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VERTEX AND SOURCE DETERMINE THE BLOCK VARIETY OF AN INDECOMPOSABLE MODULE

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Abstract. The block variety $V_{G,b}(M)$ of a finitely generated indecomposable module M over the block algebra of a p-block b of a finite group G, introduced in [15], can be computed in terms of a vertex and a source of M. We use this to show that $V_{G,b}(M)$ is connected, and that every closed homogeneous subvariety of the affine variety $V_{G,b}$ defined by block cohomology $H^*(G, b)$ (cf. [14]) is the variety of a module over the block algebra. This is analogous to the corresponding statements on Carlson's cohomology varieties in [8].

1. Introduction

The theory of the cohomology variety $V_G(M)$ for a finitely generated module M over the modular group algebra of a finite group G was developed by Carlson [6], [7], [8] and others back in the nineteen eighties. It has played a major role in recent years in the modular representation theory of finite groups. One of the problems with the theory has been that it has been much more fruitful as a way of proving theorems about the principal block than about more general blocks.

To address this problem, the second author [15] recently introduced the concept of the block variety $V_{G,b}(M)$ of a finitely generated module M for a block b of a finite group G. The definition of the ambient variety $V_{G,b}$ involves a variation of the Cartan–Eilenberg stable element method in ordinary cohomology for the Brauer pairs in the block. But the definition of $V_{G,b}(M)$ involves going through the Hochschild cohomology of the block as an intermediate step. In [17], the dependence on Hochschild cohomology for the calculation of block varieties is removed. It is shown in that paper that the block variety of M is the image in $V_{G,b}$ of the variety at the level of the defect group. But first, the module must be cut down using the idempotent for the source algebra of the block, in the sense of Puig. An example is given there, to show that this step is really necessary.

In this paper, we describe how to calculate the block variety of a module in terms of the vertex and source of the module. We show that it is given as the image in $V_{G,b}$ of the variety of the source of M as a module for the vertex. Actually, there is a subtlety here: the vertex and source are only well defined up to G-conjugacy, and we must be careful how we choose a P-conjugacy class.

Taken together with the stratification theorem for $V_{G,b}$ [17] and Carlson's theory of rank varieties, our theorem implies that $V_{G,b}(M)$ can be calculated without knowing anything about cohomology, just as was the case for $V_G(M)$.

Our theorem enables us to prove some basic properties for $V_{G,b}(M)$, including the analog of Carlson's theorem [8] that the variety of an indecomposable module is

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connected. We also show that every closed homogeneous subvariety of $V_{G,b}$ can be realised as $V_{G,b}(N)$ for a suitably chosen module N. Finally, we show that the block variety of a module survives the Green correspondence with respect to a subgroup which controls fusion of Brauer pairs in the block.

Throughout this paper, k is an algebraically closed field of prime characteristic p. Let G be a finite group. A *block of* G is a primitive idempotent b in $Z(kG)$. Following Alperin–Broué [1], a *b-Brauer pair* is a pair (Q,e) consisting of a p-subgroup Q of G and a block e of $C_G(Q)$ such that $\mathrm{Br}_Q(b)e = e$, where $\mathrm{Br}_Q : (kG)^Q \to C_G(Q)$ is the Brauer homomorphism [4]. It is shown in [1], that the set of b-Brauer pairs is a partially ordered G-set such that G acts transitively on the set of maximal b-Brauer pairs. If (P,e) is a maximal b-Brauer pair, then P is called a *defect group of* b (this notion is due to Brauer, although he did not state it in these terms). If P is a defect group of b, then P is maximal with respect to $\mathrm{Br}_P(b) \neq 0$; in particular, there is a primitive idempotent $i \in (kGb)^P$ such that $\mathrm{Br}_P(i) \neq 0$. Such an idempotent is called a *source idempotent of* b (with respect to P). By Broué–Puig [5, Theorem 1.8], for any subgroup Q of P, there is a unique block e_Q of $C_G(Q)$ such that $\mathrm{Br}_Q(i)e_Q \neq 0$. Then (Q,e_Q) is a b-Brauer pair, and e_Q is also the unique block of $C_G(Q)$ such that $(Q,e_Q) \subseteq (P,e_P)$.

We denote by $\mathcal{F}_{G,b}$ the category whose objects are the subgroups of P, and whose morphisms are the sets of group homomorphisms $\varphi : Q \to R$ for which there exists an element $x \in G$ satisfying ${}^x(Q,e_Q) \subseteq (R,e_R)$ and $\varphi(u) = xux^{-1}$ for all $u \in Q$. The *block cohomology*, introduced in [14], is defined by

$$H^*(G,b) = \varprojlim_{Q \in \mathcal{F}_{G,b}} H^*(Q,k) \ .$$

We identify $H^*(G,b)$ with the graded subalgebra of all "stable elements" $\zeta \in H^*(P,k)$ satisfying $\mathrm{res}^P_R(\zeta) = \mathrm{res}_\varphi(\zeta)$ for any morphism $\varphi : R \to P$ belonging to the category $\mathcal{F}_{G,b}$.

The algebra $H^*(G,b)$ is finitely generated graded commutative, and we denote by $V_{G,b}$ the associated affine variety, called the *block variety of* b (cf. [15]). Following [15], for any finitely generated kGb-module M, there is a canonical graded algebra homomorphism $H^*(G,b) \to \mathrm{Ext}^*_{kGb}(M,M)$, defined using Hochschild cohomology of the block, whose kernel detemines the subvariety $V_{G,b}(M)$ of $V_{G,b}$. If b is the principal block, then $H^*(G,b) \cong H^*(G,k)$, and we write then V_G and $V_G(M)$ instead of $V_{G,b}$ and $V_{G,b}(M)$, respectively (that is, in the principal block case, the block varieties coincide with Carlson's cohomology varieties, introduced in [6], [7]).

For any subgroup Q of P, the inclusion $H^*(G,b) \subseteq H^*(P,k)$ followed by the restriction $\mathrm{res}^P_Q : H^*(P,k) \to H^*(Q,k)$ is a graded algebra homomorphism, denoted by $r_Q : H^*(G,b) \to H^*(Q,k)$. We denote by $r_Q^* : V_Q \to V_{G,b}$ the induced map on varieties. We refer to Benson [3] and Evens [10] for background material on cohomology varieties of modules and Thévenaz [20] for block theory.

Theorem 1.1. *Let G be a finite group, let b be a block of G, let P be a defect group of b, and let $i \in (kGb)^P$ be a source idempotent. Let M be an indecomposable kGb-module. There is a vertex Q of M contained in P and a source U of M such that U is a direct summand of $\mathrm{Res}^P_Q(iM)$ and such that M is isomorphic to a direct summand of $kGi \underset{kQ}{\otimes} U$. For any such choice of a vertex and source of M we have*

$$V_{G,b}(M) = r_Q^*(V_Q(U)).$$

The proof of Theorem 1.1 will be given in Section 3.

Corollary 1.2. *For a finitely generated indecomposable kGb-module M, the variety $V_{G,b}(M)$ is connected.*

Proof. The variety $V_Q(U)$ is connected by Carlson's theorem [8], and hence so is its image under the map r_Q^*, which is $V_{G,b}(M)$ by Theorem 1.1. □

Corollary 1.3. *Every closed homogeneous subvariety V of $V_{G,b}$ is of the form $V_{G,b}(N)$ for some finitely generated kGb-module N.*

Proof. The inverse image in V_P of V under the map $r_P^* : V_P \to V_{G,b}$ is a closed homogeneous subvariety of V_P; thus, by Carlson [8], there is a finitely generated kP-module U such that $r_P^*(V_P(U)) = V$. Using Green's indecomposability theorem, we may write $U = \bigoplus_{j\in J} \mathrm{Ind}^P_{Q_j}(U_j)$, where J is a finite indexing set, and where for any $j \in J$, Q_j is a subgroup of P and U_j is an indecomposable kQ_j-module having Q_j as vertex. When viewed as kQ_j-kQ_j-bimodule, $ikGi$ has a direct summand isomorphic to kQ_j, where $j \in J$ (see e.g. [16, 6.1(iii)]). Thus for any $j \in J$ the left kGb-module $kGi \underset{kQ_j}{\otimes} U_j$ has an indecomposable direct summand M_j such that the restriction of iM_j to kQ_j has U_j as direct summand. Note that the kGb-module $kGi \underset{kQ_j}{\otimes} U_j$ is a direct summand of $kG \underset{kQ_j}{\otimes} U_j = \mathrm{Ind}^G_{Q_j}(U_j)$. Thus M_j has Q_j as vertex and U_j as source, and $V_{G,b}(M_j) = r_{Q_j}^*(V_{Q_j}(U_j)) = r_P^*(V_P(\mathrm{Ind}^P_{Q_j}(U_j)))$ by Lemma 2.2 below. Thus $N = \bigoplus_{j\in J} M_j$ satisfies $V_{G,b}(N) = V$. □

The next corollary states that the block variety of a module survives the Green correspondence with respect to a subgroup which controls fusion of Brauer pairs.

Corollary 1.4. *Suppose the vertex Q of M is normal P. Set $H = N_G(Q)$ and denote by c the unique block of H such that $e_Q c = e_Q$. If $\mathcal{F}_{G,b} = \mathcal{F}_{H,c}$ then $V_{G,b}(M) = V_{H,c}(f(M))$, where $f(M)$ is the Green correspondent of M.*

Proof. Since Q is normal in P, the pair (P, e_P) is also a maximal c-Brauer pair. The condition $\mathcal{F}_{G,b} = \mathcal{F}_{H,c}$ implies the equality $H^*(G,b) = H^*(H,c)$ as graded subalgebras of $H^*(P,k)$. It is well known that the Green correspondent $f(M)$ of M belongs to the block c of H. Since every vertex and source of $f(M)$ is also a vertex and source of M, the statement follows from Theorem 1.1. □

Note that if $Q = P$ is abelian, we have $\mathcal{F}_{G,b} = \mathcal{F}_{H,c}$, because in blocks with abelian defect groups, the normaliser of a defect group controls fusion (cf. Alperin–Broué [1]). Hence Corollary 1.4 applies in that situation. This particular case of 1.4 has independently been proved by H. Kawai [12], who also observed [17, 2.1] (cf. 2.3 below).

2. Auxiliary results

Lemma 2.1. *Let G be a finite group, let H be a subgroup of G and let M be a finitely generated kG-module. We have $(\mathrm{res}^G_H)^*(V_H(\mathrm{res}^G_H(M))) \subseteq V_G(M)$.*

Proof. Let $\zeta \in H^*(G,k)$ such that $\zeta \otimes \mathrm{Id}_M$ is zero in $\mathrm{Ext}^*_{kG}(M,M)$. Then $\mathrm{res}^G_H(\zeta) \otimes \mathrm{Id}_M$ is zero in $\mathrm{Ext}^*_{kH}(\mathrm{Res}^G_H(M), \mathrm{Res}^G_H(M))$. Thus the annihilator of $\mathrm{Ext}^*_{kG}(M,M)$ in $H^*(G,k)$ is contained in the inverse image in $H^*(G,k)$ of the annihilator in $H^*(H,k)$ of $\mathrm{Ext}^*_{kH}(\mathrm{Res}^G_H(M), \mathrm{Res}^G_H(M))$. Passing to varieties yields the statement. □

Lemma 2.2. *Let G be a finite group, let H be a subgroup of G, and let N be a finitely generated kH-module. We have*

$$(\mathrm{res}^G_H)^*(V_H(N)) = V_G(\mathrm{Ind}^G_H(N)).$$

Proof. This is a standard property of varieties for modules, and can be found for example in Evens [10], Proposition 8.2.4 or Benson [2], Theorem 2.26.9. □

Theorem 2.3. *Let G be a finite group, b a block of kG, P a defect group of b and i a source idempotent of b in $(kGb)^P$. The inclusion $r_P : H^*(G,b) \to H^*(P,k)$ induces a finite surjective morphism $r_P^* : V_P \to V_{G,b}$, and for any finitely generated kGb-module M we have $V_{G,b}(M) = r_P^*(V_P(iM))$, where iM is considered as kP-module.*

Proof. See Linckelmann [17], Theorem 2.1 or Kawai [12]. □

It has been shown by Puig [19] that one can recover the category $\mathcal{F}_{G,b}$ from the kP-kP-bimodule structure of the source algebra $ikGi$ of a block b (see also Linckelmann [16, 7.7] for a proof). For any two subgroups Q, R of P and any group homomorphism $\varphi : R \to Q$ we denote by ${}_\varphi kQ$ the kR-kQ-bimodule kQ endowed with the regular action of kQ on the right and the action of kR on the left obtained from restricting the regular action of kQ to kR via φ.

Proposition 2.4. *Let G be a finite group, let b be a block of G, let P be a defect group of b and let $i \in (kGb)^P$ be a source idempotent. Let Q be a subgroup of P. Every indecomposable direct summand of the source algebra $ikGi$ as kP-kQ-bimodule is isomorphic to $kP \underset{kR}{\otimes} {}_\varphi kQ$ for some subgroup R of P and some morphism $\varphi : R \to Q$ in $\mathcal{F}_{G,b}$.*

Finally, we need the following statement from Linckelmann [13], which is about a certain choice of vertices and sources of indecomposable modules (implying that vertices and sources of indecomposable modules can be detected at the source algebra level).

Proposition 2.5. *Let G be a finite group, let b be a block of G, let P be a defect group of b and let $i \in (kGb)^P$ be a source idempotent of b. Any indecomposable kGb-module M has a vertex Q contained in P and a kQ-source U such that U is a direct summand of $\mathrm{Res}^P_Q(iM)$ and M is a direct summand of $kGi \underset{kQ}{\otimes} U$.*

Proof. See 6.3 of Linckelmann [13]. □

3. Proof of Theorem 1.1

By Proposition 2.5 above there is a vertex Q of M and a kQ-source U of M such that U is a direct summand of $\mathrm{Res}^P_Q(iM)$ and such that M is a direct summand of $kGi \underset{kQ}{\otimes} U$. By Proposition 2.4, we can write

$$ikGi \cong \bigoplus_{(R,\varphi)} kP \underset{kR}{\otimes} {}_{\varphi}kQ \ ,$$

where (R,φ) runs over a family of pairs consisting of a subgroup R of P and a morphism $\varphi : R \to Q$ belonging to the category $\mathcal{F}_{G,b}$.

Then iM is, as kP-module, a direct summand of

$$ikGi \underset{kQ}{\otimes} U = \bigoplus_{(R,\varphi)} \mathrm{Ind}^P_R({}_{\varphi}U) \ ,$$

where (R,φ) is as before. Thus

$$V_P(iM) \subseteq V_P(ikGi \underset{kQ}{\otimes} U) = \bigcup_{(R,\varphi)} V_P(\mathrm{Ind}^P_R({}_{\varphi}U)) \ .$$

By Lemma 2.2, we have $V_P(\mathrm{Ind}^P_R({}_{\varphi}U)) = (\mathrm{res}^P_R)^*(V_R({}_{\varphi}U))$, for any (R,φ) as above. Using Theorem 2.3 we get

$$V_{G,b}(M) = r_P^*(V_P(iM)) \subseteq \bigoplus_{(R,\varphi)} r_R^*(V_R({}_{\varphi}U)) \ .$$

For any (R,φ) occurring in this union, we have $r_R^*(V_R({}_{\varphi}U)) = r^*_{\varphi(R)}(\mathrm{Res}^Q_{\varphi(R)}(U)) \subseteq r_Q^*(V_Q(U))$, where the first equality uses the stability of the elements in $H^*(G,b)$ with respect to morphisms in $\mathcal{F}_{G,b}$, and where the second inclusion follows from Lemma 2.1. Together, we get $V_{G,b}(M) \subseteq r_Q^*(V_Q(U))$. The other inclusion follows trivially from the fact that U is a direct summand of iM as kQ-module. □

4. A question

A theorem of Hida [11] gives a converse to Carlson's connectedness theorem. It states that given a connected closed homogeneous subvariety V of V_G, there exists an indecomposable finitely generated kG-module N such that $V_G(N) = V$. An obvious question is whether the corresponding statement holds for a connected closed homogeneous subvariety V of the block variety $V_{G,b}$. One might try to prove this statement by the method used to prove Corollary 1.3. The problem is that it is not at all clear whether there is a connected closed homogeneous subvariety of V_P whose image in $V_{G,b}$ is equal to V. An easy modification of the proof of Corollary 1.3 shows that if V is the image of such a subvariety of V_P then there exists a finitely generated indecomposable kGb-module N such that $V_{G,b}(N) = V$.

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