System Properties of Implicit Passive Electrical Networks Descriptions

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Abstract: Redesigning systems by changing elements, topology, organization, augmenting the system by the addition of subsystems, or removing parts, is a major challenge for systems and control theory. A special case is the redesign of passive electric networks which aims to change the natural dynamics of the network (natural frequencies) by the above operations leading to a modification of the network. This requires changing the system to achieve the desirable natural frequencies and involves the selection of alternative values for dynamic elements and non-dynamic elements within a fixed interconnection topology and/or alteration of the interconnection topology and possible evolution of the network (increase of elements, branches). The use of state-space or transfer function models does not provide a suitable framework for the study of this problem, since every time such changes are introduced, a new state space or transfer function model has to be recalculated. The use of impedance and admittance modeling, provides a suitable framework for the study of network properties under the process of re-engineering transformations. This paper deals with the fundamental system properties of the impedance-admittance network description which provide the appropriate framework for network re-engineering. We identify the natural topologies expressing the structured transformations linked to the impedance-graph, admittance graph-topology of the network and examine issues such as network regularity, number of finite frequencies and provide characterization of them in terms of the basic network matrices. The implicit network representation introduced provides a natural framework for expressing the different types of re-engineering transformations which can be used for the study of the natural frequencies assignment.

Keywords: Linear systems, linear networks, structured systems, system theory, system models.

1. INTRODUCTION

Electrical RLC systems modeling has become important for its application to design of mechanical networks (Smith (2002)), for the design of passive suspensions in cars as well as design of mechanical networks for earthquake buildings. The problem of redesigning autonomous (no inputs or outputs) passive electric networks (Karcanias (2008)),(Karcanias (2010)) that arises, aims to change the network (natural frequencies) by modifying the types of elements, possibly their values, the interconnection topology by possibly adding, or eliminating parts of the network. As such, this is a problem that differs considerably from a standard control problem, since it involves changing the system itself without control and, by system re-engineering, aims to achieve the desirable system properties, which may be expressed by the natural frequencies. In fact, this problem involves the selection of alternative values for dynamic elements (inductances, capacitances) and non-dynamic elements (resistances) within a fixed interconnection topology and/or alteration of the network interconnection topology and possible evolution of the network (increase of elements, branches). The use of state-space or transfer function models does not provide a suitable framework for addressing the problem of network re-engineering i.e. for studying the evolution of system properties under such transformations, since every time such changes are introduced, new versions of these models have to be recalculated. It has been realized that for RLC networks the integral-differential impedance/admittance models provide a natural setup for studying evolution of system properties under re-engineering (Karcanias (2010)), (Berger et al. (2012)). These network descriptions introduce a new implicit description for networks based on the integral-differential network operator $W(s)$. The aim of the paper is to develop some fundamental system properties of this new

implicit description, define and then demonstrate how such descriptions can provide the appropriate representation framework for the re-engineering transformations, that allows the deployment of control theoretic tools for re-engineering the properties of a given network. We use impedance and admittance modeling (Seshu and Reed (1961)), (Shearer et al. (1971)) for passive electrical networks and consider systems with no sources (autonomous descriptions), since our current interest is on the shaping of natural frequencies. Such descriptions are described by an integral-differential operator $W(p) = pC + p^{-1}L + R$ (admittance description), where C, L, R are symmetric matrices describing the topology and values of capacitances, inductances and resistances, respectively. The emphasis here is on the study of the different representations of the passive network that enable the investigation of the transformations on such models as structural transformations.
The structure of this paper is as follows: In section 2.1 we present two fundamental notions from systems modelling, i.e. the *autonomous natural impedance / admittance* models and we introduce the implicit network description. In section 2.2 we identify two natural topologies, namely, the *impedance graph* and the *admittance graph* of the network, which are linked to the specifics of the Loop and Node analysis, respectively, whereas in section 2.3 the relationship between these two models (impedance and admittance) is introduced and a preliminary result is given. In section 3 a linearization of these models that preserves the network description is introduced in terms of the network pencil $P(p)$, that allows matrix pencil theory (Gantmacher (1959)), (Karcanias and Kalogeropoulos (1986)), (Karcanias and Kalogeropoulos (1988)) to be used for the characterization of the network properties in terms of the properties of the triple (C, L, R). We investigate the link between the $W(p)$ and $P(p)$ operators and the property of network regularity linked to the invertibility of $W(p)$ is examined in section 4; this property is crucial for the ability to define transfer functions for oriented (networks with inputs and outputs) and we show that this is equivalent to a notion of graph connectivity. Furthermore, in section 5 we establish results on the number of natural frequencies of the network and relate them to the network parameters (C, L, R). The introduced implicit network description provides the natural set up for the study of effects of structural network transformations on network properties (Karcanias et al. (2014)), (Karcanias et al. (2016)). The proof of the results is given in (Karcanias et al. (2016)).

2. IMPLICIT MODELS FOR RLC NETWORKS AND THEIR RESPECTIVE TOPOLOGY

2.1 The Autonomous Natural Impedance-Admittance Model

In the network loop analysis method (Seshu and Reed (1961)), (Shearer et al. (1971)) the variables are selected such that the vertex law is automatically satisfied. Here, we consider only planar graphs with b branches and n vertices. We then consider the variables associated with each of the meshes and we define them as loop variables. The path law is then written for each mesh and substitutions are made for the across variables in terms of the loop variables using the elemental equations.
The process of working out the equations involves the selection of internal independent loops, the definition of loop currents and the transformation of current sources to equivalent voltage sources. If we denote by $\underline{f}(s) = (f_1, ..., f_q)^t$ the vector of the set of the Laplace transforms of the loop currents and by $\underline{v}(s) = (v_{s1}, ..., v_{sq})^t$ the vector of Laplace transforms of equivalent voltage sources, then the loop or impedance model is defined by [9] or by

$$Z(s)\underline{f}(s) = \underline{v_s}(s), Z(s) \in \Re^{q\times q}(s) \tag{1}$$

where: $z_{ii}(s) \in \Re[s]$ is the sum of impedances in loop i; $z_{ij}(s)$ is the sum of impedances common between loops i and j and $Z(s)$ is the network impedance matrix which is an integral-differential symmetric matrix defining the impedance model.
Similarly, in the nodal method of analysis the across variables from each vertex to some reference vertex are chosen as the unknowns in terms of which the final set of equations is formulated and are called node variables. These variables automatically satisfy the path laws. The vertex equation is then written at each node, and the through variables are then expressed directly in terms of the node variables as related by the elemental equations. The node method is the dual to the loop method and the basic steps involve the selection of internal nodes, definition of the corresponding node voltages and the transformation of the voltage sources to equivalent current sources (Nortons theorem). If we denote by $\underline{u}(s) = (u_1, u_2, ..., u_n)^t$ the Laplace transforms of the reduced node voltages and by $i_s(s) = (i_{s1}, ..., i_{sn})^t$ the vector of Laplace transforms of equivalent current sources, then the node or admittance model is defined by (Seshu and Reed (1961)), (Shearer et al. (1971)) or by:

$$Y(s)u(s) = \underline{i_s}(s), Y(s) \in \Re^{m\times m}(s) \tag{2}$$

where: $y_{ii}(s) \in \Re[s]$ is the sum of admittances in node i ; $y_{ij}(s)$ is the sum of admittances common between nodes i and j and $Y(s)$ is the network admittance matrix which defines the admittance model which again is an integral-differential symmetric matrix. The general structure of $Z(s)$ and $Y(s)$ are described by the following integral-differential symmetric operator:

$$W(s) = \begin{bmatrix} w_{11} & -w_{12} & -w_{13} & \cdots & -w_{1r} \\ -w_{12} & w_{22} & -w_{23} & \cdots & -w_{2r} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ -w_{1(r-1)} & -w_{2(r-1)} & -w_{3(r-1)} & \cdots & -w_{(r-1)r} \\ -w_{1r} & -w_{2r} & -w_{3r} & \cdots & w_{rr} \end{bmatrix} \tag{3}$$
$$\in \Re^{r\times r}(s)$$

When we consider networks with no inputs (no current, or voltage sources) in the time domain, the resulting admittance, or impedance network models may be described in a unifying way as:

$$\left\{pB + p^{-1}E + D\right\} \cdot \underline{x}(t) = \underline{0} \tag{4}$$

where p, p^{-1} are the differential, integral operators respectively and $\underline{x}(t)$ is the vector of nodal voltages, or loop currents. Such a description may be referred to as the *natural autonomous network* description and the operator $W(s) = sB + s^{-1}E + D$ (s-domain description) will be called the *natural impedance-admittance* (NI-A) operator.

Note that for the case of admittance we have that B is a matrix of capacitances, E is the matrix of inductances and D is a matrix of conductances. For the case of impedance the reverse holds true. Hence, B is the matrix of inductances, E is the matrix of capacitances and D is the matrix of resistances. The symmetric operator $W(s)$ is thus a common description of $Y(s)$ and $Z(s)$ matrices and $W(s)$ describes the dynamics of the network.

Network modeling uses the *system graph*, which is the basic topological structure that generates the system equations. Apart from the system graph we may introduce some additional topologies, which are linked to the specifics of the Node and Loop analysis. The detailed topological structures that emerge depend on the nature of the elements in the network and it is defined by the structure of the symmetric B, E, D matrices which also indicate the values of the corresponding elements (Karcanias (2010)).

2.2 The Vertex and Loop Topologies

Every network may be represented in terms of a set of vertices, or nodes and all branches between two vertices may be represented by an admittance function. Specification of the values of the across variables of the vertices defines the values of all through variables in the network. The nature of the elements in the branches of the natural vertex graph defines an element dependent topology, which is characterized by adjacency type matrices. If we set the external sources to zero, the reduced graph will be referred to as the *kernel vertex graph.* For a given kernel vertex graph we define *sub-graphs* from the respective structure of B, E, D matrices. These sub- graphs are by construction simple graphs, and the corresponding adjacency matrices are all symmetric Boolean matrices. The kernel vertex graph and its subgraphs provide a representation of the *vertex topology* of the network. The loop topology is a notion *dual* to that of the vertex topology and it is based on the following principle: Every network of m vertices and r edges (branches) may be represented by $q = m - r + 1$ loops leading to independent equations. All branches common between two loops may be represented by an impedance function. Specification of the values of through variables for the loops defines the values of all across variables in the network. In a similar way to the case of nodal analysis, we may define the *loop topology* based on the *kernel loop graph* and its *sub-graphs.*

2.3 The Relationship between Impedance and Admittance Models

We consider a network with m vertices (nodes) and q loops and let us assume that $q \geq m$. We shall refer to m and q as the *nodal, loop* cardinality respectively. We assume that the corresponding implicit Impedance and Admittance models are:

$$\begin{array}{c} Y(p)\underline{v} = \underline{0} \\ Z(p)\underline{i} = \underline{0} \end{array} \tag{5}$$

From the network topology the following result is readily established (Karcanias et al. (2016)):

Proposition 1. Let us assume that $q \geq m$. The following hold true: There exists a rational $q \times m$ matrix $T(p)$ of the type $T(p) = T_0 + pT_1 + p^{-1}T_2$, where T_0, T_1, T_2 are $q \times m$ real matrices such that

$$\underline{i} = T(p) \cdot \underline{v} \tag{6}$$

The above implies clearly that there exist a relationship between $Y(p)$ and $Z(p)$ descriptions, which needs further investigation.

3. THE LINEARIZATION OF THE AUTONOMOUS NATURAL IMPEDANCE-ADMITTANCE MODEL

Consider a network with m nodes and q loops and let us assume that $q \geq m$. We will use from now on the admittance model with a corresponding implicit description defined as:

$$\begin{array}{c} Y(p)\underline{v} = \underline{0} \\ Y(p) = pC + p^{-1}L + R \end{array} \tag{7}$$

We can define the new variables $p^{-1}\underline{v} = \underline{v}^*$ and $p\underline{v}^* = \underline{v}$ and thus the original implicit description becomes:

$$\begin{array}{c} \left\{pC + p^{-1}L + R\right\} \underline{v} = \underline{0} \\ p\underline{v}^* = \underline{v} \end{array}$$

or

$$p \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \underline{v} \\ \underline{v}^* \end{bmatrix} + \begin{bmatrix} R & L \\ -I & 0 \end{bmatrix} \begin{bmatrix} \underline{v} \\ \underline{v}^* \end{bmatrix} = \underline{0} \tag{8}$$

Clearly the vector $\xi^t = \begin{bmatrix} \underline{v}^t & \underline{v}^{*^t} \end{bmatrix}^t$ is a state vector and the above description is an implicit state space description, which is not necessarily minimal. This description preserves the *nodal structure* of the network and it will be referred to as *nodal implicit state space description* and the associated matrix pencil

$$P(s) = s \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix} + \begin{bmatrix} R & L \\ -I & 0 \end{bmatrix} = \begin{bmatrix} sC + R & L \\ -I & sI \end{bmatrix} = sF + G \tag{9}$$

will be referred to as the *nodal network pencil.* Note that the above autonomous differential description preserves the topological properties of the network. The relationship between $P(s)$ and $Y(s)$ is established in the following proposition.

Proposition 2. The following properties hold true:

(i) The determinants of $P(s)$ and $Y(s)$ are related as:
$$|Y(s)| = s^{-m} |P(s)|$$

(ii) If $Y(s) = s^{-1}Z(s)$ then:
$$|Z(s)| = \left|(s^2C + sR + L)\right| = |P(s)|$$

This allows relating the zero structure of $Y(s)$ to the zero structure of the pencil $P(s)$. In the following we examine the invariant structure properties of $P(s)$ which also characterize properties of $Y(s)$. The linearized pencil is structured, but not symmetric in the general case.

Remark 3. For the special cases where the network is characterized only by one type of dynamic elements, then the respective pencils are symmetric, preserve the network structure and inherit the passivity properties, i.e.

$$\begin{array}{c} Y(s) = sC + R \\ Z(\widehat{s}) = \widehat{s}L + R \\ \widehat{s} = s^{-1} \end{array}$$

Remark 4. The MFD factorization $Y(s) = [sI_m]^{-1}Z(s)$ is coprime at all finite s except possibly at $s = 0$. Thus the zeros of $Y(s)$ and $Z(s)$ may differ only at $s = 0$.

4. THE REGULARITY PROPERTY OF AN RLC NETWORK

The implicit description of equation 9 may be expanded to an oriented (forced) description by selecting inputs $\underline{\tau}$ and outputs $\underline{\zeta}$ which transform the model to the form:

$$\begin{gathered} Y(s)\underline{v} = Q\underline{\tau}, \underline{\zeta} = H\underline{v} \\ Y(s) = sC + s^{-1}L + R = s^{-1}Z(s) \end{gathered} \tag{10}$$

It is clear from the above that the ability to define transfer functions in a network depends on the invertibility of $Y(s)$. A network will be called *regular* if $\det[Y(s)] \neq 0$ over $\Re(s)$. Note that $Z(s) \in \Re[s]^{m\times m}$ and can always be expressed as in equation (11) where $p_{ij} \in \Re[s]$ are the polynomials resulting from the admittance functions between nodes i and j, all have positive coefficients $\widehat{p}_{ii} = \sum_{j=1}^{m} p_{ij}$. The above decomposition enables the computation of $\det[Z(s)]$. In the following we will derive criteria for the characterization of this property. The computation of the expression for this determinant allows the characterization of the regularity property in graph terms. This computation requires some definitions and notation which are introduced first.

$$\begin{aligned} Z(s) &= \\ &= \begin{bmatrix} \widehat{p}_{11} & -p_{12} & \cdots & -p_{1(m-1)} & -p_{1m} \\ -p_{12} & \widehat{p}_{22} & \cdots & -p_{2(m-1)} & -p_{2m} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -p_{1(m-1)} & -p_{2(m-1)} & \cdots & \widehat{p}_{(m-1)(m-1)} & -p_{(m-1)m} \\ -p_{1m} & -p_{2m} & \cdots & -p_{(m-1)m} & \widehat{p}_{mm} \end{bmatrix} \\ &= R(s) + T(s) = diag\{...p_{ii}...\} + T(s) \end{aligned} \tag{11}$$

Definition 5. Let us denote by $\widetilde{m} = \{1, 2, ..., m\}$ and by $\Omega_{k,m} = \{\omega_k = (i_1, i_2, ..., i_k) \in Q_{k,m}, k \le m\}$ [15], where $Q_{k,m}$ is the set of lexicographically ordered sequences of k integers from $\widetilde{m}$ and $\{p_{ij} \in \Re[s], i, j = 1, 2, ...\}$. We define:

(i) For any $\omega_k = (i_1, i_2, ..., i_k) \in \Omega_{k,m}$, $r(\omega_k) = p_{i_1 i_1} p_{i_2 i_2} \cdots p_{i_k i_k}$ and $r(\omega_m) = p_{11}p_{22}\cdots p_{mm}$.

(ii) If $\mathcal{A} = \{\rho = (j_1, j_2) \in Q_{k,m}\} = \left\{\rho_1, \rho_2, ..., \rho_\tau : \tau = \begin{pmatrix} m \\ 2 \end{pmatrix}\right\}$, then $p(\rho) = p_{j_1}p_{j_2}$ for $\rho = (j_1, j_2) \in \mathcal{A}$.

(iii) Given any $\omega_k = (i_1, i_2, ..., i_k) \in \Omega_{k,m}$, we denote by $\mathcal{A}(\omega_k)$ the subset of $\mathcal{A}$ obtained by deleting the sequences $\rho = (j_1, j_2) \in Q_{2,m}$ based on the $(i_1, i_2, ..., i_k)$ set of indices. $\mathcal{A}(\omega_k)$ has $\vartheta = (m/2) - (k/2)$ elements.

(iv) Given $\mathcal{A}(\omega_k)$ we define

$$\mathcal{B}_k(\omega_k) = \left\{ \begin{array}{l} \sigma = (\rho_{l1}, \rho_{l2}, ..., \rho_{l\nu}) \in Q_{\nu,\tau}, \\ \rho = (j_1, j_2) \in Q_{2,m} \end{array} \right\}$$

or simply $\mathcal{B}_k(\omega_k) = \{\sigma_1, \sigma_2, ...\sigma_\pi : \pi = (\tau/\nu)\}$, for $\nu \in \widetilde{m}$. The elements of $\mathcal{A}(\omega_k)$, $\mathcal{B}_k(\omega_k)$ are lexicographically ordered.

(v) Given $\omega_k = (i_1, i_2, ..., i_k) \in \Omega_{k,m}$ and the set $\mathcal{A}(\omega_k)$ we denote by $\mathcal{B}_k[\omega_k]$ the subset of $\mathcal{B}_k(\omega_k)$ that excludes all $\rho = (j_1, j_2) \in \mathcal{A}(\omega_k)$ sequences.

(vi) Let

$$\begin{aligned} \mathcal{B}_k[\omega_k] &= \left\{ \begin{array}{l} \widehat{\sigma} = (\widehat{\rho}_{l_1}, \widehat{\rho}_{l_2}, ..., \widehat{\rho}_{l_\nu}) \in \mathcal{A}(\omega_k), \\ \widehat{\rho}_{l_\kappa}(j_1, j_2) \in Q_{2,m} \end{array} \right\} = \\ &= \{\widehat{\sigma_1}, \widehat{\sigma_2}, ..., \widehat{\sigma_{\pi'}}\} \end{aligned}$$

Every element $\widehat{\sigma}$ of $\mathcal{B}_k[\omega_k]$ may be represented as

$\widehat{\sigma} = (\widehat{\rho}_{l_1}, \widehat{\rho}_{l_2}, ..., \widehat{\rho}_{l_\nu}) = (j_{l_1}1, j_{l_1}2; j_{l_2}1, j_{l_2}2; ...; j_{l_\nu}1, j_{l_\nu}2)$

The $\widehat{\sigma}$ element will be called I, if there are no more than $(k - l)$ repeated indices from the $\omega_k = (i_1, i_2, ..., i_k) \in \Omega_{k,m}$ set; otherwise the element will be called *non-proper*. The subset of proper sequences of $\mathcal{B}_k[\omega_k]$ will be denoted by $\widehat{\mathcal{B}}_k[\omega_k]$.

(vii) For any $\omega_k = (i_1, i_2, ..., i_k) \in \Omega_{k,m}$ and a proper $\widehat{\sigma} = (\widehat{\rho}_{l_1}, \widehat{\rho}_{l_2}, ..., \widehat{\rho}_{l_\nu}) = (j_{l_1}1, j_{l_1}2; j_{l_2}1, j_{l_2}2; ...; j_{l_\nu}1, j_{l_\nu}2) \in \widehat{\mathcal{B}}_k[\omega_k]$

we define as

$$r(\widehat{\mathcal{B}}_k, \omega_k) = \sum\nolimits_{\widehat{\sigma}\in\widehat{\mathcal{B}}_k[\omega_k]} p_{j_{l_1 1} j_{l_1 2}} p_{j_{l_2 1} j_{l_2 2}} \cdots p_{j_{l_\nu 1} j_{l_\nu 2}}$$

We demonstrate the above definition by an example:

Example 6. Let $\widetilde{4} = \{1, 2, 3, 4\}$. Then for $\omega_4 = (1, 2, 3, 4)$ and $r(\omega_4) = p_{11}p_{22}p_{33}p_{44}$.

(i) If $\omega_3^a = (1, 2, 3)$, then $r(\omega_3^a) = p_{11}p_{22}p_{33}$ and $r(\widehat{\mathcal{B}}_3, \omega_3^a) = p_{14} + p_{24} + p_{34}$.

(ii) If $\omega_2^a = (1, 3)$, then $r(\omega_2^a) = p_{11}p_{33}$ and

$$\begin{aligned} r(\widehat{\mathcal{B}}_2, \omega_2^a) &= p_{12}p_{14} + p_{14}p_{23} + p_{12}p_{24} + p_{14}p_{24} + \\ &+ p_{23}p_{24} + p_{12}p_{34} + p_{23}p_{34} + p_{24}p_{34}. \end{aligned}$$

(iii) If $\omega_1^a = (1)$, or $\omega_1^\beta = (2)$, or $\omega_1^\gamma = (3)$, or $\omega_1^\delta = (4)$, then $r(\omega_1^a) = p_{11}$, $r(\omega_1^\beta) = p_{22}$, $r(\omega_1^\gamma) = p_{33}$, $r(\omega_1^\delta) = p_{44}$ and

$$\begin{aligned} & r(\widehat{\mathcal{B}}_1, \omega_1^a) = r(\widehat{\mathcal{B}}_1, \omega_1^\beta) = r(\widehat{\mathcal{B}}_1, \omega_1^\gamma) = r(\widehat{\mathcal{B}}_1, \omega_1^\delta) = \\ &= p_{12}p_{13}p_{14} + p_{12}p_{13}p_{24} + p_{12}p_{13}p_{34} + p_{12}p_{14}p_{23} + \\ &+ p_{12}p_{14}p_{34} + p_{12}p_{23}p_{24} + p_{12}p_{23}p_{34} + p_{12}p_{24}p_{34} + \\ &+ p_{13}p_{14}p_{23} + p_{13}p_{14}p_{24} + p_{13}p_{23}p_{24} + p_{13}p_{23}p_{34} + \\ &+ p_{13}p_{24}p_{34} + p_{14}p_{23}p_{24} + p_{14}p_{23}p_{34} + p_{14}p_{24}p_{34} \end{aligned}$$

Lemma 7. We may express $\det\{Z(s)\}$ as a positive sum of polynomials with positive coefficients as:

$$\begin{aligned} \det\{Z(s)\} &= p_{11}p_{22}...p_{mm} + \\ &+ \sum\nolimits_{\omega\in\Omega(m-1,m)} r(\omega) r(\widehat{\mathcal{B}}_{m-1}, \omega) + \\ &+ \sum\nolimits_{\omega\in\Omega(k,m)} r(\omega) r(\widehat{\mathcal{B}}_k, \omega) + ... + \\ &+ (p_{11} + p_{22} + ... + p_{mm}) r(\widehat{\mathcal{B}}_1, \omega) \end{aligned} \tag{12}$$

Lemma 8. Let $j \in \widetilde{m}$ and for a given $\omega_k = (i_1, i_2, ..., i_k) \in \Omega_{k,m}, j \notin \omega$. Then all $p_{ji}, i \neq j, i \in \widetilde{m}$ are terms in $r(\widehat{\mathcal{B}}_k, \omega)$.

We will now state the main theorem of this paper:

Theorem 9. The network is regular if and only if the network is connected, that is there is no node i (or respectively loop) with all $p_{ij} = 0, j \in \widetilde{m}$.

Example 10. Let $\widetilde{3} = \{1, 2, 3\}$. Then:

$$\begin{aligned} \det\{Z(s)\} &= p_{11}p_{22}p_{33} + p_{11}p_{22}(p_{13} + p_{23}) + \\ &+ p_{11}p_{33}(p_{12} + p_{23}) + p_{22}p_{33}(p_{12} + p_{13}) + \\ &+ (p_{11} + p_{22} + p_{33})(p_{12}p_{13} + p_{12}p_{23} + p_{13}p_{23}) \end{aligned}$$

which is a sum of polynomials with positive coefficients. Note that if $p_{11}=0$, $p_{22}\neq 0$, $p_{33}\neq 0$, then $p_{22}p_{33}(p_{12}+p_{13})=0$ and thus $p_{12}=0$, $p_{13}=0$ and this demonstrates the result.

Note that network connectivity is equivalent to that there is no j node for which all $p_{ji}=0, \forall i\in \widetilde{m}$. Similar statement may be given for the impedance analysis. The conditions for regularity of $Z(s)$, or $Y(s)$ may be expressed on the nodal network pencil $P(s)$ and this leads to an algebraic characterization and some interesting properties of the associate admittance topology (Karcanias et al. (2016)).

5. THE ZERO STRUCTURE OF AN RLC NETWORK

We investigate now the zero structure of a regular network by examining the zero structure of the matrix pencil $P(s)$. This is also linked to the computation of the McMillan degree of $W^{-1}(s)$ (Karcanias et al. (July 7-11, 2014)). From the structure of this pencil we have the result:

Proposition 11. Consider a regular network and let us denote by $\rho_C = rank(C)$ and by $\rho_L = rank(L)$. Then, the following properties hold true:

(i) The number of zero elementary divisors is $(m-\rho_L)$ and the number of infinite elementary divisors is $(m-\rho_C)$.
(ii) If r_f denotes the number of non-zero finite zeros of $P(s)$, or $Z(s)$ then: $r_f \leq \rho_C+\rho_L$, with equality holding when all zero and infinite elementary divisors are linear.

Improved conditions for the degree of r_f may be obtained by working on the conditions defining the existence of nonlinear infinite and finite elementary divisors, which are considered next.

Definition 12. (Karcanias and Kalogeropoulos (1986)) Let $sF-G\in \Re^{p\times p}[s]$ be a regular pencil. We define:

(i) The sequence of the ∞-*Toeplitz* and 0-*Toeplitz* matrices respectively:

$$Q_1^{\infty}=[F], Q_2^{\infty}=\begin{bmatrix} F & 0\\ -G & F\end{bmatrix}, ...,$$
$$Q_k^{\infty}=\begin{bmatrix} F & 0 & 0 & \cdots & 0 & 0\\ -G & F & 0 & \cdots & 0 & 0\\ 0 & -G & F & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & F & 0\\ 0 & 0 & 0 & \cdots & -G & F\end{bmatrix} \tag{13}$$

$$Q_1^{0}=[G], Q_2^{0}=\begin{bmatrix} G & 0\\ -F & G\end{bmatrix}, ...,$$
$$Q_k^{0}=\begin{bmatrix} G & 0 & 0 & \cdots & 0 & 0\\ -F & G & 0 & \cdots & 0 & 0\\ 0 & -F & G & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & G & 0\\ 0 & 0 & 0 & \cdots & -F & G\end{bmatrix} \tag{14}$$

and we shall denote by $\mathcal{L}^{\infty}=\{\eta_1^{\infty},\eta_2^{\infty},...,\eta_k^{\infty},...\}$, $\mathcal{L}^{0}=\{\eta_1^{0},\eta_2^{0},...,\eta_k^{0},...\}$ the nullities of the corresponding matrices $\mathcal{Q}^{\infty}=\{Q_1^{\infty},Q_2^{\infty},...,Q_k^{\infty},...\}$, $\mathcal{Q}^{0}=\{Q_1^{0},Q_2^{0},...,Q_k^{0},...\}$.

(ii) We denote by $\mathcal{S}^{\infty}=\{q_1^{\infty},q_2^{\infty},...,q_{\mu}^{\infty}\}$, $\mathcal{S}^{0}=\{q_1^{0},q_2^{0},...,q_{\nu}^{0}\}$ the set of integers defining the degrees of infinite and zero elementary divisors of the pencil, which is also referred as the *Segre Characteristic at infinity* and *Segre Characteristic at zero* respectively.

Using the previous definition we have the lemma:

Lemma 13. (Karcanias and Kalogeropoulos (1986)) Let $sF-G\in\Re^{p\times p}[s]$ be a regular pencil and let us denote by $\mathcal{S}^{\infty}=\{q_1^{\infty},q_2^{\infty},...,q_{\mu}^{\infty}\}$, $\mathcal{S}^{0}=\{q_1^{0},q_2^{0},...,q_{\nu}^{0}\}$ the *Segre Characteristic at infinity* and *Segre Characteristic at zero* respectively of the pencil. Then,

$$\begin{array}{l}\eta_k^{\infty}-\eta_{k-1}^{\infty}\geq \eta_{k+1}^{\infty}-\eta_k^{\infty}\text{or}\\ \eta_k^{\infty}\geq(\eta_{k-1}^{\infty}+\eta_{k+1}^{\infty})/2, k=1,2,...\end{array} \tag{15}$$

$$\begin{array}{l}\eta_k^{0}-\eta_{k-1}^{0}\geq \eta_{k+1}^{0}-\eta_k^{0}\text{or}\\ \eta_k^{0}\geq(\eta_{k-1}^{0}+\eta_{k+1}^{0})/2, k=1,2,...\end{array} \tag{16}$$

In particular:

(i) Strict inequality holds if and only if $k\in\mathcal{S}^{\infty}$ for equation 15 and respectively $k\in\mathcal{S}^{0}$ for equation 16.
(ii) Equality in equation 15 and in equation 16 holds if $k\notin\mathcal{S}^{\infty}$ and $k\notin\mathcal{S}^{0}$ respectively.

Corollary 14. Let $sF-G\in\Re^{p\times p}[s]$ be a regular pencil. Then,

(i) If $rank(F)=\rho_{\infty}<p$ and $\eta_k^{\infty}=(\eta_{k-1}^{\infty}+\eta_{k+1}^{\infty})/2, k=1,2,...,p$ then the pencil has only $p-\rho_{\infty}$ linear infinite elementary divisors.
(ii) If $rank(G)=\rho_0<p$ and $\eta_k^{0}=(\eta_{k-1}^{0}+\eta_{k+1}^{0})/2, k=1,2,...,p$ then the pencil has only $p-\rho_0$ linear zero elementary divisors.

The above results may now be used for the network pencil

$$\begin{array}{l}P(s)=s\begin{bmatrix} C & 0\\ 0 & I\end{bmatrix}+\begin{bmatrix} R & L\\ -I & 0\end{bmatrix}=\\ =\begin{bmatrix} sC+R & L\\ -I & sI\end{bmatrix}=sF+G\in\Re^{2m\times 2m}[s]\end{array} \tag{17}$$

Proposition 15. Consider a regular network and let $P(s)=sF+G\in R^{2m\times 2m}[s]$ be the corresponding network pencil. Then, the matrices Q_k^{∞}, Q_k^{0} defined by equations 13, 14 are equivalent over $\Re$ to the matrices:

$$\begin{array}{l}\mathcal{P}_k^{\infty}=\begin{bmatrix} C & 0 & 0 & \cdots & 0 & 0\\ R & C & 0 & \cdots & 0 & 0\\ L & R & C & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & \cdots & L & R & C & 0\\ 0 & \cdots & 0 & L & R & C\end{bmatrix}\\ \mathcal{P}_k^{0}=\begin{bmatrix} L & 0 & 0 & \cdots & 0 & 0\\ R & L & 0 & \cdots & 0 & 0\\ C & R & L & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & \cdots & C & R & L & 0\\ 0 & \cdots & 0 & C & R & L\end{bmatrix}\end{array}, k=1,2,\ldots \tag{18}$$

Using the above results we may now state the criteria that characterizes the exact value of the degree of the zero polynomial.

Theorem 16. Consider a regular network defined by $P(s)$, or $W(s)$ and let us denote by $\rho_C = rank(C)$, $\eta_C = m - rank(C)$ and by $\rho_L = rank(L)$, $\eta_L = m - rank(L)$. Furthermore, let us denote by $\widetilde{\mathcal{L}}^\infty = \{\eta_1^\infty, \eta_2^\infty, ..., \eta_k^\infty, ...\}, \eta_1^\infty = \eta_C$, $\widetilde{\mathcal{L}}^0 = \{\eta_1^0, \eta_2^0, ..., \eta_k^0, ...\}, \eta_1^0 = \eta_L$ the nullities of the corresponding matrices $\mathcal{P}^\infty = \{\mathcal{P}_1^\infty, \mathcal{P}_2^\infty, ..., \mathcal{P}_k^\infty, ...\}$, $\mathcal{P}^0 = \{\mathcal{P}_1^0, \mathcal{P}_2^0, ..., \mathcal{P}_k^0, ...\}$. Then, the following properties hold true:

(i) The number of zero elementary divisors is $m - \rho_L$ and the number of infinite elementary divisors is $m - \rho_C$.
(ii) If r_f is the number of non-zero finite zeros of $P(s)$, or $Z(s)$ then $r_f = \rho_C + \rho_L$, if and only if for all $k = 1, 2, ...m$ $\eta_k^\infty = k\eta_C$ and $\eta_k^0 = k\eta_L$.

6. CONCLUSION

The paper develops some of the fundamental properties of the impedance-admittance implicit model description of an RLC network related to new natural impedance-admittance operator $W(s) = sB + s^{-1}E + D$ which provides a unifying description of the network by examining issues related to its linearization (derivation of a state space, matrix pencil description), characterizing the property of network regularity and determining the number of zeros of the network. The paradigm of passive electric networks (or analogs) has been used as the simple case that allows the investigation of the system structure evolution linked to changes in topology, nature and values of the physical elements and possibly changes in their cardinality of them. There is number of challenges that can be addressed using such representations. Amongst them are issues related to the development of methodology for assignment of natural frequencies by different types of cardinality preserving transformations and examining issues of structural invariants under cardinality changing transformations. Working directly with $W(s)$ or with the matrix pencil linearization $sF + G$ we can deploy the DAP algebro-geometric framework (Karcanias and Giannakopoulos (1984)), (Leventides and Karcanias (2009)) for the study of frequency assignment by network re-engineering. Furthermore, applications of the re-engineering framework for RLC networks may be developed in applications of passive network design such as, car suspension, damping of oscillations and earthquake protection of buildings (Marian and Giaralis (2014)). For instance, in the case of design of mass- dampers the design of the appropriate value of the mass or inerter (Smith (2002); Marian and Giaralis (2014)) may be achieved as a tuning of the value of the mass for damping oscillations.

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