation values we then obtain

$$\Delta x^2 = \langle J, \gamma, \phi |\, x^2 \,| J, \gamma, \phi \rangle - \langle J, \gamma, \phi |\, x \,| J, \gamma, \phi \rangle^2 = \frac{\hbar}{2m\omega} \left[ 1 + \tau J (\cos 2\gamma - 2\gamma \sin 2\gamma) \right], (3.14)$$

$$\Delta p^2 = \langle J, \gamma, \phi |\, p^2 \,| J, \gamma, \phi \rangle - \langle J, \gamma, \phi |\, p \,| J, \gamma, \phi \rangle^2 = \frac{\hbar m\omega}{2} \left[ 1 - \tau J (\cos 2\gamma - 2\gamma \sin 2\gamma) \right], (3.15)$$

and therefore

$$\Delta x \Delta p = \frac{\hbar}{2} + \mathcal{O}(\tau^2). \quad (3.16)$$

Thus the states $|J, \gamma, \phi \rangle$ saturate the minimal uncertainty in a simultaneous measurement of $x$ and $p$ and therefore constitute squeezed states for all values of $J$ and $\gamma$ up to first order in perturbation theory.

Using (3.10) we also verify Ehrenfest's theorem (2.6) for the operators $x$

$$i\hbar \frac{d}{dt} \langle J, \gamma + t\omega, \phi |\, x \,| J, \gamma + t\omega, \phi \rangle = \langle J, \gamma + t\omega, \phi |\, [x, h] \,| J, \gamma + t\omega, \phi \rangle, \quad (3.17)$$

$$= \langle J, \gamma + t\omega, \phi |\, \frac{i\hbar}{m} p + \frac{i\tau\omega}{2} (px^2 + x^2 p + 2xpx) \,| J, \gamma + t\omega, \phi \rangle$$

$$= -i\hbar^{3/2} \sqrt{\frac{2J\omega}{m}} \left[ \sin \hat{\gamma} + \tau \left[ \frac{1}{2} \sin \hat{\gamma} + \cos \hat{\gamma} \left( (J+1)\hat{\gamma} + \frac{3}{2} J \sin 2\hat{\gamma} \right) \right] \right]$$

and $p$

$$i\hbar \frac{d}{dt} \langle J, \gamma + t\omega, \phi |\, p \,| J, \gamma + t\omega, \phi \rangle = \langle J, \gamma + t\omega, \phi |\, [p, h] \,| J, \gamma + t\omega, \phi \rangle, \quad (3.18)$$

$$= \langle J, \gamma + t\omega, \phi |\, -i\hbar m\omega^2 x - i\tau\omega (px^2 + x^2 p) \,| J, \gamma + t\omega, \phi \rangle$$

$$= -i\sqrt{2Jm}\hbar^{3/2}\omega^{3/2} \left[ \cos \hat{\gamma} + \frac{\tau}{4} \left[ (3J+2) \cos \hat{\gamma} - 4(J+1)\hat{\gamma} \sin \hat{\gamma} - 3J \cos 3\hat{\gamma} \right] \right].$$

For convenience we abbreviated here $\hat{\gamma} := \gamma + t\omega$.

## 3.2 Obervables in the non-Hermitian system

As stated, the system we actually wish to investigate is described by the non-Hermitian Hamiltonian (3.1) with a non-trivial commutation relation (3.2) for its associated observables $X$ and $P$. In order to test the inequality (2.5) we need to compute

$$\langle J, \gamma, \Phi | \, X \, | J, \gamma, \Phi \rangle_\eta = \sqrt{\frac{2J\hbar}{m\omega}} \left[ \cos \gamma + \frac{\tau}{2} \sin \gamma (J \sin 2\gamma - 2\gamma(1 + J)) \right], \tag{3.19}$$

$$\langle J, \gamma, \Phi | \, X^2 \, | J, \gamma, \Phi \rangle_\eta = \frac{\hbar}{2m\omega} \left[ 1 + 4J \cos^2 \gamma + \tau[1 + J(2 - 2\cos 2\gamma - 6\gamma \sin 2\gamma) \right. \tag{3.20}$$
$$\left. + 2J^2 \sin 2\gamma (\sin 2\gamma - 2\gamma)] \right].$$

We note here that the actual computation has been carried out by translating first all quantities to a Hermitian setting and then following the same reasoning as in the previous subsection. Combining (3.19) and (3.20) then yields

$$\Delta X^2 = \langle J, \gamma, \Phi | \, X^2 \, | J, \gamma, \Phi \rangle_\eta - \langle J, \gamma, \Phi | \, X \, | J, \gamma, \Phi \rangle_\eta^2 \tag{3.21}$$
$$= \frac{\hbar}{2m\omega} \left[ 1 + \tau \left( 1 + J(2 - 2\gamma \sin 2\gamma - \cos 2\gamma) \right) \right].$$

The computation for the expectation values of $P$ is simpler, since the metric commutes with $p$, such that

$$\langle J, \gamma, \Phi | \, P \, | J, \gamma, \Phi \rangle_\eta = \langle J, \gamma, \phi | \, p \, | J, \gamma, \phi \rangle, \quad \text{and} \quad \langle J, \gamma, \Phi | \, P^2 \, | J, \gamma, \Phi \rangle_\eta = \langle J, \gamma, \phi | \, p^2 \, | J, \gamma, \phi \rangle, \tag{3.22}$$

and therefore

$$\Delta P^2 = \Delta p^2. \tag{3.23}$$

Expanding finally (3.21) and (3.23), we obtain

$$\Delta X \Delta P = \frac{\hbar}{2} \left[ 1 + \frac{\tau}{2} \left( 1 + 4J \sin^2 \gamma \right) \right] = \frac{\hbar}{2} \left( 1 + \hat{\tau} \langle J, \gamma, \Phi | \, P^2 \, | J, \gamma, \Phi \rangle \right). \tag{3.24}$$

This means that also in the non-Hermitian setting the minimal uncertainty product for the observables $X$ and $P$, commuting as specified in (3.2), is saturated. Thus to first order in perturbation theory also the GZ-coherent states $| J, \gamma, \Phi \rangle$ are squeezed states, remarkably this holds irrespective of the values for $J$ and $\gamma$.

Apparently this result was also obtained in [29], but our disagreement with the results presented in there is at least fourfold. Firstly, the authors used the incorrect representation for the canonical variables $X$ and $P$ as mentioned earlier. Secondly the authors computed conceptually the wrong expectations values even when using their representation. Thirdly, the authors only take the first order in (3.5) into account and therefore miss out various terms contributing to the first order calculation in $\tau$. Finally, even ignoring the previous three points and adopting all the wrong concepts used in [29], we disagree on a purely computational level with many of the expressions presented in there.

Next we also verify Ehrenfest's theorem (2.6) for the operators $X$

$$i\hbar\frac{d}{dt}\left\langle J,\gamma+t\omega,\Phi\right|X\left|J,\gamma+t\omega,\Phi\right\rangle_\eta = \left\langle J,\gamma+t\omega,\Phi\right|[X,H]\left|J,\gamma+t\omega,\Phi\right\rangle_\eta, \quad (3.25)$$

$$= \left\langle J,\gamma+t\omega,\Phi\right|\frac{i\hbar}{m}(P+\check{\tau}P^3)\left|J,\gamma+t\omega,\Phi\right\rangle_\eta$$

$$= -i\hbar^{3/2}\sqrt{\frac{2J\omega}{m}}\left[\sin\hat{\gamma}+\tau\left[(J+1)\hat{\gamma}\cos\hat{\gamma}+\frac{1}{2}\sin\hat{\gamma}(2+J-3J\cos 2\hat{\gamma})\right]\right]$$

and the operator $P$

$$i\hbar\frac{d}{dt}\left\langle J,\gamma+t\omega,\Phi\right|P\left|J,\gamma+t\omega,\Phi\right\rangle_\eta = \left\langle J,\gamma+t\omega,\Phi\right|[P,H]\left|J,\gamma+t\omega,\Phi\right\rangle_\eta, \quad (3.26)$$

$$= \left\langle J,\gamma+t\omega,\Phi\right|-im\hbar\omega^2\left(X+\frac{\check{\tau}}{2}XP^2+\frac{\check{\tau}}{2}P^2X\right)\left|J,\gamma+t\omega,\Phi\right\rangle_\eta$$

$$= -i\sqrt{2Jm}\hbar^{3/2}\omega^{3/2}\left[\cos\hat{\gamma}+\frac{\tau}{4}\left[(3J+2)\cos\hat{\gamma}-4(J+1)\hat{\gamma}\sin\hat{\gamma}-3J\cos 3\hat{\gamma}\right]\right].$$

Taking now for simplicity $\gamma=0$, differentiating (3.25) once again and combining it with (3.26) we obtain the corresponding identity to Newton's equation of motion

$$\left\langle J,t\omega,\Phi\right|\ddot{X}\left|J,t\omega,\Phi\right\rangle_\eta = -\omega^2\left\langle J,t\omega,\Phi\right|X+\frac{\check{\tau}}{2}(3XP^2+3P^2X+2PXP)\left|J,t\omega,\Phi\right\rangle_\eta, \quad (3.27)$$

The relations (3.25) and (3.26) were not recovered in [29], where the comparison between the left and right hand sides mismatched. Instead of (3.27) the authors proposed a "correspondence principle with twist". According to our argumentation this is incorrect and there is in fact no reason to assume the Newton's equation is simply the same as the one for the standard harmonic oscillator. The reason for the discrepancy are the aforementioned conceptual and computational mistakes in [29].

## 4. Fractional revival structure

Further insights into the comparison between the classical and quantum description can be obtained from the revival time for wave packets. For a general wave packet of the form $\psi = \sum c_n\phi_n$ sufficiently well localized, i.e. being governed by sub-Poissonian statistics, near a submode $n=\bar{n}$ with energy $E_{\bar{n}}$, it was argued in [30] that beside the revival of the classical-like wave packet after the classical period $T_{cl}=2\pi\hbar/|E'_{\bar{n}}|$ one may also encounter so-called fractional revivals. These partial revivals of the original wave packet occur at times $p/q(T_{rev})$, with coprime integers $p$, $q$ and the revival time given by $T_{rev}=4\pi\hbar/|E''_{\bar{n}}|$. Depending on the values of $p$ and $q$ one can interpret the emerging features as different types of superpositions of classical-like wave packets.

For the case at hand we may follow [23] and expand the energy $E_n$ in the expression for the wave packets (2.1)

$$|J,\omega t,\phi\rangle = \sum_{n=0}^{\infty} c_n(J)\exp(-itE_n/\hbar)|\phi_n\rangle, \quad (4.1)$$

with weighting function $c_n(J) = J^{n/2}/\mathcal{N}(J)\sqrt{\rho_n}$, about $\bar{n} := \langle n \rangle = Jd\ln\mathcal{N}^2(J)/dJ$. To first order in perturbation theory we then easily compute

$$\langle n \rangle = J - \tau \left( J + \frac{J^2}{2} \right) + \mathcal{O}(\tau^2), \quad \text{and} \quad \langle n^2 \rangle = J + J^2 - \tau \left( J + 3J^2 + J^3 \right) + \mathcal{O}(\tau^2), \quad (4.2)$$

such that

$$\Delta n^2 = \langle n^2 \rangle - \langle n \rangle^2 = J - \tau \left( J + J^2 \right) + \mathcal{O}(\tau^2). \quad (4.3)$$

Consequently the Mandel parameter $Q$ [31] turns out to be negative

$$Q := \frac{\Delta n^2}{\langle n \rangle} - 1 = -\frac{J\tau}{2} + \mathcal{O}(\tau^2) < 0, \quad (4.4)$$

suggesting a sub-Poissonian statistics. This implies that we have a strong localization around $\bar{n}$ required for the possibility of the revival of the classical-like sub-wave packet. According to the above expressions we compute the classical period and the revival time to

$$T_{\text{cl}} = \frac{2\pi}{\omega} - \frac{\tau}{\omega}(1 + 2J)\pi, \quad \text{and} \quad T_{\text{rev}} = \frac{4\pi}{\omega\tau}. \quad (4.5)$$

We now use these quantities to analyze the behaviour of the autocorrelation function

$$A(t) := |\langle J, \gamma, \phi \, | J, \gamma + t\omega, \phi \rangle|^2 = \left| \langle J, \gamma, \Phi \, | J, \gamma + t\omega, \Phi \rangle_\eta \right|^2. \quad (4.6)$$

In order to find a set of meaningful values for our free parameters $J$, $\tau$ and also to find an appropriate upper limit cutoff in the sum (4.1), let us first investigate the weighting function $c_n(J)$.
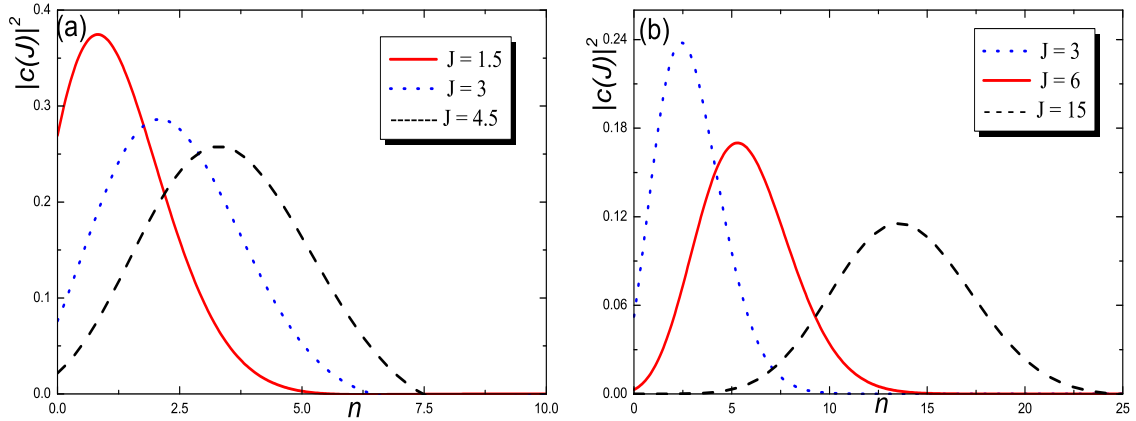


**Figure 1:** (a) Weighting function for $\tau = 0.1$ with $\langle n \rangle = 1.24, 2.25, 3.04$ for $J = 1.5, 3, 4.5$, respectively and (b) $\tau = 0.01$ with $\langle n \rangle = 2.93, 5.76, 13.72$ for $J = 3, 6, 15$, respectively.

For the chosen values we observer in figure 1 that the wave packets are well localized around $\bar{n}$ resulting from (4.2), such that the prerequisite for the validity of the analysis in [30] is given. Increasing the values of $J$ for fixed $\tau$ we observe negative values for $|c(J)|^2$ for large values of $n$, which clearly indicates that our pertubative expressions are no longer

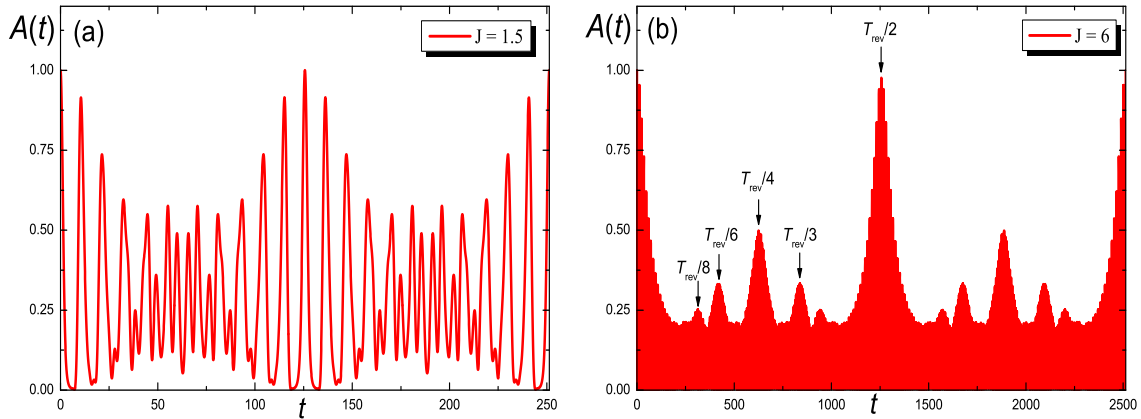**Figure 2:** (a) Autocorrelation function as a function of time for $J = 1.5$, $\tau = 0.1$, $\omega = 0.5$, $\hbar = 1$, $\gamma = 0$, $T_{cl} = 10.05$ and $T_{rev} = 251.32$; (b) Autocorrelation function as a function of time for $J = 6$, $\tau = 0.01$, $\omega = 0.5$, $\hbar = 1$, $\gamma = 0$, $T_{cl} = 11.74$ and $T_{rev} = 2513.27$.

valid in that regime. We also note that $n \approx 50$ will be a sufficiently good value to terminate the sum in the expression for the autocorrelation function (4.1) analyzed in figure 2.

In panel (a) of figure 2 we clearly observe local maxima at multiples of the classical period $T_{cl}$. As explained in [30] the first full reconstruction of the original wave packet is obtained at $T_{rev}/2$ which is clearly visible in panel (a). The fractional revivals are better observed for smaller values of $\tau$ as depicted in panel (b). In that scenario the classical periods are so small as compared to the revival time that they are no longer resolved. We clearly observe a number of fractional revivals, such as for instance $T_{rev}/4$ corresponding to the superposition of two classical-like sub-wave packets and others as indicated in the figure.

Notice that our expression and our analysis presented here differs once again from the one in [29], where for instance the mandatory revival at $T_{rev}/2$ was not observed.

## 5. Conclusions

Our central results is the construction of explicit expressions for the GZ-coherent states for a non-Hermitian system on a noncommutative space leading to a generalized version of Heisenberg's uncertainty relation. We showed that these states are squeezed for all values of $J$ and $\gamma$ as they saturate the minimal uncertainty. Crucially we established two of the nontrivial GZ-axioms are satisfied. First of all the states are shown to be temporarily stable, i.e. they remain coherent under time evolution, and second the states satisfy the action identity (2.4) allowing for a close relation to a classical description in terms of action angle variables. We also demonstrated that when using the appropriate metric Ehrenfest's theorem is satisfied for the observables $X$ and $P$. The desired resemblance of the coherent states with a classical description was further underpinned by an analysis of the revival structure exhibiting the typical quasiclassical evolution of the original wave packet. It should be noted that in the considered case the wave packet revival time (4.5)

depends explicitly on the deformation parameter $\tau$, such that a possible measurement could distinguish between a noncommutative and a standard commutative space. For instance, in the order of femtoseconds half and quarter revivals have been observed experimentally [32] for molecular wave packets described by anharmonic oscillator potentials with eigenenergies similar to (3.4). Our analysis holds to first order perturbation theory in $\tau$ and of course it would be very interesting to extend this to higher order or eventually to the exact case.

There are various other directions into which our analysis might be taken forward. For instance, different types of models might exhibit a varied behaviour. More challenging is a construction for such states in higher dimensions. A systematic comparison with different types of coherent states would be insightful, especially with rare constructions related to non-Hermitian Hamiltonians [33].

## A. Appendix

We present here a sample computation in order to make our results transparent and reproducible. In addition, we highlight the differences compared to [29]. One of the simplest computation in this context is to evaluate the expectation value of the annihilation operator (3.8). The evaluation of expectation values for different types of operators is more involved, but goes along the same lines. Using the expression for (2.1) we compute

$$\langle J, \gamma, \phi |\, a\, | J, \gamma, \phi \rangle = \frac{1}{\mathcal{N}^2} \sum_{n,m=0}^{\infty} \frac{J^{(m+n)/2} \exp\left[i\gamma(e_m - e_n)\right]}{\sqrt{\rho_m \rho_n}} \langle \phi_m |\, a\, |\phi_n \rangle. \qquad \text{(A.1)}$$

With the expansion of $|\phi_n\rangle$ to first order in $\tau$ we obtain

$$\begin{aligned}
\langle \phi_m |\, a\, |\phi_n \rangle &= \sqrt{n}\,\delta_{m,n-1} + \frac{\tau}{16}\left(\sqrt{(n+1)_4}\sqrt{n} - \sqrt{(n-3)_4}\sqrt{n-4}\right)\delta_{m,n-5} \qquad \text{(A.2)}\\
&\quad + \frac{\tau}{16}\left(\sqrt{(n+1)_4}\sqrt{n+4} - \sqrt{(n-3)_4}\sqrt{n}\right)\delta_{m,n+3},\\
&= \sqrt{n}\,\delta_{m,n-1} + \frac{\tau}{4}\sqrt{(n+1)(n+2)(n+3)}\,\delta_{m,n+3},
\end{aligned}$$

such that

$$\langle J, \gamma, \phi |\, a\, | J, \gamma, \phi \rangle = \sum_{n=1}^{\infty} \frac{\sqrt{n}\,J^{n-1/2}e^{i\gamma(e_{n-1}-e_n)}}{\mathcal{N}^2\sqrt{\rho_{n-1}\rho_n}} + \tau \sum_{n=0}^{\infty} \frac{J^{n+3/2}\sqrt{(n+1)_3}\,e^{i\gamma(e_{n+3}-e_n)}}{4\mathcal{N}^2\sqrt{\rho_{n+3}\rho_n}} + \mathcal{O}(\tau^2).$$
$$\text{(A.3)}$$

The last sum has been ignored in [29], but is an important contribution to order $\tau$. Using

$e_{n-1} - e_n = -1 - n\tau$ and $\rho_n = \rho_{n-1}e_n$ the first sum in (A.3) is evaluated as

$$\frac{e^{-i\gamma}}{\mathcal{N}^2}\sum_{n=1}^{\infty}\frac{J^{n-1/2}e^{-i\gamma n\tau}}{\rho_{n-1}\sqrt{1+\frac{\tau}{2}(1+n)}} = \frac{e^{-i\gamma}}{\mathcal{N}^2}\sum_{n=1}^{\infty}\frac{J^{n-1/2}}{\rho_{n-1}}\left[1 - \frac{\tau}{4}(1+n+4i\gamma n)\right] + \mathcal{O}(\tau^2), \text{(A.4)}$$

$$= \frac{e^{-i\gamma}}{\sqrt{J}}\left[\left(1-\frac{\tau}{4}\right)\frac{\sum_{n=1}^{\infty}\frac{J^n}{\rho_{n-1}}}{\sum_{n=0}^{\infty}\frac{J^n}{\rho_n}} - \frac{\tau}{4}(1+4i\gamma)\frac{\sum_{n=1}^{\infty}\frac{nJ^n}{\rho_{n-1}}}{\sum_{n=0}^{\infty}\frac{J^n}{\rho_n}}\right] + \mathcal{O}(\tau^2), \tag{A.5}$$

$$= \sqrt{J}e^{-i\gamma}\left[1 - \frac{\tau}{4}\left(2 + J + 4i\gamma(1+J)\right)\right] + \mathcal{O}(\tau^2). \tag{A.6}$$

For the second sum in (A.3) we use $e_{n+3} - e_n = 3 + 3\tau(2+n)$ and $\rho_{n+3} = \rho_n e_{n+3}e_{n+2}e_{n+1}$, such that it becomes

$$\tau\sum_{n=0}^{\infty}\frac{J^{n+3/2}\sqrt{(n+1)_3}e^{i\gamma(3+3\tau(2+n))}}{4\mathcal{N}^2\rho_n\sqrt{e_{n+3}e_{n+2}e_{n+1}}} + \mathcal{O}(\tau^2) = \frac{\tau J^{3/2}e^{3i\gamma}}{4\mathcal{N}^2}\sum_{n=0}^{\infty}\frac{J^n}{\rho_n} + \mathcal{O}(\tau^2) = \frac{\tau J^{3/2}e^{3i\gamma}}{4} + \mathcal{O}(\tau^2).$$
$$\tag{A.7}$$

Collecting (A.6) and (A.7) we obtain (3.8). Similarly we compute the expectation values for $x^2$, $p^2$, $x^2p^2$ etc by converting them first into expressions involving the $a$ and $a^\dagger$ and then using the arguments from above. For the noncommutative scenario we convert first to a setting involving Hermitian operators. For instance, we use

$$\langle J, \gamma, \Phi| X |J, \gamma, \Phi\rangle_\eta = \langle J, \gamma, \phi| x + \frac{\check{\tau}}{2}(p^2x + xp^2) |J, \gamma, \phi\rangle + \mathcal{O}(\tau^2) \tag{A.8}$$

and compute the right hand side as explained above.

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