Mending the broken PT-regime via an explicit time-dependent Dyson map

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Abstract: We demonstrate that non-Hermitian Hamiltonian systems with spontaneously broken PT-symmetry and partially complex eigenvalue spectrum can be made meaningful in a quantum mechanical sense when introducing some explicit time-dependence into their parameters. Exploiting the fact that explicitly time-dependent non-Hermitian Hamitonians are unobservable and not identical to the energy operators in such a scenario, we show that their corresponding non-Hermitian energy operators develop a different type of PT-symmetry from the Hamiltonians that ensures the reality of their energy spectra. For this purpose we analytically solve the fully time-dependent Dyson equation with all quantities involved being explicitly time-dependent giving rise to a time-dependent metric. The key auxiliary equation to be solved for the two level atomic system considered here is the nonlinear Ermakov-Pinney equation with time-dependent coefficients.

1. Introduction

It is well known that non-Hermitian Hamiltonians that commute with an antilinear operator for which its eigenfunctions are eigenstates [1] possess real eigenvalue spectra. $\mathcal{PT}$-symmetry [2] is a specific example for such an antilinear symmetry for which many examples have been worked out in detail, see e.g. [3]. Moreover, contrary to standard text book wisdom, such type of systems can be made quantum mechanically meaningful [4, 5, 6] by introducing new inner products for which operators associated to observables are self-adjoint. However, for systems with infinite dimensional Hilbert spaces there are also well known issues related to the boundedness of the operators involved [7, 8, 9]. For instance, while the metric operator might be bounded the inverse of the Dyson map, needed to facilitate the mapping from a non-Hermitian Hamiltonian to an isospectral Hermitian Hamiltonian, might be unbounded [7]. There is also no guarantee that time-evolution operators for time-independent non-Hermitian Hamiltonians with real eigenvalues are bounded operators [10].

Another origin for the occurrence of unbounded time-evolution operators is the spontaneously breaking of the $\mathcal{PT}$-symmetry. This scenario emerges for $\mathcal{PT}$-symmetric Hamiltonians for which its eigenstates are not eigenstates of the antilinear symmetry operator, in which case the spectrum develops complex conjugate pairs of eigenvalues. While such a situation is the most interesting one in optical settings [11, 12, 13], where different channels of gain and loss may be constructed, such systems will inevitably develop infinite grows in energy and are therefore usually discarded as being non-physical in a quantum mechanical framework. We demonstrate here that by introducing an explicit time-dependence into the parameters of the quantum Hamiltonian, such systems can be made physically meaningful. This possibility exists since in a quantum mechanical context those type of Hamiltonians are no longer associated to the observable energy operator, as that operator acquires an additional time-dependent correction term.

In order to find that correction term one needs to solve the time-dependent Dyson relation for the Dyson map. So far only few explicit solutions to these relations are known and progress has been made in various stages. The simplest scenario is to assume that only the Hamiltonian is explicitly dependent on time, but the Dyson map or the closely related metric operator are kept time-independent [14, 15]. More involved is to include the time-dependence in the latter operator with a focus on finding solutions [16, 17] without investigating the properties of the corresponding wavefunctions of the time-dependent Schrödinger equation. In [18, 19] we studied the interesting possibility to keep the non-Hermitian Hamiltonian time-independent with an explicit time-dependence in the Dyson map. This allowed us to solve time-dependent Hermitian Hamiltonian systems by transferring the time-dependence from the Hamiltonian to the Dyson map or metric operators when discussing expectation values. The corresponding solutions to the time-dependent Schrödinger equation were found to be entirely consistent for a quantum mechanical description.

Here we extend the previous analysis and consider a fully time-dependent scenario for all quantities involved, that is the non-Hermitian Hamiltonian $H(t)$ together with its Hermitian counterpart $h(t)$ both being the defining quantities in the time-dependent Schrödinger equations

$$h(t)\phi(t) = i\hbar\partial_t\phi(t), \qquad \text{and} \qquad H(t)\Psi(t) = i\hbar\partial_t\Psi(t). \tag{1.1}$$

The time-dependent invertible Dyson operator $\eta(t)$ relates the solutions of these two equations by

$$\phi(t) = \eta(t)\Psi(t), \tag{1.2}$$

as well as the two Hamiltonians via the time-dependent Dyson relation

$$h(t) = \eta(t)H(t)\eta^{-1}(t) + i\hbar\partial_t\eta(t)\eta^{-1}(t). \tag{1.3}$$

Following the standard arguments of time-independent $\mathcal{PT}$-symmetric/quasi-Hermitian quantum mechanics [2, 5, 6], by asserting that observable operators $\mathcal{O}$ in the non-Hermitian system need to be related to a self-adjoint operator $o(t)$ in the Hermitian system as $o(t) = \eta(t)\mathcal{O}(t)\eta^{-1}(t)$, this leads to the curious fact that the Hamiltonian $H(t)$, being defined as the operator satisfying the Schrödinger equation, is not observable. This feature has led to a controversy [20, 21] questioning whether it is at all possible to formulate a consistent fully time-dependent framework for non-Hermitian Hamiltonian systems. The conundrum is easily solved by making a clear distinction between the observable energy operator

$$\tilde{H}(t) = \eta^{-1}(t)h(t)\eta(t) = H(t) + i\hbar\eta^{-1}(t)\partial_t\eta(t), \tag{1.4}$$

and the unobservable Hamiltonian $H(t)$ satisfying the time-dependent Schrödinger equation. In what follows we set $\hbar = 1$. In an adiabatic approximation the energy spectrum of this operator may be dealt with consistently at each instance of time [22]. In turn, since $H(t)$ is not an observable operator this also means that its eigenvalues do not have to be real at any instance in time. It is this latter fact that we exploit to make sense of a non-Hermitian Hamiltonian with complex conjugate eigenvalues as self-consistent quantum mechanical system.

2. A two-level system with spontaneously broken PT-symmetry

To illustrate our point we consider a simple two-level spin model described by the non-Hermitian Hamiltonian

$$H = -\frac{1}{2}\left[\omega\mathbb{I} + \lambda\sigma_z + i\kappa\sigma_x\right], \tag{2.1}$$

with σ_x, σ_y, σ_z denoting the Pauli matrices, $\mathbb{I}$ the identity matrix and ω, λ, $\kappa \in \mathbb{R}$. The two eigenvalues and eigenvectors for this Hamiltonian are simply

$$E_\pm = -\frac{1}{2}\omega \pm \frac{1}{2}\sqrt{\lambda^2 - \kappa^2}, \quad \text{and} \quad \varphi_\pm = \begin{pmatrix} i(-\lambda \pm \sqrt{\lambda^2 - \kappa^2}) \\ \kappa \end{pmatrix}. \tag{2.2}$$

Using Wigner's argument [1, 2] the reality of the energy spectrum for $|\lambda| > |\kappa|$ is easily explained by identifying an antilinear symmetry operator, denoted here as $\mathcal{PT}$, that commutes with the Hamiltonian and for which $\varphi_\pm$ are simultaneous eigenstates of H and $\mathcal{PT}$

$$[\mathcal{PT}, H] = 0, \qquad \text{and} \qquad \mathcal{PT}\varphi_\pm = e^{i\phi}\varphi_\pm, \tag{2.3}$$

with $\phi \in \mathbb{R}$. When $|\lambda| > |\kappa|$ in our example the symmetry operator is easily identified as $\mathcal{PT} = \tau\sigma_z$ with τ denoting complex conjugation. When $|\lambda| < |\kappa|$ the last relation in (2.3) no longer holds and the eigenvalues become complex conjugate to each other, a scenario usually referred to as spontaneously broken $\mathcal{PT}$-symmetry. For the parameter range of the latter situation this Hamiltonian would be regarded as non-physical from a quantum mechanical point of view as it possesses channels of infinite grows in energy, such that the corresponding time evolution operators would be unbounded.

However, when one introduces an explicit time-dependence into the Hamiltonian, $H \to H(t)$, it no longer plays the role of the observable energy operator so that the complex eigenvalues do not constitute any interpretational obstacle. For a meaningful physical picture one only needs to guarantee now that the expectation values of $\tilde{H}(t)$, as defined in (1.4), are real and instead identify a new $\widetilde{\mathcal{PT}}$-symmetry to be responsible for this property

$$\left[\widetilde{\mathcal{PT}}, \tilde{H}\right] = 0, \qquad \text{and} \qquad \widetilde{\mathcal{PT}}\tilde{\varphi}_\pm = e^{i\tilde{\phi}}\tilde{\varphi}_\pm, \tag{2.4}$$

with $\tilde{\varphi}_\pm$ denoting the eigenvectors of $\tilde{H}$ and $\tilde{\phi} \in \mathbb{R}$. Notice that $\mathcal{PT}$ and $\widetilde{\mathcal{PT}}$ are only symbols here to denote different types of antilinear operators, which however to not send t to $-t$ as the time is only a real parameter in this context.

Let us therefore introduce an explicit time-dependence into the parameters of H, via $\lambda \to \alpha\kappa(t)$, $\kappa \to \kappa(t)$, and solve this problem for the time-dependent Hamiltonian

$$H(t) = -\frac{1}{2}\left[\omega\mathbb{I} + \alpha\kappa(t)\sigma_z + i\kappa(t)\sigma_x\right]. \tag{2.5}$$

To find the precise form of $\tilde{H}(t)$ we need to solve first equation (1.3) for the Dyson map $\eta(t)$. As discussed in [18, 19], this is most easily achieved by pre-selecting some concrete form for $h(t)$[1]. For simplicity we take this to be

$$h(t) = -\frac{1}{2}\left[\omega\mathbb{I} + \chi(t)\sigma_z\right], \tag{2.6}$$

with $\chi(t)$ being a general undetermined function of time. Taking $\eta(t)$ to be of the most generic Hermitian form by using the notation

$$\eta(t) = \frac{1}{2}[\eta_1(t) + \eta_4(t)]\mathbb{I} + \eta_2(t)\sigma_x + \eta_3(t)\sigma_y + \frac{1}{2}[\eta_1(t) - \eta_4(t)]\sigma_z, \tag{2.7}$$

with real functions $\eta_i(t)$, the time-dependent Dyson equation (1.3) for (2.5) and (2.6) is solved when the component functions of $\eta(t)$ satisfy the coupled first order equations

$$\dot{\eta}_1 = \frac{\kappa}{2}\eta_2, \qquad \dot{\eta}_2 = \frac{\chi + \alpha\kappa}{2}\eta_3 + \frac{\kappa}{2}\eta_1, \qquad \dot{\eta}_3 = -\frac{\chi + \alpha\kappa}{2}\eta_2, \qquad \dot{\eta}_4 = \frac{\kappa}{2}\eta_2, \tag{2.8}$$

$$\eta_1 = \eta_4, \qquad \chi = \kappa\left(\frac{\eta_3}{\eta_1} + \alpha\right). \tag{2.9}$$

The overdot denotes here as usual a differentiation with respect to time. The equations (2.8) are solved by

$$\eta_1 = \eta_4 = c\sqrt{\frac{\kappa}{\chi}}, \qquad \eta_2 = \frac{c}{\sqrt{\kappa\chi}}\left(\frac{\dot{\kappa}}{\kappa} - \frac{\dot{\chi}}{\chi}\right), \qquad \eta_3 = c\left(\sqrt{\frac{\chi}{\kappa}} - \alpha\sqrt{\frac{\kappa}{\chi}}\right), \tag{2.10}$$

with c denoting an integration constant and $\chi(t)$ satisfying the nonlinear second order equation

$$\ddot{\chi} - \frac{3}{2}\frac{\dot{\chi}^2}{\chi} + \left[\frac{3}{2}\left(\frac{\dot{\kappa}}{\kappa}\right)^2 - \frac{\ddot{\kappa}}{\kappa} + \frac{1}{2}\kappa^2(1-\alpha^2)\right]\chi + \frac{\chi^3}{2} = 0. \tag{2.11}$$

Using the parameterizations $\chi = 2/\sigma^2$ or $\kappa = 2/(\sigma^2\sqrt{\alpha^2-1})$ this equation is converted into the Ermakov-Pinney (EP) equation [23, 24] for σ

$$\ddot{\sigma} + \lambda(t)\sigma = \frac{1}{\sigma^3} \tag{2.12}$$

[1]Alternatively one may also solve the time-dependent quasi-Hermiticity relation $H^\dagger\rho(t) - \rho(t)H = i\hbar\partial_t\rho(t)$ for the metric operator $\rho(t)$ and subsequently determine $\eta(t)$ from $\rho(t) := \eta^\dagger(t)\eta(t)$. However, as argued in [19], usually this turns out to be more difficult.

with time-dependent coefficient

$$\lambda(t) = \frac{1}{2}\frac{\ddot{\kappa}}{\kappa} - \frac{3}{4}\left(\frac{\dot{\kappa}}{\kappa}\right)^2 - \frac{1}{4}\kappa^2(1-\alpha^2) \qquad \text{or} \qquad \lambda(t) = \frac{1}{2}\frac{\ddot{\chi}}{\chi} - \frac{3}{4}\left(\frac{\dot{\chi}}{\chi}\right)^2 + \frac{1}{4}\chi^2, \tag{2.13}$$

respectively. Thus either way given the time-dependent field $\kappa(t)$ in $H(t)$ or $\chi(t)$ in $h(t)$ the remaining field is constrained by the EP equation with almost identical coefficients. The EP equation emerges in many scenarios of time-dependent quantum mechanics and various areas in mathematics, see for instance [25] for an overview. The general solution for (2.12), as reported by Pinney [24], is

$$\sigma(t) = \left(Au^2 + Bv^2 + 2Cuv\right)^{1/2}, \tag{2.14}$$

where $u(t)$ and $v(t)$ are the two fundamental solutions to the equation $\ddot{\sigma} + \lambda(t)\sigma = 0$ and the constants A, B, C are constrained as $C^2 = AB - W^{-2}$ with $W = u\dot{v} - v\dot{u}$ denoting the corresponding Wronskian. Thus from the solution of the EP equation for fixed α we can obtain now a specific solution for the Dyson map (2.10). As the exceptional point at $\alpha = 1$ for H leads to qualitatively different solutions, we treat this case separately from the cases with $\alpha \neq 1$.

2.1 The $\widetilde{\mathcal{PT}}$-symmetric regimes of $\tilde{H}$, $\alpha \neq 1$

We are left with solving the EP equation so that (2.8) becomes an explicit solution to the time-dependent Dyson equation. For definiteness we assume here that $\kappa(t)$ is given and determine $\chi(t)$, but as mentioned in the previous section the reverse computation requires very little modification. Taking the time-dependent coefficient $\lambda(t)$ in the EP equation to be of the form (2.13) for $\alpha \neq 1$ we find

$$u(t) = \frac{1}{\sqrt{\kappa}}e^{\mu/2}, \quad v(t) = \frac{1}{\sqrt{\kappa}}e^{-\mu/2}, \quad \text{with } \mu(t) := \sqrt{1-\alpha^2}\int^t \kappa(s)ds, \tag{2.15}$$

such that the solution to the EP equation (2.14) becomes

$$\sigma(t) = \frac{1}{\sqrt{\kappa}}\left(Ae^{\mu} + Be^{-\mu} \pm 2\sqrt{AB - 1/(1-\alpha^2)}\right)^{1/2}. \tag{2.16}$$

Parameterizing the constants further as $A = c_1 + c_2$, $B = c_1 - c_2$ we obtain the solution

$$\chi(t) = \frac{\kappa}{\xi}, \quad \text{with } \xi := c_1\cosh\mu + c_2\sinh\mu \pm \sqrt{c_1^2 - c_2^2 - 1/(1-\alpha^2)}. \tag{2.17}$$

Thus the Dyson map is obtained from (2.10) as

$$\eta_1 = \eta_4 = \sqrt{\xi}, \quad \eta_2 = \frac{\hat{\xi}\sqrt{1-\alpha^2}}{\sqrt{\xi}}, \quad \eta_3 = \frac{1-\alpha\xi}{\sqrt{\xi}}, \quad \text{with } \hat{\xi} := c_1\sinh\mu + c_2\cosh\mu \tag{2.18}$$

Since in all relevant equations η is accompanied by it inverse we have set $c = 1$ in (2.10) without loss of generality. Noting that $\det\eta = \eta_1^2 - \eta_2^2 - \eta_3^2 = \pm 2\delta$ with $\delta := \alpha + (1-\alpha^2)\sqrt{c_2^2 - c_3^2 - 1/(1-\alpha^2)}$ the Dyson map is invertible for as long as $\alpha \neq 0$ or $c_2^2 \neq c_3^2 + 1/(1-\alpha^2)$.

Next we turn to solving the time-dependent Schrödinger equation. This is easily achieved for the first equation in (1.1) as $h(t)$ is diagonal. We find the two orthonormal solutions

$$\left|\phi_+(t)\right\rangle = e^{i\omega t/2+i\theta(t)} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \text{and} \qquad \left|\phi_-(t)\right\rangle = e^{i\omega t/2-i\theta(t)} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{2.19}$$

with $\left\langle\phi_i(t)\left|\phi_j(t)\right.\right\rangle = \delta_{ij}$ for $i,j = +,-$ and

$$\theta(t) = \frac{1}{2}\int^t \chi(s)ds = \arctan\left\{\sqrt{1-\alpha^2}\left[c_2 + \left(c_1 \mp \sqrt{c_1^2 - c_2^2 - \frac{1}{1-\alpha^2}}\right)\tanh\left[\mu(t)/2\right]\right]\right\}. \tag{2.20}$$

Having found $\eta(t)$ we obtain from (1.2) the solution for the Schrödinger equation related to the non-Hermitian Hamiltonian $H(t)$ as

$$\left|\psi_+(t)\right\rangle = \frac{-e^{i\omega t/2+i\theta(t)}}{2\delta}\begin{pmatrix} -\eta_1 \\ \eta_2 + i\eta_3 \end{pmatrix} \;\; \text{and} \;\; \left|\psi_-(t)\right\rangle = \frac{e^{i\omega t/2-i\theta(t)/}}{2\delta}\begin{pmatrix} \eta_2 - i\eta_3 \\ -\eta_1 \end{pmatrix}. \tag{2.21}$$

By construction these states are orthonormal with regard to the inner product with modified metric $\left\langle\psi_i(t)\left|\eta^2\psi_j(t)\right.\right\rangle = \delta_{ij}$ for $i,j = +,-$. Next we compute the energy operator (1.4), which acquires the form

$$\tilde{H}(t) = -\frac{1}{2}\left\{\omega\mathbb{I} + \frac{\chi}{\delta}\left[i\left(\alpha\xi - 1\right)\sigma_x + i\left(\hat{\xi}\sqrt{1-\alpha^2}\right)\sigma_y + (\xi-\delta)\sigma_z\right]\right\}. \tag{2.22}$$

Since $\tilde{H}(t)$ is related to a Hermitian Hamiltonian by a similarity transformation we expect the eigenvalues of this Hamiltonian to be real when this transformation is well defined. Indeed, it turns out that the energy expectation values for these states are real at any instance in time and simply result to

$$\tilde{E}_\pm(t) = \left\langle\psi_\pm(t)\left|\tilde{H}(t)\eta^2\psi_\pm(t)\right.\right\rangle = \left\langle\phi_\pm(t)\left|h(t)\phi_\pm(t)\right.\right\rangle = -\frac{1}{2}\left[\omega \pm \chi(t)\right]. \tag{2.23}$$

Thus as long as $\chi(t)$ is real the energy expectation values are real, which is the case when c_1,$c_2 \in \mathbb{R}$ and $c_1^2 > c_2^2 + 1/(1-\alpha^2)$ for $\alpha < 1$ or when $c_1 \in \mathbb{R}$ and $c_2 \in i\mathbb{R}$ for $\alpha > 1$. We depict the energy spectra as a function of time for some specific parameter values in figures 1 and 2.

The behaviour shown in the figures 1 and 2 is typical for non-Hermitian with an antilinear symmetry. Thus we expect for $\tilde{H}(t)$ that there exists an antilinear symmetry $\widetilde{\mathcal{PT}}$ that solves (2.4) and hence explains the reality and complexity of the eigenspectrum. Evidently the operator $\mathcal{PT}$ as introduced above is not the correct symmetry and only serves to explain the spectrum for $H(t)$. Thus we make a generic Ansatz for this operator and try to solve the first relation in (2.4). Indeed we find as the unique solution the antilinear operator

$$\widetilde{\mathcal{PT}} := \frac{1}{\sqrt{(\xi-\delta)^2 + (\alpha^2-1)\hat{\xi}^2}}\left[i\left(\sqrt{1-\alpha^2}\hat{\xi}\right)\sigma_y + (\xi-\delta)\sigma_z\right]\tau. \tag{2.24}$$

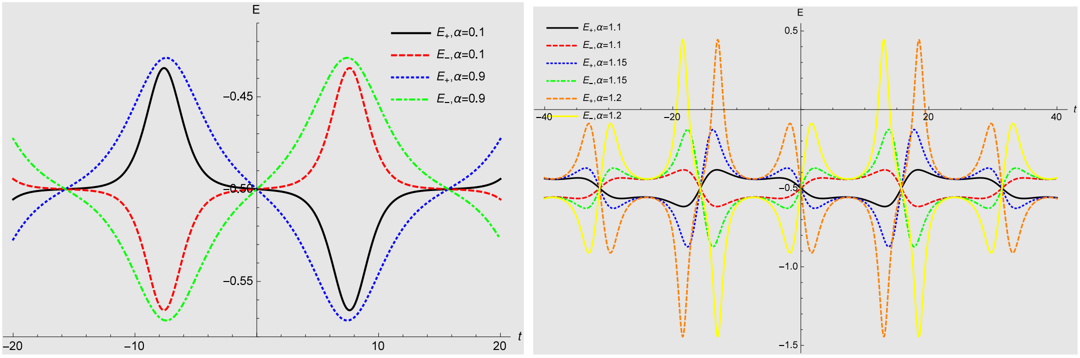

Figure 1: Real energies in the $\widetilde{\mathcal{PT}}$-symmetric phase for $\kappa(t) = \sin(t/5)$, $\omega = 1$, $c_1 = 4$ and $c_2 = 1$ (panel a) $c_2 = i$ (panel b).

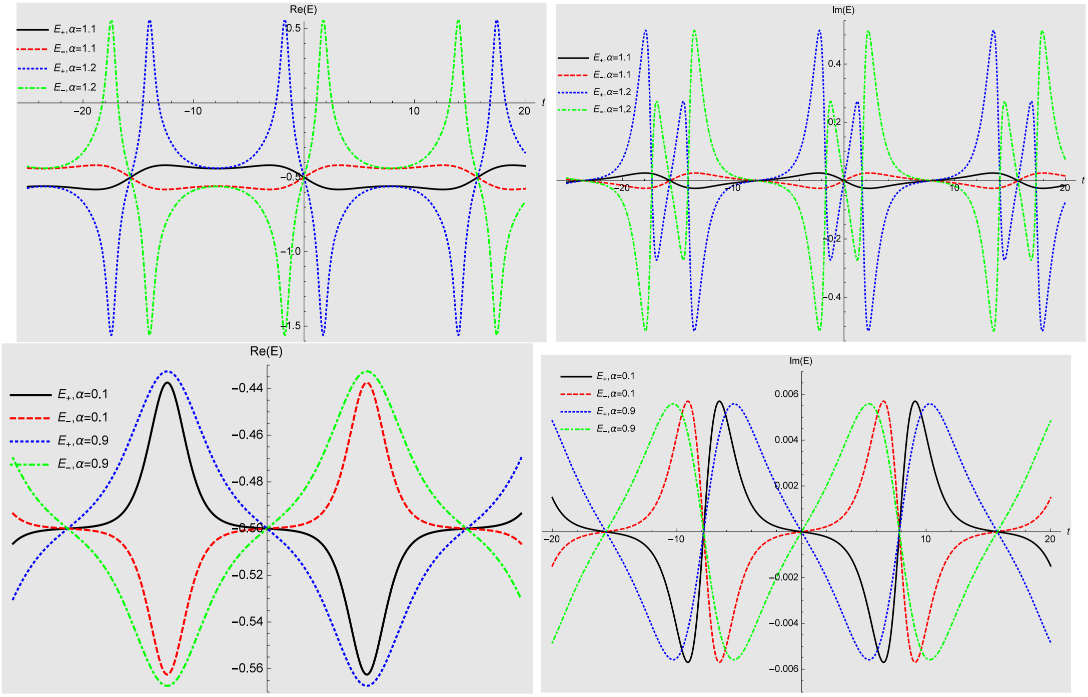

Figure 2: Complex energies in the $\widetilde{\mathcal{PT}}$-broken phase for $\kappa(t) = \sin(t/5)$, $\omega = 1$, $c_1 = 4$ and $c_2 = 1$ (panel a, b) $c_2 = i$ (panel c, d).

We verify that $\widetilde{\mathcal{PT}}$ is involutory with $\widetilde{\mathcal{PT}}^2 = \mathbb{I}$. Furthermore we verify that $\widetilde{\mathcal{PT}}\sigma_x\widetilde{\mathcal{PT}} = -\sigma^x$ and $\widetilde{\mathcal{PT}}\sigma_z\widetilde{\mathcal{PT}} \neq \sigma_z$. Thus when $\alpha \neq 0$ the new $\widetilde{\mathcal{PT}}$-symmetry is not a symmetry of $H(t)$, i.e. we have $\left[\widetilde{\mathcal{PT}}, H(t)\right] \neq 0$ but $\left[\widetilde{\mathcal{PT}}, \tilde{H}(t)\right] = 0$. In order to guarantee that this symmetry is unbroken we also need to satisfy the second equation in (2.4). We determine the eigenvectors of $\tilde{H}(t)$ as

$$\tilde{\varphi}_{\pm} \sim \begin{pmatrix} (1 \mp 1)\delta - \xi \\ \sqrt{1 - \alpha^2}\hat{\xi} + i(1 - \alpha\xi) \end{pmatrix}, \tag{2.25}$$

and verify that these vectors are indeed $\widetilde{\mathcal{PT}}$-eigenstates

$$\widetilde{\mathcal{PT}}\tilde{\varphi}_{\pm} = e^{i\tilde{\omega}_{\pm}}\tilde{\varphi}_{\pm}. \tag{2.26}$$

with

$$\tilde{\omega}_{+} = \arctan\left[\frac{2\sqrt{1-\alpha^2}(1-\alpha\xi)\hat{\xi}}{1+\xi(\xi-2\alpha+\xi\alpha^2)+(\alpha^2-1)\hat{\xi}^2}\right], \tag{2.27}$$

$$\tilde{\omega}_{-} = \arctan\left[\frac{\sqrt{1-\alpha^2}(1-\alpha\xi)\hat{\xi}}{2\delta^2-3\delta\xi+\xi^2+(\alpha^2-1)\hat{\xi}^2}\right]+\pi. \tag{2.28}$$

Thus for the regime stated above the $\widetilde{\mathcal{PT}}$-symmetry is unbroken and the eigenvalues of $\tilde{H}(t)$ are therefore guaranteed to be real. We notice that for $a > 1$ also the Hamiltonian $H(t)$ is in its $\mathcal{PT}$ -symmetric phase, but $\widetilde{\mathcal{PT}}$ is still not a symmetry for $H(t)$.

2.2 The $\widetilde{\mathcal{PT}}$-symmetric regime of $\tilde{H}$ and $\widetilde{\mathcal{PT}}$-broken regime of H, $\alpha = 0$

The value $\alpha = 0$ is special as in this case the $\widetilde{\mathcal{PT}}$-operator commutes with both $\tilde{H}(t)$ and $H(t)$, but the eigenvalues of the latter (2.2) are complex conjugate in this case. This means we expect the eigenvectors of $H(t)$ not to be eigenstates of the $\widetilde{\mathcal{PT}}$-operator. It is instructive to verify this in detail and since the formulae simplify substantially in this case, it is also useful to have a simpler example at hand. The two orthonormal solutions for $h(t)$ take on the same form as in (2.19) with

$$\theta(t) = \frac{1}{2}\int^t \chi(s)ds = \arctan\left\{c_2 + \left(c_1 \mp \sqrt{c_1^2-c_2^2-1}\tanh\left[\mu(t)/2\right]\right)\right\}, \tag{2.29}$$

and the solutions for the Schrödinger equation related to the non-Hermitian Hamiltonian $H(t)$ are also given by (2.21) with $\det\eta = 2\sqrt{c_2^2-c_3^2-1}$. The energy operator (1.4) simplifies to

$$\tilde{H}(t) = -\frac{1}{2}\omega\mathbb{I} + \frac{\chi(t)}{\sqrt{c_1^2-c_2^2-1}}\left[i\sigma_x - (c_1\sinh\mu + c_2\cosh\mu)i\sigma_y - (c_1\cosh\mu + c_2\sinh\mu)\sigma_z\right]. \tag{2.30}$$

and the $\widetilde{\mathcal{PT}}$-operator reduces to

$$\widetilde{\mathcal{PT}} := \frac{1}{\sqrt{c_1^2-c_2^2}}\left[(c_1\sinh\mu + c_2\cosh\mu)i\sigma_y + (c_1\cosh\mu + c_2\sinh\mu)\sigma_z\right]\tau. \tag{2.31}$$

Now both Hamiltonians are $\widetilde{\mathcal{PT}}$-symmetric, i.e. in addition to $\left[\widetilde{\mathcal{PT}}, \tilde{H}(t)\right] = 0$ we also have $\left[\widetilde{\mathcal{PT}}, H(t)\right] = 0$. However, whereas the eigenvectors $\tilde{\varphi}_{+} \sim \{-\eta_1, \eta_2 + i\eta_3\}$, $\tilde{\varphi}_{-} \sim \{\eta_2 - i\eta_3, \eta, \}$ of $\tilde{H}(t)$ are $\widetilde{\mathcal{PT}}$-symmetric, the eigenvectors $\varphi_{\pm} \sim \{\pm 1, 1\}$ of $H(t)$ are not eigenstates of the $\widetilde{\mathcal{PT}}$-operator. Hence we have

$$\widetilde{\mathcal{PT}}\varphi_{\pm} \neq e^{i\omega_{\pm}}\varphi_{\pm} \qquad \text{and} \qquad \widetilde{\mathcal{PT}}\tilde{\varphi}_{\pm} = e^{i\tilde{\omega}_{\pm}}\tilde{\varphi}_{\pm}. \tag{2.32}$$

Concretely we identity

$$\tilde{\omega}_\pm = \arctan\left[\pm\frac{c_2^2 - c_1^2 - (c_1\cosh\mu + c_2\sinh\mu)\sqrt{c_1^2 - c_2^2 - 1}}{c_1\sinh\mu + c_2\cosh\mu}\right]. \tag{2.33}$$

Thus the $H(t)$ system is always in the spontaneously broken $\widetilde{\mathcal{PT}}$-symmetry phase whereas $\tilde{H}(t)$ is $\widetilde{\mathcal{PT}}$-symmetric as long as $c_1^2 - c_2^2 > 1$.

2.3 The exceptional point of $H(t)$ at $\alpha = 1$

The value $\alpha = 1$ is an exceptional point for $H(t)$ as it marks the transition from real to complex conjugate eigenvalues and at the same time the two eigenvectors coalesce. For $\tilde{H}$ it also indicates the boundary of the real eigenvalues, but they do not become complex conjugate to each other and the two eigenvectors remain different. The EP equation admits a qualitatively different solution in this case. Taking the time-dependent coefficient to be of the form (2.13) for $\alpha = 1$ with given κ we find

$$u(t) = \frac{1}{\sqrt{\kappa}}, \quad v(t) = \frac{1}{\sqrt{\kappa}}\mu, \quad \text{with } \mu := \int^t \kappa(s)ds, \tag{2.34}$$

such that the solution to the EP equation (2.14) becomes

$$\sigma(t) = \sqrt{\frac{\mu}{\kappa}}\left(A\mu + B\mu^{-1} \pm 2\sqrt{AB-1}\right)^{1/2}. \tag{2.35}$$

so that

$$\chi(t) = \frac{\kappa}{\xi}, \quad \text{with } \xi := \frac{1}{2}\left(B + A\mu^2 \pm 2\mu\sqrt{AB-1}\right). \tag{2.36}$$

Using these expressions the Dyson map is obtained from (2.10) as

$$\eta_1 = \eta_4 = \sqrt{\xi}, \quad \eta_2 = \frac{\hat{\xi}}{\sqrt{\xi}}, \quad \eta_3 = \frac{1}{\sqrt{\xi}} - \sqrt{\xi}, \qquad \text{with } \hat{\xi} := A\mu + \sqrt{AB-1}. \tag{2.37}$$

In this case we compute $\det\eta = \eta_1^2 - \eta_2^2 - \eta_3^2 = \pm 2\delta$ with $\delta = A - 1$.

The energy operator (1.4) acquires the form

$$\tilde{H}(t) = -\frac{1}{2}\left\{\omega\mathbb{I} + \frac{\chi}{\delta}\left[i(\xi-1)\sigma_x + i\hat{\xi}\sigma_y + (\xi-\delta)\sigma_z\right]\right\} \tag{2.38}$$

Similarly as above we construct the antilinear symmetry operator for this operator

$$\widetilde{\mathcal{PT}} := \frac{1}{\sqrt{(\xi-\delta)^2 + \hat{\xi}^2}}\left[i\hat{\xi}\sigma_y + (\xi-\delta)\sigma_z\right]\tau. \tag{2.39}$$

The eigenvectors of $\tilde{H}(t)$ are computed to

$$\tilde{\varphi}_\pm \sim \begin{pmatrix} (1\mp 1)\delta - \xi \\ \hat{\xi} + i(1-\xi) \end{pmatrix}, \tag{2.40}$$

which are indeed $\widetilde{\mathcal{PT}}$-eigenstates, that is we have $\widetilde{\mathcal{PT}}\tilde{\varphi}_{\pm} = e^{i\tilde{\omega}_{\pm}}\tilde{\varphi}_{\pm}$ with

$$\tilde{\omega}_{+} = \arctan\left[\frac{(1-\xi)\hat{\xi}}{1-(1+A)\xi+\xi^{2}}\right]+\pi, \tag{2.41}$$

$$\tilde{\omega}_{-} = \arctan\left[\frac{\sqrt{1-\alpha^{2}}(1-\alpha\xi)\hat{\xi}}{3+2A(A-2)-(3-A)\xi+\xi^{2}}\right]. \tag{2.42}$$

This means as long as $AB > 1$ the energy operator $\tilde{H}(t)$ is $\widetilde{\mathcal{PT}}$-symmetric with regard to (2.39).

3. Conclusions

We have demonstrated that a non-Hermitian Hamiltonian in its spontaneously broken $\mathcal{PT}$-symmetric phase allows for a self-consistent quantum mechanical description when an explicit time-dependence is introduced into its parameters. This is possible as the Hamiltonian that satisfies the time-dependent Schrödinger equation becomes unobservable and instead the energy operator develops real eigenvalues at any instance in time. We identified the new antilinear operator $\widetilde{\mathcal{PT}}$ that explains the reality of the spectrum of the energy operator in parts of the parameter regime.

We have solved for the first time the time-dependent Dyson equation in conjunction with the time-dependent Schrödinger equation in complete generality. Previously only special cases were considered, e.g. one of the Hamiltonians was kept time-independent or just the time-dependent Dyson equation was solved without further elaboration on whether the solutions obtained can be used in the solutions to the time-dependent Schrödinger equation.

Naturally it would be interesting to investigate different types of models and in particular extend the analysis to systems with infinite dimensional Hilbert spaces.

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