The rational status of quantum cognition

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Abstract
Classic probability theory (CPT) is generally considered the rational way to make inferences, but there have been some empirical findings showing a divergence between reasoning and the principles of classical probability theory (CPT), inviting the conclusion that humans are irrational. Perhaps the most famous of these findings is the conjunction fallacy (CF). Recently, the CF has been shown consistent with the principles of an alternative probabilistic framework, quantum probability theory (QPT). Does this imply that QPT is irrational or does QPT provide an alternative interpretation of rationality? Our presentation consists of three parts. First, we examine the putative rational status of QPT using the same argument as used to establish the rationality of CPT, the Dutch Book (DB) argument, according to which reasoners should not commit to bets guaranteeing a loss. We prove the rational status of QPT by formulating it as a particular case of an extended form of CPT, with separate probability spaces produced by changing context. Second, we empirically examine the key requirement for whether a CF can be rational or not; the results show that participants indeed behave rationally, at least relative to the representations they employ. Finally, we consider whether the conditions for the CF to be rational are applicable in the outside (non-mental) world. Our discussion provides a general and alternative perspective for rational probabilistic inference, based on the idea that contextuality requires either reasoning in separate CPT probability spaces or reasoning with QPT principles.

Keywords
Decision making, conjunction fallacy, Dutch Book Theorem, rationality.

1. Introduction

One of the most evocative findings in the decision literature is Tversky and Kahneman's (1983) so-called conjunction fallacy (CF). Participants were told of a hypothetical person, Linda (the most famous hypothetical personage in decision theory), described as a feminist (*F*) and not like a bank teller (*BT*). Participants were asked to rank order the likelihood of statements about Linda, the critical ones being *BT* versus $F\&BT$. The extensively replicated result (Busemeyer et al., 2011) is typically interpreted as showing that participants consider $Prob(F\&BT) > Prob(BT)$. This finding has epitomized irrational decision making, not just because it is superficially incorrect, but because it seems impossible to have $Prob(F\&BT) > Prob(BT)$. The CF is a statement along the lines that the counts for *A*&*B* (e.g., that it rains and snows, over a certain period) can be higher than the counts for just *A* (e.g., that it just rains) – superficially impossible. Moreover, the CF is persistent, in that the CF seems an intuitive judgment, even after the relevant principles have been explained (cf. Gilboa, 2009). Can human intuition be sometimes so at odds with normative prescription?

The principles at stake are those of classical probability theory (CPT), which we operationalize in terms of the Kolmogorov axioms, regardless of whether probabilities correspond to subjective degrees of belief (Bayesian approach) or are computed as frequencies (frequentist approach). CPT has been the predominant descriptive and normative framework for decision making. CPT cognitive models have enjoyed widespread success and, indeed, CPT principles are highly intuitive, which is a salutary observation, given that it is human intuition we are trying to model. In a well-known quote by Laplace (1816, cited in Perfors et al., 2011) "… [CPT] is nothing but common sense reduced to calculation." Additionally, CPT is argued to embody rational decision making (Oaksford & Chater, 2009; see also Griffiths et al., 2010; Tenenbaum et al., 2011). But there have also been reports of persistent divergence between CPT prescription and human behavior (e.g., Tversky & Kahneman, 1983). We develop this article focused on the CF, but arguments are intended as general. Also, we use the label Linda paradigm to refer the basic procedural details of all related decision tasks (and analogously for the *BT*, *F* conjuncts).

We are interested in whether the CF is indeed a fallacy or not and whether it necessarily reflects irrational decision making. If the CF is not a fallacy, then it would be more accurate to talk about a conjunction effect (CE), which we do when a CE reflects correct decision making.

Note, some authors have argued that naïve participants do not understand the 'and' in the Linda paradigm as a conjunction, in which case the CF would not necessarily be a fallacy. Even though the relevant empirical results do not all go in one direction, the emerging picture is that when manipulations are introduced to disambiguate the meaning of 'and' in the conjunction, a CF persists (Bar-Hillel & Neter, 1993; Bonini et al., 2004; Tentori & Crupi, 2012; but see Mellers et al., 2001). A related issue is whether the *BT* conjunct is understood as the (intended) marginal or as a statement like *BT*&~*F* .Understanding the conjunct as *BT*&~*F* absolves the CE of fallacy (to use Dulany & Hilton's, 1991, phrase). There is good motivation for considering the possibility that participants interpret the *BT* conjunct as *BT*&~*F*. Dulany and Hilton (1991; Hilton, 1995; Hilton & Slugoski, 2001; see also Adler, 1984) proposed that social rules governing communication, such as the various maxims of conversational implicatures (Grice, 1975), ought to be taken into account when considering how a logical statement is understood. They suggested that, because of the rich information provided about Linda, participants are likely to assume that omitting the ~*F* information in the *BT* conjunct just indicates ~*F* anyway (cf. Levinson, 1983). The implication is that when the conjunct is properly disambiguated, the CF should be reduced or eliminated (Dulany & Hilton, 1984; Macdonald & Gilhooly, 1990). However, other studies manipulating the way the *BT* conjunct is disambiguated still reported large CF rates (e.g., Agnoli & Krantz, 1989, and Messer & Griggs, 1993, as well as the original Tversky & Kahneman, 1983, study). Moreover, some researchers directly introduced the conjunction *BT*&~*F* in the list of options and still reported large CF rates (Tentori et al., 2004; Wedell & Moro, 2008).

Overall, it seems clear that conversational implicatures play an important role in the Linda paradigm (Hilton, 1995). Equally, it is also the case that in many cases when the conjunct should be sufficiently disambiguated there is a residual CF (Moro, 2009).

The CF is incorrect according to CPT, under standard assumptions. CPT has come to dominate beliefs for probabilistic inference to such an extent, that it is hard to even consider alternative intuitions for probabilistic inference. However, mathematicians have long known that CPT is not the only probability theory (Narens, 2015). It is clearly possible that the same judgment may be correct according to one system and incorrect according to another. Quantum Probability Theory (QPT), the rules for assigning probabilities to events from quantum mechanics, without the physics, is a framework for probabilistic inference, alternative to CPT. Some researchers have been exploring the value of QPT in cognitive modelling, partly because of the inherent *contextuality* in QPT probabilistic inference. If in some cases probabilistic inference shows contextuality, then maybe QPT is an appropriate framework for providing corresponding cognitive models (Busemeyer & Bruza, 2011; Pothos & Busemeyer, 2013). QPT cognitive models have been proposed for the CF, other probabilistic fallacies in decision making (e.g., violations of the law of total probability), and other classically problematic results (e.g., violations of logic in conceptual combinations; Aerts & Aerts, 1995; Atmanspacher & Filk, 2010; Busemeyer et al., 2011; Haven & Khrennikov, 2013).

In QPT, $Prob(Y\&X) > Prob(Y)$ can be appropriate. So, we have a situation where, superficially, the same judgment is correct according to one probabilistic framework (QPT), but incorrect according to another (CPT). Is it possible to determine which framework is more appropriate for evaluating the CE in the Linda paradigm? Moreover, given such conflicts in corrective prescription, can we reach any conclusion regarding the rational status or not of the CE?

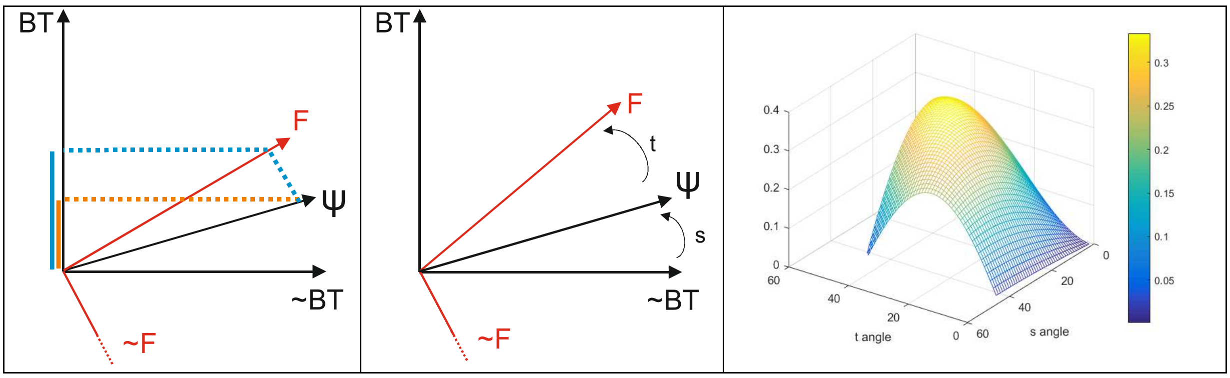
2. The conjunction fallacy, QPT, and CPT with multiple probability spaces

The CE is possible in QPT for incompatible questions. For incompatible questions, a joint probability distribution is impossible and answering one question typically leads to uncertainty about the other. For incompatible questions, conjunctions must be assessed sequentially, for example, $Prob(F\&then\ BT)$, and each question creates a unique perspective or context for answering subsequent incompatible ones. So, regarding the CE, the intuitive explanation is that the *BT* possibility is unlikely in isolation but, once the *F* possibility is accepted, the *BT* one becomes more likely, since feminists can have all sorts of professions, including the *BT* one.

We can demonstrate simply how a CE arises (full details in Busemeyer et al., 2011). We represent the mental state as a column vector $|\psi\rangle$ and different questions are represented as subspaces. Each subspace is associated with a projector operator, determining the projection of a vector along the subspace. If *A* is a subspace corresponding to a question about $|\psi\rangle$, then $Prob(A) = |P_A|\psi\rangle|^2$, where P_A is the projector to *A*, i.e. probability is the squared length of the projection of $|\psi\rangle$ along the *A* subspace. The conjunction of incompatible questions *F*, *BT* is computed as a sequential projection (which can be thought of as evaluating the two questions one after the other), $Prob(F \wedge then\ BT) = |P_{BT} \cdot P_F \cdot |\psi\rangle|^2$, where P_{BT} and P_F are the projectors corresponding to the *BT*, *F* questions. It then follows that $|P_{BT}|\psi\rangle|^2 = |P_{BT}P_F|\psi\rangle|^2 + |P_{BT}P_{\sim F}|\psi\rangle|^2 + \delta$, where $P_{\sim F}$ is the projector for the question of whether Linda is not a feminist and δ is an interference term. Since δ can be negative, a CE is possible. Figure 1a is a toy illustration of the QPT model, in which the *F*, *BT* questions are one-dimensional in a two-dimensional space. The state vector $|\psi\rangle$ is close to the *F* and ~*BT* subspaces, as per the initial description of Linda. Then, $Prob(F \wedge then\ BT) = |P_{BT} \cdot P_F \cdot |\psi\rangle|^2$ is the sequential projection first to the *F* subspace and then to the *BT* one, illustrated as the solid blue vertical line in Figure 1a. Clearly, the direct projection from the mental state to the *BT* subspace has a shorter length, that is, $Prob(F \wedge then\ BT) > Prob(BT)$, which is a graphical demonstration of the intuition that the *BT* question is unlikely from the baseline perspective of the Linda description, but more likely from the perspective of accepting Linda as a *F*. Note, incompatibility in QPT requires specific assumptions regarding the relevant questions (Figure 1b), which in turn guide predictions for the CE (Figure 1c).

To a physicist, this discussion should not be surprising, since CEs can occur with physical systems. Consider electron spin (the argument generalizes to higher dimensional spaces). The questions regarding the vertical and horizontal spin of a particle are incompatible. Consider an electron prepared in a state such that vertical spin is up. Then, $Prob(Down|C_V) = |\langle D|\psi\rangle|^2 = 0$ (since the electron is in the up direction and we are computing the probability that it is in the down direction). But, suppose we measure horizontal spin first. Because of incompatibility, this measurement creates uncertainty regarding the vertical spin of the electron, so that $Prob(Right\ and\ then\ Down|C_{HV}) > 0$ [Footnote 1].

Are we to conclude that mother nature is irrational? Nature is neither rational or irrational, but it does work according to quantum rules (at the microphysical level). But of course psychologists would hardly be concerned with the microphysical world. The point of the spin example is to challenge the perception that a CE is 'impossible'. Clearly, there are (microphysical) situations that produce probabilities consistent with the CE. The challenge for the psychologist is to understand the source of the CE in terms relevant to decision making and explore whether the circumstances that produce the CE in physical systems have some analogues for decision making situations.



a	*b*	*c*

Figure 1a. An illustration of how the CE emerges in a QPT approach to Tversky and Kahneman's (1983) Linda paradigm. The two-dimensional representation is a caricature and the QPT model proper must be specified with subspaces of arbitrary dimensionality (Busemeyer et al., 2011). Figure 1b. In a QPT model basis sets need be precisely related. Decreasing the angle *s*+*t* implies an assumption that a prototypical feminist is less likely to be a *BT*. Figure 1c. Plotting the quantity $|P_B P_F|\psi\rangle|^2 - |P_B|\psi\rangle|^2 = sin^2(t+s) \cdot cos^2 t - sin^2(s)$, which is the size of the CE in this simple model.

In decision terms, incompatibility means *contextuality*, so that the same question in isolation has to be treated as a different question compared to when processed in the context of other, incompatible ones. In the Linda paradigm, we can denote the *BT* question in isolation as just *BT* and this same question in the context of the *F* question as BT_F (BT_F is not *BT* conditioned on *F* true, it is just different from *BT*). Then, if *F*, *BT* are incompatible, $BT \neq BT_F$ and so it is clearly possible that $Prob(F\&BT_F) > Prob(BT)$. For compatible questions, this contextual dependence does not exist, $BT \equiv BT_F$. A set of compatible questions forms a Boolean algebra, but incompatible questions form a partial Boolean algebra, which is a set of Boolean algebras combined together (Hughes, 1989).

The idea that contextuality can alter the meaning of superficially identical questions is not unique to QPT; it can be motivated from alternative both non-technical and technical perspectives. Regarding the former, consider the literature on conversational implicatures in relation to CE tasks. It has already been proposed that context-dependent interpretations of a question would obviate a need for consistency across question repetition (review in Hilton, 1995). How does this translate in a Linda paradigm? Part-whole contrasts refer to the finding that e.g. if a person is asked about romantic life and life in general, the second question would be treated as referring to any aspects of life excluding romantic life. Such an assumption follows from the conversational maxim of quantity, that a questioner should be provided with new information (Grice, 1975). It relates to the given-new contract, which is an assumption that a subsequent question should involve new information, compared to a previous one, e.g., are there more Fords or are there more cars in the parking lot? (Adler, 1984; Hilton, 1995). In the Linda paradigm, we can then speculate that the *BT* question is about the characteristics that make a person like a *BT*. By contrast, in the context of the $F\&BT_F$ conjunction, the BT_F question might acquire a different meaning, corresponding to the characteristics that make a person a *BT*, over and above any characteristics of being a *BT* that relate to feminism. Such a proposal plausibly follows from the maxim of quantity and the given-new contract, in the sense that in asking about $F\&BT_F$ the experimenter might be interpreted as intending information about *BT*, that is new in relation to the *F* one, thus $BT \neq BT_F$.

Regarding alternative mathematical frameworks which can motivate the assumption $BT \neq BT_F$, Dzafarov and Kujala's (2013; see also Hammond, 2011) context by default idea is that whenever a question *A* is asked in the context of another question *X*, one has to index the random variable *A* by *X*, so that it has a different meaning compared to when evaluated in isolation. This proposal is consistent with QPT decision models, for incompatible questions, but note that it has not been developed into a specific, empirically testable model for the CE.

Importantly, the idea that contextuality can change the meaning of superficially identical questions can be expressed classically too, since we can keep track of different meanings, through conditionalization. In the Linda paradigm, assume CPT and consider the *BT*, *F* questions contextual, so that $BT \neq BT_F$. Then, we have an extended CPT distribution, with two probability spaces, one for *BT* alone {*BT*, ~*BT* | *BT* alone} and another for *F* and BT_F asked together {$F\&BT_F$, $\sim F\&BT_F$, $F\&\sim BT_F$, $\sim F\&\sim BT_F$ | *F*, *BT* together}. Now the judgment $Prob(F \wedge BT_F|F, BT\ together) > Prob(BT|BT\ alone)$ satisfies all CPT axioms.

So, for both QPT and CPT a CE can be absolved of fallacy, if we consider the *BT*, *F* questions as contextual and so assume $BT \neq BT_F$. However, there is an important qualification. In QPT, incompatibility goes hand in hand with specific principles for relating different probability spaces and informing the exact difference between BT, BT_F (cf. Figure 1b). Thus, the QPT CE approach can be extended in a reasonably constrained way to cover related findings, e.g., disjunction fallacies (Bar-Hillel & Neter, 1993; Carlson & Yates, 1989), order effects in the Linda paradigm (Stolarz-Fantino et al., 2003), and unpacking effects (Rottenstreich & Tversky, 1997), and allows new predictions, notably a constraint on decision order effects (Wang et al., 2014). Instead, in CPT, contextuality requires what is typically a post hoc conditionalization and so is less appealing theoretically. Indeed, CPT theorists rarely consider the possibility of contextuality in the CE (one exception is Hartmann, & Meijs, 2012). The standard CPT approach in the Linda paradigm is to employ CPT with a single probability space to evaluate all possibilities, {$F\&BT$, $\sim F\&BT$, $F\&\sim BT$, $\sim F\&\sim BT$}. The judgment that $Prob(F\&BT) > Prob(BT)$ is then incorrect. Other CPT approaches have been developed, but none that consider the *F*, *BT* questions to be contextual. CPT plus noise accounts assume the conjunction fallacy is produced purely by noisy estimates (Costello & Watts, 2014), but the relevant normative principles correspond to CPT ones in a single probability space. Such accounts have to conclude that the CF is a judgment error. Inductive confirmation accounts (Tentori et al., 2013) assume that naïve observers compute probabilistic functions other than CPT conjunctions when faced with a conjunction.

There is some evidence that the distinction between compatible and incompatible representations is what drives the CE in the Linda paradigm. In some cases, Linda paradigm manipulations can be interpreted as encouraging compatible representations and in these cases the CE rate should be reduced. For example, Agnoli and Krantz (1989) found that brief training on the algebra of sets reduced the CE rate. Such training should bias participants to employ compatible representations for the *F*, *BT* questions, because it would reinforce a single classical space representation for these questions (see also Yamagishi, 2003, who employed nested-sets instruction, and Wolfe & Reyna, 2010, who trained participants to use 2x2 tables for computing joint probabilities). Nilsson (2008) reported fewer CEs when participants were trained with feedback to compute *Prob*(*A*&*B*) directly, instead of computing *Prob*(*A*), *Prob*(*B*) and then combining the two. In QPT terms, 'direct' computation would be more likely to reflect the concurrent evaluation of questions, which is a characteristic of compatible questions. Nilsson et al. (2008) also reported that increased familiarity, which is thought to make compatible representations more likely (cf. Trueblood & Pothos, 2014), reduced the rate of CEs. All these results are broadly supportive of the QPT assumption regarding incompatibility as the source of the CE (further discussion in Busemeyer et al., 2015).

Determining whether two questions are incompatible in the first place is a challenge to the QPT CE approach and we summarize the progress to date (Busemeyer et al., 2011, 2015). A marker of incompatibility would be order effects, that is the extent to which assessing *X* and then *Y* produces the same responses as assessing *Y* and then *X*. Note, order effects are linked to an important test of quantum structure, the QQ equality (Wang et al., 2014). Another marker of incompatibility is evidence for constructive influences in human behavior. QPT cognitive models require that any 'measurement' (e.g., a judgment) changes the 'system' (e.g., the mental state of a participant) in a specific way. Some researchers have taken advantage of this requirement to produce and validate corresponding empirical predictions (e.g., White et al., 2014; Yearsley & Pothos. 2014). Finally, considering two incompatible questions must require serial rather than parallel processing. Though this is an aspiration for future work, in principle it should be possible to develop empirical tests of serial vs. parallel processing (e.g., using Systems Factorial Technology; Townsend & Nozawa, 1995).

Overall, we think the QPT perspective of when $Prob(Y\&X_Y)$ can be greater than $Prob(X)$ is reasonable and well supported. However, the fact that a CE is allowed within one system of rules hardly entails that the CE can be considered *rational* as well (Section 3). Moreover, even if the rationality issue can be resolved in an abstract way, we have yet to consider whether participants really reason in the Linda paradigm assuming $BT \neq BT_F$ (Section 4) and whether incompatibility has any relevance to the outside (non-mental) world (Section 5).

3. The Dutch Book theorem and the rational status of the conjunction effect

Rationality can be defined in many ways, even leaving aside its everyday language meaning (the Oxford Online Dictionary definition of rational is "based on or in accordance with reason or logic"). One way to approach rationality is consistency and correctness according to a particular system of rules. A problem with such an approach is that it becomes difficult to discriminate between different systems of rules, that is, to understand whether decisions according to one system of rules might be considered more or less rational compared to decisions according to another system of rules (or how to deal with situations when the prescription from two different rule systems conflicts). Another approach to rationality is consistency with a particular criterion. With this approach, different decisions, even produced across different systems of rules, can be evaluated against the chosen criterion.

There has been intense debate in psychology and other sciences over how exactly to select an appropriate criterion for rationality. In psychology, for situations of risk neutrality, an influential perspective (e.g., Oaksford & Chater, 2009) is that rational decisions in relation to a set of gambles are those for which a sure loss is not possible. Put differently, if a person assigns probabilities to gambles in such a way that he/she will always lose money, regardless of how events turn out, then clearly there is something wrong with the probabilistic assignment – it is not rational. This idea is the Dutch Book (DB) criterion (though note that risk averse decisions are vulnerable to DBs).

An incredible theorem, the Dutch Book (DB) theorem, proves that consistency with the principles of CPT protects reasoners from assigning probabilities to gambles in a way that leaves them vulnerable to a sure loss (de Finetti et al., 1993). We can so conclude that decisions consistent with CPT should be considered rational (Oaksford & Chater, 2009; see also Griffiths et al., 2010; Tenenbaum et al., 2011).

The DB theorem is simple (we follow Howson & Urbach, 1993). Consider a combination of bets, concerning a hypothesis *h*. The stake, *S*, is an amount of money to be obtained, if *h* turns out to be true (0 otherwise). To enter the bet, the amount $p \cdot S$ has to be paid, where *p* is the betting ratio that *h* is true (the higher the betting ratio, the more likely it is that the *h* is true). So, the net outcome if *h* is true is (*S*-*pS*) and -*pS* otherwise. At this point, we have not stated where betting ratios come from. However, whichever system they come from, betting ratios in a gambling situation must satisfy a set of properties, otherwise a sure loss will be incurred. One can easily prove (e.g., Howson & Urbach, 1993) what these properties have to be: betting ratios have to be positive and between 0 and 1 (the former for impossible events, the latter for tautologies); the betting ratio for the union of two mutually exclusive and exhaustive events must be the sum of the betting ratios for each bet separately; finally, if the event *A* has already occurred first, then the betting ratio for another event *B* must be the betting ratio that *A* and then *B* occur, divided by the betting ratio that *A* occurs. As an example, consider how a sure loss can arise if we employ betting ratios violating one of the above properties, that the betting ratio for a tautology should be 1. Since *h* is necessarily true, the payoff is (*S*-*pS*). So unless *p*=1 a sure loss will be incurred for either the party entering the bet or providing the payoffs.

The important point is that all these properties required of betting ratios are consistent with the CPT axioms. Therefore, if betting ratios are CPT probabilities, then there is no danger of a sure loss and likewise for any probabilistic decision consistent with CPT (see Howson & Urbach, 1993, for an accessible introduction). This is the DB theorem for CPT.

But, what can be inferred regarding QPT, noting that QPT axioms are very different from CPT ones? In CPT, probabilistic assessment essentially depends on generalized volumes, with events as subsets of an overall sample space (cf. Isham, 1989). By contrast, in QPT, probabilistic assessment involves projection onto subspaces and depends on the geometric relations between these subspaces in an overall vector space. It is surprising that QPT probabilities satisfy all the above properties required of betting ratios too, that is, QPT decisions are consistent with the DB criterion in exactly the same way as CPT ones are, in principle at least (Appendix 1 provides a more complete presentation) [Footnote 2]. This last qualification is a key aspect of this paper, since QPT concerns questions that are incompatible. In the outside macroscopic world, incompatible questions are contextual ones. If there are no contextual questions in the outside world, then it becomes irrelevant to consider the implications from QPT for rationality. Note, in QPT there is an additional requirement of non-commutativity for conjunctions, which goes beyond the properties listed above. However, both CPT and QPT allow $Prob(A\&B|condition\ 1) \neq Prob(A\&B|condition\ 2)$ (Section 2), and in addition in QPT there is a technical framework for relating the two.

Let us return to the CF. Note first that the CF persists (albeit sometimes at reduced rates), even when the Linda task is presented in a betting framework (Sides et al., 2002; see also Bar-Hillel & Neter, 1993; Bonini et al., 2004). This is an interesting finding, since the DB criterion is a standard for rationality rooted in a coherence/ consistency argument within a betting framework.

In a single (classical) probability sample space, it is clearly irrational to bet more on $F\&BT$ than *BT*. This is shown in Appendix 2 (see also Gilio & Over, 2012), but the intuitive reason is this: Suppose you bet on the conjunction being true and a marginal being false. Then it is easy to identify stakes, probabilities that produce a sure loss for all states of nature, since there are no circumstances of the conjunction being true, without both marginals being true. If *F*, *BT* are considered contextual, then $Prob(F\&BT_F) > Prob(BT)$ can be correct and readily translates into an arrangement which obviates DB vulnerability. Table 1 shows how the truth table without contextuality can be extended when assuming that *BT*, *F* are contextual. In the latter case, there is a separate truth table for *BT* and BT_F, so that a person may hold a belief that Linda is a *BT,* independently of the belief of whether she is a $F \wedge$

BT_F. In a corresponding DB table, it is now possible for the conjunction to be true, even when one of the marginals is false, and these additional possibilities protect from a sure loss for all states of nature (Appendix 3 provides a step by step account for how a sure loss is avoided).

Overall, both CPT and QPT provide an equivalent perspective on whether the CE is rational, namely this is so when $BT \neq BT_F$, i.e., when the *BT, F* questions are contextual. However, thinking classically, we are not used to considering the same question as different in different contexts and the assumption $BT \neq BT_F$ can only be realized through a post hoc conditionalization of the relevant probabilities; CPT researchers rarely adopt this approach. In QPT, an assumption of incompatibility naturally entails contextuality.

BT(alone)	*F*	BT_F bank teller in the context of feminism	$F \wedge BT_F$
T	T	T	T
T	T	F	F
T	F	T	F
T	F	F	F
F	T	T	T
F	T	F	F
F	F	T	F
F	F	F	F

Table 1. The shaded cells correspond to the classical truth table for the CE, in a single probability space. The non-shaded cells show how the truth table must be expanded, when assuming that the *BT*, *F* questions are contextual and we wish to evaluate bets involving *BT* in isolation and in the conjunction $F \wedge BT_F$.

4. Empirical examination of the $BT \neq BT_F$ assumption

To recap, both QPT and the CPT produce exactly the same perspective on the rational status of the CE, namely that when the *F*, *BT* questions are contextual, then $BT \neq BT_F$ and a CE does not produce a sure loss – it is consistent with (DB) normative prescription. In this section we explore whether participants in a Linda paradigm indeed represent $BT \neq BT_F$, since if they do a CE is rational relative to their representations. That the same linguistic element can have multiple meanings in different contexts is a familiar idea in linguistics (e.g., Duffy et al., 1988; Morris, 1994). However, there have been no demonstrations in decision making that the contexts induced by different combinations of questions can alter question meaning. This brief empirical demonstration explores this possibility, in the Linda paradigm.

Regarding methods, some researchers have argued that a frequentist presentation in the Linda paradigm encourages participants to make more accurate judgments (Tversky & Kahneman, 1983, themselves pointed this out; Gigerenzer, 1994; Reeves & Lockhart, 1993; but not all evidence is consistent, see Weddell & Moro, 2007). Additionally, there is some evidence that employing an estimation response mode, instead of a ranking task, reduces the CE rate, which indicates that participants make more accurate probability judgments (Hertwig & Chase, 1998; Morier & Borgida, 1984). Presently, we employ frequentist presentation and an estimation task, since we seek evidence that $BT \neq BT_F$, under circumstances which typically promote accurate probabilistic reasoning. However, note also that asking participants to estimate BT vs. BT_F may not produce a pure measure of equivalence between the two questions, since responses might be influenced by general knowledge biases and assumptions participants might be making about the questions.

4.1 Participants and design

We recruited 1,838 participants from Amazon Turk, offering $0.40 for a task which on average took approximately five minutes. Of the 1,811 participants who responded to a question about whether payment was adequate 1,420 responded that it was and 294 felt it was generous. Mean participant age was 34.1 years and the gender balance was 978 males, 849 females (11 participants did not respond). Participants were recruited from the USA and the UK. The experiment had received ethics approval from the Department of Psychology, City University London and all participants provided informed consent.

The intended sample size was 1,800, to be split between six between-participant conditions. Regarding the Linda paradigm (Tversky & Kahneman, 1983), the two corresponding conditions were identical but for the fact that in one participants were asked to evaluate the probability of the *BT*, *F* properties for Linda individually and in the other there was a single screen, requiring participants to respond to the *BT*, *F* questions individually and together with the conjunction $F\&\ BT$. In this latter case, we assume that the *BT* question corresponds to BT_F. Two more analogous conditions were employed for Tversky and Kahneman's (1983) Bill paradigm and yet two more conditions for a new paradigm, referred to as the Julie paradigm. Because we opted for recruitment over Amazon Turk, we did not carry out a power analysis, rather decided a priori to halt recruitment at 1,800 (because of the way Amazon Turk works, actual sample size can exceed the intended one by a small percentage).

4.2 Materials and procedure

The experiments were designed in Qualtrics. Regarding the Linda paradigm, participants first saw a screen informing them that they would read a paragraph about a hypothetical person, Linda, and that they would have to answer some questions about Linda. Participants also read that there are no correct or wrong answers and that they should use their intuition. The next screen presented the information about Linda, as in Tversky and Kahneman (1983; Appendix 4). Then, in the condition we shall henceforth call the 'individual' condition, participants were shown one by one (randomized order) eight questions about Linda (in all conditions, conjunctions were shown together with the questions for the relevant marginals on a single screen). Six of these questions were fillers (mostly from Tversky & Kahneman, 1983). The two critical questions were the *F* question and the *BT* question. The format of each question was identical. Participants were told to evaluate the probability that Linda is e.g. a *F.* They were told: Imagine 100 women, whose description fits exactly that of Linda. How many women like Linda (out of 100) do you think are *F*? Participants had to indicate their answer on a slider scale from 0 to 100, with labels every 10 percentile points. After responding, participants saw the next question, without feedback. Note, the information about Linda was shown with each separate question. The condition we shall henceforth call the 'conjunction' condition was identical to the individual condition, but for the fact that instead of having the F, BT questions separately, the F, BT, and $F\&\,BT$ were all shown together on the same screen, one after the other (F first, then BT, then the conjunction). There was a separate response scale for each question, but the scale labels were shared. Once participants finished the experimental questions, they were debriefed and thanked.

The two Bill conditions, individual and conjunction, were set up in an identical way, but instead of using the Linda story and Linda questions we employed the Bill story and Bill questions from Tversky and Kahneman (1977). Bill is described as someone unlikely to have creative urges. The key questions concern whether he is an accountant (A; likely given the story) and whether he plays Jazz (J; unlikely). We also provided an additional example of a question which we think better illustrates how the meaning of the same question can vary across conjunction decision contexts. In the two Julie conditions, participants were told of Julie, a fit, young lady, with a black belt in Karate, growing up in a city. She is waiting alone for the bus at a bus stop on a quiet suburban street, after a late school class. The question is whether she is scared or not (S). Consider two separate contexts for the scared question. In one context, participants are also asked whether Julie sees a lost kitten (K). This context is about whether Julie has confidence in relation to her environment and so, in this context, it is unlikely that Julie is scared (we label the scared question in the kitten context S_K). In another context, participants are also asked whether there is a speeding car about to crush her against the bus stop (C). Such a context relates to the inevitability of fateful disasters beyond one's control and makes Julie's skills and implied street-smarts irrelevant. Would participants consider Julie to be scared now? (We label the question of whether Julie is scared in the car context S_C.) Note, in both contexts, the questions S_K and S_C are exactly the same: How many women like Julie (out of 100) do you think are scared? Moreover, the S_K and S_C marginals are clearly distinguished from the conjunctions. For example, in the S_C case, participants were asked three questions (on the same screen), how many women like Julie (out of 100) are likely to (1) see an out of control speeding car about to crush her against the bus stand, (2) be scared, (3) see an out of control speeding car about to crush her against the bus stand and be scared ($C\&S_C$).

4.3 Results

We are interested in whether the same two questions acquire different meaning in different contexts of a CF decision making problem. We examine this by comparing the (frequentist) probability of e.g. the BT, BT_F questions (and analogously for the other conditions). Note, these questions were responded to in between participant conditions; we think it is not be possible to explore this issue within participants, since participants may feel constrained to respond in the same way to the same question, regardless of context. Also, the use of an estimation task allows equality in BT, BT_F ratings.

The dataset consisted of 1,837 responses, but not all of these responses were complete and so a small percentage had to be dropped. In addition, in the debriefing part of the experiment, we asked participants whether they had taken part in previous similar experiments, emphasizing that the answer would not affect payment. We removed an additional 13 participants in this way. No other trimming of the data set was conducted (Figure 3 shows final sample size per condition).

The critical pairs of questions were BT, BT_F in the Linda conditions, J, J_A in the Bill conditions, and S_K, S_C in the Julie conditions (note, in all cases there was evidence for a CE). In the Linda and Bill conditions, the BT, J questions were unlikely and so we expected responses to be positively skewed, which was the case (skewness was examined within each condition). The S_K question was also positively skewed, but this was less so for the S_C one, again as expected. In all cases, the assumption of normality was violated, using the Shapiro-Wilk test (Appendix 5). Because of the high degree of positive skewness in most conditions, and so as to have a uniform approach to data processing, we transformed all variables using the function $log_{10}(value + 1)$ (Howell, 2007). We compared the pairs of relevant variables using independent samples t-tests, also computing Cohen's *d*. Means and standard deviations are shown in Figure 3. In the Linda and Bill conditions, the contextual property was judged to be more likely than the non-contextual one, as expected. When comparing *BT*, BT_F, we found t(598)=2.024, *p*=.043, *d*=0.17. When comparing *J*, J_A, we found t(610)=2.174, *p*=.03, *d*=0.18. Finally, S_C was higher than S_K, as expected, t(601)=2.583, *p*=.01, *d*=0.22.

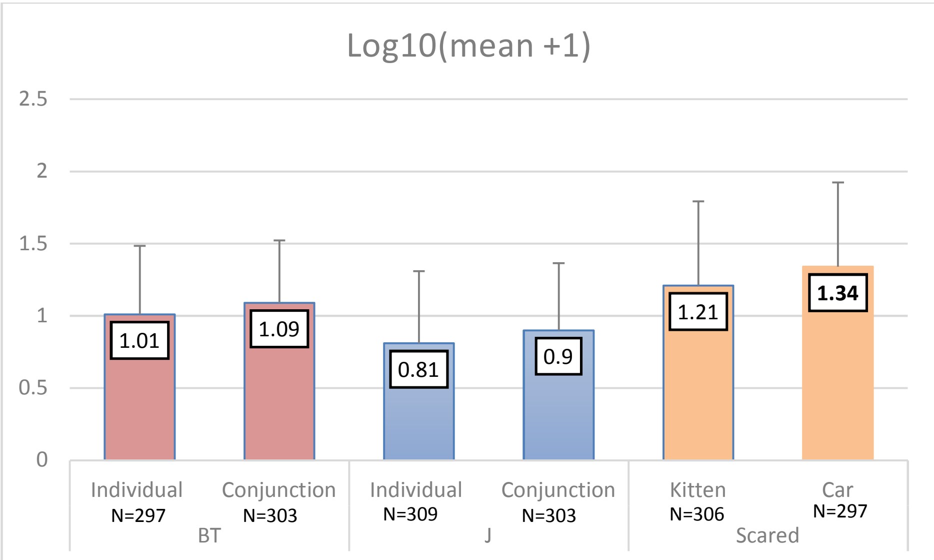


Figure 2. The mean value of the transformed versions of all variables of interest. We have included error bars for standard deviations, to show the large variability. The mean for each variable is shown within each bar. All conditions are between participants.

5. Implications for real world decision making

Section 4 provided some evidence that participants consider $BT \neq BT_F$ in the Linda paradigm, which indicates that a CE is rational, relative to the mental representations employed by participants (a caveat is that a conclusive empirical demonstration would require an impractical within participants design, so the empirical result can only be taken as suggestive). The Section 4 result and the evidence for QPT cognitive models generally (e.g., Busemeyer & Pothos, 2013) suggests that some questions are mentally represented as contextual, regardless of whether they are really contextual in the outside world or not. That is, the computational procedure on representations that leads to a CE can be rational (Section 3, the DB Theorem for QPT), given the belief $BT \neq BT_F$, but the belief $BT \neq BT_F$ in itself may not be rational. Indeed, it is clear that there are questions which are *not* contextual in the outside world.

Consider Tentori et al.'s (2013) report of a CF concerning whether a Scandinavian person has blonde hair (BH) and blue eyes (BE), $Prob(BE\&BH) > Prob(BH)$. The *BH*, *BE* questions are not contextual, since a person either has *BH* or *BE* or not, there is no sense in which the biological meaning of these questions is modified depending on context and so this CE is a CF. Thus, it appears that decision makers wrongly interpret some questions as contextual (such as *BE*, *BH*) and then possibly employ QPT, perhaps by *analogy* to other situations when contextuality is appropriate. This may reflect bounded rationality (Simon, 1955). It is plausible to think of QPT representations sometimes employed as shortcuts, since they typically require a lower dimensionality than equivalent classical ones. Also, questions represented in an incompatible way typically have to be processed sequentially, while compatible questions concurrently (cf. Evans et al., 2007; Elqayam & Evans, 2013; Fernbach & Sloman, 2009; Sloman, 1996; Kahneman, 2001). The former process is more consistent with fast, intuitive decisions, and the latter with thoughtful ones and indeed there has been some evidence that high CRT (Cognitive Reflection Test; Frederick, 2005) respondents are more likely to employ compatible representations (Yearsley et al., 2016).

The adoption of contextual representations for outside world questions by analogy at best provides a bounded rationality justification for using such representations, but cannot address the key question regarding normative prescription and rationality: Do certain questions in the outside world warrant the contextuality required by the DB criterion to absolve the CE of fallacy? If the answer is yes, then the present discussion does provide an alternative perspective of rationality, such that for contextual questions the CE can be rational. We offer three classes of situations where this is plausibly the case and speculate on the frequency of such situations in everyday decision making.

The first class concerns situations where a decision maker shares with a judge (who judges the outcome for e.g. a bet) an assumption that two questions are contextual. If both the judge and a decision maker share the same contextual representations the decision maker will e.g. avoid a DB from a CE. Note, this is not about a judge and a decision maker both sharing any arbitrary system for reasoning (e.g., if they both reason on the basis of the representativeness heuristic, they will both eventually fall prey to a DB). We know that it is not uncommon for decision makers to employ contextual representations, as demonstrated by the various QPT models for probabilistic fallacies (note, it will not be possible to explain all probabilistic fallacies in quantum terms; cf. Wang et al., 2014). What would be the frequency of such situations? It has been suggested that the transition from quantum-like to classical representations occurs with increasing familiarity (e.g., Trueblood & Pothos. 2014). We can then speculate that the relative proportion of familiar combinations of questions relative to unfamiliar (but plausible) combinations is modestly high. Note, if we interact a lot with fellow humans whose representations are contextual, then quantum-like reasoning by analogy may be an appropriate strategy. Then, the question becomes why might humans have such representations *frequently*. This is a question for evolutionary psychology. It is possible that, even when unwarranted in normative terms, there are adaptive reasons for employing contextual representations (e.g., in terms of the reduced dimensionality of the space needed to represented quantum vs, classical questions).

For this first class of contextual questions we are still in the realm of mental representations shared between humans. Are there questions in the outside world and outside the mental realm of any judge that are contextual? This may look like a tall order; as noted by Sloman for the Linda paradigm (personal communication, March 2016) "in any world I know, there is the equivalent of a registry for people. We have an independent means to decide if someone is a *BT* or a *F* so there's no normative sense in which the relevant questions are incompatible." Surprising as this may seem, we think there are several ways for contextual questions to arise.

The second class of contextual questions concerns situations where a previous question can disturb a system (and so affect subsequent questions); i.e., the disturbing influence of the previous question produces the context. A fundamental aspect of QPT is that a measurement (e.g., a decision) has to change the system in a specific way, so that measurements create different contexts for subsequent questions. Psychologically, there is evidence that decisions can alter the mental state (Schwarz, 2007; White et al., 2014; Yearsley & Pothos, 2016). More importantly, questions can have a disturbing influence on macroscopic systems, thus creating contextual pairs of questions, for which a CE would be rational. Consider a baby, Eve. Her parents bet about whether Eve will be sleeping on her front or back at two times, 10pm and 11pm; call these variables Position_{10} and Position_{11}. They determine how Eve is sleeping by checking on her, but this wakes her up and she moves around before going back to sleep. Now suppose Eve tends to sleep on her front with a probability of .90, stationary over the night. Then, $Prob(Position_{10} = front) = .9 = Prob(Position_{11} = front)$ and $Prob(Position_{10} = back) = .1 = Prob(Position_{11} = back)$, when there is no checking. But, $Prob(Position_{10} = front\ \ \&Position_{11} = back) \gg .1$, since the 11pm question in the conjunction assumes a prior check at 10pm and measuring at 10pm disturbs Eve and makes her likely to roll over.

An observer wishing to make a decision regarding baby Eve could construct a single probability space representation with $Position_{10} \equiv Position_{11}$ (assuming the 11pm question is evaluated after a prior check, but not the 10pm one). But this will lead to incorrect judgments, since it misses important structure in the problem. The correct representation of the $Position_{10}$, $Position_{11}$ questions is contextual, one has to employ either CPT and conditionalization or the more detailed framework for contextuality of QPT, and a CE is rational. We can also recast Tversky and Kahneman's (1983) Borg example (Table 3) in a way that the questions are disturbing. Suppose Mary is a tenacious tennis player who is really motivated to win after losing an early game. Then, *Prob*(*Mary loses the first game & $Mary_1$ wins the set*) > *Prob*(*$Mary_2$ wins the set*), because *$Mary_1$* and *$Mary_2$* should be considered different individuals (for the purpose of these questions), one approaching the game with a casual attitude and another really driven to win.

Another example is *Prob*(*Samsung smartphone explodes* & *the value of $Apple_1$ shares go up*) > *Prob*(*the value of $Apple_2$ shares go up*). Plausibly, Apple should be considered as a different competitor when the issue of exploding smartphones is introduced. A trader would not be surprised if *$Apple_1$* is considered less competitive than *$Apple_2$*, but the additional points here concern whether this contextuality should be modeled with CPT or QPT and the implications for the rational status of CEs. Note, the main condition for determining whether QPT can capture disturbing influences is that resolving a question changes the system state to identify itself with one of the 'eigenstates' of the question, that is, one of the possible answers of the question, and that subsequent questions are incompatible with the original one. It is hard to estimate the frequency of such situations, but a reasonable speculation is that while not infrequent they would be less frequent than Class 1 ones.

The third class of outside world contextuality concerns questions which can have different meaning, depending on context. There is much research on how contextuality can affect meaning (e.g., Duffy et al., 1988; Morris, 1994; Rottenstreich & Tversky, 1997), but the implications for rationality have been less considered. Consider a biology student, Peter, who overhears his parents discussion. Dad asks mum, 'How likely is it that Bubbles will suffer from a heart attack?' (*H*). Peter's default assumption is that Bubbles refers to his pet fish, as he is very fond of it and often talks about it. Since fish's hearts lack coronary vessels, they are unlikely to develop coronary disease and suffer from heart attacks. Hence, *Prob*(*H*)~very small. Suppose instead dad asks mum "How likely is it that Bubbles will go to the supermarket and how likely is it that Bubbles will suffer from a heart attack?" ($S\&H_S$). The first question in this conjunction makes it clear that Bubbles refers to Peter's granddad, whose nickname is Bubbles because he likes scuba diving. It is straightforward to fill in details so that a CE is plausible (e.g., Bubbles the grandad may have high cholesterol and find going to the supermarket frustrating). In this example $Prob(H) < Prob(S\&H_S)$ is clearly sensible. The Heart question has two separate meanings, one for which heart attacks are extremely unlikely (heart of a fish) and another for which they are fairly likely (heart attacks are a common cause of premature death in humans, especially amongst high cholesterol individuals).

We further illustrate the sense in which a CE is rational, in the Hearts example. Table 2 shows that if the counts for the *H*, H_S questions are collapsed, using question H_{IG}, then it is impossible for the conjunction to be more likely than the marginal. However, doing so is clearly incorrect, since it misses important structure in the problem, namely that $H \neq H_S$. Assuming $H \neq H_S$, there is no fallacy in believing that $Prob(H_S|S)$ is changed in the *S* context, to become higher than is classically allowed, so that $Prob(H_S|S) \cdot Prob(S) = Prob(S \wedge H_S) > Prob(H)$ (Figure 3).

As another analogous example, consider Bob, an employee of Sue. Sue decides on employee raises, with two principles: raises are extremely unlikely; everything must be done to prevent employees from being poached by competing companies. Then, the rule for evaluating the question of Bob's raise in isolation is different from the rule if there is a possibility that Bob has an offer by a competitor (i.e., the *raise* question has two different meanings), which leads to $Prob(raise) = low$ and $Prob\left(raise_{offer}|offer\right)Prob(offer) = Prob\left(offer\&raise_{offer}\right) = high$. The analogue of Figure 3 is a diagram where the $raise_{offer}$ and *raise* sets are separate. One *could* combine these sets, so assuming $raise_{offer} \equiv raise$, making it impossible to have $Prob(offer\&raise) > Prob(raise)$. But this would be a misrepresentation of the situation, since it ignores the information that the $raise_{offer}$, *raise* questions are evaluated using different rules. One can also think of less subtle examples, e.g., *Prob*(*John is a double agent* & *there is a* bug_1 *under his plate*) > *Prob*(*there is a* bug_2 *under John's plate*), where $bug_1 \neq bug_2$. Regarding the frequency of such Class 3 situations in decision making, we can speculate that it is analogous to that for Class 2 ones.

H	*H_S*	*S*
T(0.05)	T(0.30)	T(0.40)
F(0.95)	F(0.70)	F(0.60)

Explore 100 fishes and 100 humans:

H	*H_S*	*H_IG*	*S*	*S&H_S*	*S&H_IG*
T(5)	T(30)	T(35)	T(40)	T(15)	T(15)
F(95)	F(70)	F(165)	F(60)	F(85)	F(185)

Table 2. Probabilities and frequencies for the H, H_S, S questions. The H, S questions are contextual and it is wrong to collapse the H, H_S counts. We do so for illustration using the H_{IG} variable (IG: ignorant). H_{IG}=H+H_S, with no individuals identical, and $S\&H_{IG}$=$S\&H_S$ + $S\&H$, where $S\&H$=0.

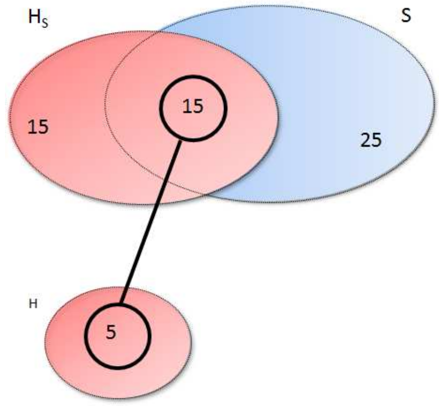


Figure 3. An illustration of the CE in the Bubbles example. The circled frequencies illustrate $Prob(S \wedge H_S) > Prob(H)$. S: go to the supermarket; H: fish heart attack; H_S: grandad heart attack.

Note in QPT there is no distinction between Class 2 and Class 3 situations: two questions are either compatible or incompatible (contextual). According to QPT, the Class 2 vs. Class 3 distinction is a process level one, about the different ways in which incompatible questions may exist. Briefly, Class 2 situations are when measurements have a disturbing influence. Class 3 ones occur when there is some sort of ‘communication’ between different cognitive variables (e.g., in the Bubbles example, the meaning of the Hearts question and the Supermarket question). A fundamental point is that QPT provides a way to think about *all* contextual situations (and model them with the same mathematics), without having to know the details of the process giving rise to contextuality. Note this is both unexpected and profound. It is easy to find cases of contextuality that cannot be accounted for by QPT (cf. the QPT bounds for the Bell inequality or the temporal Bell inequality; Nielsen & Chuang 2000), so it is surprising that so far contextuality cases relevant to psychology can be handled using QPT.

Overall, we claim that *all* the outside world situations in which the CE is rational involve contextual questions, as above, where essentially different variables correspond to the conjunction and the conjunct. An important future objective is to estimate more precisely the extent of contextual questions in everyday reasoning, noting that a consideration of prominent CEs in the literature reveals them as CFs (Table 3). However, equally, readily dismissing all CEs as irrational may miss important structure in a problem. Do decision makers have some intuition that for Class 2 or 3 situations non-classical reasoning (e.g., order effects, CEs) is not as incorrect as for cases of (single probability space) classical representations? This is another objective for future work.

Scenario	Judgment	Evaluation
Linda is a described as a prototypical *F*. [A]	$Prob(F\&BT) > Prob(BT)$	Resolving the *BT* question seems a simple matter of checking an e.g. public record. So, this is an error of representing noncontextual questions as contextual (possibly by analogy).

Bill is described as uncreative. [A]	$Prob(A\&J) > Prob(J)$ *A*: accountant *J*: playing jazz for a hobby	The *J* question can have different meanings. It can mean that one is passionate about Jazz or that one occasionally plays Jazz, out of obligation. If it can be established exactly how $J_A \neq J$, then the judgment is rational.
Borg reaches the Wimbledon finals in 1981. [A]	$Prob(WM\& LFS) > Prob(LFS)$ WM: win match LFS: lose first set	This judgment is irrational, though consider $Prob(LFS\& WM_{LFS})$ vs. $Prob(WM)$. For a particular tennis player, losing the first set may have a huge motivational boost (a 'disturbing influence', as in the Eve example), which makes it more likely to overall win.
A Scandinavian individual. (discussion in B)	$Prob(BE\&BH) > Prob(BH)$ *BH*: blonde hair *BE*: blue eyes	This judgment is irrational, since there is no sense in which the meaning of the *BH* question can depend on whether the context involves the *BE* question or not.

Table 3. A consideration of whether some well-known CEs should be considered CFs. A: Tversky and Kahneman (1983); B: Tentori et al. (2013).

6. Summary and concluding comments

The seminal finding of the CE in Tversky and Kahneman (1983) has posed an immense descriptive and normative challenge for understanding decision making and behavior generally. Influential approaches for explaining the CE include Tversky and Kahneman's (1983) representativeness heuristic, averaging models (Fantino et al., 1997), CPT plus noise models (Costello & Watts, 2014), and assumptions that rather than conjunctions participants in Linda paradigm experiments compute different probability functions, such as inductive confirmation (Tentori et al., 2013). The present work proposed a novel perspective on whether committing the CE should be considered irrational, motivated by the recent model for the CE based on QPT.

In QPT, questions can be compatible or incompatible. Compatible questions allow the use of a single common probability space. Incompatible questions are contextual; they cannot be represented in the same probability space and certainty for one makes us uncertain (changes our perspective) for the other. Contextuality is the key idea, which allowed a descriptive account of why the CE occurs in the Linda paradigm. The present paper is concerned with the rational status of decisions involving contextual questions, as in the case of the CE.

A commonly used standard for rational behavior, especially in situations of risk neutrality, is the DB criterion, that reasoners should not commit to gambles leading to a loss, for all states of nature. The implication from QPT regarding the CE is that, if the *BT*, *F* questions are incompatible and so contextual, then *BT* in the context of feminism, BT_F, should be considered a different question from *BT* in isolation (or in some other context). If $BT \neq BT_F$ then *BT* can be false but $F\&BT_F$ true, so that a reasoner could consider $F\& BT_F$ more likely than *BT*, without falling prey to a DB. Even though the assumption that $BT \neq BT_F$ can be implemented classically too, only the QPT approach requires this assumption to explain the CE. Also, it is only QPT that provides a principled approach to understanding why and how *BT*, BT_F can be different (cf. Avis et al., 2008; Dzhafarov & Kujala, 2013; Hammond, 2011).

Can it be the case that exactly the same question, 'is Linda a *BT*?', has different meanings in isolation and with the *F* question (cf. Dulany & Hilton, 1991; Hilton, 1995)? In a large sample not confined to university undergraduates and employing frequentist probability assessments, regarding the Linda and the Bill scenarios from Tversky and Kahneman (1983) and the new Julie scenario, there were statistically significant effects, indicating that BT_F is considered by participants more likely than *BT* (analogously for the other scenarios). This finding indicates that participants committing the CE are rational relative to their representation. However, most observers would agree that there usually is a single record that can be used to determine whether Linda is a *BT* and so this question is noncontextual with the *F* one. Under such circumstances, the CE is a fallacy (and the mental error is that questions that ought to be noncontextual are represented as contextual, perhaps by analogy). But, we presented examples of questions which can be resolved in some objective sense and are still incompatible/contextual. In decision making, we are not used to thinking that questions that sound the same are different – there is no way to linguistically distinguish between *H*, H_S in the Bubbles example (or *BT*, BT_F etc.). We hope that this work will motivate a search for contextual questions, for which committing the CE is rational.

In conclusion, CEs should not be readily dismissed as irrational. One should check the conditions for evaluating the relevant questions. These conditions may reveal incompatibility/contextuality, in which case a CE would certainly not be irrational and may reflect the optimal way of responding. QPT decision models have enjoyed good descriptive success, but until now implications for rationality have been unclear. By demonstrating the consistency of QPT with the DB theorem and discussing the possibility of contextual questions in the outside world, we provide a powerful argument for the rational status of QPT models and an extended perspective for the rational status of CPT ones.

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Footnotes

1. For readers familiar with matrix algebra, these computations are straightforward: One pair of eigenvectors, $|U\rangle = \begin{pmatrix}1\\0\end{pmatrix}$, $|D\rangle = \begin{pmatrix}0\\1\end{pmatrix}$, is used to measure spin vertically and another pair, $|R\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$, $|L\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$, horizontally. The question of whether the spin is up is assessed with the projector $P_U = |U\rangle\langle U| = \begin{pmatrix}1\\0\end{pmatrix} \cdot (1 \quad 0) = \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}$ and analogously for the other questions. For a particle in the up direction, $|\psi\rangle = |U\rangle = \begin{pmatrix}1\\0\end{pmatrix}$. In the C_V measurement condition, we directly measure vertical spin, i.e. $Prob(Down|C_V) = |\langle D|\psi\rangle|^2 = 0$. In the C_{HV} measurement condition we first measure horizontal spin and then vertical spin, i.e. $Prob(Right\ and\ then\ Down|C_{HV}) = |\langle D|R\rangle\langle R|\psi\rangle|^2 = \frac{1}{4}$.

2. Note, regarding QPT, some subtleties arise when considering post-selection or weak measurements in QPT (Tamir & Cohen, 2013). However, as long as one is consistent in considering questions in different experimental conditions as different (which we do), then such issues can be ignored.

Appendices

Appendix 1. The DB Theorem for QPT.

We closely follow Howson and Urbach's (1993) notation and our analyses regarding CPT reproduce Howson and Urbach's (1993) ones. The analyses for QPT were developed by close analogy. DB arguments are developed in terms of betting ratios, which can be probabilities or something else. If the betting ratios are specified as probabilities, then one can examine whether the decision maker is vulnerable to a sure loss or not. Because in the following discussion we are just concerned with DB adherence for CPT or QPT probabilities, for simplicity we will just use the term probabilities, and not betting ratios.

For a bet *on* a hypothesis (probability p, stake S), the betting information is simply presented as in Table A1a, which is typically simplified to Table A1b, since it is not necessary to show the payoff for B. Table A1c shows the payoff when A bets *against* a hypothesis. To understand Table A1c, note that, if a player were betting on hypothesis h, with probability p, stake S, then she would pay player B Sp to enter the bet, hoping to win S-Sp net, if h turns out to be true. When betting against h, the probability that h is true is still p. So, player A has to pay player B $S(1-p)$ (since she wants the hypothesis to be false) to enter the bet, hoping to make $S-S(1-p)$, if h is false, but lose $-S(1-p)$, if h is true. Finally, a conditional bet on a given b is defined as a bet on a, which can only be evaluated if b turns out to be true, otherwise the bet is cancelled (Table A1d; Howson & Urbach, 1993, p.81).

h	Payoff to A	Payoff B
T	$S-pS$	$-S+pS$
F	$-pS$	pS

Table A1a. Payoffs betting *on* a hypothesis.

h	Payoff to A
T	$(1-p)S$
F	$-pS$

Table A1b. Payoffs just for player A betting *on* a hypothesis.

h	Payoff to A
T	$(p-1)S$
F	pS

Table A1c. Payoffs betting against a hypothesis.

a b	Payoff
T T	$S(1-p)$
F T	$-pS$

F		cond. is off

Table A1d. Payoffs for a conditional bet on hypothesis $a|b$.

We can now proceed with the DB theorem. The axioms of CPT (cf. Kolmogorov, 1933) for finite spaces are (1) there is a universal set U, called the sample space, containing N unique outcomes, (2) an event is a subset of U, (3) the collection of all events forms a Boolean algebra; the probability p is a function from events to real numbers such that it satisfies (4) $p(a)\geq 0$, (5) $p(\mathrm{U})=1$, (6) if a, b are mutually exclusive then $p(a \cup b)=p(a)+p(b)$ (additivity), and (7) the conditional rule is $p(a|b)=p(a \wedge b)/p(b)$, for any pair of events a, b. These axioms imply that if a, b are mutually exclusive, $a \wedge b=\varnothing$ and then $p(a \wedge b)=p(\varnothing)=0$. Note, event U is the certain event and $\varnothing$ the impossible event.

The axioms of QPT (cf. Gudder, 1988) for finite spaces are (1) there is a vector space U, spanned by N orthonormal basis vectors, (2) an event is a subspace of U, (3) the collection of all events forms a partial Boolean algebra; S is a vector in U and q is probability, such that (4) $q(a)=|P_a S|^2$, where P_a is the projector for subspace a, (5) $q(U)=|P_U S|^2=|S|^2=1$, (6) if a, b are mutually exclusive, then $P_a * P_b=0$, (7) the conditional rule is $q(a|b)=|P_a * P_b \mathrm{S}|^2/q(b)$, for any pair of events, a, b (note, strictly speaking one should be using double vertical lines, indicating the norm of a vector, instead of single vertical lines, indicating complex number modulus, but for this discussion this distinction is not important). These axioms imply additivity, because if a, b are mutually exclusive, then $P_a * P_b = 0$, so that $q(a \vee b) =|(P_a + P_b) S|^2=|P_a S|^2+|P_b S|^2=q(a)+q(b)$, where $a \vee b$ is the span of the union of subspaces a, b (corresponding to the event a or b).

A demonstration of the DB theorem typically proceeds by showing how, for each of the CPT axioms, it is only consistency with the axiom that avoids a DB.

A negative probability $p<0$ violates CPT axiom $p(a) \geq 0$. If player A chooses $p<0$ then player B can choose $S<0$, so that player A is guaranteed to lose, thus allowing B to make a DB against A (Table A1b). In QPT, probabilities cannot be negative, because they are computed as squared lengths of projections. Thus, a DB cannot be made for QPT probabilities in this way. Suppose next that player A assigns $p<1$ to event U (which is certain to be true). Player B can choose $S<0$, guaranteeing a loss for A (Table A1b). So, a DB is possible against A, unless $p(\mathrm{U})=1$. Both CPT and QPT share the axiom $p(\mathrm{U})=1$. Finally, classically, a DB can be made for disjunctions of mutually exclusive events, unless $p(a \text{ or } b) = p(a) + p(b)$ (Table A2).

Col. 1	Col. 2	Col. 3	Col. 4	Col. 5	Col. 6	Col. 7
Bet (i) on a	Bet (ii) on b	Bet (iii) against $a \vee b$	Bet(i)	Bet(ii)	Bet(iii)	Summing bets (i), (ii), (iii)
S, p (stake, prob)	S, q	S, r				
T	T	undefined				
T	F	T	$(1-p)S$	$-qS$	$-(1-r)S$	$(r-(p+q))S$
F	T	T	$-pS$	$(1-q)S$	$-(1-r)S$	$(r-(p+q))S$
F	F	F	$-pS$	$-qS$	rS	$(r-(p+q))S$

Table A2. DB argument regarding disjunction of mutually exclusive events (following Howson & Urbach, 1993, p. 80).

Note that the truth table for $a \vee b$ is simply the classical truth table for disjunction, for mutually exclusive possibilities. Column 7 adds the payoffs for the different bets, e.g., for the second row we have (1-(*p*+*q*)-(1-*r*))*S*=(*r*-(*p*+*q*))*S*. Column 7 illustrates that, unless we have *r*=*p*+*q*, as required by CPT, then a DB is possible. Both CPT theory and QPT satisfy this additive property for mutually exclusive events, so a DB cannot be made against a disjunction of mutually exclusive hypotheses, assessed by classical or quantum probabilities.

Finally, regarding a conditional bet on *a* given *b*, we specify a series of bets, whose combination reveals a DB, unless conditional probabilities are defined according to $p(a|b) = \frac{p(a \wedge b)}{p(b)}$. Consider the specification of bets, in columns 1 to 4 in Table A3. We aim to show that *p*=*q*/*r* is the only choice which avoids a DB (this is what motivates the stake, probability for the against bet on Column 4). The payoffs for the final combination of bets, in Column 7, is 0 only as long as *p*=*q*/*r*, that is $p(a|b) = \frac{p(a \wedge b)}{p(b)}$. Note, the conjunction, $p(a \wedge b)$, is always commutative in CPT. Conditional quantum probabilities are defined as $q(A|B) = \frac{|P_A P_B|\psi\rangle|^2}{|P_B|\psi\rangle|^2}$, where P_A, P_B indicate projectors. The sequential projection $|P_A P_B|\psi\rangle|^2$ is the (sequential) conjunction $q(B \wedge \ then\ A)$. This reveals a fundamental distinction between CPT and QPT. In CPT, all events are compatible, they can be answered concurrently, and conjunction is commutative (classically, $p(A \wedge B)$ refers to the concurrent assessment of A and B, so there is no assessment order). Quantum events are sometimes incompatible, which are impossible to assess concurrently. Since assessment order matters, it could be the case that *q*(A ∧ then B) ≠ *q*(B ∧ then A). Incompatible events require the use of the more general partial Boolean algebra in QPT, because incompatible events cannot be defined within a single Boolean sample space. In QPT, certainty for one event changes the sample space for evaluating other (incompatible) events. Regarding DB arguments, if only a single assessment sequence is considered, then QPT is consistent with CPT, because *p*(B ∧A|B evaluated first) = *q*(B ∧ then A). Therefore, as for CPT, no DB can be formed against a person following QPT rules for conditional bets with a single order. If the order of assessing A and B matters, then the classical conjunction needs be modified anyway (written as $a \wedge b|\ order$).

Col. 1	Col. 2	Col. 3	C. 4	Column 5	Column 6	Column 7
on $a \wedge b$	against *b*	against $a\|b$	on *b*	first two columns	add the third column to Col. 5	add the fourth column to Col. 6
r, q (stake, prob)	*q, r*	*r, p*	*q-pr, r*			
T	T	T	T	*r*(*1-q*)-*q*(*1-r*) = *r-q*	*r-q-r*(*1-p*)=*rp-q*	*rp-q*+(1-*r*)(*q-pr*) = *r*(*pr*-q)
F	T	F	T	*-rq-q*(*1-r*) = *-q*	*-q+rp*= *rp-q*	*r*(*pr*-q)
F	F	cond. is off	F	*-rq+qr=0*	0 (the bet $a\|b$ is off when *b* is false)	0-*r*(*q-pr*) = *r*(*pr*-q)

Table A3. DB argument regarding conditional probabilities (following Howson & Urbach, 1993, p.83).

References
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Appendix 2. Conjunction fallacy with classical probabilities in a single probability space.

We show that a decision maker committing the CF and representing the two questions in a single probability space will fall foul of a DB, since, for each state of nature, a relevant combination of bets produces a sure loss. We use a notation corresponding to Tversky and Kahneman's (1983) Linda example, so that *F* corresponds to the possibility that Linda is a feminist, *BT* that she is a bank teller, and the behavior of interest is $Prob(F \wedge BT_F) > Prob(BT)$ (but note here we are assuming $BT \equiv BT_F$). Recall, betting on *h*, with stake *S*, probability *p*, gives payoff (1-*p*)*S* when *h* it true, –*pS* when *h* is false. A DB is evident in column 7 in Table A4, since, using the empirical result (*q*>*r*), for all states of nature a loss is guaranteed. A DB could be avoided only if we set *pr*=*q*, as required by CPT here, and contrary to the empirical result.

Col. 1	Col. 2	Col. 3	C. 4	Column 5	Column 7
against *BT*	against $F\|BT$	on $F \wedge BT_F$	on *BT*	overall payoff	using empirical result *q*>*r*
q, r (stake, prob)	*r, p*	*r, q*	*q-pr, r*		
T	T	T	T	(*r-1*)*q*+(*p-1*)*r*+*r*(*1-q*)+(*q-pr*)(*1-r*)= *r*(*pr-q*)	-ve (since if q>r then q>r*(number less than 1)
T	F	F	T	(*r-1*)*q*+*pr-qr*+(*q-pr*)(*1-r*)= *r*(*pr-q*)	-ve
F	cond. is off	F	F	*qr-qr*+(*pr-q*)*r*= *r*(*pr-q*)	-ve

Table A4.

Reference
Tversky, A., & Kahneman, D. (1983). Extensional versus intuitive reasoning: The conjuctive fallacy in probability judgment. Psychological Review, 90, 293-315.

Appendix 3. Avoiding a DB when committing the conjunction fallacy.

There are several ways to proceed. We first adopt an approach which is, in some ways, most analogous to that in Table A4. Table A5 below is as directly analogous as possible to Table A4. The only difference is that the truth table for *BT* is now decoupled from that for the conjunction, since $BT \neq BT_F$, and it is this decoupling which avoids a DB. Noting that shaded rows in Table A5 are identical to those in Table A4, it is straightforward to show that some rows necessarily lead to a positive payoff, others to a negative payoff, given the empirical result of *q*>*r*. Note, Lüder's postulate requires that $Prob(BT_F|F) = \frac{Prob(F \wedge then\ BT_F)}{Prob(F)}$ (analogously for Bayes's law). However, in the table below, *BT* is not the same as BT_F and so we no longer require *q*=*pr* (*p* is the probability of *F*|*BT*, *q* of $F \wedge BT_F$, and *r* of *BT*). Note, what prevents the DB is the last row, according to which we can have *BT* as false and the conjunction $F \wedge BT_F$ as true; this configuration maps well to the Linda paradigm (since the *BT* question is unlikely, but the conjunction likely).

Col. 1	Col. 2	Col. 3	C. 4	Column 5	Column 6
against *BT* stake, prob	against *F*|*BT*	on $F \wedge BT_F$	on *BT*	overall payoff	using the empirical result *q*>*r*
q, r	*r, p*	*r, q*	*q-pr, r*		
T	T	T	T	*r*(*pr-q*)	-ve
T	T	F	T	(*r-1*)*q*+(*p-1*)*r*-*qr*+(*q-pr*)(*1-r*)= *r*(*pr-q-1*)	-ve (since pr-q is –ve)
T	F	F	T	*r*(*pr-q*)	-ve
F	cond. is off	F	F	*r*(*pr-q*)	-ve
F	cond. is off	T	F	*qr*+(1-*q*)*r*-(*q-pr*)*r*=*r*+(*pr-q*)*r*	+ve (since this is the sum of a number minus a fraction of the same number)

Table A5.

We can simply illustrate this point alternatively. Let us set up a DB table, involving $F \wedge BT_F$, *F*, and BT_F|*F.* It is pertinent to consider this range of events, since in the Linda paradigm the QPT assumption is that the *F* question is evaluated first and then the *BT* one (Busemeyer et al., 2011). The main point now is that it is not appropriate to map the empirical result that the conjunction is greater than the marginal to $F \wedge BT_F$ and BT_F, since the *BT* question in isolation is different than the BT_F one. Therefore, we have to introduce an additional bet against *BT*, with a truth table uncoupled to that of the conjunction $F \wedge BT_F$ and the conditional BT_F|*F*. Using the empirical result *q*>*t*, it is trivially the case that a DB no longer obtains (Table A6). In Table A6, note that *q*=*pr* is required by both Bayes law and Lüder's postulate, for the BT_F question, since this equation links BT_F and $F \wedge BT_F$ in a single probability space. In Table A4, we tried to set *q*>*r* (the empirical finding) in a single probability space and got a DB.

Col. 0	Col. 1	Col. 2	Col. 3	C. 4	Column 5	Column 7
against *BT*	against *F*	against BT_F|*F*	on $F \wedge BT_F$	on *F*	overall payoff columns 0 to 4	using *q*=*pr* and empirical result *q*>*t*
s, t stake, prob	*q, r*	*r, p*	*r, q*	*q-pr, r*		
T	T	T	T	T	*r*(*pr-q*)+(*t-1*)*s*	-ve
T	T	F	F	T	*r*(*pr-q*)+(*t-1*)*s*	-ve

T	F	cond. is off	F	F	*r*(*pr*-*q*)+(*t*-*1*)*s*	-ve
F	T	T	T	T	*r*(*pr*-*q*) + *ts*	+ve
F	T	F	F	T	*r*(*pr*-*q*) + *ts*	+ve
F	F	cond. is off	F	F	*r*(*pr*-*q*) + *ts*	+ve

Table A6.

If we wish, we may generalize these basic ideas slightly by appealing to the consistent histories approach to QPT (Griffiths, 2003). To avoid a DB, we are restricting ourselves to single experimental conditions – the same verbal question in different experimental conditions is treated as a different question (e.g., *BT* and BT_F). The reason for doing so is that, in QPT, $Prob(A \wedge B) + Prob(\sim A \wedge B) \neq Prob(B)$, because of interference. The restriction to single experimental conditions, to avoid a DB, means that the various probabilities obey the correct sum rules. But, this is exactly what is guaranteed by the conditions for decoherence of histories in the consistent histories approach. In this context, decoherence refers to the vanishing of interference between different possible histories of a system. This may happen dynamically, for example as a result of interactions between a system and its environment, or for structural reasons, such as for histories which involve different values of a conserved quantity. Thus, the restriction to a single experimental condition can be replaced with a restriction to a single consistent family of histories. Specifically, with such an approach, instead of different experimental conditions, one would consider different families of histories. If $Prob(A \wedge B \wedge \ldots.)$ is a consistent family, then all the marginals can be included in the same family (and all the sum rules satisfied), and no DB is possible. This is true even though this family contains probabilities referring to many different experimental conditions. Then, families of histories which are not consistent can be treated in the same way as we employed different experimental conditions above.

This approach also makes clear the relationship between QPT seen as an extended version of CPT and standard CPT; if all the histories we wish to consider lie within a single consistent family, then we may drop the reference to the family of histories and recover a CPT description of this set of measurements. Again, this is true even though the family contains probabilities referring to many experimental conditions.

References

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Appendix 4. The materials for the experimental demonstration.

For the Linda conditions, the information participants saw was:

"Linda is 31 years old, single, outspoken and very bright. She majored in philosophy. As a student, she was deeply concerned with issues of discrimination and social justice, and also participated in anti-nuclear demonstrations."

<u>The questions in the Linda individual condition were:</u>
(In this and the other conditions each question was presented individually on a screen, with the exception of the conjunction questions, which were shown together with the corresponding marginals on the same screen; e.g., Figure A1.)
1. A teacher in elementary school.
2. Works in a bookstore.
Takes Yoga classes.
Works in a bookstore and takes Yoga classes.
3. Is active in the feminist movement. (*F*)
4. Is a woman.
5. Is a psychiatric social worker.
6. Is a member of the League of Women Voters.
7. Is a bank teller. (*BT*)
8. Is an insurance salesperson.

<u>The questions in the Linda conjunction condition were:</u>
1. A teacher in elementary school.
2. Works in a bookstore.
Takes Yoga classes.
Works in a bookstore and takes Yoga classes.
3. Is a woman (catch).
4. Is a psychiatric social worker.
5. Is a member of the League of Women Voters.
6. Is an insurance salesperson.
7. Is active in the feminist movement.
Is a bank teller. (BT_F)
Is active in the feminist movement and a bank teller. ($F\&BT_F$) (Note, order of *F, BT* reversed relative to original Tversky & Kahneman, 1983, paradigm.)

For the Bill conditions, the information participants saw was:
"Bill is 34 years old. He is intelligent, but unimaginative, compulsive, and generally lifeless. In school, he was strong in mathematics but weak in social studies and humanities."

<u>The questions in the Bill individual condition were:</u>
1. Bill is a physician.
Bill plays poker for a hobby.
Bill is a physician and plays poker for a hobby.
2. Bill is an architect.

3. Bill is an accountant. (*A*)
4. Bill plays jazz for a hobby. (*J*)
5. Bill surfs for a hobby.
6. Bill is a reporter.
7. Bill climbs mountains for a hobby.
8. Bill is a man.

<u>The questions in the Bill conjunction condition were:</u>
1. Bill is a physician.
Bill plays poker for a hobby.
Bill is a physician and plays poker for a hobby.
2. Bill is an architect.
3. Bill surfs for a hobby.
4. Bill is a reporter.
5. Bill is an accountant.
Bill plays jazz for a hobby. (J_A)
Bill is an accountant and plays jazz for a hobby. (A&J_A)
6. Bill climbs mountains for a hobby.
7. Bill is a man.

<u>For the Julie conditions, the information participants saw was:</u>
"Julie is 17 years old. She has grown up in a large city and is familiar with its dangers. She is fit and has a black belt in Karate. She has just finished a late class (it is now dark) at her high school and waits for the bus to take her home. The bus stop is on a quiet, suburban, well-to-do street. All questions you will be asked concern Julie while waiting for the bus stop."

<u>The questions in the Julie Kitten condition were:</u>
1. Encounters a lost kitten. (*K*)
Is scared. (*S*)
Encounters a lost kitten and is scared (K&S_K)
2. Sees an out of control speeding car about to crush her against the bus stand. (*C*)
3. Starts reading a book to pass the time.
4. Is a woman.
5. Ends up waiting for longer than she expected, because the bus is late.
6. Is on her own at the bus stop.

<u>The questions in the Julie Car condition were:</u>
1. Encounters a lost kitten. (*K*)
2. Sees an out of control speeding car about to crush her against the bus stand. (*C*)
Is scared. (*S*)
Sees an out of control speeding car about to crush her against the bus stand and is scared (C&S_C)
3. Starts reading a book to pass the time.

4. Is a woman.
5. Ends up waiting for longer than she expected, because the bus is late.
6. Is on her own at the bus stop.

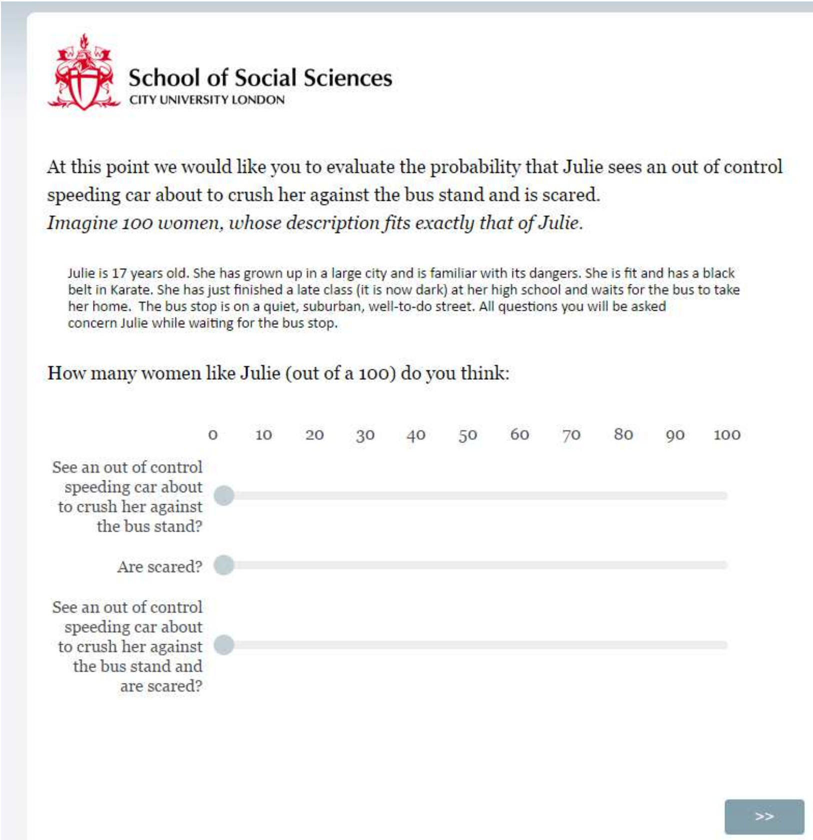

Figure A1. An example of how the questions for a conjunction (in the Julie Car condition) would be presented to participants. The format for presenting individual questions (e.g., S instead of S_C) was identical, but a single question was presented, instead of three, as in this example. Note that the information about the relevant individual (Julie in this case) was presented together with every single question.

Appendix 5. Some supplementary analyses regarding the experimental demonstration.

All variables were positively skewed, as shown in Figure A2.

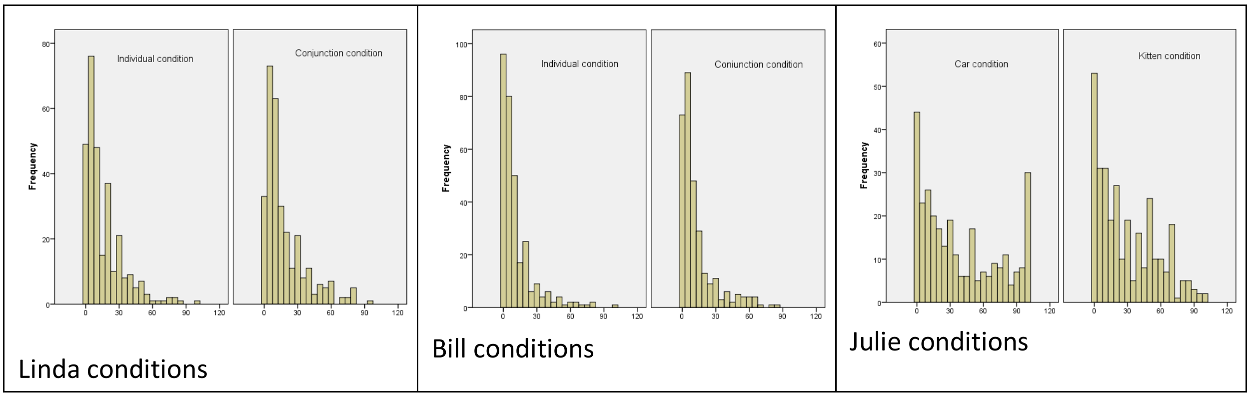

Figure A2. Histograms for the BT, BT_F in the Linda conditions, J, J_A in the Bill conditions, and S_K, S_C in the Julie conditions, showing the positive skewness of the variables.

		Normality tests		
	Skewness	Statistic	df	Sig.
Linda individual	1.837	0.806	297	<.0005
Linda conjunction	1.701	0.804	303	<.0005
Bill individual	2.58	0.691	309	<.0005
Bill conjunction	2.128	0.723	303	<.0005
Julie car	0.554	0.879	297	<.0005
Julie kitten	0.706	0.902	306	<.0005

Table A7. Skewness values for the six conditions. The normality test employed is the Sharipo-Wilk one. The null hypothesis that the data come from a normal distribution is easily rejected in all cases.