When Micro Prudence increases Macro Risk: The Destabilizing Effects of Financial Innovation, Leverage, and Diversification

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Abstract

By exploiting basic common practice accounting and risk management rules, we propose a simple analytical dynamical model to investigate the effects of micro-prudential changes on macro-prudential outcomes. Specifically, we study the consequence of the introduction of a financial innovation that allow reducing the cost of portfolio diversification in a financial system populated by financial institutions having capital requirements in the form of VaR constraint and following standard mark-to-market and risk management rules. We provide a full analytical quantification of the multivariate feedback effects between investment prices and bank behavior induced by portfolio rebalancing in presence of asset illiquidity and show how changes in the constraints of the bank portfolio optimization endogenously drive the dynamics of the balance sheet aggregate of financial institutions and, thereby, the availability of bank liquidity to the economic system and systemic risk. The model shows that when financial innovation reduces the cost of diversification below a given threshold, the strength (due to higher leverage) and coordination (due to similarity of bank portfolios) of feedback effects increase, triggering a transition from a stationary dynamics of price returns to a non stationary one characterized by steep growths (bubbles) and plunges (bursts) of market prices.

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1 Introduction

In most standard economic models, financial institutions are viewed as passive players and credit does not have any macroeconomic effect. Yet, a growing body of empirical literature consistently finds that an acceleration of credit growth is the single best predictor of future financial instability (see Gourinchas et al., 2001; Mendoza and Terrones, 2008; Borio and Drehmann, 2009; Reinhart and Rogoff, 2009, 2011; Schularick and Taylor, 2012). These empirical results confirm that the balance sheet dynamics of financial intermediaries, far from being passive and exogenous, is instead the "endogenous engine" that drives the boom-bust cycles of funding and liquidity and hence the dynamics of systemic risk. As stated by Adrian and Shin (2010): "balance sheet aggregates such as total assets and leverage are the relevant financial intermediary variables to incorporate into macroeconomic analysis". In fact, a change in the total assets of the financial institutions has important consequences in driving the financial cycles through their influence on asset pricing, the availability of credit, and funding of real activities. In this way changes in the total asset and leverage of financial intermediaries play a key role in determining the level of real activity. However, while the proximate cause for crises is very often an expansion of the balance sheets of financial intermediaries the ultimate causes for these dynamics remain unclear.

In this paper, by exploiting basic common practice accounting and risk management rules, we propose a simple analytical dynamical framework to investigate the effects of micro-prudential changes on macro-prudential outcomes. Specifically, we study the consequence of the introduction of a financial innovation that allows reducing the cost of portfolio diversification in a financial system populated by financial institutions having capital requirements in the form of VaR constraint and following standard mark-to-market and risk management rules. We provide a fully analytical description of the dynamics of the multivariate feedback induced by portfolio rebalancing and trasmitted over the bipartite network of investment prices and bank assets. We show quantitatively how changes in the constraints of the bank portfolio optimization endogenously drive the dynamics of assets prices and that of the balance sheets of financial institutions and, thereby, the availability of bank liquidity to the economic system and systemic risk.

In building our model we try to keep behavioral assumptions at minimum, exploiting instead the implications of "objective" constraints imposed by regulatory institutions and standard market practice. We then start from a simple portfolio optimization problem in presence of cost of diversification and VaR constraint[1] showing how a reduction in the costs of diversification (due, for instance, to financial innovations such as securization) leads to an increase in both leverage and diversification.

So a first result is that financial innovation which, by increasing the optimal level of diversification, reduces idiosyncratic risks, actually increases the exposure to undiversifiable macro risks by increasing the optimal leverage of a VaR constrained investor. Moreover, a higher level of diversification, by increasing the overlap among bank portfolios, increases the correlation among them. Thus, the combined increase in risk exposure and correlation of financial institutions will expose the economy to higher level of systemic risk.

We then link these results to the literature on the portfolio rebalancing induced by the mark-to-market accounting rules and VaR constraint (see for instance Adrian and Shin (2010); Greenwood et al. (2012); Duarte and Eisenbach (2013); Adrian and Shin (2014)). In this balance sheet models an increase in the value of the assets, increases the amount of equity leading to surplus of capital with respect to the VaR requirements which is adjusted by expanding the asset side through borrowing i.e. by raising new debt (typically done with repos contracts). Hence, VaR capital requirements, induce a perverse demand function: financial institution will buy more assets if their price rises and (with an analogous mechanism but with reversed sign) sell more assets when their price falls. Therefore, a VaR constrained financial institution will have positive feedback effect on the prices of the assets in his portfolio.[2] The intensity and coordination (among financial institutions) of these portfolio rebalancing feedbacks will depend, respectively, on the degree of leverage and diversification.

In the second part of the paper, we then analyze the multivariate dynamics of the endoge-

[1]Note that VaR type of constraints arise from the capital requirements contained in Basel I and II bank regulations but also from margin on collateralized borrowing imposed by creditors (see Brunnermeier and Pedersen 2008), rating agencies, and internal risk management models.

[2]This type of active balance sheet management is particularly utilized by investment banks, ABS issuers, security broker-dealers, i.e. by the so called market-based financial intermediaries or shadow banking system.

nous asset price determined by the impact of supply and demand generated by the financial institutions rebalancing their portfolio. The analytical results obtained in this second part of the paper by applying the multivariate dynamic framework are manifolds: (i) higher overlap induced by lower diversification costs increases both the variance and correlation of the investment demands of FIs rebalancing their portfolios; (ii) the feedback between investment prices and bank asset induced by the multi-round portafoglio rebalances of VaR constrained banks, leads to a multivariate VAR process whose maximum eigenvalue depends on the degree of leverage and average illiquidity of the assets; (iii) lower level of diversification costs or capital requirements can lead to dynamic instability of the system; (iv) the VAR process can be represented as a combination of many idiosyncratic AR processes around a single common AR process of the average values (i.e. the market); (v) the endogenous feedback induced by portfolio rebalancing introduces an additional component to the variance, covariance, and correlation of both the individual investment and the bank portfolios for which we derive closed–form expressions; (vi) reduction in diversification costs monotonically increase variance and correlation of individual investments thus acting as a “multiplier” of market risk; (vii) the variance of portfolios shows, however, a non-monotonic relation with respect to diversification costs; (viii) the endogenous feedback makes historical estimation of variance covariance to be overestimated during periods of increasing leverage and underestimated during periods of deleveraging thus providing a rationale for countercyclical capital requirements; (ix) in presence of endogenous feedbacks, an exogenous shock will trigger a sequence of portfolio rebalances which will amplify its initial impact; (x) reduction in diversification costs, by increasing the strength and coordination of individual feedbacks, increases the variability of bank total asset, which governs the supply of credit and liquidity to financial system

1.1 Related literature

In addition to the literature on the portfolio rebalancing induced by the mark-to-market accounting rules and VaR constraint (Adrian and Shin, 2010; Greenwood et al., 2012; Duarte and Eisenbach, 2013; Adrian and Shin, 2014) already mentioned, our paper tries to combine several strands of literature: (i) the one on the impact of the imposition of capital requirements on the behavior of financial institutions and their possible procyclical effects (Danıelsson et al., 2004; Danielsson et al., 2009; Adrian and Shin, 2009; Adrian et al., 2011; Adrian and Boyarchenko, 2012; Tasca and Battiston, 2012); (ii) the literature on distressed selling and its impact on the market price dynamics (Shleifer and Vishny, 1992; Kyle and Xiong, 2001; Shleifer and Vishny, 2011; Cont and Wagalath, 2011; Thurner et al., 2012; Cont and Wagalath, 2012; Caccioli et al., 2012); in particular, it extends the theoretical models underpinning the systemic risk measure that quantifies the vulnerability of the financial system to fire-sale spillover (Greenwood et al., 2012; Duarte and Eisenbach, 2013) to a dynamic multi-round liquidity spillover framework; (iii) the literature on the effects of diversification and overlapping portfolios on systemic risk (Wagner, 2011; Tasca and Battiston, 2011; Caccioli et al., 2012; Lillo and Pirino, 2015); Differently from the paper of Wagner (2011) which also proposes a model where higher level of diversification might increase aggregate risk, we identify a different mechanism for this effect. In addition to the synchronization of portfolio rebalancing among banks (as in Wagner 2011), we also consider the impact of diversification costs on bank leverage and through that on the intensity of those portfolio rebalancing and provide a fully analytical description of the resulting time series dynamics of assets prices and bank total assets. (iv) the literature on the risks of financial innovation (Brock et al., 2009; Caccioli et al., 2009; Haldane and May, 2011); (v) the literature on the determinants of the dynamics of balance sheet aggregates and credit supply of financial institutions (Stein 1998, Bernanke and Gertler 1989, Bernanke, Gertler and Gilchrist 1996, 1999 and Kiyotaki and Moore 1997).

Our contribution is to propose a simple model that, by combining these different streams of literature, provides a fully analytical quantification of the links between micro prudential rules and macro prudential outcomes in a multivariate context which considers both the presence of endogenous feedback caused by portfolio rebalancing and the impact of financial innovations on the cost of diversification.

The paper is organized as follows. Section 2 presents the model set up and the analytical results by first describing the portfolio decision problem of financial institutions facing VaR constraints and diversification costs and then analyzing its macroeconomic consequences in the dynamic case which considers the impact of investor demands on the asset dynamics.

Section 3 analyzes the systemic risk implications of our model both a static setting without feedback and in a dynamic setting with the endogenous feedback generated by portfolio rebalancing. Based on those analytical results, Section 4 discusses the macro-prudential consequences of the introduction of financial innovations reducing diversification costs. Section 5 summarizes and concludes.

2 The model

2.1 Portfolio decisions

We begin by considering a financial institution endowed with a given amount of initial equity capital E and we model its portfolio selection across a collection of risky investments $i = 1, ..., M$. In general, these might be individual investments or asset classes. In order to keep the subsequent dynamic model fully analytical, we assume that, from the point of view of the financial institutions, all the risky investments are ex-ante statistically equivalent. As a consequence, financial institutions adopt a simple investment strategy consisting in forming an equally weighted portfolio[3] by randomly selecting m risky investments from the whole collection of M available investment assets.

Financial institutions, correctly perceive that risky investment entails both an idiosyncratic (diversifiable) risk component and a systematic (undiversifiable) risk component, i.e. the expected variance[4] of the risky investment i, σ_i^2, can be decomposed as $\sigma_i^2 = \sigma_s^2 + \sigma_d^2$ where σ_s^2 is the perceived systematic risk and σ_d^2 is the perceived diversifiable risk component. Hence, the expected mean and volatility per dollar invested in the portfolio chosen by a given institution are μ and $\sigma_p = \sqrt{\sigma_s^2 + \frac{\sigma_d^2}{m}}$, respectively.

Because of the presence of transaction costs, firms specialization and other type of fric-

[3]Theoretical and empirical advantages of the naive equally weighted strategy are provided in Benartzi and Thaler (2001); Pflug et al. (2012); DeMiguel et al. (2009); Tu and Zhou (2011). Equally weighted portfolios are also popular among practitioners as they are robust to specification errors in the dynamics of individual asset and provides performance in line (if no better) than those from more sophisticated Markowitz-type approaches.

[4]Which, in general, might be different from the realized one since we remain agnostic on the process of expectation formation of the financial institutions.

tions, we assume the existence of "costs of diversification" (see Constantinides, 1986) which, in general, can prevent each institution to achieve full diversification of its portfolio (precisely the existence of these costs in real markets spurred the developments of financial innovation products as we will discuss in the next sections).

Let r_L be the per dollar average interest expense on the liability side, then the Net Interest Margin (NIM) of the financial institution is $\mu - r_L$. The NIM is therefore a measure of the overall profitability of a financial institution.

In line with the recent theoretical and empirical literature on bank behavior (Brunnermeier and Pedersen, 2009; Adrian and Shin, 2010, 2014), financial institutions are confronted with a Value at Risk (VaR) type of constraints. The VaR constraint is typically computed as some multiple of the standard deviation of the portfolio of assets A.

With σ_p the expected holding period volatility per dollar of asset A and α a scaling constant, the VaR constraint faced by the financial institution is

$$VaR = \alpha \sigma_p A \leq E. \tag{1}$$

As empirically shown by Adrian and Shin (2010) financial institutions adjust their asset side rather than raising or redistributing equity capital. In agreement with these empirical observations, we will consider the equity capital of the financial institutions to be fixed. Notice that this does not prevent the value of equity to change over time as in fact happens as a consequence of the bank profits and losses. It only assumes that, in managing their VaR capital requirements, financial institutions prefer buying and selling activities in their asset sides rather than rising new equity or redistributing the one in excess.

Summarizing, given their NIM and level of equity E, financial institutions, facing cost of diversification and VaR constraints, choose the level of total asset A and degree of diversification m which maximize their returns from the risky investments. That is, assuming cost of diversification proportional to m, financial institutions maximize

$$\max_{A,m} \quad A(\mu - r_L) - \tilde{c}m \qquad \text{s.t.} \quad \alpha A \sqrt{\sigma_s^2 + \frac{\sigma_d^2}{m}} \leq E. \tag{2}$$

where $\tilde{c}$ is the cost for investment (assumed to be the same across all investments). Dividing by E and defining $c = \tilde{c}/E$, we can express the maximization problem in terms of the leverage

$\lambda = \frac{A}{E}$,

$$\max_{\lambda,m} \quad \lambda(\mu - r_L) - cm \qquad \text{s.t.} \quad \alpha\lambda\sqrt{\sigma_s^2 + \frac{\sigma_d^2}{m}} \leq 1. \tag{3}$$

Hence, each institution chooses the optimal leverage $\lambda^* = A^*/E$ and the optimal number of investments m^* which maximizes its Return On Equity (ROE) under its VaR constraints. It is convenient to transform the constraint by squaring both sides so that the Lagrangian can be written as

$$L = \lambda(\mu - r_L) - cm - \frac{1}{2}\gamma\left(\alpha^2\lambda^2\left(\sigma_s^2 + \frac{\sigma_d^2}{m}\right) - 1\right). \tag{4}$$

where γ is the Lagrange multiplier for the VaR constraint. The first order condition for λ is

$$(\mu - r_L) - \gamma\alpha^2\sigma_p^2\lambda = 0 \qquad \Rightarrow \qquad \lambda = \frac{1}{\gamma}\,\frac{1}{\alpha^2}\,\frac{\mu - r_L}{\left(\sigma_s^2 + \frac{\sigma_d^2}{m}\right)} \tag{5}$$

Substituting in the constraint we obtain the Lagrange multiplier or shadow price of the VaR constraint γ

$$\gamma = \frac{1}{\alpha}\frac{\mu - r_L}{\sqrt{\sigma_s^2 + \frac{\sigma_d^2}{m}}} = \frac{1}{\alpha}\frac{\mu - r_L}{\sigma_p} \tag{6}$$

which is proportional to the Sharpe ratio. The optimal number of investments m^* is then,

$$m^* = \frac{\sqrt{\gamma}\alpha\lambda\sigma_d}{\sqrt{2c}} = \lambda\sigma_d\sqrt{\frac{\alpha}{2c}\frac{\mu - r_L}{\sigma_p}} \tag{7}$$

which shows that, as expected, the level of diversification chosen is inversely related to the cost of diversification c. For the leverage we have,

$$\lambda^* = \frac{1}{\alpha\sqrt{\sigma_s^2 + \frac{\sigma_d^2}{m}}} = \frac{1}{\alpha\sigma_p} \tag{8}$$

thus, the optimal leverage is inversely related to the volatility of the asset portfolio. In the following, we will drop the star symbol on the optimal values for notational convenience, i.e. we will denote the target leverage λ^* and diversification m^* simply as λ and m, respectively.

Figure 1 reports the numerical solutions for the optimal leverage as a function of different levels of diversification costs (and for a given choice of the set of the remaining parameters in the model). Each line corresponds to different levels of systematic to idiosyncratic noise

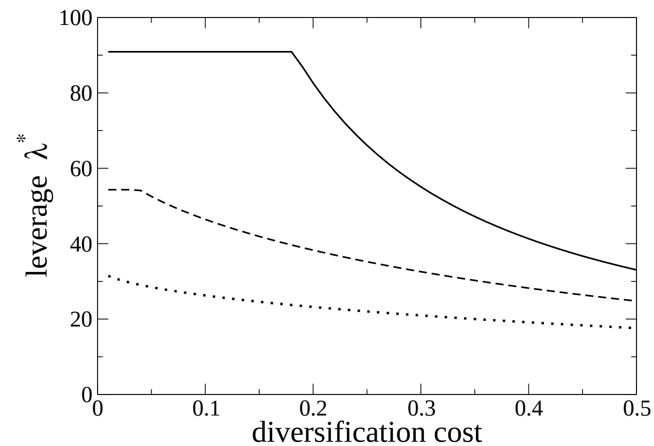

Figure 1: Relation between the optimal leverage λ^* and the diversification cost c, obtained by solving numerically Eq.s (7) and (8). The used parameters are: $M = 20$, $\alpha = 1.64$ (corresponding to a 5% VaR in a Gaussian setting), $\mu - r_L = 0.08$, $\sigma_d = 0.03$. We then choose σ_s equal to 0 (solid line), 0.009 (dashed line), and 0.018 (dotted line).

ratio ($\sigma_s/\sigma_d = \{0,\ 0.3,\ 0.6\}$). A reduction of diversification costs, by increasing the level of diversification and hence relaxing the VaR constraint, allows the financial institution to increase the optimal leverage, especially for lower level of the systematic to idiosyncratic noise ratio. Note that below a given cost the optimal leverage becomes constant due to the saturation of diversification reached when the portfolio becomes perfectly diversified across all the M available investments. The sensitivity of the optimal leverage to diversification costs is higher for lower systematic to idiosyncratic noise ratios.

2.2 Overlapping portfolios

We now assume that our economy is composed by a group of N financial institutions labeled with $j = 1, ..., N$ and investing in the M risky investments as described above. The portfolio holdings of the N banks can be represented by using a bipartite graph, where the first set of nodes is composed by the N banks and the second set of nodes is composed by the M risky investments; i.e. each bank j investing in the investment asset i can be represented by a link in the bipartite network connecting the bank node j with the investment node i (see

Duarte and Eisenbach 2013 and Di Gangi et al. 2015 for the properties of such network in the US banking system).

As in the standard CAPM framework, we make the conventional assumption that banks have homogeneous expectation over the investments assets and thus each bank solves a similar optimization problem identifying the same optimal degree of diversification m. However, we will assume that each bank chooses randomly and independently the investments across the set of the M available ones, so that the selected portfolios will be different for different banks. A realization of portfolio choices of all the banks leads to a specific instance of the bipartite graph characterized by a $N \times M$ matrix of portfolio weights $\mathbf{W}$. In the following we will consider average values for these realizations of the bipartite graph configurations.

The number of banks n having a specific risky investment in their portfolio is a random variable described by the binomial distribution

$$P(n; N, M, m) = \binom{N}{n} \left(\frac{m}{M}\right)^{n} \left(1 - \frac{m}{M}\right)^{N-n} \tag{9}$$

whose mean value is clearly $E[n] = mN/M$.

Taken two banks, we can define the overlap o of their portfolios as the number of risky investments in common in the two portfolios. Also o is a random variable and it is distributed as an hypergeometric distribution

$$P(o; M, m) = \frac{\binom{m}{o}\binom{M-m}{m-o}}{\binom{M}{m}} \qquad 0 \leq o \leq m. \tag{10}$$

Its mean value is $E[o] = m^2/M$ and its variance is $V[o] = m(M-m)^2/(M^2 - M)$. Finally, the fractional overlap of two portfolios $o_f = o/m$ is a number between 0 and 1 describing which fraction of the portfolio is in common between the two banks. Clearly, the mean fractional overlap is $\bar{o} \equiv E[o_f] = m/M$, therefore the value of the portfolio size m is also a measure of the average fractional overlap $\bar{o}$ between portfolios and viceversa.

The left panel of Figure 2 shows the numerical solutions of the fractional overlap, coming from the optimal portfolio decision, as a function of different levels of diversification costs (again, each line corresponds to different levels of systematic to idiosyncratic noise ratio, $\sigma_s/\sigma_d = \{0,\ 0.3,\ 0.6\}$). The figure shows how reducing the costs of diversification, by the introduction of some new form of financial products for example, increases the degree of overlap and hence correlation, between the portfolio of financial institutions.

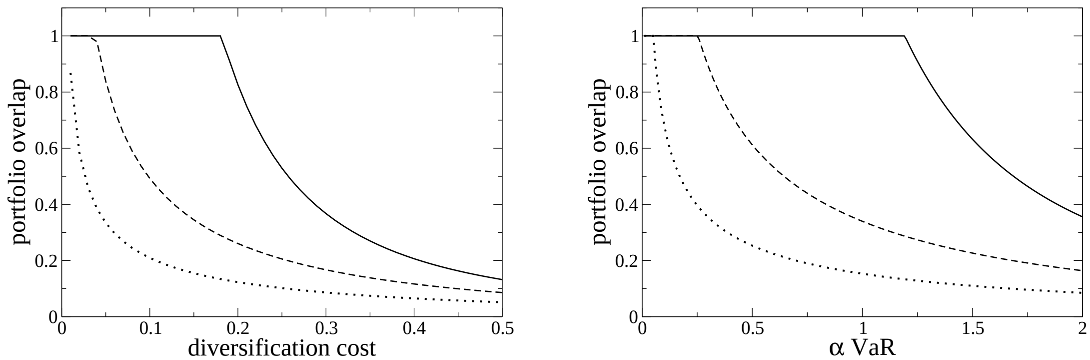

Figure 2: The left panel shows the mean fractional overlap $\bar{o}$ between two portfolios versus the diversification cost c and the right panel shows $\bar{o}$ versus the α parameter of the VaR constraint. (see Fig. 1 for the parameters). In the the right panel we set $c = 0.25$.

The fractional overlap resulting from the portfolio choices of financial institutions, can also be represented as a function of the tightness of the imposed capital requirements. This relation, depicted in the right panel of Figure 2, implies that regulator could tune the required capital ratio α so to reach a given level of overlap, and hence correlation, among financial institutions.

2.3 Asset demand from portfolio rebalancing

Having identified their optimal leverage, financial institution periodically rebalance their portfolios in order to maintain the desired target leverage. The rebalancing of the portfolio of individual bank j at time t, is given by the difference between the desired amount of asset $A^*_{j,t} = \lambda E_{j,t}$ and the actual one $A_{j,t}$,[5] i.e. $\Delta R_{j,t} \equiv A^*_{j,t} - A_{j,t}$. By defining the realized return portfolio $r^p_{j,t}$, $\Delta R_{j,t}$ can be written as (see Appendix A)

$$\Delta R_{j,t} = (\lambda - 1) A^*_{j,t-1} r^p_{j,t}, \tag{11}$$

that is, any profit or loss from investments in the chosen portfolio ($r^p_{j,t} A^*_{j,t-1}$) will directly result in a change in the asset value amplified by the current degree of leverage (being

[5]As clearly shown by Adrian and Shin (2010), the balance sheet adjustments are typically performed by expanding or contracting the asset side rather than the level of equity.

$\lambda > 1$). Hence, a VaR constrained financial institution will have a positive feedback effect on the prices of the assets in his portfolio.

The total demand of the risky investment i at time t will be simply the sum of the individual demand of the financial institutions who picked investment i in their portfolio.

Being more convenient to work with matrices and vectors, let us define $\mathbf{R}_t$ the $M \times 1$ vector of investment returns and $\mathbf{Q}_{t-1} = diag[(\lambda - 1)A^*_{j,t-1}]$ a $N \times N$ diagonal matrix. Finally, let us consider $\mathbf{W}$ the $N \times M$ matrix of portfolio weights characterizing the banks-investments bipartite network. Then, the $M \times 1$ vector of demand $\mathbf{D}_t$ is

$$\mathbf{D}_t = \mathbf{W}'\mathbf{Q}_{t-1}\mathbf{W}\mathbf{R}_t \tag{12}$$

From this expression the linear character of the relation between the demand of each investment and the return of all other investments is evident.

2.4 Risky asset dynamics with endogenous feedbacks

In this section we study the dynamics of the model in the case where the return of the risky investments are endogenously influenced by the former period demands coming from the portfolio rebalancing of financial institutions. In presence of rebalancing feedbacks, the return process will now be made of two components:

$$r_{i,t} = e_{i,t-1} + \varepsilon_{i,t} \tag{13}$$

the exogenous component $\varepsilon_{i,t}$ coming from the external shocks and the endogenous component $e_{i,t-1}$ coming from the previous period portfolio rebalancing of the financial institutions.

We assume that the exogenous component has a multivariate factor structure

$$\varepsilon_{i,t} = f_t + \epsilon_{i,t}, \tag{14}$$

with the factor f_t and the idiosyncratic noise $\epsilon_{i,t}$ uncorrelated and distributed with mean zero and constant volatility, respectively σ_f and σ_ϵ (the same for all investments). Thus, the variance of the exogenous component of the risky investment i is $V(\varepsilon_i) = \sigma_f^2 + \sigma_\epsilon^2$.

Assuming a standard linear price impact function, the endogenous component of the return of investment i at time t becomes[6]

$$e_{i,t} = \frac{D_{i,t}}{\gamma_i C_{i,t}} \tag{15}$$

where $C_{i,t} = \sum_{j=1}^{N} I_{\{i\in j\}} \frac{A^*_{j,t-1}}{m}$ is a proxy for market capitalization of investment i, and γ_i is a parameter expressing the market liquidity of the investment i.

Given the homogeneity of investments, we can assume that all have the same market capitalization which, since on average there are Nm/M banks investing in i, is equal to

$$C_{i,t} \simeq C_t = \frac{N}{M}\bar{A}^*_{t-1} \tag{16}$$

where $\bar{A}^*_{t-1} \equiv N^{-1}\sum_{j=1}^{N} A^*_{j,t-1}$ is the average bank asset size (assumed to exist).

Substituting Equations (12), (13), and (16) in (15) and using matrix notation we obtain the following Vector Autoregressive (VAR) dynamics of the vector of the endogenous components

$$\boldsymbol{e_t} = \boldsymbol{\Phi}\, \boldsymbol{r_t} = \boldsymbol{\Phi}\left(\boldsymbol{e_{t-1}} + \boldsymbol{\varepsilon_t}\right) \tag{17}$$

with

$$\boldsymbol{\Phi} \equiv \frac{M}{N\bar{A}^*_{t-1}}\boldsymbol{\Gamma}^{-1}\mathbf{W}'\mathbf{Q}_{t-1}\mathbf{W} = \frac{M}{N}\boldsymbol{\Gamma}^{-1}\mathbf{W}'\tilde{\mathbf{Q}}\mathbf{W} \tag{18}$$

where $\boldsymbol{\Gamma}$ is a $M \times M$ diagonal matrix with diagonal elements $\gamma_{i,i}$ (the market liquidity of investment i) and $\tilde{\mathbf{Q}} = \mathbf{Q}_{t-1}/\bar{A}^*_{t-1}$ a $N \times N$ diagonal matrix with diagonal elements $\tilde{Q}_{jj} = (\lambda_j - 1)A^*_{j,t-1}/\bar{A}^*_{t-1}$. The matrix $\tilde{\mathbf{Q}}$ is assumed to be independent from t, since the leverage is fixed and the fraction of total asset of a bank is assumed not to change in the investigated period[7].

2.4.1 Random matrix approach

In order to proceed with the computation of the dynamical properties of returns, we take expectations over the ensemble of the random matrices $\mathbf{W}$ and study the model determined

[6]A stochastic component coming from the exogenous demands of traders not actively rebalancing their portfolio could be added at the cost of complicating the subsequent computations.

[7]These assumptions on the constancy of the matrix $\boldsymbol{\Gamma}$ and $\tilde{\mathbf{Q}}$ are invoked in order to obtain a standard VAR(1) with constant autoregressive matrix. They could be relaxed at the price of obtaining a dynamic VAR(1) with time varying autoregressive matrices.

by the expectation of the matrix $\mathbf{B} \equiv \mathbf{W}'\tilde{\mathbf{Q}}\mathbf{W}$. Depending on the quantity of interest, this approximation is more or less reliable and we later use numerical simulations to investigate this point.

In order to have analytical tractability of the problem, from now on we assume that the investment selection process is a series of M independent Bernoullian draws each with probability $\frac{m}{M}$. The parameter m represents the average degree of diversification of portfolios[8]. In other words the number of investments of each bank is a Binomial variable with mean m and, to keep the model general, we also assume that each bank has a leverage λ_j (possibly related to the outcome of the Binomial).

Under these assumptions $\mathbf{W}$ is then a random matrix where each entries is an independent Bernoullian random variable $X_{j,i}$ with probability m/M "normalized" by the sum $s_j = \sum_i X_{j,i}$, i.e. each generic element of the matrix $\mathbf{W}$ is $W_{j,i} = \frac{X_{j,i}}{s_j}$. Clearly, $\sum_i W_{j,i} = 1$.

The generic element of $\mathbf{B}$ is

$$B_{ij} = \sum_{k=1}^{N} \tilde{Q}_{k,k} W_{k,i} W_{k,j}. \tag{19}$$

Being able to compute (see Appendix B)

$$E[W_{k,i}^2] \simeq \frac{1}{mM} \qquad\qquad E[W_{k,i}, W_{k,j}] \simeq \frac{1}{M^2 c}. \tag{20}$$

We have,

$$E[B_{ii}] \simeq \frac{1}{mM} \sum_k \tilde{Q}_{kk} \qquad\qquad E[B_{ij}] \simeq \frac{1}{M^2 c} \sum_k \tilde{Q}_{kk}, \quad i \neq j, \tag{21}$$

where $c \equiv 1 + \frac{1}{m} - \frac{1}{M}$.

In conclusion, the average matrix $\bar{\mathbf{\Phi}} \equiv E[\mathbf{\Phi}]$ of the VAR(1) is

$$\bar{\mathbf{\Phi}} \simeq (\bar{\lambda} - 1)\mathbf{\Gamma}^{-1}\mathbf{\Psi} \qquad \text{with} \qquad \mathbf{\Psi} = \begin{bmatrix} \frac{1}{m} & \frac{1}{Mc} & \cdots & \frac{1}{Mc} \\ \frac{1}{Mc} & \frac{1}{m} & \cdots & \frac{1}{Mc} \\ \vdots & & \ddots & \vdots \\ \frac{1}{Mc} & \frac{1}{Mc} & \cdots & \frac{1}{m} \end{bmatrix}. \tag{22}$$

[8]The choice of treating the diversification as a random variable simplifies the analytical computations. A model with fixed m can be developed analytically in a simplified and symmetric setting or, via numerical simulations, in a general setting. It is possible to show that the conclusions of the papers do not depend on this choice and for this reason we choose to treat the number of risky investment as a random variable.

where

$$\bar{\lambda} = \frac{\sum_{j=1}^{N} \lambda_j A^*_{j,t-1}}{\sum_{j=1}^{N} A^*_{j,t-1}} \tag{23}$$

is the asset weighted average leverage of the financial system. Notice that if all the banks have the same leverage, the matrix Φ is independent from the bank asset size distribution (provided the mean exists).

The dynamics of such VAR(1) process is determined by the eigenvalues of the matrix $\bar{\mathbf{\Phi}}$. The maximum eigenvalue of $\bar{\mathbf{\Phi}}$, dictating the dynamics of the VAR(1) process, becomes (see Appendix C)

$$\Lambda_{max} \simeq (\bar{\lambda} - 1)\overline{\gamma^{-1}} \tag{24}$$

where $\overline{\gamma^{-1}}$ is the average of all the γ_i^{-1}. Hence, the maximum eigenvalue depends on the degree of leverage and on the average illiquidity of the investments.

When the maximum eigenvalue is greater than one, the return processes become non-stationary and explosively accelerating. It is important to remark that even a reduction in the liquidity of only one risky investment (by changing the average illiquidity of the investments) impacts the dynamics of all the traded investments and can potentially drive the whole financial system towards instability. In fact, depending on the average of the $\frac{1}{\gamma_i}$, the maximum eigenvalue (and thus the dynamical properties of the whole system) will be highly sensitive to illiquid investments, i.e. to investment having a small γ.

Figure 3 shows the maximum eigenvalue $\Lambda_{\max}$ as a function of the diversification cost c (left panel) and as a function of the mean portfolio overlap $\bar{o}$ (right panel). We notice that a reduction of the diversification cost tends to reinforce the feedback induced by portfolio rebalancing which can lead to dynamic instability of the system (for $\Lambda_{\max} > 1$) when the diversification costs decrease below a certain threshold (which is higher for smaller ratio of systematic to idiosyncratic volatility). Analogously, we can analyze the dependence of the maximum eigenvalue of the dynamical system from the degree of portfolio overlap among the financial institutions. A higher level of coordination in portfolio rebalancing, due to similarities in the portfolio compositions, also reinforces the aggregate feedback between market prices and balance sheet values pushing the system toward the region of instability $\Lambda_{\max} > 1$. Note however, that the transition to a non stationary process is not achieved

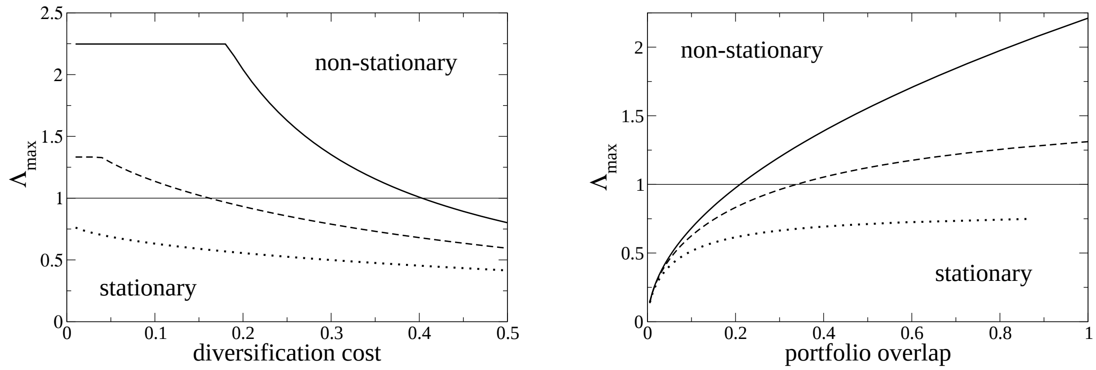

Figure 3: The left panel shows the maximum eigenvalue $\Lambda_{\max}$ as a function of the diversification cost, while the right panel shows $\Lambda_{\max}$ as a function of the mean fractional overlap $\bar{o}$ between two portfolios (see Fig. 1 for the parameters). We set $\gamma = 40$. The horizontal solid line shows the condition $\Lambda_{\max} = 1$, therefore the return dynamics is stationary below this line and non stationary above it.

when the portfolio overlap is equal to one, but, depending on the other parameters, also a moderate value of the portfolio overlap can lead to market instability.

Λ_{max} is the maximum eigenvalue of the average matrix $\bar{\mathbf{\Phi}}$, while the maximum eigenvalue of $\mathbf{\Phi}$ is a random variable depending on N and the bank size distribution. It is known (see Boyd and Vandenberghe (2004)) that for a symmetric real matrix the maximum eigenvalue is a convex function. Because of the Jensen inequality, the expectation of the maximum eigenvalue over the random matrix ensemble is larger than or equal to the maximum eigenvalue of the mean matrix $\bar{\mathbf{\Phi}}$. Therefore, the derived value of Λ_{max} is a lower bound of the average maximum eigenvalue and when $\Lambda_{max} > 1$, indicating a non stationary dynamics, also the average maximum eigenvalue will be larger than one. Morever, since the Jensen correction is function of the variance of the random variable which in turn is inversely related to N in our case, we expect that this Jensen correction will decrease when the number of banks N increases. In the next section we investigate numerically how this difference between Λ_{max} and the average maximum eigenvalue depends on the heterogeneity of bank asset size and on the number of banks.

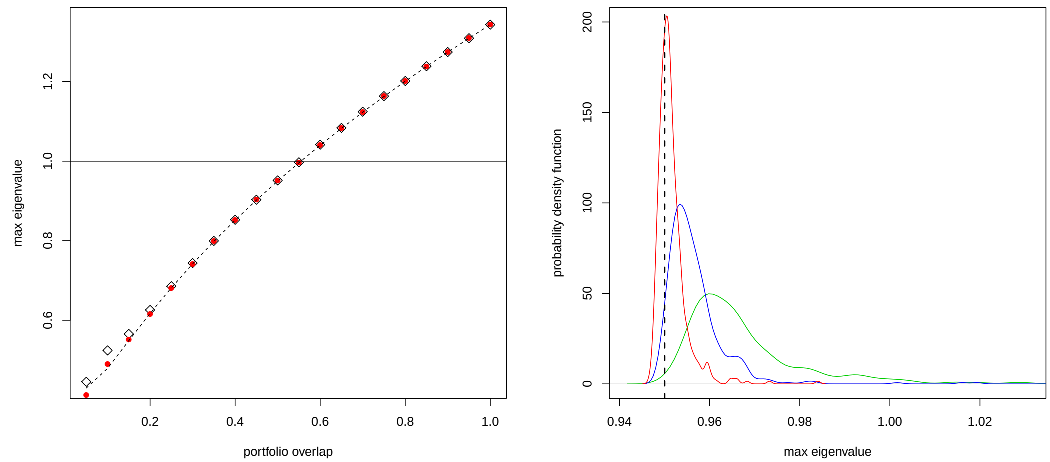

Figure 4: Left panel. Maximum eigenvalue Λ_{max} as a function of the mean fractional overlap $\bar{o}$ between two portfolios when $N = 1,000$. The dashed black line is the result of Eq. 24. Red circles and black diamonds are the mean maximum eigenvalue over 500 simulations when banks have homogeneous asset size and asset size drawn from lognormal distribution with $\mu = 11.7$ and $\sigma^2 = 1.8$ (as in Janicki and Prescott (2006)), respectively. Right panel. Estimation of the probability density function of the maximum eigenvalue in the case of lognormal asset size distribution, $m = 10$, and $N = 250$ (green), 500 (blue), and $1,000$ (red). The vertical dashed line is the theoretical value of Eq. 24.

2.4.2 Numerical simulations and the role of bank size heterogeneity

We investigate numerically the approximations made in the previous calculations considering the role of bank size heterogeneity and of the number of banks N. For simplicity we will show the results for $M = 20$ investment assets, $\alpha = 1.64$, $\sigma_d = 0.03$ $\sigma_s = 0$, and $\gamma = 54$. Similar results are observed for other parameters values.

First we set the number of banks equal to $N = 1,000$ and we assume that they have all the same asset size. We perform 500 numerical simulations of the matrix $\mathbf{W}$ and for each of them we compute the maximum eigenvalue. We then compare its mean value over the simulations with the theoretical value (see red circles in the left panel of Figure 4). As a comparison, we also consider a realistic bank asset size distribution. It is well known that bank size is very heterogeneous and different functional forms have been proposed. Here we consider the empirical results of Janicki and Prescott (2006) who fitted the asset size of the roughly $10,000$ US banks with a lognormal distribution with parameters $\mu = 11.7$ and $\sigma^2 = 1.8$. Left panel of Figure 4 shows that both in the homogeneous and in the heterogeneous case the agreement between simulations and the theoretical value is excellent. This fact confirms that in the large N limit bank size distribution is irrelevant and the approximations leading to Eq. 24 are very good.

We then consider the role of finite size corrections, investigating the distribution of the maximum eigenvalue for fixed fractional overlap and variable number of banks N. Specifically, we consider the lognormal distribution of bank size and we fix the value of $m = 10$. The right panel of Fig. 4 reports the estimation of the probability density function of the maximum eigenvalue for $N = 250, 500, 1,000$. As expected from the convexity argument, form small value of N the probability density function of the maximum eigenvalue has considerable mass above the theoretical (and asymptotic) value of Eq. 24 (vertical dashed line). Notice however that even for $N = 250$ the mode of the distribution is only 1% larger than the theoretical value, confirming again that our numerical approximation is very good.

In conclusion this result indicates the asymptotic (in N) nature of our analytical approximation. As suggested above, the finite sample bias is due to the nonlinear and convex nature of the maximum eigenvalue. In fact, numerical simulations and t-tests confirm that the mean value of B_{ij} is correctly described by Eq. 21. In any case it is worth noticing that our numerical simulations indicate that for finite samples, bank size heterogeneity makes the financial system more unstable as compared to the homogeneous case. In fact, all else being equal, for finite and small N the maximum eigenvalue is larger for the heterogenous than for the homogeneous case, making the system closer to the transition between the stationary and the non-stationary dynamics.

2.5 Properties of risky asset dynamics

Here we give an exact description of the dynamics of investment assets computing in closed form the variance-covariance matrix of asset returns. In fact, using the average representation of Eq. 22, we notice that $m\mathbf{\Psi}$ can be written as

$$m\boldsymbol{\Psi} = (1-b)\mathbf{I} + b\boldsymbol{\iota}\boldsymbol{\iota}' \tag{25}$$

with the scalar $b = \frac{m}{Mc}$, identity matrix $\mathbf{I}$, and the column vector of ones $\boldsymbol{\iota}$. Hence, the VAR for the vector of endogenous components in equation (17) can be rewritten as

$$\boldsymbol{e_t} = (1-b)\,\mathcal{A}(\boldsymbol{e_{t-1}} + \boldsymbol{\varepsilon_t}) + b\,M\boldsymbol{\mathcal{A}\iota}(\bar{e}_{t-1} + \bar{\varepsilon}_t) \tag{26}$$

with matrix $\mathcal{A} \equiv \frac{\bar{\lambda}-1}{m}\boldsymbol{\Gamma}^{-1}$ and scalars $\bar{e}_t \equiv \frac{1}{M}\sum_{k=1}^{M} e_{k,t}$ and $\bar{\varepsilon}_t \equiv \frac{1}{M}\sum_{k=1}^{M} \varepsilon_{k,t}$. The scalar $\bar{e}_t$ can be interpreted as the endogenous return of the market portfolio. Thus, the endogenous component of an individual investment becomes

$$e_{i,t} = (1-b)\,a_i(e_{i,t-1} + \varepsilon_{i,t}) + b\,Ma_i(\bar{e}_{t-1} + \bar{\varepsilon}_t) \tag{27}$$

with scalar $a_i = \frac{\bar{\lambda}-1}{m\gamma_i}$.

Therefore, the process for $e_{i,t}$ can be rewritten as a linear combination of a standard univariate AR(1) process and a dynamic process depending on the averages of previous period endogenous components and shocks. In this way, $e_{i,t}$ is a mixture of a perfectly idiosyncratic process (i.e. uncorrelated with the others investment processes) receiving weight $1-b$ and a perfectly correlated process with weight b. Being $b = \frac{m}{Mc}$, the higher is the value of m, the higher is the weight given to the perfectly correlated component of mixture and, hence, the higher the correlations among the endogenous components of the different investments.

Moreover, assuming $a_i = a\ \forall i$ (i.e. all investments have the same liquidity), the process for $\bar{e}_t$ becomes:

$$\bar{e}_t \;=\; a(1-b+bM)(\bar{e}_{t-1} + \bar{\varepsilon}_t) \equiv \phi\,(\bar{e}_{t-1} + \bar{\varepsilon}_t) \tag{28}$$

with $\phi \equiv a(1-b+bM)$. Therefore, the dynamics of the average process $\bar{e}_t$ is also an autoregressive of order one; its variance, assuming stationarity of e_t, is (see Appendix D)

$$V(\bar{e}_t) = \frac{\Lambda_{\max}^2}{1-\Lambda_{\max}^2}\,V(\bar{\varepsilon}_t) \tag{29}$$

with $V(\bar{\varepsilon}_t) = \left(\sigma_f^2 + \frac{\sigma_\epsilon^2}{M}\right)$.

Finally, defining the distance of the endogenous component of investment i from the average as $\Delta e_{i,t} \equiv e_{i,t} - \bar{e}_t$, we also have that

$$\Delta e_{i,t} = (1-b)\,a(\Delta e_{i,t-1} + \Delta\varepsilon_{i,t}) \tag{30}$$

where $\Delta\varepsilon_{i,t} \equiv \varepsilon_{i,t} - \bar{\varepsilon}_t$. So that the dynamics of the individual distance of the endogenous component of investment i from the average value $\bar{e}_t$ is also an autoregressive process of order one.

We can then interpret the dynamics of the endogenous components of each individual investment as an idiosyncratic AR(1) process around a common process for the average value also following an AR(1) and where the amplitude of the idiosyncratic component is inversely related to the portfolio diversification. In other words, the dynamics of endogenous returns can be described as a multivariate "ARs around AR". When the process is stationary, the mean market behavior is described by a mean reverting process. In turn, each investment performs a mean reverting process around the market mean. It can be shown that the time scale for mean reversion of the market is always larger than the time scale of reversion of an investment toward the market mean behavior. Moreover, when m increases the time scale of reversion of individual investment declines, which means that investments become more quickly synchronized with the mean market behavior. Finally, notice that this type of multivariate "ARs around AR" dynamics is also followed by the endogenous component of portfolio returns $e_t^p \equiv \frac{1}{m}\sum_{k=1}^{m} e_{k,t}$ where the number of assets in portfolio is $m < M$.

Importantly, this representation clearly shows that, as for the exogenous component, also the variability of the endogenous component of returns can be decomposed into a systematic component associated with the volatility of $\bar{e}_t$ and an idiosyncratic one. Therefore, both the exogenous and endogenous components contain a diversifiable and undiversifiable source of risk, so that also the total risk of the investments return is composed of these two type of risk, σ_s and σ_d, as perceived by the financial institutions.

Thanks to this representation we are able to explicitly compute the variance and covariances of the process for the endogenous components $e_{i,t}$, which are reported in Appendix (D). It can be shown that a larger leverage increases both the variances and the covariances of $e_{i,t}$, while a greater degree of diversification reduces the variances and increases the covariances. Both are positively related with correlations. In particular, it can be shown (see Appendix D) that the correlations among the endogenous returns tend to one as $m \to M$.[9]

[9]Notice that the endogenous correlations would not tend to one in presence of an additional stochastic component in the price impact function (Eq. 15) coming from the exogenous demand of traders not actively rebalancing their portfolio (see Footnote 6).

Taking into account the feedback induced by the portfolio rebalancing introduces a new endogenous component in the variance of the investment asset given by the variance of the endogenous component

$$V(r_{i,t}) = V(e_{i,t}) + V(\varepsilon_{i,t}) \tag{31}$$

where the exogenous variance $V(\varepsilon_{i,t}) = \sigma_f^2 + \sigma_\epsilon^2$ and the explicit expression for the endogenous variance $V(e_{i,t})$ is given in Appendix D. This expression shows that the endogenous component of return leads to an increase of the volatility of an investment above its "bare" level $V(\varepsilon_{i,t})$. This volatility increase is at the end due to the finite liquidity of the investments and disappears when $\gamma \to \infty$ and it can therefore be seen as an "illiquidity induced contribution to volatility".

Analogously, the covariance between the returns of two investments is enhanced by the contribution coming from the covariance between the endogenous components (see Appendix D)

$$Cov(r_{i,t}, r_{j,t}) = Cov(e_{i,t}, e_{j,t}) + \sigma_f^2. \tag{32}$$

Figure 5 shows the variance of returns and the correlation between the endogenous component of returns of two investments as a function of diversification cost c. We see that when cost is high, variance and correlations are low. By decreasing cost, variance of returns increases as well as correlations. If the market factor is not strong enough, there is a value of c for which variance diverges, corresponding to the case where the maximum eigenvalue $\Lambda_{\max}$ becomes equal to one. In this limit, correlations become closer and closer to one.

As a consequence the variance of portfolio returns in presence of the rebalancing feedbacks becomes

$$\begin{aligned} V(r_t^p) &= \frac{V(e_{i,t})}{m} + \frac{m-1}{m} Cov(e_{i,t}, e_{j,t}) + \sigma_f^2 + \frac{\sigma_\epsilon^2}{m} \\ &= V(e_p) + \sigma_f^2 + \frac{\sigma_\epsilon^2}{m}, \end{aligned} \tag{33}$$

which, as for investment returns, means that the endogenous component (and therefore the illiquidity of the assets) increases the volatility of portfolio by a illiquidity induced contribution. The left panel of Figure 6 shows the variance of portfolios as a function of

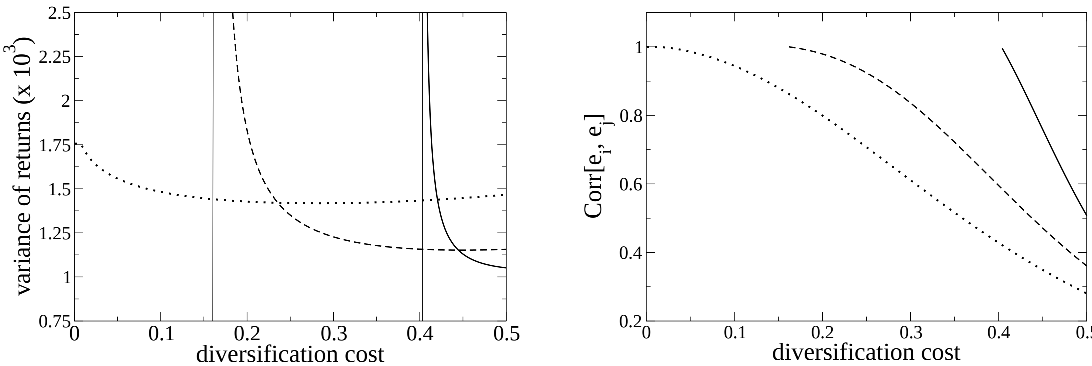

Figure 5: The left panel shows the variance of investment returns, $V(r_{i,t})$, of Eq. (31) as a function of the diversification cost c, while the right panel shows the correlation of the endogenous component of returns between two investments, $Corr(e_{i,t}, e_{j,t})$ as a function of c (see Fig. 1 for the parameters). The vertical lines in the left panel indicate where the variance of returns diverges and below these values correlations in the right panel are clearly not defined.

diversification cost c for different values of the ratio σ_s/σ_d. It is important to notice that by reducing diversification cost, the variance of portfolios initially declines. This means that in this regime, financial innovation makes portfolios less risky and it is therefore beneficial. However, the variance of the portfolio reaches a minimum for a given value of c and by reducing further the diversification cost, one gets closer and closer to the critical condition $\Lambda_{\max} = 1$ and the variance increases without bounds. In this regime, even small variations of the cost lead to huge increases of the riskiness of the portfolios.

It is also interesting to note that the variance of the observed market portfolio (the one containing all the M investments with equal weights) is

$$\begin{aligned} V(r_t^M) &= V(\bar{e}) + \sigma_f^2 + \frac{\sigma_\epsilon^2}{M} \equiv V(\bar{e}) + \sigma_{p_M}^2 = \frac{\Lambda_{\max}^2}{1-\Lambda_{\max}^2}\,\sigma_{p_M}^2 + \sigma_{p_M}^2 \\ &= \frac{1}{1-\Lambda_{\max}^2}\,\sigma_{p_M}^2 \end{aligned} \tag{34}$$

with $\sigma_{p_M}^2 \equiv \sigma_f^2 + \frac{\sigma_\epsilon^2}{M}$ being the market portfolio return when feedback due to impact is not present, i.e. corresponds to the case of an infinitely liquid market. So, the factor $\frac{1}{1-\Lambda_{\max}^2}$ representing the magnification of the exogenous variance due to the portfolio rebalancing, can then be termed the "variance multiplier" of the endogenous component. Clearly, for larger values of the maximum eigenvalue of the VAR process, the variance multiplier will increase exploding for $\Lambda^2_{\max} \to 1$.

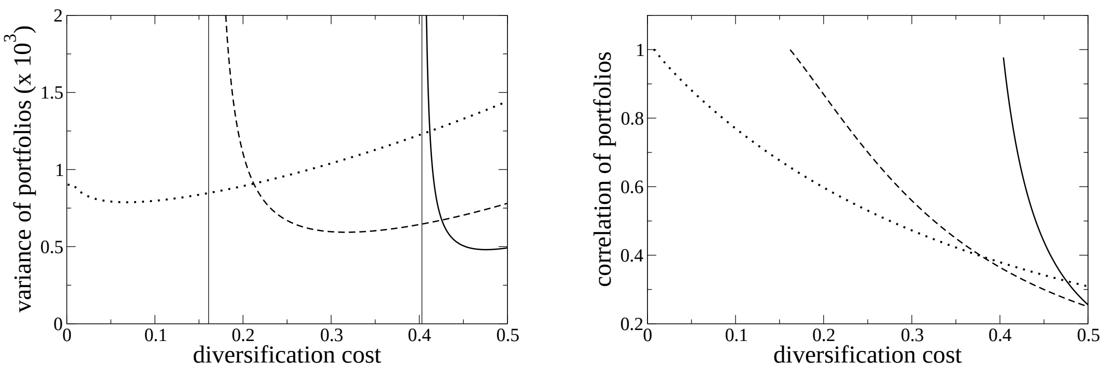

Figure 6: The left panel shows the variance of portfolios, $V(r^p_t)$, of Eq. (33) as a function of the diversification cost c, while the right panel shows their correlation, ρ^e_p, of Eq. (36) (see Fig. 1 for the parameters). The vertical lines in the right panel indicate where the variance of portfolios diverges and below these values correlations in the right panel are clearly not defined.

Similarly, the covariance between two portfolios containing m assets becomes

$$
\begin{aligned}
Cov(r^p_{h,t}, r^p_{k,t}) &= Cov(e_{i,t}, e_{j,t}) + \sigma^2_f + m\frac{m}{M}\left(\frac{V(e_{i,t}) - Cov(e_{i,t}, e_{j,t}) + \sigma^2_\epsilon}{m^2}\right) \\
&= \frac{V(e_{i,t})}{M} + \frac{M-1}{M} Cov(e_{i,t}, e_{j,t}) + \sigma^2_f + \frac{\sigma^2_\epsilon}{M} \\
&= V(\bar{e}) + \sigma^2_f + \frac{\sigma^2_\epsilon}{M} \\
&= \frac{1}{1-\Lambda^2_{\max}}\,\sigma^2_{p_M}. \qquad (35)
\end{aligned}
$$

In fact, given the factor structure of $e_{i,t}$, with the factor being $\bar{r}_t = \bar{e}_{t-1} + \bar{\varepsilon}_t$, the covariance $Cov(e_{p,t}, \bar{e}_t)$ is equal to the variance of the factor $V(\bar{e})$ (as for the exogenous covariance).

Finally, the correlation between two portfolios in presence of endogenous feedback can be written as

$$\rho_p = \frac{V(\bar{e}) + \sigma_f^2 + \frac{\sigma_\epsilon^2}{M}}{V(e_p) + \sigma_f^2 + \frac{\sigma_\epsilon^2}{m}} = \frac{V(e_p)\frac{V(\bar{e})}{V(e_p)} + \left(\sigma_f^2 + \frac{\sigma_\epsilon^2}{m}\right)\frac{\sigma_f^2 + \frac{\sigma_\epsilon^2}{M}}{\sigma_f^2 + \frac{\sigma_\epsilon^2}{m}}}{V(e_p) + \sigma_f^2 + \frac{\sigma_\epsilon^2}{m}}$$

$$= \frac{\sigma_e^2 \rho_e + \sigma_\varepsilon^2 \rho_\varepsilon}{\sigma_e^2 + \sigma_\varepsilon^2} \tag{36}$$

where $\sigma_e^2 \equiv V(e_p)$, $\sigma_\varepsilon^2 \equiv \sigma_f^2 + \frac{\sigma_\epsilon^2}{m}$, $\rho_e \equiv \frac{Cov(e_{p,t}, \bar{e}_t)}{V(e_p)} = \frac{V(\bar{e})}{V(e_p)}$, and $\rho_\varepsilon \equiv \frac{\sigma_f^2 + \frac{\sigma_\epsilon^2}{M}}{\sigma_f^2 + \frac{\sigma_\epsilon^2}{m}}$. That is, the portfolio correlation in presence of active asset management is a weighted average of the endogenous correlations between e_p and $\bar{e}$, i.e. ρ_e, and the correlation between the exogenous shocks, (i.e. ρ_ε), with weights the respective variances σ_e^2 and σ_ε^2. Since both the endogenous ρ_e and exogenous ρ_ε correlations tend to one as $m \to M$, also the total correlation of the portfolio ρ_p tends to one as $m \to M$.

The right panel of Figure 6 shows the correlation between portfolios as a function of the diversification cost c. Correlation between portfolios steadily increases by reducing diversification costs essentially because the overlap between portfolios increases. It is important to notice that the condition of divergence of the variance does not imply a perfect overlap between portfolio. For example, with the given parameters the transition to infinite variance and non stationary portfolios occurs at $\bar{o} = 0.34$ when $\sigma_s/\sigma_d = 0.3$ and at $\bar{o} = 0.21$ when $\sigma_s/\sigma_d = 0$. Thus correlation between portfolios can become very close to one even if the portfolio overlap is relatively small.

2.6 Bank asset dynamics

The dynamics of the rebalanced bank asset $A^*_{i,t}$, can be written as

$$A^*_{j,t} = \lambda_j E_{j,t} = \lambda_j \left(E_{j,t-1} + r^p_{j,t} A^*_{j,t-1}\right) = A^*_{j,t-1} + \lambda_j r^p_{j,t} A^*_{j,t-1} \tag{37}$$

thus, the relative change of the bank j total asset $r^A_{j,t}$ is simply given as

$$r^A_{j,t} \equiv \frac{A^*_{j,t} - A^*_{j,t-1}}{A^*_{j,t-1}} = \lambda_j r^p_{j,t}. \tag{38}$$

Therefore, the variance and covariance of the relative change of bank assets $r^A_{j,t}$ are simply

$$V(r^A_{j,t}) = \lambda_j^2 V(r^p_{j,t}) \tag{39}$$

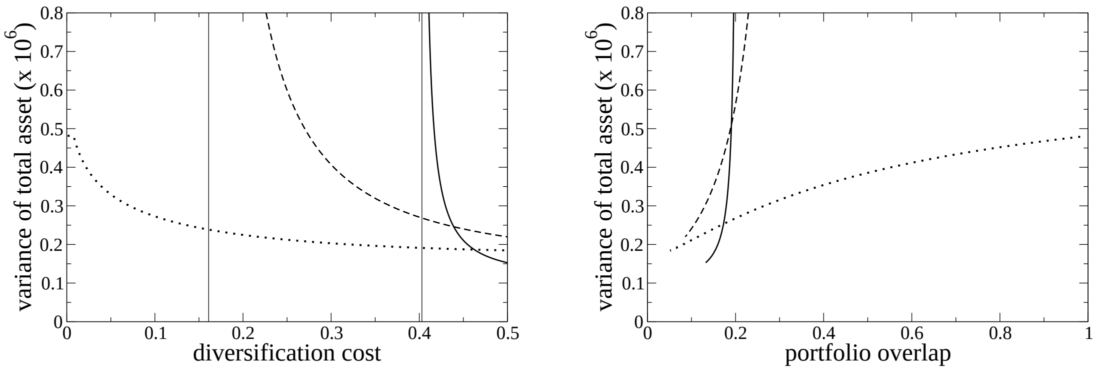

Figure 7: Variance of the total asset, $\sum_{j=1}^{N} r^p_{j,t}$, of the whole banking sector as a function of the diversification cost c (left panel) and of the mean fractional overlap $\bar{o}$ between portfolios (right panel) (see Fig. 1 for the parameters) The vertical lines indicate where the variance of total asset diverges.

and

$$Cov(r^A_{h,t}, r^A_{k,t}) = \lambda_h \lambda_k Cov(r^p_{h,t}, r^p_{k,t}), \tag{40}$$

where the expression for $V(r^p_{j,t})$ and $Cov(r^p_{h,t}, r^p_{k,t})$ are given in equation (33) and (35), respectively. The properties of the bank assets dynamics are then dictated by those of the portfolio (with its exogenous and endogenous components) and further amplified by the degree of leverage.

We can finally compute the variance of the total asset of the whole banking sector

$$\begin{aligned} V\left(\sum_{j=1}^{N} r^A_{j,t}\right) &\simeq NV\left(r^A_{j,t}\right) + N(N-1)Cov\left(r^A_{h,t}, r^A_{k,t}\right) \simeq N\bar{\lambda}^2 V\left(r^p_{j,t}\right) + N(N-1)\bar{\lambda}^2 Cov\left(r^p_{h,t}, r^p_{k,t}\right) \\ &= \bar{\lambda}^2 V\left(\sum_{j=1}^{N} r^p_{j,t}\right) \end{aligned} \tag{41}$$

where for simplicity we have assumed that all the banks have the same leverage $\bar{\lambda}$. Moreover $V\left(\sum_{j=1}^{N} r^p_{j,t}\right)$ is explicitly given in terms of the original variables in Appendix D where it is also shown that for $m \to M$ it reduces to

$$V\left(\sum_{j=1}^{N} r^p_{j,t}\right) \xrightarrow[m \to M]{} \frac{N^2 \sigma^2_{p_M}}{1 - \Lambda_{\max}}. \tag{42}$$

These analytical results allows us to analyze the determinants of the variability of total asset of the banking sector which governs the expansion and contraction of the supply of credit and liquidity to financial system.

Figure 7 shows the variance of the total asset of the whole banking sectors as a function of the diversification cost (left panel) and of the mean fractional overlap between portfolios (right panel). We observe that the variance of the total asset monotonically increases when one decreases diversification cost or increases the overlap between portfolios. As one of these two related variables leads the system close to the critical point, the variance of the total asset of the banking sector explodes. Moreover, close to the transition point, the variance of the total asset increases dramatically when one changes slightly the typical overlap between portfolios.

3 Systemic risk

We now analyze the systemic risk implications of our model first in the static setting without feedback and then in a setting with the endogenous feedback of investor demands on the asset dynamics.

3.1 Static analysis

First, as previously shown, when the diversification m increases, the correlation between the portfolio returns of two financial institutions will increase, with $\rho_p \xrightarrow[m \to M]{} 1$, which, ceteris paribus, tends to increase the probability of a systemwide contagion during a crisis event.

Second, given a negative realization of the systematic (exogenous and endogenous) component $s_t = \bar{e}_t + f_t$, the portfolio return distribution conditioned on this systematic shock s_t^{shock} is (considering, for simplicity, a normal distribution for portfolio returns with zero mean):

$$r_{i,t}^p | s_t^{shock} \sim N\left(s_t^{shock},\ \frac{\sigma_d^2}{m}\right). \tag{43}$$

where $r_{i,t}^p = \sum_{j=1}^m \frac{r_{i,j,t}}{m}$ is the portfolio return of bank i at time t.

Consequently, the probability of default of a financial institution given a systematic shock

s_t^{shock} is

$$\begin{aligned} PD_{i,t-1} &= P\left(\left[r_{i,t}^{p}|s_t^{shock}\right] \leq -\alpha\sqrt{\sigma_s^2 + \frac{\sigma_d^2}{m}}\right) \\ &= \Phi\left(\frac{-\alpha\sqrt{\sigma_s^2 + \frac{\sigma_d^2}{m}} - s_t^{shock}}{\sqrt{\frac{\sigma_d^2}{m}}}\right) \xrightarrow[m\to M,\ M\to\infty]{} 1 \quad \forall\ s_t^{shock} < -\alpha\sigma_s, \end{aligned} \tag{44}$$

where Φ is the standard normal cdf. Therefore, for any negative shock of the systematic component larger than its VaR, i.e. $\alpha\sigma_s$, the probability of default increases with the degree of diversification m.

In summary, higher degree of diversification increases both the probability of default of single institutions (in case of large systematic shocks) and the correlations among them, thus exposing the economy to a higher level of systemic risk.

3.2 Dynamic analysis

The results of the previous section show that the endogenous return dynamics adds an additional component to both the variance and covariance of the risky investments. If such endogenous components were not accounted for in the evaluation of portfolio volatility for the VaR, obviously, there would be an underestimation of each agent's risk, leading to an under capitalization of the banking sector and, hence, to an higher fragility of the system. Nevertheless, the practice of empirically estimating variances and covariances of risky assets from past data, automatically considers both the exogenous and endogenous components of volatility.

However, contrary to the case without endogeneity, the investments variances and covariances now depend on the level of diversification and, in particular, the degree of leverage (through the dynamics of the endogenous component). Therefore, a change, say, in the degree of leverage will cause a structural shift in the future level of variances and covariances which will not be captured by the empirical estimation on past data.

In particular, in periods when leverage increases, portfolio volatility estimated on historical past data will tend to underestimate future risk (coming from stronger rebalancing feedback) leading to an increase of systemic risk. On the contrary, in periods of decreasing leverage, future volatility will tend to decrease (reduced feedback intensity) so that future realized volatility will tend to be lower than the historical one. Therefore, the results of our model provide a theoretical support for countercyclical capital requirements as often advocated in the aftermath of the recent financial crises.

Moreover, it is important to notice that a given negative realization of the exogenous factor f_t, will trigger a sequence of portfolio rebalances causing the price of all risky assets to decay for several periods. Within our framework, we can explicitly compute the expected impact on the future return dynamics triggered by a given common shock.

Being $\boldsymbol{e_t} = \boldsymbol{\Phi r_t}$ (from equation 17), the vector of investment returns also follows a VAR(1)

$$\boldsymbol{r_t} \;=\; \boldsymbol{e_{t-1}} + \boldsymbol{\iota} f_t + \boldsymbol{\epsilon_t} = \boldsymbol{\Phi r_{t-1}} + \boldsymbol{\iota} f_t + \boldsymbol{\epsilon_t}. \tag{45}$$

The total future impact of the shocks over the next h periods will be given by the h-period cumulative mean return conditioned on the factor shock f_t^{shock}, which is (for h sufficiently large)

$$E\left[\boldsymbol{r_{t:t+h}} | f_t^{shock}\right] \simeq \left(\mathbf{I} - \boldsymbol{\Phi}\right)^{-1} \boldsymbol{\iota} f_t^{shock}. \tag{46}$$

Hence, the larger the maximum eigenvalue of $\boldsymbol{\Phi}$ the larger will be the magnitude and persistence of future adjustments leading to a larger cumulative impact that the financial system will have to absorb. So the larger the maximum eigenvalue the higher will be the probability that the system, because of capital or liquidity constraints, will not be able to absorb the initial shock. This also means that systemic risk is positively related to the magnitude of the eigenvalues of the matrix $\boldsymbol{\Phi}$.

4 Discussion: Introducing financial innovations

The results on the dynamics of the asset prices can be summarized as follows. When the costs of diversification c are high, the degree of diversification i.e. the number of asset m randomly selected and the degree of leverage are small. Thus the portfolio of the financial institutions are heterogeneous and little leveraged. Therefore, the endogenous feedbacks, coming from the amplification of individual demands induced by leverage targeting (as illustrated in the previous section), are of moderate size and uncoordinated. Thus, an amplification mechanism at the aggregate level between asset values and prices of risky investments will not tend to arise.

We now discuss the effect of the introduction of financial innovation products (such as securitization of mortgages or ABS products) that permits to reduce the cost of diversification c. Despite the simplicity of our framework, the introduction of financial innovation has several important consequences. First, a financial innovation which reduces the cost of diversification c, by increasing the optimal level of diversification m, reduces the volatility of the portfolio which in turns increase the leverage of the institution. In this way, financial innovation will tend to increase the degree of leverage in the system. By increasing the leverage, the individual exposition to the undiversifiable macro factor risk increases; i.e., although each individual is more resilient to idiosyncratic shocks, they become more sensitive to the shocks in the macro factor.

Second, by increasing m, the overlap in the portfolios of the different financial institutions will be larger, increasing the "similarity" of the portfolio choices among the investors and, thereby, increasing the correlations among portfolio returns and balance sheet dynamics.

Third, an increase in leverage will heavily affect the dynamics of the risky investments by increasing both their variances, covariances and correlations, through a strengthening of the endogenous component.

As a consequence of these effects, individual reactions in terms of asset demands will be more aggressive (due to higher leverage) and more coordinated (because of the larger correlation in the profits-losses realizations). This rise in the strength and coordination of the individual reactions will make more likely to have aggregate feedback in which the rise of the price of some investments leads to an excess of equity (by the realized capital gains) and, hence, to an expansion of the balance sheets driving new demands for the asset which pushes the price up and so on. The very same mechanism will operate also in the opposite direction during market crisis when the aggregate feedback will aggravate price declines and balance sheet contractions. When the diversification cost falls below a given threshold (implying the maximum eigenvalue of the vector return process exceeding one) the aggregate feedback will produce price and balance sheet dynamics that become explosive (see also Corsi and Sornette 2011).

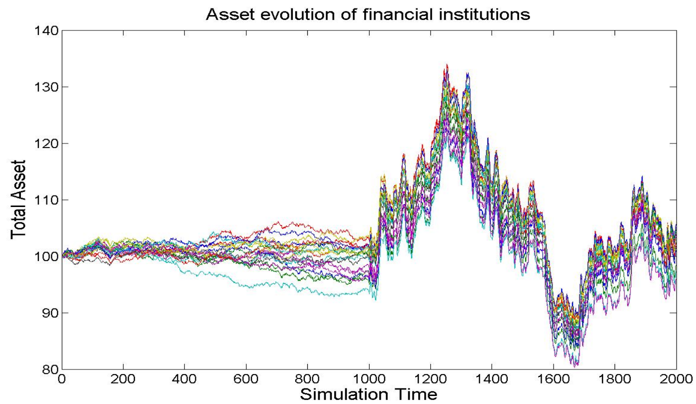

Figure 8: Numerical simulation of the dynamics of individual total asset of financial institutions before and after a structural break (at simulation time 1000) on the diversification costs that induces an increase of leverage and diversification. Leverage goes from 10 to 60 and fractional overlap from 0.1 to 0.8 .

These feedbacks could be reinforced even further by endogenizing the dynamics of financial innovation or, as in Brunnemeier and Pedersen (2008), that of the market liquidity. For instance, following the intuition of Brunnemeier and Pedersen (2008), the market liquidity could be assumed to inversely depends on the past realized volatility so that an increase in the endogenous feedback, by increasing volatility also increase the market impact of the portfolio rebalancing (through the reduction in market liquidity), thus further reinforcing the feedback.

Therefore, through these mechanisms reinforcing the endogenous feedback, financial innovation can give rise to a steep growth (bubble) and plunge (burst) of market prices and banking sector total assets. As explained by Adrian and Shin (2010), the total asset of the banking sector is the relevant variable for the determination of the amount of credit supplied to the financial and real sector. Hence, an increase in the variability of the total asset of the banking sector will have major consequences on the availability of funding to the economy causing the instability to be transmitted from the financial sector to the real one.

To visually illustrate the impact on the dynamics of financial intermediaries total asset of a shift in the degree of leverage and diversification induced by a reduction in the diversification costs, we simulate the bank asset dynamics with a structural break represented by a sudden increase (at simulation time 1000) in the degree of leverage and diversification (see Figure 8). Going from a low level of leverage and diversification to a high level we observe: (i) a dramatic increase in the correlation and amplitude of the changes in the total asset of individual financial institutions, and (ii) a sudden shift in the total banking sector assets (i.e. simply the sum of all the individual bank total assets), which will imply going from an approximately constant supply of credit to a regime with wide swings in the credit supply.

5 Conclusions

In this paper we investigate the determinants of the balance sheet dynamics of financial intermediaries by modeling the dynamic interaction between asset prices and bank behavior induced by regulatory constraints and multi-round portafoglio rebalances. Standard capital requirements, in the form of Value–at–Risk (VaR) constraints, together with the level of diversification costs (related to the availability of derivatives products), determine bank decisions on diversification and leverage which, in turns, strongly affect the dynamics of traded assets through the bank strategies of portfolio rebalances in presence of a finite asset liquidity. We show how changes in the constraints of the bank portfolio optimization (such as changes in the prevailing cost of diversification or changes in the micro-prudential policies) endogenously drive the dynamics of bank balance sheets, asset prices, and systemic risk.

The analytical results obtained by applying our simple framework are manifolds: (i) a reduction of diversification costs, by increasing the level of diversification and hence relaxing the VaR constraint, allows the financial institutions to increase the optimal leverage; (ii) it also increases the degree of overlap, and thereby correlation, between the portfolios of financial institutions; (iii) even in absence of feedback effects, higher degree of diversification increases both the probability of default of single institutions (in case of large systematic shocks) and the correlations among them, thus exposing the economy to a higher level of systemic risk; (iv) the higher overlap induced by a reduction in diversification costs increases both the variance and correlation of the investment demands of financial institutions rebalancing their portfolios; (v) the dynamic interaction between investment prices and bank behavior induced by portfolio rebalancing leads to a multivariate VAR process whose maximum eigenvalue depends on the degree of leverage and on the average illiquidity of the assets; (vi) higher diversification, by increasing the strength and coordination of individual feedbacks, can lead to dynamic instability of the system; (vii) the VAR process can be represented as a combination of many idiosyncratic AR processes around a single common AR process of the average values; (viii) the endogenous feedback introduces an additional component to the variance, covariance and correlation of both the individual investment assets and the bank portfolios; (ix) both the variance and correlation of individual investments monotonically increase with a reduction in the diversification costs; (x) a simple variance multiplier exists for the variance of the observed market portfolio. (xi) the relation between the variance of the portfolio and diversification costs is non-monotonic as it initially declines with costs while then rapidly increases when the reduction of diversification costs makes the system approaching its critical point causing the variance to explode; (xii) the effects of the endogenous feedback make historical estimation of variances and covariances to be overestimated during periods of increasing leverage and underestimated during periods of deleveraging, thus providing a rationale for the adoption of countercyclical capital requirements; (xiii) in presence of endogenous feedbacks, a negative realization of the systematic component will trigger a sequence of portfolio rebalances which will amplify, over time, its initial impact; (xiv) the variability of total asset of the banking sector, which governs the expansion and contraction of the supply of credit and liquidity to financial system, is highly sensitive to variation in the costs of diversification.

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Appendices

A Portfolio rebalance

Given that $A^*_{j,t} = \lambda E_{j,t}$, the difference between the desired amount of asset ($A^*_{j,t}$) and the actual one ($A_{j,t}$) can be written as (dropping the sub-index j for sake of notation simplicity):

$$\begin{aligned}
\Delta R_t &\equiv A^*_t - A_t \\
&= \lambda E_t - A_t \\
&= \lambda (E_{t-1} + r^p_t A^*_{t-1}) - (1 + r^p_t) A^*_{t-1} \\
&= \lambda \left(\frac{A^*_{t-1}}{\lambda} + r^p_t A^*_{t-1} \right) - (1 + r^p_t) A^*_{t-1} \\
&= (\lambda - 1) r^p_t A^*_{t-1}
\end{aligned}$$

B Derivation of equation (21)

We compute the expectation of the matrix $\mathbf{B} = \mathbf{W}'\tilde{\mathbf{Q}}\mathbf{W}$, where the expectation is taken over the ensemble of random matrices $\mathbf{W}$. The generic element is

$$B_{ij} = \sum_{k=1}^{N} \tilde{Q}_{k,k} W_{k,i} W_{k,j} \tag{47}$$

with $W_{k,i} = \frac{X_{k,i}}{s_k}$ and $s_k = \sum_i X_{k,i}$.

Knowing that $X_{k,i}$ is a Bernoullian variables with probability m/M, we have

$$\begin{aligned}
\mu_x \equiv E[X_{k,i}] &= \frac{m}{M} && (48)\\
\sigma^2_x \equiv V[X_{k,i}] &= \frac{m}{M}(1 - \frac{m}{M}) && (49)\\
E[X^2_{k,i}] &= \frac{m}{M} && (50)\\
E[X_{k,i} X_{k,j}] &= \frac{m^2}{M^2}. && (51)
\end{aligned}$$

Moreover, knowing that $s_k = \sum_i X_{k,i}$ is a $Bin(M, m/M)$, we have

$$\begin{aligned}
\mu_s \equiv E[s_k] &= m && (52)\\
\sigma^2_s \equiv V(s_k) &= m(1 - \frac{m}{M}) && (53)\\
E[s^2_k] &= m^2(1 + \frac{1}{m} - \frac{1}{M}) \equiv m^2 c && (54)\\
\mathrm{Cov}[X_{k,i}, s_k] &= \sigma^2_x && (55)
\end{aligned}$$

We need now to compute $E[W_{k,i}]$, $E[W_{k,i}^2]$, and $E[W_{k,i}W_{k,j}]$. For $E[W_{k,i}]$ notice that by symmetry we have

$$1 = E\left[\frac{\sum_i X_{k,i}}{s_k}\right] = \sum_i E\left[\frac{X_{k,i}}{s_k}\right] \Rightarrow E\left[\frac{X_{k,i}}{s_k}\right] = \frac{1}{M} \tag{56}$$

For $E[W_{k,i}^2]$ we apply the formula for the variance of the ratio of two random variables (obtained by a Taylor expansion around the expected values)

$$\mathrm{Var}\left[\frac{X_{k,i}}{s_k}\right] \simeq \frac{\mu_x^2}{\mu_s^2}\left[\frac{\sigma_x^2}{\mu_x^2} - \frac{2\mathrm{Cov}[X_{k,i}, s_k]}{\mu_x\mu_s} + \frac{\sigma_s^2}{\mu_s^2}\right] \tag{57}$$

obtaining

$$E[W_{k,i}^2] \simeq \frac{1}{mM} - \frac{1}{mM^2} + \frac{1}{M^3} \simeq \frac{1}{mM} \tag{58}$$

While for $E[W_{k,i}W_{k,j}]$ we have (again from a Taylor approximation)

$$E[W_{k,i}W_{k,j}] = E\left[\frac{X_{k,i}X_{k,j}}{s_k^2}\right] \simeq \frac{E[X_{k,i}X_{k,j}]}{E[s_k^2]} = \frac{E[X_{k,i}]E[X_{k,j}]}{E[s_k^2]} = \frac{1}{m^2c}\frac{m^2}{M^2} = \frac{1}{M^2c}. \tag{59}$$

Therefore, the expectations of B_{ij} are

$$E[B_{ii}] \simeq \frac{1}{mM}\sum_k \tilde{Q}_{kk} \qquad\qquad E[B_{ij}] \simeq \frac{1}{M^2c}\sum_k \tilde{Q}_{kk}, \quad i \neq j \tag{60}$$

i.e. Eq. (21).

C VAR Eigenvalues

In this Appendix we derive the eigenvalues of $\mathbf{\Psi}$ and (in an approximate form) of $(\bar{\lambda}-1)\mathbf{\Gamma}^{-1}\mathbf{\Psi}$. First of all we notice that the matrix $\mathbf{\Psi}$ has all diagonal elements equal to $d = 1/m$ and all the off diagonal elements equal to $d_{\text{off}} = \frac{1}{Mc}$. Thus, $(\bar{\lambda}-1)\mathbf{\Gamma}^{-1}\mathbf{\Psi} - \mathbf{I}\Lambda$ can be rewritten as

$$(\bar{\lambda}-1)\mathbf{\Gamma}^{-1}\mathbf{\Psi} - \mathbf{I}\Lambda = A + uv' \tag{61}$$

where

$$A = diag[g_i(d - d_{\text{off}}) - \Lambda] \tag{62}$$

$$u = (g_1 \ldots g_M)' \tag{63}$$

$$v = (d_{\text{off}} \ldots d_{\text{off}})' \tag{64}$$

where we have set $g_i = (\bar{\lambda}-1)\gamma_i^{-1}$. In order to compute the characteristic polynomial of the matrix we can use the Sherman-Morrison formula

$$\det(A+uv') = (1+v'A^{-1}u)\det A = \left(1 + d_{\text{off}}\sum_{i=1}^{M}\frac{g_i}{g_i(d-d_{\text{off}})-\Lambda}\right)\prod_{i=1}^{M}[g_i(d-d_{\text{off}})-\Lambda] \tag{65}$$

Setting this expression to zero and solving for Λ gives the eigenvalues, but the equation cannot be solved analytically in general.

If all the liquidity parameters γ_i are equal to γ, the above expression simplifies to

$$[g(d-d_{\text{off}})-\Lambda]^{M-1}[g(d-d_{\text{off}})-\Lambda+Mgd_{\text{off}}]=0 \tag{66}$$

Thus in the degenerate case, the spectrum is composed by $M-1$ degenerate (and small) eigenvalues equal to $g\left(\frac{1}{m}-\frac{1}{Mc}\right)$ and one large eigenvalue equal to $g\left(\frac{1}{m}+\frac{M-1}{Mc}\right)\simeq(\bar{\lambda}-1)\gamma^{-1}$.

When the liquidity parameters are different, we can expect that the spectrum has the same characteristics and the large eigenvalue is determined by setting to zero the first term in brackets of Equation 65, i.e.

$$1+d_{\text{off}}\sum_{i=1}^{M}\frac{g_i}{g_i(d-d_{\text{off}})-\Lambda}=0 \tag{67}$$

Since the eigenvalue is large, we can approximate this equation with $1-d_{\text{off}}\sum_i g_i/\Lambda\simeq 0$, i.e.

$$\Lambda\simeq d_{\text{off}}\sum_{i=1}^{M}g_i\simeq(\bar{\lambda}-1)\overline{\gamma^{-1}} \tag{68}$$

where

$$\overline{\gamma^{-1}}=\frac{1}{M}\sum_{i=1}^{M}\frac{1}{\gamma_i} \tag{69}$$

is the average of the inverse of the liquidity parameters. For a discussion of the validity of this approximation, see Lillo and Mantegna (2005)

D Endogenous variance and covariance

Recalling that

$$\begin{aligned} e_{i,t} &= (1-b)\,a(e_{i,t-1}+\varepsilon_{i,t})+b\,Ma(\bar{e}_{t-1}+\bar{\varepsilon}_t), \\ \bar{e}_t &= a(1-b+bM)(\bar{e}_{t-1}+\bar{\varepsilon}_t)\equiv\phi(\bar{e}_{t-1}+\bar{\varepsilon}_t), \end{aligned}$$

with scalar $a=\frac{\bar{\lambda}-1}{m\gamma}$, $b=\frac{1}{Mc}$, and $\phi=a(1-b+bM)$, and that stationarity of e_t implies $\gamma>\bar{\lambda}-1$, we have that

$$V(\bar{e}_t)=\frac{\phi^2V(\bar{\varepsilon})}{1-\phi^2}=\frac{(\bar{\lambda}-1)^2(\sigma_f^2+\frac{\sigma_\epsilon^2}{M})}{\gamma^2-(\bar{\lambda}-1)^2}=\frac{\Lambda_{\max}^2}{1-\Lambda_{\max}^2}\left(\sigma_f^2+\frac{\sigma_\epsilon^2}{M}\right) \tag{70}$$

and

$$\begin{aligned}
Cov(e_{i,t}, \bar{e}_t) &= \phi ab M V(\bar{e}_t) + \phi a(1-b) Cov(e_{i,t}, \bar{e}_t) + \phi^2(\sigma_f^2 + \frac{\sigma_\epsilon^2}{M}) \\
&= \frac{\phi ab M V(\bar{e}_t) + \phi^2(\sigma_f^2 + \frac{\sigma_\epsilon^2}{M})}{1 - \phi a(1-b)} \\
&= \frac{(\bar{\lambda}-1)^2(\sigma_f^2 + \frac{\sigma_\epsilon^2}{M})}{\gamma^2 - (\bar{\lambda}-1)^2} \\
&= \frac{\Lambda_{\max}^2}{1-\Lambda_{\max}^2} \left(\sigma_f^2 + \frac{\sigma_\epsilon^2}{M}\right) \\
&= V(\bar{e}_t). \qquad (71)
\end{aligned}$$

Hence, the variance of $e_{i,t}$ reads

$$\begin{aligned}
V(e_{i,t}) \;=\;& a^2(1-b)^2 V(e_{i,t}) + a^2b^2M^2 V(\bar{e}_t) + 2a^2b(1-b)M Cov(e_{i,t}, \bar{e}_t) + \\
&+\; a^2(1-b)^2(\sigma_f^2 + \sigma_\epsilon^2) + a^2(b^2M^2 + 2b(1-b)M)(\sigma_f^2 + \sigma_\epsilon^2/M) \\
=\;& \frac{\left(a^2b^2M^2 + 2a^2b(1-b)M\right) V(\bar{e}_t) + a^2(1-b)^2(\sigma_f^2 + \sigma_\epsilon^2) + a^2(b^2M^2 + 2b(1-b)M)(\sigma_f^2 + \frac{\sigma_\epsilon^2}{M})}{1 - a^2(1-b)^2}
\end{aligned} \qquad (72)$$

and the covariance between two different stocks is

$$\begin{aligned}
Cov(e_{i,t}, e_{j,t}) =\;& a^2(1-b)^2 Cov(e_{i,t}, e_{j,t}) + a^2b^2M^2 V(\bar{e}_t) + 2a^2b(1-b)M Cov(e_{i,t}, \bar{e}_t) \\
&+ a^2(b^2M^2 + 2b(1-b)M)(\sigma_f^2 + \frac{\sigma_\epsilon^2}{M} + a^2(1-b)^2\sigma_f^2) \\
=\;& \frac{\left(a^2b^2M^2 + 2a^2b(1-b)M\right) V(\bar{e}_t) + a^2(b^2M^2 + 2b(1-b)M)(\sigma_f^2 + \frac{\sigma_\epsilon^2}{M}) + a^2(1-b)^2\sigma_f^2}{1 - a^2(1-b)^2}
\end{aligned} \qquad (73)$$

By substituting back a, b, ϕ, and $V(\bar{e}_t)$, and defining $\tilde{\lambda} = \bar{\lambda} - 1$ we get the expression of the variance and covariance of $e_{i,t}$ in terms of the original variables. The expression are quite long and for this reason we do not report them here. Instead we report their value under the approximation $b \simeq \frac{m-1}{M-1}$ which is very good and also is the value obtained in a simplified model where banks are homogeneous and invest exactly in m assets. The approximated expressions are:

$$V(e_{i,t}) \simeq -\frac{\tilde{\lambda}^2\Big(m^2\Big(\sigma_\epsilon^2\left(\tilde{\lambda}^2 - \gamma^2(M-1)\right) + \sigma_f^2\left(\tilde{\lambda}^2 - \gamma^2(M-1)^2\right)\Big) + 2m\Big(M\left(\sigma_\epsilon^2\left(\gamma^2 - \tilde{\lambda}^2\right) - \tilde{\lambda}^2\sigma_f^2\right) - \gamma^2\sigma_\epsilon^2\Big) + M\Big(M\left(\sigma_\epsilon^2\left(\tilde{\lambda}^2 - \gamma^2\right) + \tilde{\lambda}^2\sigma_f^2\right) + \gamma^2\sigma_\epsilon^2\Big)\Big)}{\left(\gamma^2 - \tilde{\lambda}^2\right)\left(m^2\left(\gamma^2(M-1)^2 - \tilde{\lambda}^2\right) + 2\tilde{\lambda}^2 mM - \tilde{\lambda}^2M^2\right)} \qquad (74)$$

$$Cov(e_{i,t}, e_{j,t}) \simeq -\frac{\tilde{\lambda}^2\left(m^2\left(\sigma_f^2\left(\tilde{\lambda}^2 - \gamma^2(M-1)^2\right) - \gamma^2(M-2)\sigma_\epsilon^2\right) - 2m\left(\tilde{\lambda}^2M\sigma_f^2 + \gamma^2\sigma_\epsilon^2\right) + M\left(\tilde{\lambda}^2M\sigma_f^2 + \gamma^2\sigma_\epsilon^2\right)\right)}{\left(\gamma^2 - \tilde{\lambda}^2\right)\left(m^2\left(\gamma^2(M-1)^2 - \tilde{\lambda}^2\right) + 2\tilde{\lambda}^2 mM - \tilde{\lambda}^2M^2\right)} \qquad (75)$$

and that of the correlations

$$Corr(e_{i,t}, e_{j,t}) \simeq \frac{m^2\left(\gamma^2(M-2)\sigma_\epsilon^2 + \sigma_f^2\left(\gamma^2(M-1)^2 - \tilde{\lambda}^2\right)\right) + 2m\left(\tilde{\lambda}^2M\sigma_f^2 + \gamma^2\sigma_\epsilon^2\right) - M\left(\tilde{\lambda}^2M\sigma_f^2 + \gamma^2\sigma_\epsilon^2\right)}{m^2\left(\sigma_\epsilon^2\left(\gamma^2(M-1) - \tilde{\lambda}^2\right) + \sigma_f^2\left(\gamma^2(M-1)^2 - \tilde{\lambda}^2\right)\right) + 2m\left(M\left(\sigma_\epsilon^2\left(\tilde{\lambda}^2 - \gamma^2\right) + \tilde{\lambda}^2\sigma_f^2\right) + \gamma^2\sigma_\epsilon^2\right) + M(M(\sigma_\epsilon^2(\gamma^2 - (\lambda-1)^2) - \tilde{\lambda}^2\sigma_f^2) - \gamma^2\sigma_\epsilon^2)} \qquad (76)$$

for which we can prove $Corr(e_{i,t}, e_{j,t}) \xrightarrow[m \to M]{} 1$.

Finally, we compute the variance of the sum of all the portfolios held by the N banks. Using equation (33) and (35) for $V\left(r^p_{j,t}\right)$ and $Cov\left(r^p_{h,t}, r^p_{k,t}\right)$, we have

$$
\begin{aligned}
V\left(\sum_{j=1}^{N} r^p_{j,t}\right) =& NV\left(r^p_{j,t}\right) + N(N-1)Cov\left(r^p_{h,t}, r^p_{k,t}\right) \\
&\simeq N(m^3\left(\tilde{\lambda}^2\sigma_\epsilon\left(-\gamma^2+\tilde{\lambda}^2+N\left(\gamma^2(M-1)^2-\tilde{\lambda}^2\right)\right)+\gamma^2MN\sigma_f\left(\gamma^2(M-1)^2-(\lambda-1)^2\right)\right)+m^2M \\
&\left(\sigma_\epsilon\left(3\tilde{\lambda}^2\left(\gamma^2-(\lambda-1)^2\right)+N\left(3\tilde{\lambda}^4-\gamma^2\tilde{\lambda}^2\left(M^2-2M+2\right)+\gamma^4(M-1)^2\right)\right)+2\gamma^2\tilde{\lambda}^2MN\sigma_f\right) \\
&-\tilde{\lambda}^2mM^2\left(\gamma^2MN\sigma_f+\sigma_\epsilon\left(3\left(\gamma^2-\tilde{\lambda}^2\right)+N\left(3\tilde{\lambda}^2-2\gamma^2\right)\right)\right)+\tilde{\lambda}^2M^3(N-1)\sigma_\epsilon\left(\tilde{\lambda}^2-\gamma^2\right))/ \\
&mM\left(\gamma^2-\tilde{\lambda}^2\right)\left(m^2\left(\gamma^2(M-1)^2-\tilde{\lambda}^2\right)+2\tilde{\lambda}^2mM-\tilde{\lambda}^2M^2\right)
\end{aligned}
\tag{77}
$$

which reduces to

$$
V\left(\sum_{j=1}^{N} r^p_{j,t}\right) \xrightarrow[m \to M]{} \frac{\gamma^2 N^2(\sigma_f + \frac{\sigma_\epsilon}{M})}{\left(\gamma^2 - \tilde{\lambda}^2\right)} = \frac{N^2\sigma^2_{p_M}}{1-\Lambda_{\max}} \tag{78}
$$