Degenerate multi-solitons in the sine-Gordon equation

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Abstract: We construct various types of degenerate multi-soliton and multi-breather solutions for the sine-Gordon equation based on Bäcklund transformations, Darboux-Crum transformations and Hirota's direct method. We compare the different solution procedures and study the properties of the solutions. Many of them exhibit a compound like behaviour on a small timescale, but their individual one-soliton constituents separate for large time. Exceptions are degenerate cnoidal kink solutions that we construct via inverse scattering from shifted Lamé potentials. These type of solutions have constant speed and do not display any time-delay. We analyse the asymptotic behaviour of the solutions and compute explicit analytic expressions for time-dependent displacements between the individual one-soliton constituents for any number of degeneracies. When expressed in terms of the soliton speed and spectral parameter the expression found is of the same generic form as the one formerly found for the Korteweg de-Vries equation.

1. Introduction

Previously [1, 2, 3] we constructed compound soliton solutions to the Korteweg de-Vries (KdV) equation that were composed of a fixed number of one-soliton solutions. These solutions have the interesting property that in a large regime at a small timescale the one-solitons constituents travel simultaneously at the same speed and with the same amplitude. Due to this property the collection of them could be regarded as an almost stable compound. In this regime the solutions behave similar to the famous tidal bore phenomenon which consists of multiple wave amplitudes of heights up to several meters traveling jointly upstream a river covering distances of up to several hundred kilometers, see e.g. [4]. However, the solutions we obtained display different behaviour at different timescales. At very large time the individual one-soliton constituents separate from each other with a time-dependent displacement, which could be computed exactly in closed analytical form for any number of one-solitons contained in the solution. For the nonlinear Schrödinger equation degenerate solutions were recently studied in [5].

Technically these solutions were obtained by regularizing well-known multi-soliton solutions when considering complex $\mathcal{PT}$-symmetric solutions with real energies. This complexification was necessary as the standard limit of one velocity coinciding with another leads to cusp type solutions tending to infinity. The procedure was made conceptually possible as the value of the overall energy was shown to remain real maintaining the same value.

Here we argue that for the sine-Gordon equation such type of compound multi-kink solutions also exist and the limit leading to degeneracy can be carried out even for the real valued solutions. We study here the sine-Gordon equation in light-cone variables

$$\phi_{xt} = \sin\phi, \tag{1.1}$$

which we denote by x, t and are related the original time and coordinate T, X, respectively, as $x = (X + T)/2$ and $t = (X - T)/2$. Non-degenerate solutions to the sine-Gordon equations (1.1) are known for a long time [6]. A convenient compact expression may be found, when transformed to light-cone variables, for instance in [7],

$$\phi_{\alpha_1\alpha_2\ldots\alpha_n}(x,t) = \arccos\left\{1 - 2\left[\ln\left(\det M\right)\right]_{xt}\right\} \tag{1.2}$$

with

$$M_{ij} = \frac{(\alpha_i\alpha_j)^{1/2}}{\alpha_i + \alpha_j}\left[e^{\eta_i} + (-1)^{i+j}e^{-\eta_j}\right], \qquad \eta_i = x\alpha_i + t/\alpha_i + \mu_i, \tag{1.3}$$

and constants α_i, μ_i, $\alpha_i \neq \alpha_j$ for $i,j = 1,\ldots,n$. The solution (1.2) demonstrates the problem that arises when one tries to obtain degenerate solutions from the previous expressions by naively setting the parameters $\alpha_i = \alpha_j$, as then the determinant of M evidently vanishes for $n > 3$ or just a constant for $n = 2$. Different versions, all plagued by a similar problem, may be found in [6, 8, 9]. We abbreviate solutions with a m-fold degeneracy in α as $\phi_{m\alpha\alpha_{m+1}\ldots\alpha_n}(x,t)$, i.e. this denotes an n-soliton solution with $\alpha_1 = \alpha_2 = \ldots = \alpha_m = \alpha$.

As is well known [10, 11, 12], the scattering of one-solitons within standard multi-soliton solutions leads to displacements in space or depending on the type of interaction delays or advances in time, which can be used to compute quantum mechanical scattering matrices in semi-classical approximations [13]. In [3] we observed for the degenerate KdV-solutions that these displacements become time-dependent. Here we find a similar asymptotic behaviour for the majority of our solutions and notice that the expressions for the delays take on a universal model independent form when expressed in terms of the soliton speed v and the constant spectral parameter α. For a N-soliton or N-kink solution ϕ_{Na} with N parameterized as $N = 2n + 1 - \kappa$ we find the universal formula for the time-dependent displacements between the different one-soliton contributions

$$\Delta_{n,\ell,\kappa}(t) := \frac{1}{\alpha}\ln\left[\frac{(n-\ell)!}{(n+\ell-\kappa)!}\left(4\alpha\left|tv\right|\right)^{2\ell-\kappa}\right] \quad n = 1,\ldots;\ \ell = \kappa,\ldots,n;\ \kappa = 0,1, \tag{1.4}$$

with speeds $v = \alpha^2$ and $v = -1/\alpha^2$ in the KdV-solutions and sine–Gordon kink solutions, respectively. We also construct degenerate solutions based on cnoidal kink solutions, which have constant speed and do not display displacements at any time.

Our manuscript is organized as follows: In sections 2, 3 and 4 we construct degenerate multi-soliton solutions based on Bäcklund transformations, Darboux-Crum transformations and Hirota's direct method, respectively. In section 5 we analyze the asymptotic behaviour of the solutions and compute displacements between the different one-soliton solutions deriving our universal formula (1.4).

2. Degenerate multi-solitons from Bäcklund transformations

It is well-known that in general the standard superposition principle for linear wave differential equations is replaced by model dependent functional relations involving more than two solutions for nonlinear wave equations. In the case of the sine-Gordon equation this equation arise from the combination of four sets of *Bäcklund transformations* where each of them relate two different solutions of (1.1), say ϕ and ϕ', as

$$\frac{\phi_x + \phi'_x}{2} = \frac{1}{\kappa} \sin\left[\frac{\phi - \phi'}{2}\right], \qquad \text{and} \qquad \frac{\phi_t - \phi'_t}{2} = \kappa \sin\left[\frac{\phi + \phi'}{2}\right]. \tag{2.1}$$

Evidently, the two solutions in (2.1) can not be arbitrary and need to be selected such that κ is a constant. Taking now four solutions ϕ, ϕ', ϕ'', ϕ''' and relate them pairwise as indicated in the Lamb-Bianchi diagram [14, 15] in figure 1 one obtains four sets of Bäcklund transformations.

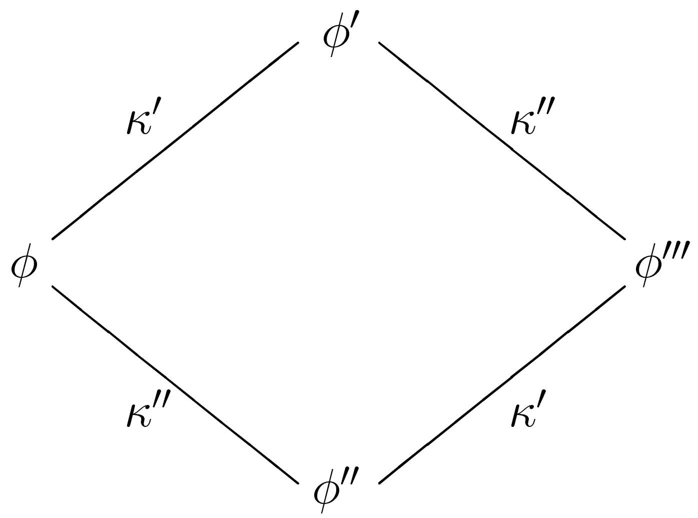

Figure 1: Lamb-Bianchi diagram for four arbitrary solutions ϕ, ϕ', ϕ'', ϕ''' of the sine-Gordon equation, with each link representing a Bcklund transformation (2.1) with constant κ chosen as indicated.

Eliminating all differentials leads to the *nonlinear superposition principle* for the sine-Gordon equation

$$\phi''' = 4 \arctan\left[\frac{\kappa' + \kappa''}{\kappa' - \kappa''} \tan\left(\frac{\phi' - \phi''}{4}\right)\right] + \phi. \tag{2.2}$$

Hence a new solution ϕ''' can be constructed in the functional way as described by (2.2) from three known solutions ϕ, ϕ', ϕ''. Moreover, one can easily see that one can not directly obtain degenerate solutions from (2.2) as that would require $\phi' = \phi''$ and $\kappa' = \kappa''$. We will now demonstrate how the limits may be taken appropriately, thus leading to degenerate multi-soliton solutions. Subsequently we study the properties of this new type of solutions.

At first we construct an n-soliton solution with an $(n-1)$-fold degeneracy. For this purpose we start by relating four solutions to the sine-Gordon equation as depicted in the Lamb-Bianchi in figure 1 with the choice $\phi = \phi_{(n-2)\alpha}$, $\phi' = \phi_{(n-1)\alpha}$, $\phi'' = \phi_{(n-2)\alpha\beta}$ and constants $\kappa' = 1/\alpha$ and $\kappa'' = 1/\beta$, such that by (2.2) we obtain

$$\phi_{(n-1)\alpha\beta} = 4\arctan\left[\frac{\alpha+\beta}{\alpha-\beta}\tan\left(\frac{\phi_{(n-2)\alpha\beta}-\phi_{(n-1)\alpha}}{4}\right)\right] + \phi_{(n-2)\alpha}. \tag{2.3}$$

Using the identity (see appendix for a derivation)

$$\lim_{\beta\to\alpha}\frac{\alpha+\beta}{\alpha-\beta}\tan\left(\frac{\phi_{(n-2)\alpha\beta}-\phi_{(n-1)\alpha}}{4}\right) = \frac{\alpha}{2(1-n)}\frac{d\phi_{(n-1)\alpha}}{d\alpha}, \tag{2.4}$$

we can perform the non-trivial limit $\beta \to \alpha$ in (2.3), obtaining in this way the recursive equation

$$\phi_{n\alpha} = \phi_{(n-2)\alpha} - 4\arctan\left(\frac{\alpha}{2(n-1)}\frac{d\phi_{(n-1)\alpha}}{d\alpha}\right), \qquad \text{for } n \geq 2, \tag{2.5}$$

with $\phi_{n\alpha} = 0$ for $n \leq 0$. In principle equation (2.5) is sufficient to the degenerate solutions $\phi_{n\alpha}$ recursively. However, it still involves a derivative term which evidently becomes more and more complex for higher order. We eliminate this term next and replace it with combinations of just degenerate solutions. By iterating the Bäcklund transformation (2.1) we compute the derivatives with respect to x and t to

$$(\phi_{n\alpha})_x = 2\alpha\sum\nolimits_{k=1}^{n}(-1)^{n+k}\sin\left(\frac{\phi_{k\alpha}-\phi_{(k-1)\alpha}}{2}\right), \tag{2.6}$$

$$(\phi_{n\alpha})_t = \frac{2}{\alpha}\sum\nolimits_{k=1}^{n}\sin\left(\frac{\phi_{k\alpha}+\phi_{(k-1)\alpha}}{2}\right). \tag{2.7}$$

Using the relation (see appendix for a derivation)

$$\alpha\,(\phi_{n\alpha})_\alpha = x\,(\phi_{n\alpha})_x - t\,(\phi_{n\alpha})_t\,, \tag{2.8}$$

we convert this into the derivative with respect to α required in the recursive relation (2.5)

$$\frac{\alpha}{2}\frac{d\phi_{(n-1)\alpha}}{d\alpha} = \sum_{k=1}^{n}\left[(-1)^{n+k}x\alpha\sin\left(\frac{\phi_{k\alpha}-\phi_{(k-1)\alpha}}{2}\right) - \frac{t}{\alpha}\sin\left(\frac{\phi_{k\alpha}+\phi_{(k-1)\alpha}}{2}\right)\right]. \tag{2.9}$$

Therefore for $n \geq 2$ equation (2.5) becomes

$$\phi_{n\alpha} = \phi_{(n-2)\alpha} - 4\arctan\left[\tfrac{1}{1-n}\sum_{k=1}^{n-1}\left[(-1)^{n+k}x\alpha\sin\left(\tfrac{\phi_{k\alpha}-\phi_{(k-1)\alpha}}{2}\right) + \tfrac{t}{\alpha}\sin\left(\tfrac{\phi_{k\alpha}+\phi_{(k-1)\alpha}}{2}\right)\right]\right]. \tag{2.10}$$

This equation can be solved iteratively with an appropriate choice for the initial condition ϕ_α. Taking this to be the well-known kink solution

$$\phi_\alpha = 4\arctan\left(e^{\xi_+}\right), \qquad \text{with } \xi_\pm := t/\alpha \pm x\alpha, \tag{2.11}$$

we compute from (2.5) the degenerate multi-soliton solutions

$$\phi_{\alpha\alpha} = 4\arctan\left(\frac{\xi_-}{\cosh\xi_+}\right), \tag{2.12}$$

$$\phi_{\alpha\alpha\alpha} = 4\arctan\left(\frac{\xi_+\cosh\xi_+ - \xi_-^2\sinh\xi_+}{\xi_-^2+\cosh^2\xi_+}\right) + \phi_\alpha, \tag{2.13}$$

$$\phi_{\alpha\alpha\alpha\alpha} = 4\arctan\left[\frac{-\xi_-}{3\cosh\xi_+}\,\frac{\xi_-^4+3\xi_+^2-(3+2\xi_-^2)\cosh^2\xi_+ + 3\xi_+\sinh 2\xi_+}{\xi_-^4+\xi_+^2+2\xi_-^2+\cosh^2\xi_+ - 2\xi_-^2\xi_+\tanh\xi_+}\right] + \phi_{\alpha\alpha}\,. \tag{2.14}$$

We notice that unlike standard soliton solutions these solutions are governed by different timescales, being exponential and polynomial. Snapshots of these solutions at two specific values in time are depicted in figure 2 for some concrete values of α. The N-kink solution with $N = 2n$ exhibits n almost identical solitons traveling at nearly the same speed at small time scales. When $N = 2n + 1$ the solutions do not vanish asymptotically and have an additional kink at large values of x.

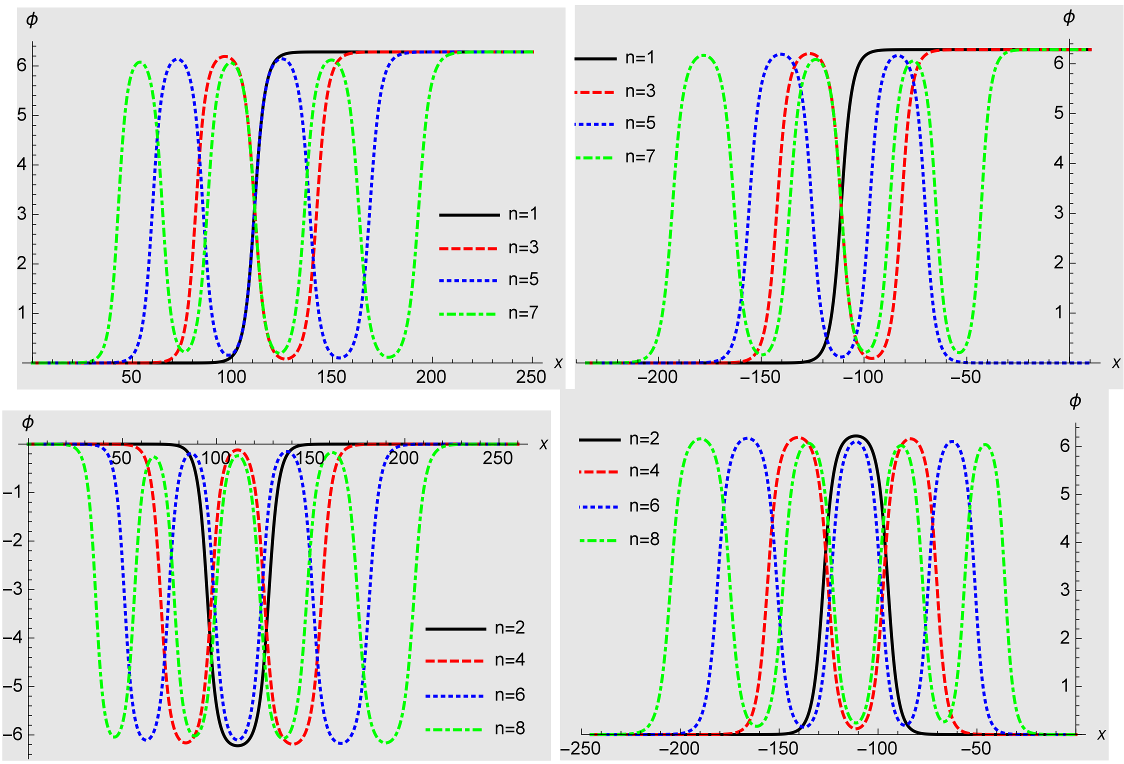

Figure 2: Sine-Gordon degenerate n-kink solutions at different times $t = -10$ (panel a,c) and $t = 10$ (panel b,d) for spectral parameter $\alpha = 0.3$.

It is clear that solutions constructed in this manner, i.e. by iterating (2.10), will be of a form involving sums over arctan-functions, which does not immediately allow to study properties such as the asymptotic behaviour we are interested in here. Of course one may combine these functions into one using standard identities, which become however increasingly nested for larger n in $\phi_{n\alpha}$. This can be avoided by deriving a recursive relation directly for the argument of just one arctan-function. Defining for this purpose the functions τ_n via the relation

$$\phi_{n\alpha} = 4\arctan\tau_n, \tag{2.15}$$

we convert the recursive relation (2.5) in ϕ into a recursive relation in τ as

$$\tau_n = \frac{\tau_{n-2}(1+\tau_{n-1}^2) - \frac{2\alpha}{n-1}\frac{d\tau_{n-1}}{d\alpha}}{1+\tau_{n-1}^2 + \frac{2\alpha}{n-1}\tau_{n-2}\frac{d\tau_{n-1}}{d\alpha}}. \tag{2.16}$$

Similarly as for the derivatives with respect to ϕ we may also compute them for the functions τ using (2.15). Computing first

$$(\tau_n)_x = \alpha(1+\tau_n^2)\sum\nolimits_{k=1}^{n}(-1)^{n+k}\frac{(\tau_k-\tau_{k-1})(1+\tau_{k-1}\tau_k)}{(1+\tau_{k-1}^2)(1+\tau_k^2)}, \tag{2.17}$$

$$(\tau_n)_t = \frac{1}{\alpha}(1+\tau_n^2)\sum\nolimits_{k=1}^{n}\frac{(\tau_k+\tau_{k-1})(1-\tau_{k-1}\tau_k)}{(1+\tau_{k-1}^2)(1+\tau_k^2)}, \tag{2.18}$$

and using $\alpha\,(\tau_n)_\alpha = x\,(\tau_n)_x - t\,(\tau_n)_t$ we obtain the derivative of τ with respect to α

$$\frac{\alpha}{(1+\tau_n^2)}\frac{d\tau_n}{d\alpha} = \sum_{k=1}^{n}\left[(-1)^{n+k}x\alpha\frac{(\tau_k-\tau_{k-1})(1+\tau_{k-1}\tau_k)}{(1+\tau_{k-1}^2)(1+\tau_k^2)} - \frac{t}{\alpha}\frac{(\tau_k+\tau_{k-1})(1-\tau_{k-1}\tau_k)}{(1+\tau_{k-1}^2)(1+\tau_k^2)}\right], \tag{2.19}$$

which we use to convert (2.16) into

$$\tau_n = \frac{\tau_{n-2} - \frac{2}{n-1}\sum\limits_{k=1}^{n-1}\left[(-1)^{n+k-1}x\alpha\frac{(\tau_k-\tau_{k-1})(1+\tau_{k-1}\tau_k)}{(1+\tau_{k-1}^2)(1+\tau_k^2)} - \frac{t}{\alpha}\frac{(\tau_k+\tau_{k-1})(1-\tau_{k-1}\tau_k)}{(1+\tau_{k-1}^2)(1+\tau_k^2)}\right]}{1+\frac{2\tau_{n-2}}{n-1}\sum\limits_{k=1}^{n-1}\left[(-1)^{n+k-1}x\alpha\frac{(\tau_k-\tau_{k-1})(1+\tau_{k-1}\tau_k)}{(1+\tau_{k-1}^2)(1+\tau_k^2)} - \frac{t}{\alpha}\frac{(\tau_k+\tau_{k-1})(1-\tau_{k-1}\tau_k)}{(1+\tau_{k-1}^2)(1+\tau_k^2)}\right]}. \tag{2.20}$$

Using the variables $\xi_\pm$ instead of x, t we obtain

$$\tau_n = \frac{\tau_{n-2} - \frac{2}{n-1}\sum\limits_{k=1}^{n-1}\left[\frac{\tau_k(\tau_{k-1}^2-1)\xi_{(-1)^{n+k}} + \tau_{k-1}(\tau_k^2-1)\xi_{(-1)^{n+k+1}}}{(1+\tau_{k-1}^2)(1+\tau_k^2)}\right]}{1+\frac{2\tau_{n-2}}{n-1}\sum\limits_{k=1}^{n-1}\left[\frac{\tau_k(\tau_{k-1}^2-1)\xi_{(-1)^{n+k}} + \tau_{k-1}(\tau_k^2-1)\xi_{(-1)^{n+k+1}}}{(1+\tau_{k-1}^2)(1+\tau_k^2)}\right]}. \tag{2.21}$$

These relations lead to simpler compact expressions allowing us to study the asymptotic properties of these functions more easily. Iterating (2.16) we obtain the first solutions as

$$\tau_1 = e^{\xi_+}, \tag{2.22}$$

$$\tau_2 = \frac{2\xi_-\tau_1}{1+\tau_1^2} = \frac{\xi_-}{\cosh\xi_+}, \tag{2.23}$$

$$\tau_3 = \frac{(1+2\xi_+ + 2\xi_-^2)\tau_1 + \tau_1^3}{1+(1-2\xi_+ + 2\xi_-^2)\tau_1^2}, \tag{2.24}$$

$$\tau_4 = \frac{4\xi_-(3+3\xi_+ + \xi_-^2)\tau_1 + 4\xi_-(3-3\xi_+ + \xi_-^2)\tau_1^3}{3+(6+12\xi_+^2 + 4\xi_-^4)\tau_1^2 + 3\tau_1^4}, \tag{2.25}$$

$$\tau_5 = \frac{3c_+\tau_1 + 2d_+\tau_1^3 + 9\tau_1^5}{9+2\tau_1^2 d_- + 3c_-\tau_1^4}, \tag{2.26}$$

with

$$c_\pm = 6\xi_+^2 \pm 12\xi_+\xi_-^2 \pm 12\xi_+ + 2\xi_-^4 + 18\xi_-^2 + 3,$$
$$d_\pm = \pm 18\xi_+^3 + 18\xi_+^2\xi_-^2 - 9\xi_+^2 \mp 6\xi_+\xi_-^4 \mp 36\xi_+\xi_-^2 \pm 18\xi_+ + 2\xi_-^6 + 3\xi_-^4 + 27\xi_-^2 + 9.$$

It is now straightforward to compute the τ_n for any larger value of n in this manner.

It is clear that by setting up the nonlinear superposition equation (2.3) for different types of solutions will produce recurrence relations for new types of degenerate multi-soliton solutions. We will not pursue this here, but instead compare the results obtained in this section with those obtained from different type of methods.

3. Degenerate multi-solitons from Darboux-Crum transformations

Classical integrability is build into the zero curvature condition, or AKNS-equation [16], for two operators U and V, which can be expressed equivalently in terms of two linear first order differential equations for an auxiliary function Ψ

$$\partial_t U - \partial_x V + [U,V] = 0 \qquad \Leftrightarrow \qquad \Psi_t = V\Psi,\ \Psi_x = U\Psi. \tag{3.1}$$

When [17] taking the matrix valued functions U, V and the column vector Ψ to be of the form

$$U = \begin{pmatrix} \frac{i}{2}\phi_x & \frac{\alpha}{2} \\ \frac{\alpha}{2} & -\frac{i}{2}\phi_x \end{pmatrix}, \qquad V = \begin{pmatrix} 0 & \frac{1}{2\alpha}e^{i\phi} \\ \frac{1}{2\alpha}e^{-i\phi} & 0 \end{pmatrix}, \qquad \Psi = \begin{pmatrix} \psi \\ \varphi \end{pmatrix}, \tag{3.2}$$

the zero curvature condition (3.1) holds when the field ϕ satisfies the sine-Gordon equation (1.1). Thus we have the four auxiliary equations

$$\psi_x = \frac{i}{2}\phi_x\psi + \frac{\alpha}{2}\varphi, \quad \varphi_x = \frac{\alpha}{2}\psi - \frac{i}{2}\phi_x\varphi, \quad \psi_t = \frac{1}{2\alpha}e^{i\phi}\varphi, \quad \varphi_t = \frac{1}{2\alpha}e^{-i\phi}\psi. \tag{3.3}$$

Computing ψ_{xx} and φ_{xx} decouples the first order equations (3.3) into two Schrödinger equations

$$-\psi_{xx} + V_+\psi = -\frac{\alpha^2}{4}\psi, \qquad -\varphi_{xx} + V_-\varphi = -\frac{\alpha^2}{4}\varphi \tag{3.4}$$

with potentials

$$V_{\pm} = -\frac{1}{4}(\phi_x)^2 \pm \frac{i}{2}\phi_{xx}. \tag{3.5}$$

We notice that for real sine-Gordon fields ϕ the potentials are complex, whereas real potentials may be obtained from purely imaginary fields ϕ. However, even for complex potentials the invariance of these equations under the antilinear involution maps, or $\mathcal{PT}$-symmetries,

$$\mathcal{PT}_{\mu} : x \to -x,\ t \to -t,\ i \to -i,\ \phi \to -\phi,\ \chi \to e^{i\mu}\chi, \qquad \text{for } \chi = \psi, \varphi, \tag{3.6}$$

will guarantee the spectral parameter α to be real [18, 19, 20, 21]. Note that μ might be different for the two equations in (3.4).

Taking now ψ_{α_0}, φ_{α_0} and ψ_{α_1}, φ_{α_1} to be solutions of (3.3) with spectral parameters α_0 and α_1, respectively, but the same sine-Gordon field $\phi = \phi^{(0)}$, the Darboux transformations [22, 23] yield the new solutions for the two Schrödinger equations (3.4)

$$\psi^{(1)}_{\alpha_0,\alpha_1} = \alpha_0\varphi_{\alpha_0} - \alpha_1\frac{\varphi_{\alpha_1}}{\psi_{\alpha_1}}\psi_{\alpha_0}, \quad \text{and} \quad \varphi^{(1)}_{\alpha_0,\alpha_1} = \alpha_0\psi_{\alpha_0} - \alpha_1\frac{\psi_{\alpha_1}}{\varphi_{\alpha_1}}\varphi_{\alpha_0}, \tag{3.7}$$

accompanied by the new solution for the sine-Gordon equation

$$\phi^{(1)}_{\alpha_0,\alpha_1} = \phi^{(0)} - 2i\ln\frac{\varphi_{\alpha_1}}{\psi_{\alpha_1}}, \tag{3.8}$$

involving two spectral parameters.

The iteration of this procedure is usually referred to as the Darboux–Crum transformation [24, 23] leading to the new solutions

$$\psi^{(n)}_{\alpha_0,\alpha_1,\ldots,\alpha_n} = \frac{W\left[\psi_{\alpha_0},\psi_{\alpha_1},\ldots,\psi_{\alpha_n}\right]}{W\left[\psi_{\alpha_1},\ldots,\psi_{\alpha_n}\right]}, \qquad \varphi^{(n)}_{\alpha_0,\alpha_1,\ldots,\alpha_n} = \frac{W\left[\varphi_{\alpha_0},\varphi_{\alpha_1},\ldots,\varphi_{\alpha_n}\right]}{W\left[\varphi_{\alpha_1},\ldots,\varphi_{\alpha_n}\right]}, \tag{3.9}$$

together with

$$\phi^{(n)}_{\alpha_0,\alpha_1,\ldots,\alpha_n} = \phi^{(0)} - 2i\ln\frac{W\left[\varphi_{\alpha_0},\varphi_{\alpha_1},\ldots,\varphi_{\alpha_n}\right]}{W\left[\psi_{\alpha_1},\ldots,\psi_{\alpha_n}\right]}. \tag{3.10}$$

Here $W[\cdot]$ denotes the Wronskians, e.g. $W[f,g] = fg_x - gf_x$ or in general $W[f_1,f_2,\ldots,f_n] = \det w$ with $w_{ij} = d^{i-1}f_j/dx^{i-1}$ for $i,j = 1,\ldots,n$.

Next we carry out the limit to the degenerate case by using Jordan states in the Wronskian similarly as discussed in more detail for the KdV-equation in [2]. In general *Jordan states* $\Xi^{(k)}_{\lambda}$ are defined as the solutions of the iterated Schrödinger equation

$$\hat{H}^{k+1}\Xi^{(k)}_{\lambda} = \left[-\partial_x^2 + V - E(\lambda)\right]^{k+1}\Xi^{(k)}_{\lambda} = 0, \tag{3.11}$$

with eigenvalue $E(\lambda)$ depending on the spectral parameter λ and potential V. This means for $k = 0$ the Jordan states are just the eigenfunction of the Schrödinger equation, namely $\Xi^{(0)}_{\lambda} = \psi_{\lambda}$ or $\Xi^{(0)}_{\lambda} = \phi_{\lambda}$ with ϕ_{λ} being the second fundamental solution to the same eigenvalue $E(\lambda)$. It can be constructed from Liouville's formula $\phi_{\lambda}(x) = \psi_{\lambda}(x)\int^{x}\left[\psi_{\lambda}(s)\right]^{-2}ds$ from the first solution ψ_{λ}. The general solution to (3.11) is

$$\Xi^{(k)}_{\lambda} = \sum_{l=0}^{k} c_l\chi^{(l)}_{\lambda} + \sum_{l=0}^{k} d_l\Omega^{(l)}_{\lambda}, \qquad c_l, d_l \in \mathbb{R}, \tag{3.12}$$

with $\chi^{(k)}_{\lambda} := \partial^k\psi_{\lambda}/\partial E^k$ and $\Omega^{(k)}_{\lambda} := \partial^k\phi_{\lambda}/\partial E^k$.

3.1 Kinks, antikinks, breather and imaginary cusps from vanishing potentials

We start by solving the four linear first order differential equations (3.3) to the lowest level in the Darboux-Crum iteration procedure for some specific choices of $\phi^{(0)}$. Considering the simplest case of vanishing potentials $V_{\pm} = 0$, by taking $\phi^{(0)} = 2\pi\lambda$ with $\lambda \in \mathbb{Z}$, the equations in (3.3) are easily solved by

$$\psi_\alpha(x,t) = c_1 e^{\xi_+/2} + c_2 e^{-\xi_+/2} \quad \text{and} \quad \varphi_\alpha(x,t) = c_1 e^{\xi_+/2} - c_2 e^{-\xi_+/2}. \tag{3.13}$$

Evidently, the constants c_1, $c_2 \in \mathbb{C}$ are taking care about the boundary conditions. Imposing the $\mathcal{PT}_\mu$-symmetry as defined in (3.6) on each of these solutions selects out some specific choices for the constants. For instance, for $\lambda = 0$ and $c_1 = c_2$ the fields in (3.13) obey the symmetries $\mathcal{PT}_0 : \psi_\alpha \to \psi_\alpha$, $\mathcal{PT}_\pi : \varphi_\alpha \to -\varphi_\alpha$ and we obtain from the Darboux transformation (3.8) the purely imaginary cusp solution

$$\phi_\alpha^c = -2i \ln\left(\tanh\left|\xi_+\right|/2\right). \tag{3.14}$$

Imposing instead the symmetries $\mathcal{PT}_{-\pi/2} : \psi_\alpha \to i\psi_\alpha$, $\mathcal{PT}_{-\pi/2} : \varphi_\alpha \to -i\varphi_\alpha$ on the fields in (3.13) the kink and antikink solutions

$$\phi_\alpha = 4\arctan\left(e^{\xi_+}\right) \qquad \text{and} \qquad \phi_{\bar{\alpha}} = 4\,\text{arccot}\left(e^{\xi_+}\right), \tag{3.15}$$

are obtained from $\lambda = 1$ and $c_2 = ic_1 \in \mathbb{R}$ and $\lambda = 0$ and $c_1 = ic_2 \in \mathbb{R}$, respectively. We notice that the remaining constant c_1 or c_2 cancel out in all solutions in (3.14) and (3.15). Iterating these results leads for instance to

3.1.1 Degenerate kink solutions

For the choice $c_2 = ic_1$ we obtain the degenerate solutions

$$\phi_{na} = -2i \ln \frac{W\left[\varphi_\alpha, \partial_\alpha\varphi_\alpha, \partial_\alpha^2\varphi_\alpha, \ldots, \partial_\alpha^{n-1}\varphi_\alpha\right]}{W\left[\psi_\alpha, \partial_\alpha\psi_\alpha, \partial_\alpha^2\psi_\alpha, \ldots, \partial_\alpha^{n-1}\psi_\alpha\right]}, \tag{3.16}$$

which when evaluated explicitly coincide precisely with the expressions previously obtained in (2.12)-(2.14).

3.1.2 Degenerate complex cusp solutions

In a similar way we can construct degenerate purely complex cusp solutions. Such type of solutions are also well-known in the literature, see for instance [25] for an early occurrence. These solutions appear to be non-physical at first sight, but they find applications for instance as an explanation for the entrainment of air [26]. For the choice $c_1 = c_2$ we obtain

$$\phi_{2a}^c = -2i \ln \frac{W\left[\varphi_\alpha, \partial_\alpha\varphi_\alpha\right]}{W\left[\psi_\alpha, \partial_\alpha\psi_\alpha\right]} = -2i \ln\left(\frac{\sinh\xi_+ + \xi_-}{\sinh\xi_+ - \xi_-}\right), \tag{3.17}$$

$$\begin{aligned} \phi_{3a}^c &= -2i \ln \frac{W\left[\varphi_\alpha, \partial_\alpha\varphi_\alpha, \partial_\alpha^2\varphi_\alpha\right]}{W\left[\psi_\alpha, \partial_\alpha\psi_\alpha, \partial_\alpha^2\psi_\alpha\right]} \\ &= -2i \ln\left(\frac{\cosh\left(3\xi_+/2\right) + 2\xi_+ \sinh\left(\xi_+/2\right) - (1+2\xi_-^2)\cosh\left(\xi_+/2\right)}{\sinh\left(3\xi_+/2\right) - 2\xi_+ \cosh\left(\xi_+/2\right) + (1+2\xi_-^2)\sinh\left(\xi_+/2\right)}\right). \end{aligned} \tag{3.18}$$

Similarly we can proceed to obtain the solutions ϕ_{na}^c for $n > 3$.

3.1.3 Degenerate breathers

Breather solutions may be obtained in various ways. An elegant real solution can be constructed as follows: Taking as the starting point the two-kink solution with two distinct spectral parameters α and β

$$\phi_{\alpha\beta} = -2i \ln \frac{W\left[\varphi_\alpha, \varphi_\beta\right]}{W\left[\psi_\alpha, \psi_\beta\right]} = 4 \arctan\left[\frac{\alpha+\beta}{\alpha-\beta}\frac{\sinh\left[(\frac{1}{2\beta}-\frac{1}{2\alpha})(t-x\alpha\beta)\right]}{\cosh\left[(\frac{1}{2\beta}+\frac{1}{2\alpha})(t+x\alpha\beta)\right]}\right], \tag{3.19}$$

we obtain a breather by converting one of the functions in the argument into a trigonometric function. Taking first $\beta \to 1/\alpha$ we obtain

$$\phi_{\alpha,1/\alpha} = 4 \arctan\left[\frac{\alpha^2+1}{\alpha^2-1}\frac{\sinh\left[\frac{1}{2}(\alpha-\frac{1}{\alpha})(t-x)\right]}{\cosh\left[\frac{1}{2}(\alpha+\frac{1}{\alpha})(t+x)\right]}\right]. \tag{3.20}$$

Thus by demanding that $(\alpha^2+1)/(\alpha^2-1) = i\theta$ and $(\alpha - \frac{1}{\alpha})/2 = -i\tilde{\theta}$ for some constants for $\theta, \tilde{\theta} \in \mathbb{R}$ we obtain an oscillatory function in the argument of the arctan. Solving for instance the first relation gives $\alpha = \tilde{\alpha} =: (\theta - i)/\sqrt{1+\theta^2}$ so that $\tilde{\theta} = 1/\sqrt{1+\theta^2}$. The corresponding breather solution then results to

$$\phi_{\tilde{\alpha},1/\tilde{\alpha}} = 4 \arctan\left[\theta \frac{\sin\left[(t-x)/\sqrt{1+\theta^2}\right]}{\cosh\left[\theta(t+x)/\sqrt{1+\theta^2}\right]}\right]. \tag{3.21}$$

This solution evolves with a constant speed -1 modulated by some overall oscillation resulting from the sin-function. Similarly we can construct a two-breather solution from two degenerated kink-solutions

$$\phi_{\alpha\alpha\beta\beta} = -2i \ln \frac{W\left[\varphi_\alpha, \partial_\alpha\varphi_\alpha, \varphi_\beta, \partial_\beta\varphi_\beta\right]}{W\left[\psi_\alpha, \partial_\alpha\psi_\alpha, \psi_\beta, \partial_\beta\psi_\beta\right]}, \tag{3.22}$$

by using the same parameterization $\phi_{\tilde{\alpha},\tilde{\alpha},1/\tilde{\alpha},1/\tilde{\alpha}}$.

3.2 Cnoidal kink solutions from shifted Lamé potentials

The sine-Gordon equation also admits a solution in terms of the Jacobi amplitude $\mathrm{am}(x,m)$ depending on the parameter $0 \leq m \leq 1$ in the form

$$\phi_{cn}^{(0)} = 2\,\mathrm{am}\left(\frac{x-t}{\sqrt{\mu}}, \mu\right) \tag{3.23}$$

for any $0 < \mu(m) < 1$. Denoting the Jacobi elliptic functions as $\mathrm{cn}(x,m) = \cos[\mathrm{am}(x,m)]$, $\mathrm{sn}(x,m) = \sin[\mathrm{am}(x,m)]$ and $\mathrm{dn}(x,m) = (1 - m\sin^2[\mathrm{am}(x,m)])^{1/2}$ this follows directly from $\mathrm{am}'(x,m) = \mathrm{dn}(x,m)$, $\mathrm{dn}'(x,m) = -m\,\mathrm{sn}(x,m)\,\mathrm{cn}(x,m)$ and the double angle identity for the sin-function. The potentials (3.5) following from the solution (3.23) are

$$V_{\pm}^{cn} = -\frac{1}{\mu}\,\mathrm{dn}^2\left(\frac{x-t}{\sqrt{\mu}}, \mu\right) \mp i\,\mathrm{cn}\left(\frac{x-t}{\sqrt{\mu}}, \mu\right)\mathrm{sn}\left(\frac{x-t}{\sqrt{\mu}}, \mu\right) \tag{3.24}$$

$$= \frac{\sqrt{m}}{2}\,\mathrm{sn}^2\left(\frac{x-t}{2m^{1/4}} \mp \frac{i}{2}K', m\right) - \frac{1}{4}(m^{1/2} + m^{-1/2}), \tag{3.25}$$

where we used the parameterization $\mu = 4\sqrt{m}/(1+\sqrt{m})^2$ with $K(m)$ denoting the complete elliptic integral of the first kind and $K'(m) = K(1-m)$. Notice that this is a complex shifted and scaled Lamé potential [27, 28] invariant under any $\mathcal{PT}$-symmetry as defined in (3.6). Such type of potentials emerge in various contexts, e.g. they give rise to elliptic string solutions in AdS_3 and dS_3 [29] or the study of the origin of spectral singularities in periodic $\mathcal{PT}$-symmetric systems [30].

Next we need to find the solutions ψ and φ to the AKNS-equations (3.3) corresponding to the sine-Gordon solution $\phi_{cn}^{(0)}$. We need to distinguish the two cases $0 \leq \alpha \leq 1$ and $\alpha > 1$. In the first case we parameterize $\alpha = m^{1/4}$ finding the solutions

$$\psi_m^<(x,t) = c\,\mathrm{cn}\left[\frac{x-t}{2m^{1/4}} - \frac{i}{2}K', m\right], \quad \varphi_m^<(x,t) = -ic\,\mathrm{cn}\left[\frac{x-t}{2m^{1/4}} + \frac{i}{2}K', m\right] \tag{3.26}$$

and for the second case we parameterize $\alpha = m^{-1/4}$ obtaining the solutions

$$\psi_m^>(x,t) = ic\,\mathrm{dn}\left[\frac{x-t}{2m^{1/4}} - \frac{i}{2}K', m\right], \quad \varphi_m^>(x,t) = c\,\mathrm{dn}\left[\frac{x-t}{2m^{1/4}} + \frac{i}{2}K', m\right], \tag{3.27}$$

with integration constant c. For real values of c we observe the $\mathcal{PT}$-symmetries $\mathcal{PT}_0\psi_m^< = \psi_m^<$, $\mathcal{PT}_\pi\varphi_m^< = \varphi_m^<$, $\mathcal{PT}_\pi\psi_m^> = \psi_m^>$, $\mathcal{PT}_0\varphi_m^> = \varphi_m^>$.

The Darboux transformation (3.8) then yields the real solutions for the sine-Gordon equation

$$\phi_{<,m}^{(1)}(x,t) = 2\,\mathrm{am}\left(\frac{x-t}{\sqrt{\mu}}, \mu\right) - 4\arctan\left[\frac{\mathrm{dn}\left(\frac{x-t}{2m^{1/4}}, m\right)\mathrm{sn}\left(\frac{x-t}{2m^{1/4}}, m\right)}{\mathrm{cn}\left(\frac{x-t}{2m^{1/4}}, m\right)}\right] - \pi, \tag{3.28}$$

$$\phi_{>,m}^{(1)}(x,t) = 2\,\mathrm{am}\left(\frac{x-t}{\sqrt{\mu}}, \mu\right) - 4\arctan\left[\sqrt{m}\frac{\mathrm{cn}\left(\frac{x-t}{2m^{1/4}}, m\right)\mathrm{sn}\left(\frac{x-t}{2m^{1/4}}, m\right)}{\mathrm{dn}\left(\frac{x-t}{2m^{1/4}}, m\right)}\right] - \pi, \tag{3.29}$$

after using the addition theorem for the Jacobi elliptic functions, the following properties $\mathrm{cn}\,(iK'/2, m) = \sqrt{1+\sqrt{m}}/m^{1/4}$, $\mathrm{sn}\,(iK'/2, m) = i/m^{1/4}$, $\mathrm{dn}\,(iK'/2, m) = \sqrt{1+\sqrt{m}}$, and the well known relation between the ln and the arctan-functions. Notice that the cn-function can be vanishing for real arguments, such that $\phi_<^{(1)}$ is a discontinuous function. Furthermore, we observe that this solution has a fixed speed and does not involve any variable spectral parameter, which is vital for the construction of multi-solutions, see e.g. [2]. For this reason we construct a different type of solution also related to $\phi_{cn}^{(0)}$ that involves an additional parameter.

These type of solutions can be obtained from

$$\Psi_{m,\beta}^\pm(x) = \frac{H(x \pm \beta)}{\Theta(x)} e^{\mp xZ(\beta)}, \qquad \Phi_{m,\beta}^\pm(x) = \frac{\Theta(x \pm \beta)}{\Theta(x)} e^{\mp xY(\beta)}, \tag{3.30}$$

solving the Schrödinger equation involving the Lamé potential V_L

$$-\Psi_{xx} + V_L\Psi = E_\beta\Psi, \quad \text{with } V_L = 2m\,\mathrm{sn}\,(x,m)^2 - (1+m), \tag{3.31}$$

with $E_\beta = -m \operatorname{sn}(\beta, m)^2$ and $E_\beta = -1/\operatorname{sn}(\beta, m)^2$, respectively. The functions H, Θ, Y and Z are defined in terms of Jacobi's theta functions $\vartheta_i(z,q)$ with $i = 1,2,3,4$, $\kappa = \pi/(2K)$ and nome $q = \exp(-\pi K/K')$ as

$$H(z) := \vartheta_1(z\kappa, q), \quad \Theta(z) := \vartheta_4(z\kappa, q), \quad Y(z) := \kappa \frac{H'(z\kappa)}{H(z\kappa)}, \quad Z(z) := \kappa \frac{\Theta'(z\kappa)}{\Theta(z\kappa)}. \tag{3.32}$$

With a suitable normalization factor and the introduction of a time-dependence the function $\Psi^{\pm}(x)$ can be tuned to solve the equations (3.3). We find

$$\Psi^{<}_{\pm,m,\beta}(x,t) = \Psi^{\pm}_{m,\beta}\left(\frac{x-t}{2m^{1/4}} - \frac{i}{2}K'\right) e^{\pm \frac{t}{2m^{1/4}} \frac{\operatorname{cn}(\beta,m)\operatorname{dn}(\beta,m)}{\operatorname{sn}(\beta,m)}}, \tag{3.33}$$

$$\Phi^{<}_{\pm,m,\beta}(x,t) = \mp e^{\pm iK'Z(\beta) \pm i\beta\kappa} \Psi^{\pm}_{m,\beta}\left(\frac{x-t}{2m^{1/4}} + \frac{i}{2}K'\right) e^{\pm \frac{t}{2m^{1/4}} \frac{\operatorname{cn}(\beta,m)\operatorname{dn}(\beta,m)}{\operatorname{sn}(\beta,m)}}, \tag{3.34}$$

for $\alpha = m^{1/4} \operatorname{sn}(\beta, m)$ and

$$\Psi^{>}_{\pm,m,\beta}(x,t) = \Phi^{\pm}_{m,\beta}\left(\frac{x-t}{2m^{1/4}} - \frac{i}{2}K'\right) e^{\mp \frac{t}{2m^{1/4}} \frac{\operatorname{cn}(\beta,m)\operatorname{dn}(\beta,m)}{\operatorname{sn}(\beta,m)}} \tag{3.35}$$

$$\Phi^{>}_{\pm,m,\beta}(x,t) = \mp e^{\pm iK'Y(\beta) \pm i\beta\kappa} \Phi^{\pm}_{m,\beta}\left(\frac{x-t}{2m^{1/4}} + \frac{i}{2}K'\right) e^{\mp \frac{t}{2m^{1/4}} \frac{\operatorname{cn}(\beta,m)\operatorname{dn}(\beta,m)}{\operatorname{sn}(\beta,m)}} \tag{3.36}$$

for $\alpha = 1/(m^{1/4} \operatorname{sn}(\beta, m))$. The corresponding solutions for the sine-Gordon equation resulting from the Darboux transformation (3.8) are

$$\phi^{\ell(1)}_{\pm,m,\beta}(x,t) = \phi^{(0)}_{cn} \pm 2\beta\kappa - 4\arctan\left[i\frac{M^{\ell}_{\pm} - (M^{\ell}_{\pm})^*}{M^{\ell}_{\pm} + (M^{\ell}_{\pm})^*}\right], \qquad \ell = <,> \tag{3.37}$$

with the abbreviations

$$M^{<}_{\pm} = H\left(\frac{x-t}{2m^{1/4}} \pm \beta + \frac{i}{2}K'\right)\Theta\left(\frac{x-t}{2m^{1/4}} - \frac{i}{2}K'\right), \tag{3.38}$$

$$M^{>}_{\pm} = \Theta\left(\frac{x-t}{2m^{1/4}} \pm \beta + \frac{i}{2}K'\right)\Theta\left(\frac{x-t}{2m^{1/4}} - \frac{i}{2}K'\right). \tag{3.39}$$

We depict this solution in figure 3. We notice that the two solutions depicted are qualitatively very similar and appear to be just translated in amplitude and x. However, these translations are not exact and even the approximations depend nontrivially on β and m.

Taking the normalization constants in (3.26) and (3.27) respectively as $c = \pm m^{1/4}/(1-m)^{1/4}$ and $c = i/(1-m)^{1/4}$ we recover the simpler solution with constant speed parameter from the limits

$$\lim_{\beta\to K} \Psi^{\ell}_{\pm,m,\beta}(x,t) = \psi^{\ell}_{m}(x,t), \quad \lim_{\beta\to K} \Phi^{\ell}_{\pm,m,\beta}(x,t) = \varphi^{\ell}_{m}(x,t), \quad \lim_{\beta\to K} \phi^{\ell}_{\pm,m,\beta}(x,t) = \phi^{\ell}_{m}(x,t),$$

such that (3.33) and (3.36) can be viewed as generalizations of those solutions.

It is interesting to compare these type of solutions and investigate whether they can be used to obtain Bäcklund transformations. It is clear that since $\phi^{(0)}_{cn}$ does not contain any

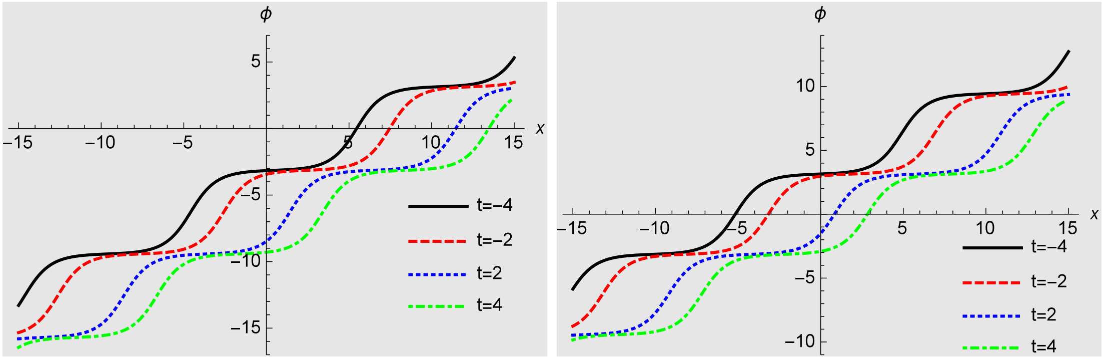

Figure 3: Sine-Gordon cnoidal kink solution $\phi^{>(1)}_{+,m,\beta}(x,t)$ (panel a) and degenerate cnoidal kink solution $\phi^{c>}_{+,m,\beta\beta}(x,t)$ (panel b) for spectral parameter $\beta = 0.9$ and $m = 0.3$ at different times.

spectral parameter it can not be employed in the nonlinear superposition (2.2). However, taking $\phi_0 = \phi^{\ell(1)}_{\pm,m,\alpha}(x,t)$, $\phi_1 = \phi^{\ell(1)}_{\pm,m,\beta}(x,t)$ and $\phi_2 = \phi^{\ell(1)}_{\pm,m,\gamma}(x,t)$ we identify from (2.1) the constants $\kappa_1 = \pm m^{1/4}\,\mathrm{sn}\,(\alpha-\beta,m)$ and $\kappa_2 = \pm m^{1/4}\,\mathrm{sn}\,(\alpha-\gamma,m)$, such that by (2.2) we obtain the new three soliton solutions

$$\phi^{\ell(3)}_{\pm,\alpha\beta\gamma,m} = \phi^{\ell(1)}_{\pm,m,\alpha} + 4\arctan\left[\frac{\mathrm{sn}\,(\alpha-\beta,m)+\mathrm{sn}\,(\alpha-\gamma,m)}{\mathrm{sn}\,(\alpha-\beta,m)-\mathrm{sn}\,(\alpha-\gamma,m)}\tan\left(\frac{\phi^{\ell(1)}_{\pm,m,\beta}-\phi^{\ell(1)}_{\pm,m,\gamma}}{4}\right)\right]. \tag{3.40}$$

As is most easily seen in the simpler solutions (3.28) and (3.29) the solutions for $\ell =>$ are also regular in the cases with spectral parameter. We will not explore here how the limit $\beta,\gamma \to \alpha$ can be taken in these expressions and how one might iterate these solutions in a similar way as explained in section 2. Instead we present the same approach as in the previous sections.

3.2.1 Degenerate cnoidal kink solutions

Using the solution $\phi^{(0)}_{cn}$ as initial solutions and the solutions (3.33) and (3.36) to the AKNS equations we are now in a position to compute the degenerate cnoidal kink solutions using the Darboux transformation involving Jordan states from

$$\phi^{c\ell}_{\pm,m,\beta\beta} = \phi^{(0)}_{cn} - 2i\ln\frac{W\left[\Phi^{\ell}_{\pm,m,\beta},\partial_\beta\Phi^{\ell}_{\pm,m,\beta}\right]}{W\left[\Psi^{\ell}_{\pm,m,\beta},\partial_\beta\Psi^{\ell}_{\pm,m,\beta}\right]}. \tag{3.41}$$

A lengthy calculation yields

$$\phi^{c\ell}_{\pm,m,\beta\beta} = \phi^{(0)}_{cn} \pm 4\beta\kappa - 4\arctan\left[i\frac{N^{\ell}_{\pm}-(N^{\ell}_{\pm})^*}{N^{\ell}_{\pm}+(N^{\ell}_{\pm})^*}\right]. \tag{3.42}$$

where we defined the quantities

$$N^{<}_{\pm} = \Theta^2\left(\frac{x-t}{2m^{1/4}}-\frac{i}{2}K'\right)\left\{H^2\left(\frac{x-t}{2m^{1/4}}\pm\beta+\frac{i}{2}K'\right)W_\beta\left[\partial_\beta\Theta\left(\beta\right),\Theta\left(\beta\right)\right]\right. \tag{3.43}$$
$$\left. + \Theta^2\left(\beta\right)W_\beta\left[H\left(\frac{x-t}{2m^{1/4}}\pm\beta+\frac{i}{2}K'\right),\partial_\beta H\left(\frac{x-t}{2m^{1/4}}\pm\beta+\frac{i}{2}K'\right)\right]\right\},$$

$$N_{\pm}^{>} = \Theta^2\left(\frac{x-t}{2m^{1/4}} - \frac{i}{2}K'\right)\left\{\Theta^2\left(\frac{x-t}{2m^{1/4}} \pm \beta + \frac{i}{2}K'\right)W_\beta\left[\partial_\beta H(\beta), H(\beta)\right]\right. \quad (3.44)$$
$$\left. + H^2(\beta)\,W_\beta\left[\Theta\left(\frac{x-t}{2m^{1/4}} \pm \beta + \frac{i}{2}K'\right), \partial_\beta\Theta\left(\frac{x-t}{2m^{1/4}} \pm \beta + \frac{i}{2}K'\right)\right]\right\}.$$

Notice that the argument of the arctan is always real. These functions are regular for real values of β. Furthermore we observe that the additional speed spectral parameter is now separated from x and t, so that the degenerate solution has only one speed, i.e. the degenerate solution is not displaced at any time. We depict this solution in figure 3.

4. Degenerate multi-solitons from Hirota's direct method

Finally we explore how the degenerate solutions may be obtained within the context of Hirota's direct method [31]. The key idea of this solution procedure is to convert the original nonlinear equations into bilinear forms, which can be solved systematically. When parameterizing $\phi(x,t) = 2i\ln[g(x,t)/f(x,t)]$ the sine-Gordon equation (1.1) was found [6, 31] to be equivalent to the two equations

$$D_x D_t f \cdot f + \frac{1}{2}(g^2 - f^2) = \lambda f^2, \quad \text{and} \quad D_x D_t g \cdot g + \frac{1}{2}(f^2 - g^2) = \lambda g^2, \qquad (4.1)$$

with D_x, D_t denoting the Hirota derivatives. Explicitly we have $D_x D_t f \cdot f = 2f^2(\ln f)_{xt}$. Taking $g = f^*$ the equations (4.1) become each others conjugate and with $\lambda = 0$ can be solved by the Wronskian

$$f = W[\psi_{\alpha_1}, \psi_{\alpha_2}, \ldots, \psi_{\alpha_N}], \qquad (4.2)$$

where

$$\psi_\alpha = e^{\xi_+/2} + i c_\alpha e^{-\xi_+/2}. \qquad (4.3)$$

For simplicity we ignore here an overall constant that may be canceled out without loss of generality and also do not treat the possibility $\xi_+ \to -\xi_+$ separately. This gives rise to the real valued N-soliton solutions

$$\phi = 2i\ln\frac{f^*}{f} = 4\arctan\left(i\frac{f^* - f}{f^* + f}\right) = 4\arctan\frac{f_i}{f_r}, \qquad (4.4)$$

where $f = f_r + i f_i$ with $f_r, f_i \in \mathbb{R}$. For instance the one, two and three-soliton solution obtained in this way are

$$\phi_\alpha = 4\arctan(c_\alpha e^{-\xi_+^\alpha}), \qquad (4.5)$$

$$\phi_{\alpha\beta} = 4\arctan\left[\Gamma_{\alpha\beta}\frac{c_\beta e^{\xi_+^\beta} - c_\alpha e^{\xi_+^\alpha}}{1 + c_\alpha c_\beta e^{\xi_+^\alpha + \xi_+^\beta}}\right], \qquad (4.6)$$

$$\phi_{\alpha\beta\gamma} = 4\arctan\left[\frac{c_\alpha c_\beta c_\gamma + c_\alpha\Gamma_{\alpha\beta}\Gamma_{\alpha\gamma}e^{\xi_+^\beta + \xi_+^\gamma} + c_\beta\Gamma_{\beta\alpha}\Gamma_{\beta\gamma}e^{\xi_+^\alpha + \xi_+^\gamma} + c_\gamma\Gamma_{\gamma\alpha}\Gamma_{\gamma\beta}e^{\xi_+^\alpha + \xi_+^\beta}}{c_\beta c_\gamma\Gamma_{\alpha\beta}\Gamma_{\alpha\gamma}e^{\xi_+^\alpha} + c_\alpha c_\gamma\Gamma_{\beta\alpha}\Gamma_{\beta\gamma}e^{\xi_+^\beta} + c_\alpha c_\beta\Gamma_{\gamma\alpha}\Gamma_{\gamma\beta}e^{\xi_+^\gamma} + e^{\xi_+^\alpha + \xi_+^\beta + \xi_+^\gamma}}\right], \qquad (4.7)$$

where $\Gamma_{xy} := (x+y)/(x-y)$. We kept here the constants $c_\alpha, c_\beta, c_\gamma$ generic as it was previously found [2] that they have be chosen in a specific way to render the limits to the degenerate case finite.

Following the procedure outlined in [2] we replace the standard solutions to the Schrödinger equation in the non-degenerate solution by Jordan states in the computation of f in (4.2) as

$$f = W[\psi_\alpha, \partial_\alpha \psi_\alpha, \partial_\alpha^2 \psi_\alpha, \ldots, \partial_\alpha^N \psi_\alpha]. \tag{4.8}$$

We then recover from (4.4) the degenerate kink solution $\phi_{\alpha\alpha}$ and $\phi_{\alpha\alpha\alpha}$ in (2.12) and (2.13), respectively, with $c_\alpha = -1$ in (4.3). Unlike as in the treatment of the Korteweg de-Vries equation [2] the equations are already in a format that allows to carry out the limits $\lim_{\beta\to\alpha} \phi_{\alpha\beta} = \phi_{\alpha\alpha}$ and $\lim_{\beta,\gamma\to\alpha} \phi_{\alpha\beta\gamma} = \phi_{\alpha\alpha\alpha}$ with the simple choices $c_\alpha = c_\beta = 1$ and $c_\alpha = c_\beta = c_\gamma = -1$, respectively.

5. Time-dependent displacements

Let us now compute the time-dependent displacements by tracking the one-soliton solution within a degenerate multi-soliton solution as explained in [3].

5.1 Time-dependent displacements for multi-kink solutions

Unlike as for standard multi-soliton solutions one can not track the maxima or minima for the kink-solutions as they might have maximal or minimal amplitudes extending up to infinity. However, they have many intermediate points in-between the extrema that are uniquely identifiable. For instance, for the solutions constructed in sections 2. and 3.1 a suitable choice is the point of inflection at half the maximal value, that is at $\phi_{n\alpha} = \pi$ corresponding to $\tau_{n\alpha} = 1$. For an N-soliton solution $\phi_{N\alpha}$ with N parameterized as $N = 2n + 1 - \kappa$ we find that these N points are reached asymptotically

$$\lim_{t\to\infty} \tau_{2n+1-\kappa} \left(-\frac{t}{\alpha^2} \pm \Delta_{n,\ell,\kappa}, t \right) = 1, \qquad n = 1, 2, \ldots;\ \ell = \kappa, \ldots, n-1, n;\ \kappa = 0, 1 \tag{5.1}$$

for the time-dependent displacements

$$\Delta_{n,\ell,\kappa}(t) := \frac{1}{\alpha} \ln \left[\frac{(n-\ell)!}{(n+\ell-\kappa)!} \left(\frac{4t}{\alpha} \right)^{2\ell-\kappa} \right] \tag{5.2}$$

For example, given a 5-soliton solution ϕ_{5a} we have $n = 2$, $\kappa = 0$ and $\ell = 0, 1, 2$, so that we can compare it with five laterally displaced one-soliton solutions as depicted in figure 4.

Let us now derive this expression for the first examples. Introducing the notation $x_\pm := -t/\alpha^2 \pm 1/\alpha \ln \delta$, assuming that $\delta \sim t^\mu$, $\mu \geq 1$ and taking y to be a polynomial in t, we obtain the useful auxiliary limits

$$\lim_{t\to\infty} \lim_{x\to x_\pm} \left(y + \xi_+ \right) = \lim_{t\to\infty} \left(y \pm \ln \delta \right) \approx \lim_{t\to\infty} y, \tag{5.3}$$

$$\lim_{t\to\infty} \lim_{x\to x_\pm} \left(y + \xi_- \right) = \lim_{t\to\infty} \left(y + \frac{2t}{\alpha} \mp \ln \delta \right) \approx \lim_{t\to\infty} \left(y + \frac{2t}{\alpha} \right), \tag{5.4}$$

$$\lim_{t\to\infty} \lim_{x\to x_\pm} \tau_1 = \lim_{t\to\infty} \left(\delta^{\pm 1} \right). \tag{5.5}$$

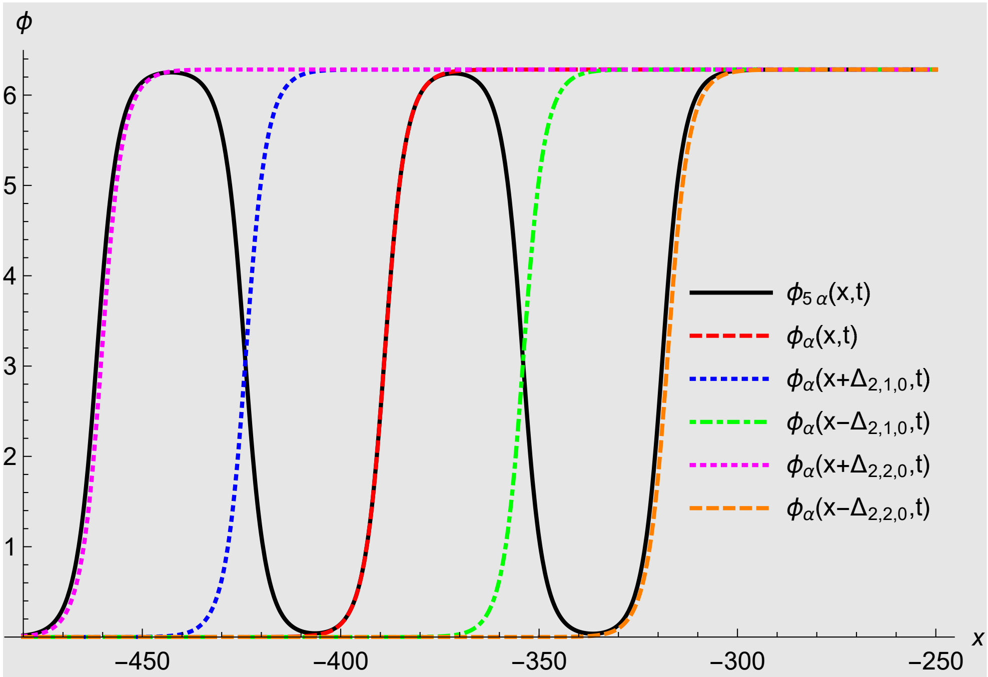

Figure 4: Degenerate 5-soliton solution compared with 5 time-dependently laterally displaced one-soliton solutions for $\alpha = 0.3$ at time $t = 35$.

Using these expressions in (2.22) - (2.26) and the notation $\delta_{n,\ell,\kappa} = \alpha \exp(\Delta_{n,\ell,\kappa})$, $T = 2t/\alpha$ we derive the asymptotic expressions for the N-soliton solution for the lowest values of N

$$\lim_{\substack{t\to\infty \\ x\to x_\pm}} \tau_2 \approx \lim_{t\to\infty} \frac{2T\delta^\pm}{1+(\delta^\pm)^2} = \lim_{t\to\infty} \frac{2T}{\delta^\pm} = 1 \quad \text{for } \delta^\pm = 2T = \delta_{1,1,1},$$

$$\lim_{\substack{t\to\infty \\ x\to x_\pm}} \tau_3 \approx \lim_{t\to\infty} \frac{2T^2+(\delta^\pm)^2}{2T^2\delta^\pm} = 1 \qquad \text{for } \delta^\pm = 2T^2 = \delta_{1,1,0} \text{ or } \delta^\pm = 1,$$

$$\lim_{\substack{t\to\infty \\ x\to x_\pm}} \tau_4 \approx \lim_{t\to\infty} \frac{4T^3\left[1+(\delta^\pm)^2\right]}{4T^4\delta^\pm + 3(\delta^\pm)^3} = 1 \quad \text{for } \delta^\pm = T = \delta_{2,1,1} \text{ or } \delta^\pm = \frac{4}{3}T^3 = \delta_{2,2,1},$$

$$\lim_{\substack{t\to\infty \\ x\to x_\pm}} \tau_5 \approx \lim_{t\to\infty} \frac{4T^6\delta^\pm + 9(\delta^\pm)^3}{4T^6 + 6T^4(\delta^\pm)^2} = 1 \quad \text{for } \delta^\pm = \frac{2}{3}T^4 = \delta_{2,2,0} \text{ or } \delta^\pm = \frac{2}{3}T^2 = \delta_{2,1,0} \text{ or } \delta^\pm = 1,$$

The limits need to be carried out in consecutive order, i.e. first replace $x \to x_\pm$ and then compute $t \to \infty$. These are the first explicit examples for the asymptotic values all confirming (5.2). Similarly we have computed examples for higher values of N that may also be cast into the general formula (5.2). So far we have not obtained a generic proof valid for any N.

Of course one may easily convert the shifts from light-cone to the original variables. Having computed the lateral displacement Δ_x the time-displacement is obtained as usual from $\Delta_t = -\Delta_x/v$, where $v = 1/\alpha^2$ in our case. Then one simply obtains $\Delta_X = \Delta_x(1-\alpha^2)$ and $\Delta_T = \Delta_x(1+\alpha^2)$.

5.2 Time-dependent displacements for breather solutions

For the breathers it is even less evident what point in the solution is suitable for tracking due to the overall oscillation. However, since we are only interested in the net movement we can neglect the internal oscillation and determine the displacement for an enveloping function that surrounds the breather and moves with the same overall speed. For the one-breather solution an enveloping function is obtained by setting the sin-function in (3.21) to 1, obtaining

$$\phi^{\text{env}}_{\tilde{\alpha},1/\tilde{\alpha}} = 4\arctan\left[\frac{\theta}{\cosh\left[\theta(t+x)/\sqrt{1+\theta^2}\right]}\right]. \tag{5.6}$$

This function is depicted together with the breather solution in figure 5 having a clearly identifiable maximum at $4\arctan\theta$ which we can track.

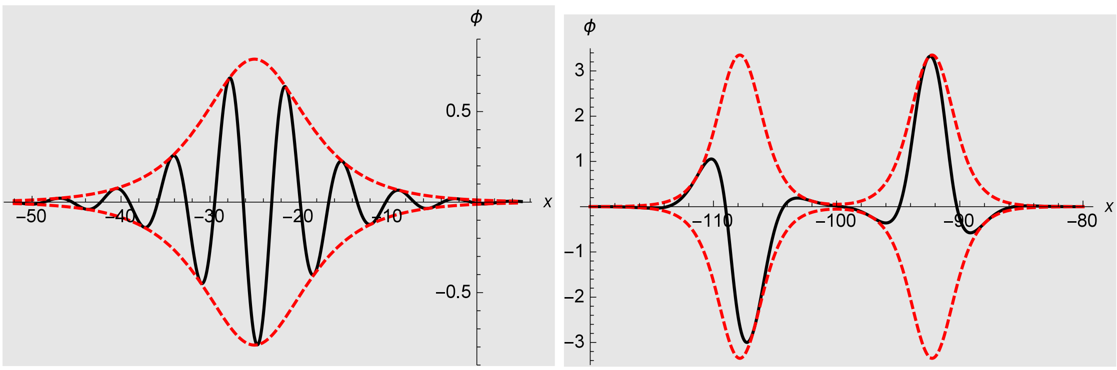

Figure 5: One-breather solution surrounded by enveloping function $\pm\phi^{\text{env}}_{\tilde{\alpha},1/\tilde{\alpha}}$ at $t = 25$ for $\theta = 1/5$ panel (a) and degenerate two-breather solution surrounded by enveloping function $\pm\phi^{\text{env}}_{\tilde{\alpha},\tilde{\alpha},1/\tilde{\alpha},1/\tilde{\alpha}}$ at $t = 100$ for $\theta = 4/3$ panel (b).

We compare this now with the breather solution $\phi_{\tilde{\alpha},\tilde{\alpha},1/\tilde{\alpha},1/\tilde{\alpha}}$ constructed in section 3.1.3. Taking for that solution $\sin\left[(t-x)/\sqrt{1+\theta^2}\right] \to 0$ and $\cos\left[(t-x)/\sqrt{1+\theta^2}\right] \to 1$ we obtain the enveloping function

$$\phi^{\text{env}}_{\tilde{\alpha},\tilde{\alpha},1/\tilde{\alpha},1/\tilde{\alpha}} = 4\arctan\left[\frac{4(t-x)\theta\sqrt{1+\theta^2}\cosh\left(\frac{(t+x)\theta}{\sqrt{1+\theta^2}}\right)}{2\theta^2(t-x)^2 + 2(t+x)^2 + 1 + \theta^{-2} + (1+\theta^{-2})\cosh\left(\frac{2\theta(t+x)}{\sqrt{1+\theta^2}}\right)}\right]. \tag{5.7}$$

This function tends asymptotically to the maximal value of the one-breather enveloping function

$$\lim_{t\to\infty} \phi^{\text{env}}_{\tilde{\alpha},\tilde{\alpha},1/\tilde{\alpha},1/\tilde{\alpha}}\left(-t \pm \Delta_{\tilde{\alpha},\tilde{\alpha},1/\tilde{\alpha},1/\tilde{\alpha}}, t\right) = 4\arctan(\theta), \tag{5.8}$$

when shifted appropriately with the time-dependent displacement

$$\Delta_{\tilde{\alpha},\tilde{\alpha},1/\tilde{\alpha},1/\tilde{\alpha}}(t) = \frac{1}{\theta}\sqrt{1+\theta^2}\ln\left(\frac{4\theta^2 t}{\sqrt{1+\theta^2}}\right). \tag{5.9}$$

Similarly we may compute the displacements for the solutions $\phi_{\tilde{\alpha},\tilde{\alpha},\tilde{\alpha},1/\tilde{\alpha},1/\tilde{\alpha},1/\tilde{\alpha}}$ etc.

6. Conclusions

We have constructed various types of degenerate multi-soliton solutions from three different methods commonly used in the context of classical nonlinear integrable systems. Using the recurrence relations constructed in section 2 from Bäcklund transformations is the most efficient way to obtain N-soliton solutions for large values of N. These equations are easily implemented in computer calculations. By just requiring a simple solution to the original nonlinear equation they also have a relatively easy starting point. However, the equations are less universal as those presented in section 3 using Jordan states in Darboux-Crum transformations. The disadvantage of this method is that they need in addition the solutions to the AKNS equations. For large values of N the computations become more involved than the recurrence relations for the Bäcklund transformations. Finally, Hirota's direct method turned out to be the simplest approach for the sine-Gordon model as the limit to the degenerate case could be taken directly from the N-soliton solutions with different spectral parameters. However, as discussed for the Korteweg de-Vries equation in [2] this is not a general feature and one might have to tune the arbitrary constants involved in a very specific and nontrivial way, especially when one wishes to implement shift parameters.

We computed the explicit analytic expression for the asymptotic time-dependent displacements between the one-soliton constituents. It turns out that formula (1.4) is shared by multi-kink solutions in the sine-Gordon equation and the multi-soliton solutions of the Korteweg de-Vries equation when expression part of the combinations of the spectral parameter in terms of the appropriate speed in the model. It seems obvious to conjecture that this might also hold for other models, which would be interesting to investigate. For the breather solutions we computed the displacements for the enveloping functions. Interestingly we also found that it is possible to construct compound solutions that travel uniformly and do not display any displacement at any time.

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A. Appendix

In this appendix we present some derivations of identities used in the manuscript.

First we present a derivation of identity (2.4). We start by considering the limit for $n = 2$

$$\lim_{\beta\to\alpha} \frac{\alpha+\beta}{\alpha-\beta} \tan\left(\frac{\phi_\beta - \phi_\alpha}{4}\right) \tag{A.1}$$

by taking $\beta = \alpha + h$ and letting h tend to zero. So (A.1) may be written as

$$\begin{aligned} \lim_{h\to 0} \frac{2\alpha+h}{-h} \tan\left(\frac{\phi_{\alpha+h} - \phi_\alpha}{4}\right) &= -2\alpha \lim_{h\to 0} \frac{1}{\cos\left[\frac{\phi_{\alpha+h}-\phi_\alpha}{4}\right]} \frac{\sin\left[\frac{\phi_{\alpha+h}-\phi_\alpha}{4}\right]}{h}, \\ &= -2\alpha \lim_{h\to 0} \frac{\phi_{\alpha+h} - \phi_\alpha}{4h}, \\ &= -\frac{\alpha}{2}\frac{d\phi_\alpha}{d\alpha}. \end{aligned} \tag{A.2}$$

Viewing ϕ as a function of α we identified in the last equality the standard expression for the derivative. Using this expression the non-degenerate and degenerate two-soliton solutions mav be written as

$$\phi_{\alpha\beta} = 4\arctan\left[\frac{\alpha+\beta}{\alpha-\beta}\tan\left(\frac{\phi_\beta-\phi_\alpha}{4}\right)\right], \qquad \text{and} \qquad \phi_{\alpha\alpha} = -4\arctan\left[\frac{\alpha}{2}\frac{d\phi_\alpha}{d\alpha}\right], \tag{A.3}$$

respectively.

Next we considering the limit for $n = 3$

$$\lim_{\beta\to\alpha} \frac{\alpha+\beta}{\alpha-\beta} \tan\left(\frac{\phi_{\alpha\beta} - \phi_{\alpha\alpha}}{4}\right) \tag{A.4}$$

We re-write this expression as

$$\lim_{\beta\to\alpha}\lim_{\gamma\to\alpha} \frac{\beta+\gamma}{\beta-\gamma} \tan\left(\frac{\phi_{\alpha\gamma} - \phi_{\alpha\beta}}{4}\right) = \lim_{\beta\to\alpha}\lim_{\gamma\to\beta} \frac{\beta+\gamma}{\beta-\gamma} \tan\left(\frac{\phi_{\alpha\gamma} - \phi_{\alpha\beta}}{4}\right), \tag{A.5}$$

which when setting $\gamma = \beta + h$ becomes

$$
\begin{aligned}
\lim_{\beta\to\alpha}\lim_{h\to 0}\frac{2\beta+h}{-h}\tan\left(\frac{\phi_{\alpha\beta+h}-\phi_{\alpha\beta}}{4}\right) &= -2\lim_{\beta\to\alpha}\beta\lim_{h\to 0}\frac{1}{\cos\left[\frac{\phi_{\alpha\beta+h}-\phi_{\alpha\beta}}{4}\right]}\frac{\sin\left[\frac{\phi_{\alpha\beta+h}-\phi_{\alpha\beta}}{4}\right]}{h} \\
&= -2\lim_{\beta\to\alpha}\beta\lim_{h\to 0}\frac{\phi_{\alpha\beta+h}-\phi_{\alpha\beta}}{4h} \\
&= -\frac{\alpha}{2}\lim_{\beta\to\alpha}\frac{d\phi_{\alpha\beta}}{d\beta} \\
&= -\frac{\alpha}{4}\frac{d\phi_{\alpha\alpha}}{d\alpha}
\end{aligned}
\tag{A.6}
$$

The last equality follows from a direct calculation using the ecpressions in (A.3). We compute

$$
\frac{1}{2}\frac{d\phi_{\alpha\alpha}}{d\alpha} = -\frac{\frac{d\phi_\alpha}{d\alpha}+\alpha\frac{d^2\phi_\alpha}{d\alpha^2}}{1+(\alpha/2)^2\left(\frac{d\phi_\alpha}{d\alpha}\right)^2} \tag{A.7}
$$

and

$$
\lim_{\beta\to\alpha}\frac{d\phi_{\alpha\beta}}{d\beta} = \frac{32\alpha(\phi_\beta-\phi_\alpha)+16(\alpha^2-\beta^2)\frac{d\phi_\beta}{d\beta}}{16(\alpha-\beta)^2+(\alpha+\beta)^2(\phi_\alpha-\phi_\beta)^2}. \tag{A.8}
$$

Replacing $\phi_\beta \to \phi_\alpha + h\frac{d\phi_\alpha}{d\alpha} + h^2/2\frac{d^2\phi_\alpha}{d\alpha^2},\ \beta \to \alpha + h$ and identifying the derivitives as above, this becomes precisely the right had side of (A.7). It is now clear how to proceed for larger n. We have verified the identity up to $n = 6$.

Identity (2.8) is a simple variable transformation based on the assumption that $\phi_{n\alpha}(x,t)$ can alway be expressed as $\phi_{n\alpha}(\xi_+,\xi_-)$. So we compute

$$
\frac{\partial\phi_{n\alpha}}{\partial x} = \alpha\frac{\partial\phi_{n\alpha}}{\partial\xi_+} - \alpha\frac{\partial\phi_{n\alpha}}{\partial\xi_-}, \tag{A.9}
$$

$$
\frac{\partial\phi_{n\alpha}}{\partial t} = \frac{1}{\alpha}\frac{\partial\phi_{n\alpha}}{\partial\xi_+} + \frac{1}{\alpha}\frac{\partial\phi_{n\alpha}}{\partial\xi_-}, \tag{A.10}
$$

$$
\frac{\partial\phi_{n\alpha}}{\partial\alpha} = (x-\frac{t}{\alpha^2})\frac{\partial\phi_{n\alpha}}{\partial\xi_+} - (x+\frac{t}{\alpha^2})\frac{\partial\phi_{n\alpha}}{\partial\xi_-}. \tag{A.11}
$$

Comparing (A.9), (A.10) and (A.11) we can eliminate the derivtives with respect to ξ_+, ξ_- and obtain (2.8).