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A RECIROCITY FOR SYMMETRIC ALGEBRAS

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Abstract. The aim of this note is to show, that the reciprocity property of group algebras in [5, (11.5)] can be deduced from formal properties of symmetric algebras, as exposed in [1], for instance.

Let \( \mathcal{O} \) be a commutative ring. By an \( \mathcal{O} \)-algebra we always mean a unitary associative algebra over \( \mathcal{O} \). Given an \( \mathcal{O} \)-algebra \( A \), we denote by \( A^0 \) the opposite algebra of \( A \). An \( A \)-module is a unitary left module, unless stated otherwise. A right \( A \)-module can be considered as a left \( A^0 \)-module. If \( A, B \) are \( \mathcal{O} \)-algebras, we mean by an \( A \)-\( B \)-bimodule always a bimodule whose left and right \( \mathcal{O} \)-module structure coincide; in other words, any \( A \)-\( B \)-bimodule can be regarded as \( A \otimes \mathcal{O} B^0 \)-module.

For an \( A \)-\( A^0 \)-bimodule \( M \) we set \( M^A = \{ m \in M \mid am = ma \text{ for all } a \in A \} \). In particular, \( A^A = Z(A) \), the center of \( A \). If \( A, B, C \) are \( \mathcal{O} \)-algebras, \( M \) is an \( A \)-\( B \)-bimodule and \( N \) is a \( A \)-\( C \)-bimodule, we consider the space \( \text{Hom}_A(M, N) \) of left \( A \)-module homomorphisms from \( M \) to \( N \) as \( B \)-\( C \)-bimodule via \( (b.\varphi.c)(m) = \varphi(mb)c \). Similarly, if furthermore \( N' \) is a \( C \)-\( B \)-bimodule, we consider the space \( \text{Hom}_{B^0}(M, N') \) of right \( B \)-module homomorphisms from \( M \) to \( N' \) as \( C \)-\( A \)-bimodule via \( (c.\psi.a)(m) = c\psi(am) \).

An \( \mathcal{O} \)-algebra \( A \) is called symmetric if \( A \) is finitely generated projective as \( \mathcal{O} \)-module and if \( A \) is isomorphic to its \( \mathcal{O} \)-dual \( A^* = \text{Hom}_\mathcal{O}(A, \mathcal{O}) \) as \( A \)-\( A \)-bimodule. The image \( s \in A^* \) of \( 1_A \) under any \( A \)-\( A \)-isomorphism \( \Phi : A \cong A^* \) fulfills \( \Phi(a) = a.s = s.a \) for all \( a \in A \); that is, \( s \) is symmetric and the map \( a \mapsto a.s \) is a bimodule isomorphism \( A \cong A^* \). Any such linear form is called a symmetrising form of \( A \). The choice of a symmetrising form on \( A \) is thus equivalent to the choice of a bimodule isomorphism \( A \cong A^* \).

**Theorem 1.** Let \( A, B \) be symmetric \( \mathcal{O} \)-algebras and let \( M, N \) be \( A \)-\( B \)-bimodules which are finitely generated projective as left and right modules. We have a bifunctorial \( \mathcal{O} \)-linear isomorphism

\[
(M^* \otimes_A N)^B \cong (N \otimes_B M^*)^A
\]

which is canonically determined by the choice of symmetrising forms of \( A \) and \( B \).

**Proof.** Let \( s \in A^* \) and \( t \in B^* \) be symmetrising forms on \( A \) and \( B \), respectively. It is well-known (see [1] or also the appendix in [3]) that there is an isomorphism of
\[ B \text{-} A \text{-bimodules} \]
\[
\left\{
\begin{array}{l}
\text{Hom}_A(M, A) \cong M^* \\
\quad f \\
\quad s \circ f
\end{array}
\right.
\]
which is functorial in \( M \). Moreover, since \( M \) and \( N \) are finitely generated projective as left and right modules, we have an isomorphism of \( B \text{-} B \text{-bimodules} \)
\[
\left\{
\begin{array}{l}
\text{Hom}_A(M, A) \otimes A N \cong \text{Hom}_A(M, N) \\
\quad f \otimes n \\
\quad (m \mapsto f(m)n)
\end{array}
\right.
\]
which is functorial in both \( M \) and \( N \). Taking \( B \text{-} \text{fixpoints} \) yields \((M^* \otimes N)^B \cong (\text{Hom}_A(M, A) \otimes A N)^B \cong (\text{Hom}_A(M, N))^B = \text{Hom}_{A \otimes B^0}(M, N)\). Similarly, there is an isomorphism of \( B \text{-} A \text{-bimodules} \)
\[
\left\{
\begin{array}{l}
\text{Hom}_{B^0}(M, B) \cong M^* \\
\quad g \\
\quad t \circ g
\end{array}
\right.
\]
and we have an isomorphism of \( A \text{-} A \text{-bimodules} \)
\[
\left\{
\begin{array}{l}
N \otimes A \text{Hom}_{B^0}(M, B) \cong \text{Hom}_{B^0}(M, N) \\
\quad n \otimes g \\
\quad (m \mapsto ng(m))
\end{array}
\right.
\]
As before, taking \( A \text{-} \text{fixpoints} \) yields \((N \otimes M^*)^A \cong (N \otimes \text{Hom}_{B^0}(M, B))^A \cong (\text{Hom}_{B^0}(M, N))^A = \text{Hom}_{A \otimes B^0}(M, N)\).  □

Remark. The proof of Theorem 1 shows, that the two expressions in the statement of Theorem 1 are isomorphic to \( \text{Hom}_{A \otimes B^0}(M, N) \). In particular, for \( M = N \), this induces algebra structures on \((M^* \otimes M)^B\) and \((M \otimes M^*)^A\).

Taking derived functors of the fixpoint functors in Theorem 1 yields the following consequence on Hochschild cohomology.

Corollary. With the notation and assumptions of Theorem 1, we have an isomorphism of graded \( \mathcal{O} \text{-modules} \) \( HH^*(B, M^* \otimes N) \cong HH^*(A, N \otimes M^*) \).

Proof. Let \( P \) be a projective resolution of \( M \) as \( A \text{-} B \text{-bimodule} \). Then \( P^* = \text{Hom}_\mathcal{O}(P, \mathcal{O}) \) is an \( \mathcal{O} \text{-injective} \) resolution of \( M^* \). Thus \( N \otimes B^* \) and \( P^* \otimes N \) are \( \mathcal{O} \text{-injective} \) resolutions of \( N \otimes M^* \) and \( M^* \otimes N \), respectively. Using Theorem 1, we have isomorphisms of cochain complexes \( \text{Hom}_{B^0}(B, P^* \otimes N) \cong (P^* \otimes N)^B \cong (N \otimes P^*)^A \cong \text{Hom}_{A \otimes B^0}(A, N \otimes P^*) \). Taking cohomology yields the statement. □

Let \( A \) be an \( \mathcal{O} \text{-algebra} \). Following the terminology in [2], [3] (which generalises [4]), an \( \text{interior} \) \( A \text{-algebra} \) is an \( \mathcal{O} \text{-algebra} B \) endowed with a unitary algebra homomorphism \( \sigma : A \to B \). If \( A, B \) are \( \mathcal{O} \text{-algebras} \), \( C \) is an interior \( B \text{-algebra} \) and \( M \) an \( A \text{-} B \text{-bimodule} \), we set \( \text{Ind}_M(C) = \text{End}_\mathcal{O}(M \otimes B) \), considered as interior \( A \text{-algebra} \) via the homomorphism \( A \to \text{Ind}_M(C) \) sending \( a \) to the \( \mathcal{O}^0 \text{-endomorphism} \) given by left multiplication with \( a \) on \( M \otimes B \).
Theorem 2. Let $A$, $B$ be symmetric $O$-algebras and let $M$ be an $A$-$B$-bimodule which is finitely generated projective as left and right module. There is a canonical anti-isomorphism of $O$-algebras

$$(\text{Ind}_M(B))^A \cong (\text{Ind}_{M^*}(A))^B.$$  

Proof. We have $\text{Ind}_M(B) = \text{End}_{B^0}(M)$ and $\text{Ind}_{M^*}(A) = \text{End}_{A^0}(M^*)$. Since taking $O$-duality is a contravariant functor, this algebra is isomorphic to $\text{End}_A(M)^0$. Taking fixpoints completes the proof. □

The group algebra $OG$ of a finite group $G$ is a symmetric algebra. More precisely, $OG$ has a canonical symmetrising form, namely the form $s : OG \to O$ mapping a group element $g \in G$ to zero if $g \neq 1$ and to 1 if $g = 1$. Following the terminology of Puig [4], an interior $G$-algebra is an $O$-algebra endowed with a group homomorphism $\sigma : G \to A^X$. Such a group homomorphism extends uniquely to an $O$-algebra homomorphism $OG \to A$, and thus $A$ becomes an interior $OG$-algebra (and vice versa). If $H$ is a subgroup of $G$ and $B$ an interior $H$-algebra, the induced algebra $\text{Ind}_H^G(B)$ defined in [4] is the $O$-module $OG \otimes_B OG$ endowed with the multiplication

$$\otimes_H \quad (x \otimes b \otimes y)(x' \otimes b' \otimes y') = (x \otimes byx'b' \otimes y')$$  

provided that $yx' \in H$, and 0 otherwise, where $x, y, x', y' \in G$ and $b, b' \in B$. The algebra $\text{Ind}_H^G(B)$ is viewed as interior $G$-algebra with the structural homomorphism $x \in G$ to $\sum_{y \in [G/H]} xy \otimes 1_B \otimes y^{-1}$.

For $B = OH$, we have the obvious identification $\text{Ind}_H^G(OH) = OG \otimes_{OH} OG$, with multiplication given by $(x \otimes y)(x' \otimes y') = x \otimes yx'y'$ if $yx' \in H$ and 0 otherwise, where $x, y, x', y' \in G$. The previous notion of algebra induction is consistent with this concept:

**Lemma.** Let $G$ be a finite group, $H$ a subgroup of $G$ and let $B$ be an interior $H$-algebra. Set $M = OG_H$. There is an isomorphism of $O$-algebras

$$\left\{ \begin{array}{ll}
\text{Ind}_H^G(B) & \cong \text{Ind}_M(B) \\
(x \otimes b \otimes y) & \mapsto (z \otimes c \mapsto x \otimes byzc \text{ if } yz \in H \text{ and } 0 \text{ otherwise}) ,
\end{array} \right.$$  

where $x, y, z \in G$ and $b, c \in B$.

Proof. Straightforward verification. □

**Theorem 3.** (Stalder [5]) Let $G$ be a finite group, let $H, K$ be subgroups of $G$. Consider $OG$ as $OH$-$OK$-bimodule via multiplication in $OG$. Then there is an isomorphism of $O$-algebras

$$\left\{ \begin{array}{ll}
(\text{Ind}_H^G(OH))^K & \sim (\text{Ind}_K^G(OK))^H \\
\sum_{k \in [K/K_{(x \otimes y)}]} kx \otimes yk^{-1} & \mapsto \sum_{h \in [H/H_{(x^{-1} \otimes y^{-1})}]} hx^{-1} \otimes y^{-1}h^{-1} ,
\end{array} \right.$$  

where $K_{(x \otimes y)}$ is the stabilizer in $K$ of $x \otimes y \in \text{Ind}_H^G(OH)$ under the action of $K$ and $H_{(x^{-1} \otimes y^{-1})}$ is the stabilizer in $H$ of $x^{-1} \otimes y^{-1} \in \text{Ind}_K^G(OK)$ under the action of $H$. 

There are (at least) three ways to go about the proof of Theorem 3: by explicit verification or by interpreting Theorem 3 as special case of either Theorem 1 or Theorem 2. We sketch the three different proofs.

Proof 1 of Theorem 3. The image of the set $G \times G$ in \( \text{Ind}_H^G(\mathcal{O}H) = \mathcal{O}G \otimes_{\mathcal{O}H} \mathcal{O}G \) is an $\mathcal{O}$-basis which is permuted under the action of $K$ by conjugation. Thus the subalgebra \( (\text{Ind}_H^G(\mathcal{O}H))^K \) of $K$-stable elements has as $\mathcal{O}$-basis the set of relative traces \( Tr^K_{1_K(z \otimes y)}(x \otimes y) \), where $x, y \in G$. If $x, x', y, y' \in G$ and $k \in K$ such that

\[
xk \otimes yk^{-1} = x' \otimes y'
\]

in \( \text{Ind}_H^G(\mathcal{O}H) \), there is a (necessarily unique) $h \in H$ such that $kx = x'h^{-1}$ and $yk^{-1} = hy'$, which in turn is equivalent to the equality

\[
hx^{-1} \otimes y^{-1}h^{-1} = (x')^{-1} \otimes (y')^{-1}
\]

in \( \text{Ind}_H^G(\mathcal{O}K) \). Thus the map $x \otimes y \mapsto x^{-1} \otimes y^{-1}$ induces a bijection between the sets of $K$-orbits and of $H$-orbits of the images of $G \times G$ in \( \text{Ind}_H^G(\mathcal{O}H) \) and \( \text{Ind}_K^G(\mathcal{O}K) \), respectively. In particular, with the notation above, we have $k \in K(z \otimes y)$ if and only if $h \in H(z^{-1} \otimes y^{-1})$, and the correspondence $k \mapsto h$ induces a group isomorphism \( K(z \otimes y) \cong H(z^{-1} \otimes y^{-1}) \). From this follows that the map given in Theorem 3 is an $\mathcal{O}$-linear isomorphism. It remains to verify that this is an algebra homomorphism. In \( \text{Ind}_H^G(\mathcal{O}H) \), multiplication is given by $(x \otimes y)(z \otimes t) = x \otimes yzt$, if $yz \in H$ and 0, otherwise, where $x, y, z, t \in G$. If $yz \in H$, then in \( \text{Ind}_K^G(\mathcal{O}K) \), the elements $(yz)z^{-1} \otimes t^{-1}(yz)^{-1}$ and $z^{-1} \otimes t^{-1}$ are in the same $H$-orbit, and the multiplication in \( \text{Ind}_K^G(\mathcal{O}K) \) yields $(x^{-1} \otimes y^{-1})(yz^{-1} \otimes t^{-1}(yz)^{-1}) = x^{-1} \otimes t^{-1}y^{-1}$, and this corresponds precisely to the bijection between the sets of $K$-orbits and $H$-orbits of the images of the set $G \times G$ in \( \text{Ind}_K^G(\mathcal{O}K) \) and \( \text{Ind}_H^G(\mathcal{O}H) \), respectively. \( \square \)

Proof 2 of Theorem 3. We are going to apply Theorem 1 to the particular case where $A = \mathcal{O}H$, $B = \mathcal{O}K$, $M = N = \mathcal{O}G$ viewed as $A$-$B$-bimodule (through the inclusions $H \subseteq G$, $K \subseteq G$). This yields an $\mathcal{O}$-linear isomorphism

\[
((\mathcal{O}G)^* \otimes_{\mathcal{O}H} \mathcal{O}G)^K \cong (\mathcal{O}G \otimes_{\mathcal{O}K} (\mathcal{O}G)^*)^H.
\]

Composing this with the canonical isomorphism \( (\mathcal{O}G)^* \cong \mathcal{O}G \) mapping $f \in (\mathcal{O}G)^*$ to $\sum_{x \in G} f(x^{-1})x$ yields the isomorphism in Theorem 3. \( \square \)

Proof 3 of Theorem 3. Applying Theorem 2 and the above Lemma to $A = \mathcal{O}K$, $B = \mathcal{O}H$ and $M = \mathcal{O}G$ as $A$-$B$-bimodule yields an anti-isomorphism \( (\text{Ind}_H^G(\mathcal{O}H))^K \cong (\text{Ind}_K^G(\mathcal{O}K))^H \). The map sending $x \otimes y$ to $y^{-1} \otimes x^{-1}$ is an anti-automorphism of \( \text{Ind}_H^G(\mathcal{O}H) \) which induces an anti-automorphism of \( (\text{Ind}_H^G(\mathcal{O}H))^K \). Composing both maps yields again the isomorphism in Theorem 3. \( \square \)

Remark. The proof 3 of Theorem 3 is essentially the proof given in [5, §11].
A reciprocity for symmetric algebras

References

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