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Anomalous dimensions of finite size field strength operators in $\mathcal{N} = 4$ SYM

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ABSTRACT: In the $\mathcal{N} = 4$ super Yang-Mills theory, we consider the higher order anomalous dimensions $\gamma_L(g)$ of purely gluonic operators $\mathrm{Tr}\,\mathcal{F}^L$ where $\mathcal{F}$ is a component of the self-dual field strength. We propose compact closed expressions depending parametrically on L that reproduce the prediction of Bethe Ansatz equations up to five loop order, including transcendental dressing corrections. The size dependence follows a simple pattern as the perturbative order is increased and suggests hidden relations for these special operators.

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1. Introduction

Integrable structures emerge as a deep property of four dimensional Yang-Mills theories in the 't Hooft planar limit. In the simplest context, integrability underlies and governs the scale evolution of renormalized composite operators belonging to specific subsectors of the theory [1].

Historically, this intriguing phenomenon was discovered in the study of planar QCD, definitely a non-trivial quantum theory [2]. At one-loop, suitable maximal helicity Wilson operators admit a peculiar renormalization mixing matrix, the dilatation operator. It can be identified with the Hamiltonian of integrable XXX spin chains with $\mathfrak{sl}(2,\mathbb{R})$ symmetry. This is a light-cone subalgebra of the full four dimensional conformal algebra $\mathfrak{so}(4,2)$.

¿From a modern perspective, conformal symmetry, unbroken in QCD at one-loop, does not appear to be a necessary condition for integrability, as discussed in [3, 4, 5, 6]. Nevertheless, it plays an important role by imposing selection rules and multiplet structures and is helpful to clarify the origin and details of integrability. The same reasoning applies to supersymmetric extensions of QCD with $\mathcal{N} = 1, 2, 4$ supercharges. In particular, multiplets of composite operators are greatly simplified in the maximal superconformal $\mathcal{N} = 4$ theory [7]. Also, intermediate level integrability is achieved in various orbifold reductions of $\mathcal{N} = 4$ SYM reducing the number of supercharges [8].

As is well known, in the maximal $\mathcal{N} = 4$ case another conceptual tool is available to deepen the investigation, namely Maldacena AdS/CFT duality [9] [1]. It relates $\mathcal{N} = 4$ SYM and $AdS_5 \times S^5$ superstring which is classically integrable [12]. Currently, a lot is known about the duality between the integrability properties of the two sides of the correspondence with a continuous very stimulating back and forth feeding. In particular, AdS/CFT duality has been a crucial ingredient to arrive at the higher loop proposal for the S-matrix of $\mathcal{N} = 4$ SYM [13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24].

Forgetting for a while the string side, we can ask what kind of understanding can be gained from integrability in the gauge theory. This is a natural issue if we are ultimately interested in low-energy physical applications to hadronic phenomenology. As a first step, we can honestly postpone the important problem of taming conformal and supersymmetry breaking and the precise link with QCD [25]. Working within the $\mathcal{N} = 4$ SYM theory, we can examine the outcomes and limitations of integrability as far as it is currently understood.

¿From this point of view, integrability can be regarded as a tool for multi-loop computations, although this attitude could be admittedly narrow-minded. The typical object that is computed are higher order corrections to the anomalous dimension of specific composite operators. In all cases, these operators are single traces of the general form

$$\mathcal{O} = \text{Tr}\left(\prod_{i=1}^{L} D^{n_i}\, X_i\right) + \text{permutations}, \tag{1.1}$$

where X_i are elementary fields in certain subsectors of the full $\mathcal{N} = 4$ SYM and covariant derivatives generically appear to close the renormalization mixing.

In the most favorable cases, we are able to write down Bethe Ansatz equations providing the anomalous dimension of $\mathcal{O}$ as a perturbative series in the 't Hooft coupling g

$$\gamma_{\mathcal{O}}(g) = \sum_{n\geq 0} c_n(\mathcal{O})\, g^{2\,n}. \tag{1.2}$$

Such formidable results face a first and major limitation, namely the well-known *wrapping problem* (see [26, 27] for recent developments). The coefficients c_n are reliable up to a maximum order $n_{\max}$ that typically depends linearly on L. This means that $\gamma_{\mathcal{O}}(g)$ is actually calculable up to, say, $\mathcal{O}(g^{2L})$ terms - a stumbling wall to any extrapolation to the genuine strong coupling regime. A notable exception occurs in the $L \to \infty$ thermodynamical limit. Then, wrapping is absent and resummations of Eq. (1.2) can be attempted to match string duality predictions [24, 28].

A more subtle limitation appears when we try to investigate the dependence on L and $\{n_i\}$ at fixed perturbative order. Apart from very special cases, the Bethe equations do not provide the expansion coefficients as functions of L and $\{n_i\}$, but just provide sequences of numerical (sometimes rational) values for each given operator. This is an unwanted situation as can be appreciated in the sector of the so-called twist operators [29]. These are operators with a certain phenomenological origin in the QCD case. The length L is

[1] For those aspects of the duality that most concern our analysis, we refer the reader to [10, 11].

fixed and one would like to know the analytic dependence of $\gamma(g, S)$ on the spin quantum number $S = \sum_i n_i$. For instance, this is a standard procedure to analyze BFKL physics of pomeron exchange [30].

Very recently, intense work on twist-2 and 3 operators has led to higher order conjectures for the functions $c_n(S)$ appearing in the expansion $\gamma(g, S) = \sum_n c_n(S)\, g^{2\,n}$ [26, 31]. Proofs are however missing, at least beyond one-loop. It seems that new tools are needed to derive them rigorously from the Bethe Ansatz equations.

If we give up exact results and turn to approximation, systematic methods can be applied to extract the large L, n_i corrections. Indeed, this is a thermodynamical limit of the underlying spin chain where both the length and the number of magnons grows to infinity. Various techniques are available and have been successfully applied to rank-1 $\mathfrak{su}(2)$ and $\mathfrak{sl}(2)$ subsectors [32, 33, 34, 35, 36]. For higher rank sectors (see also [37, 38, 21] for an analysis in the rank-2 $\mathfrak{su}(3)$ sector) the techniques developed in [39] could be useful. Indeed, in the recent [40] an integral equation describing finite size corrections to the full nested Bethe ansatz was derived.

On the string side of the AdS/CFT correspondence, there is an analogously intense ongoing discussion on how finite size effects of the string world-sheet could modify the solvability of the string sigma-model in $\mathrm{AdS}_5 \times S^5$ by means of a Bethe ansatz [41, 13, 14, 42, 16, 43]. The currently known Bethe equations for quantum strings in $\mathrm{AdS}_5\times\mathrm{S}_5$ are asymptotic and describe the string spectrum with an exponential accuracy as long as the string length is sufficiently large [44]. The breakdown of the asymptotic approximation via exponential terms, firstly described from a field theory point of view in [45], has been determined for the spectrum of spinning strings in the $\mathfrak{su}(2)$ and $\mathfrak{sl}(2)$ sector [46, 44] and for the giant magnon [47] dispersion relation in [48, 49]. In particular, the exponential term in the finite size correction to the giant magnon dispersion relation has been recently and nicely rederived in [27] via a generalization of known results in relativistic quantum field theory, and there is a general and deep interest in obtaining exact results which should be valid for any value of the string lenght, which is in turn proportional to the lenght (R-charge) of the corresponding gauge operator.

In this paper, we contribute to the above general discussion and consider, in the gauge field theory context, a special class of operators where finite size corrections can be given in closed form. In other words, we provide the coefficients c_n in Eq. (1.2) as exact functions of the operator length L. This is a seemingly unique result which, although peculiar, is very interesting and puzzling and certainly deserves some attention.

The considered operators have a complicated mixing pattern and reduce at one loop to the purely gluonic higher dimensional condensates of the form

$$\mathcal{O}_L = \mathrm{Tr}\, \mathcal{F}^L, \tag{1.3}$$

where $\mathcal{F}$ is one component of the self-dual Yang-Mills field strength. The operators $\mathcal{O}_L$ are exact eigenstates of the one-loop dilatation operator and can be mapped to the ferromagnetic states of an integrable spin $S = 1$ chain [50, 51]. As such, their one-loop dependence on the length L is trivial and (including the classical dimension), it is known that [50, 51, 52, 28]

$$\gamma_{\mathcal{O}_L} = L\,(2 + 3\,g^2 + \cdots), \tag{1.4}$$

Beyond one-loop, the analysis of [28] provide efficient computational tools to derive the sequence $\{c_n(L)\}$ for any given L, although not parametrically. It turns out immediately that $c_n(L)$ is not linear in L as far as $n \geq 2$. So, starting at two-loops, non-trivial finite size corrections appear.

In this paper, we analyze the sequences $\{c_n(L)\}$ at fixed n as L is varied up to large values. By a careful investigation of the (infinite precision) numerics, we conjecture and provide closed expressions for $c_n(L)$ valid up to 5 loops, including the transcendental terms coming from the S-matrix dressing phase [17]. To give an example, the two-loop anomalous dimension takes the remarkably simple form

$$\gamma_L(g) = 2\,L + 3\,L\,g^2 + \left(-\frac{51}{8} + \frac{9}{8}\,\frac{1}{(-1)^L\,2^{L-1}+1}\right)\,L\,g^4 + \cdots\,, \tag{1.5}$$

with exponentially suppressed corrections to the trivial linear scaling with L. We have been able to extend the above equation up to five loops. The detailed results will be illustrated in the main text. Here, we just anticipate the large L limit which reads

$$\frac{\gamma_L(g)}{L} = f_0(g) + g^4\,h(g\,L)\,e^{-L\,\log\,2} + \mathcal{O}(e^{-2\,L\,\log\,2}), \tag{1.6}$$

where $f_0(g)$ has been computed in [28]. The function $h(z)$ is regular around $z = 0$ and does not receive contributions from the dressing phase, at least up to five loops. The size corrections to the thermodynamical limit are thus characterized by a finite specific correlation length $\xi = 1/\log\,2$. In the final Section of the paper, we shall try to argue why this correlation length abruptly appears at two-loops breaking the trivial linear dependence on L.

2. One-loop ferromagnetic multi-gluon operators in the chiral sector

In this Section, we introduce the special class of $\mathcal{N} = 4$ multi-gluon operators that we are going to analyze. For completeness, we also review their one-loop integrability properties and, in particular, the reduction of the mixing matrix to the Hamiltonian of an integrable XXX_1 chain.

In the planar limit, the most general purely gluonic local gauge invariant operators are easily identified. They are single trace operators built with covariant derivatives of the field strength

$$\mathrm{Tr}\,(D^{n_1}F_{\mu_1\nu_1}\cdots D^{n_L}F_{\mu_L\nu_L})\,. \tag{2.1}$$

The anomalous dimension matrix Γ and the would-be spin chain Hamiltonian H are related by

$$\Gamma = \mu\frac{\partial}{\partial\mu}\,\log\,Z \equiv g^2\,H, \qquad g^2 = \frac{g^2_{\mathrm{YM}}\,N_c}{8\pi^2}, \tag{2.2}$$

where Z is the renormalization matrix and g is the scaled 't Hooft coupling kept fixed in the planar limit $N_c \to \infty$.

The one-loop operator H takes the form of a nearest neighbor Hamiltonian conserving the length L in Eq. (2.1). It can be written

$$H = \sum_{n=1}^{L} H_{n,n+1}, \tag{2.3}$$

where the link Hamiltonian $H_{n,n+1}$ acts on the fields at positions n and $n+1$ and is independent on n. For the complete $\mathcal{N}=4$ SYM theory, the elementary fields are included in the singleton multiplet V [10] and the link Hamiltonian reads [53]

$$H^{\mathcal{N}=4} = 2\sum_{j=0}^{\infty} h(j)\, P_j^{\mathcal{N}=4}, \qquad h(j) = \sum_{n=1}^{j}\frac{1}{n}, \tag{2.4}$$

where $P_j^{\mathcal{N}=4}$ is a projector onto the irreducible superconformal multiplets appearing in the decomposition of the two-site states $V\otimes V$.

To restrict the analysis to purely gluonic operators it is convenient to adopt the conformal analysis exploited in the QCD reduction described in [50, 51] (see also [54]). We first split into irreducible components the field strength $F_{\mu\nu}$ transforming as $(1,0)\oplus(0,1)$ under the $\mathfrak{so}(3,1)=\mathfrak{su}(2)\oplus\mathfrak{su}(2)$ Lorentz algebra. This is achieved by means of the 't Hooft symbols [55] projecting $F_{\mu\nu}$ onto the self-dual $(1,0)$ and anti-self-dual $(0,1)$ components

$$F_{\mu\nu} = \eta^A_{\mu\nu}\, f^A + \overline{\eta}^A_{\mu\nu}\, \overline{f}^A, \qquad A=1,2,3. \tag{2.5}$$

The purely chiral gluon operators are the subset of Eq. (2.1) built using only the self-dual part of $F_{\mu\nu}$

$$\mathrm{Tr}\left\{D^{n_1} f^{A_1}\cdots D^{n_L} f^{A_L}\right\}. \tag{2.6}$$

At one loop, they close under renormalization mixing. The relevant link Hamiltonian can be obtained by restriction of $H^{\mathcal{N}=4}$. To this aim, it is convenient to organize the various covariant derivatives of f^A in a $\mathcal{N}=0$ conformal infinite dimensional multiplet

$$V^f = \{D^n f\}_{n\geq 0}. \tag{2.7}$$

Two-site states decompose in irreducible multiplets labeled by the conformal spin j according to

$$V^f\otimes V^f = \bigoplus_{j=-2}^{\infty} V_j^{ff}. \tag{2.8}$$

Also, the conformal splitting of the full $\mathcal{N}=4$ projector $P_j^{\mathcal{N}=4}$ turns out to involve the conformal projector P_{j-2}^{ff} only. This leads to the following purely gluonic link Hamiltonian in the chiral sector

$$H = 2\sum_{j=-2}^{\infty} h(j+2)\, P_j^{ff}. \tag{2.9}$$

Finally, if we further restrict to operators without derivatives, one can prove that the only modules appearing in the r.h.s. of Eq. (2.8) are those with $j=-2,-1,0$ [51]. To make contact with the spin chain interpretation, we introduce spin $S = 1$ $\mathfrak{su}(2)$ operators $\{S^i\}$ acting on the three components f^A as

$$(S^i\, f)^A = i\, \varepsilon^{iAB}\, f^B. \tag{2.10}$$

Then, the modules P_j^{ff} with $j = -2, -1, 0$ can be shown to be associated with the $\mathfrak{su}(2)$ representations with $S = 0, 1, 2$ respectively, appearing in the decomposition $1 \otimes 1 = 2 \oplus 1 \oplus 0$. The link Hamiltonian $H_{n,n+1}$ in Eq. (2.9) can be written as a polynomial in $\mathbf{S}_n \cdot \mathbf{S}_{n+1}$ with the result

$$H = 3L + \frac{1}{2}\sum_n \left[\mathbf{S}_n \cdot \mathbf{S}_{n+1} - (\mathbf{S}_n \cdot \mathbf{S}_{n+1})^2\right]. \tag{2.11}$$

This is an anti-ferromagnetic integrable spin-chain that can be diagonalized by Bethe Ansatz [56]. The ground state is highly non-trivial, but the maximally excited states are a trivial ferromagnetic multiplet. A convenient representative is the operator

$$\mathcal{O}_L = \mathrm{Tr}(\mathcal{F}^L), \tag{2.12}$$

where $\mathcal{F} = f^+$ is the maximal eigenstate of S^z. The anomalous dimension of this state (including the classical dimension) is simply

$$\gamma_L(g) = 2\,L + 3\,L\,g^2 + \mathcal{O}(g^2). \tag{2.13}$$

The linear dependence on L follows from the uniform structure of the ferromagnetic state.

In this paper, we shall be working on the conformal/field-theory side of the AdS/CFT correspondence. However, it must be mentioned that the natural candidate for a semi-classical string state dual to $\mathcal{O}_L$ has been proposed in [57, 58]. It describes a rigid circular string rotating simultaneously in two orthogonal spatial planes of AdS_5 with equal spins $S \equiv L$. At large S, the weak-coupling extrapolation for the energy is given by

$$\begin{aligned} E &= p(\lambda)\,S + q(\lambda) + \dots, \\ p_{\lambda\gg 1} &= p_0 + \frac{p_1}{\sqrt{\lambda}} + \dots, \qquad\qquad q_{\lambda\gg 1} = \sqrt{\lambda}q_0 + q_1 + \dots . \end{aligned} \tag{2.14}$$

The results for the three-level and 1-loop coefficients of the solution have been calculated in [58] within a stability region $0.4 \lesssim \frac{S}{\sqrt{\lambda}} \gtrsim 1.17$, corresponding to a fixed value ($m = 1$) of the winding number. It should be noticed that, since in the semiclassical approximation λ is large on the string side, the interval of stability for the solution does include large values of S, allowing the comparison to large S, large λ asymptotics of the exact anomalous dimension. The linear dependence on S exhibited by the solution (2.14) supports the identification of the gauge theory operator $\mathcal{O}_L$ with this particular rigid spinning string solution.

3. Higher loop extension of the scaling field $\mathcal{O}_L$

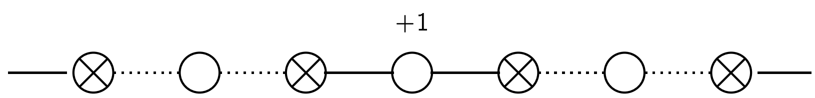
At more than one-loop, the operator $\mathcal{O}_L$ ceases to be an eigenstate of the dilatation operator because the purely gluonic chiral sector does not close under mixing anymore. The higher order scaling operator receives corrections and contributions from the other sectors of the full $\mathfrak{psu}(2,2|4)$ theory and is uniquely defined by the boundary condition of being $\mathcal{O}_L$ at one-loop. In the following, we shall not be pedantic about this distinction and keep naming $\mathcal{O}_L$ the multi-loop extension of $\mathrm{Tr}\mathcal{F}^L$.

A great deal of information about $\mathcal{O}_L$ has been obtained in [28] in the framework of the long-range Bethe Ansatz equations. In this Section, we quickly summarize these results with some additional investigation of the finite but large L Bethe roots. This will fix the setup for the computation of $\gamma_L(g)$.

3.1 Dynkin diagrams and Bethe roots

As is well known, several choices are available for the Dynkin diagram of a Lie superalgebra. In the case of $\mathfrak{psu}(2,2|4)$, the one loop analysis of the operators $\mathcal{O}_L$ is almost trivial with the Kac distinguished form. Indeed, $\mathcal{O}_L$ is the vacuum state and no calculation is needed.

On the other hand, the all-loop Bethe equations are known for a limited set of (different) choices of the Dynkin diagram [16]. In particular, we shall work with the following one

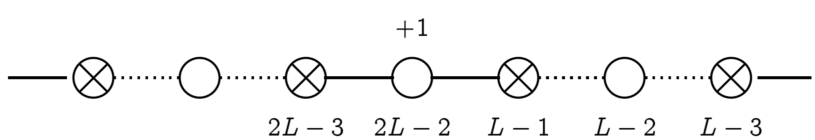


(3.1)

With respect to this Dynkin diagram, the vacuum is the BPS state Tr $\mathcal{Z}^L$ and $\mathcal{O}_L$ is a highly excited state with many excitations, whose momenta have to be diagonalized by solving the Bethe Ansatz equations in order to reproduce the correct energy. The excitation pattern of Bethe roots for $\mathcal{O}_L$ is

$$(K_1, K_2, K_3, K_4, K_5, K_6, K_7) = (0, 0, 2L-3, 2L-2, L-1, L-2, L-3) \tag{3.2}$$

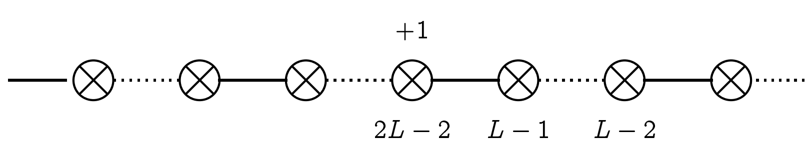
where K_i is the excitation number of the i-th node of the Dynkin diagram



(3.3)

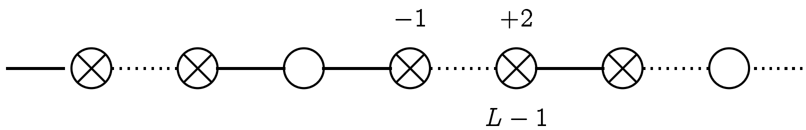
All but the first two nodes are highly excited.

All the one-loop Bethe equations can be exhibited as roots of explicit polynomial by means of the dualization procedure illustrated in [59] to which we defer the reader for more details. It is instructive to describe the procedure in graphical terms. Dualizing first at nodes 3 and 7, we obtain



(3.4)

Dualizing at nodes 4 and 6, we obtain



(3.5)

The Bethe equations are thus reduced to the simple equation

$$\left(\frac{u_{5,k}+i}{u_{5,k}-i}\right)^L = 1, \qquad k = 1, \dots, L-1, \tag{3.6}$$

which is solved by

$$u_{5,k} = \cot\frac{\pi\,k}{L}. \tag{3.7}$$

The dualization process can be inverted step by step providing exact polynomials whose roots are the Bethe roots at any finite L. In particular, one finds for the roots at node 4 and 6 the explicit result [28]

$$\begin{aligned}
Q_4(u) &= \left(u+\frac{i}{2}\right)^L\left[\left(u+\frac{3\,i}{2}\right)^L-\left(u-\frac{i}{2}\right)^L\right]+\left(u-\frac{i}{2}\right)^L\left[\left(u-\frac{3\,i}{2}\right)^L-\left(u+\frac{i}{2}\right)^L\right]. \\
Q_6(u) &= \left(u+\frac{3\,i}{2}\right)^L+\left(u-\frac{3\,i}{2}\right)^L-\left(u+\frac{i}{2}\right)^L-\left(u-\frac{i}{2}\right)^L.
\end{aligned} \tag{3.8}$$

Of course, from the knowledge of $Q_4(u)$ one can prove again the one-loop result Eq. (2.13).

Going over to higher orders, we have to work with the long range Bethe equations which are a deformation of the one-loop ones. They involve the standard quantities

$$x(u) = \frac{u}{2}\left(1+\sqrt{1-\frac{2\,g^2}{u^2}}\right), \qquad x^\pm = x\left(u\pm\frac{i}{2}\right), \tag{3.9}$$

and read

$$\begin{aligned}
1 &= \prod_{j=1}^{2L-2}\frac{x_{3,k}-x^+_{4,j}}{x_{3,k}-x^-_{4,j}}, \\
\left(\frac{x^+_{4,k}}{x^-_{4,k}}\right)^L &= \prod_{\substack{j=1\\ j\neq k}}^{2L-2}\frac{x^+_{4,k}-x^-_{4,j}}{x^-_{4,k}-x^+_{4,j}}\,\frac{1-g^2/2\,x^+_{4,k}x^-_{4,j}}{1-g^2/2\,x^-_{4,k}x^+_{4,j}}\,\sigma^2(u_{4,k},u_{4,j}) \\
&\quad\times\prod_{j=1}^{2L-3}\frac{x^-_{4,k}-x_{3,j}}{x^+_{4,k}-x_{3,j}}\prod_{j=1}^{L-1}\frac{x^-_{4,k}-x_{5,j}}{x^+_{4,k}-x_{5,j}}\prod_{j=1}^{L-3}\frac{1-g^2/2\,x^-_{4,k}x_{7,j}}{1-g^2/2\,x^+_{4,k}x_{7,j}}, \\
1 &= \prod_{j=1}^{L-2}\frac{u_{5,k}-u_{6,j}+\frac{i}{2}}{u_{5,k}-u_{6,j}-\frac{i}{2}}\prod_{j=1}^{2L-2}\frac{x_{5,k}-x^+_{4,j}}{x_{5,k}-x^-_{4,j}}, \\
1 &= \prod_{\substack{j=1\\ j\neq k}}^{L-2}\frac{u_{6,k}-u_{6,j}-i}{u_{6,k}-u_{6,j}+i}\prod_{j=1}^{L-1}\frac{u_{6,k}-u_{5,j}+\frac{i}{2}}{u_{6,k}-u_{5,j}-\frac{i}{2}}\prod_{j=1}^{L-3}\frac{u_{6,k}-u_{7,j}+\frac{i}{2}}{u_{6,k}-u_{7,j}-\frac{i}{2}}, \\
1 &= \prod_{j=1}^{L-2}\frac{u_{7,k}-u_{6,j}+\frac{i}{2}}{u_{7,k}-u_{6,j}-\frac{i}{2}}\prod_{j=1}^{2L-2}\frac{1-g^2/2\,x_{7,k}x^+_{4,j}}{1-g^2/2\,x_{7,k}x^-_{4,j}},
\end{aligned} \tag{3.10}$$

where $\sigma^2(u_k, u_j)$ is the dressing phase to be discussed later.

It is possible to perform a *partial* dualization of these equations and obtain reduced long-range equations involving roots at nodes 4, 5, 6 only. These are

$$\begin{aligned}
\left(\frac{x^+_{4,k}}{x^-_{4,k}}\right)^L &= \prod_{\substack{j=1\\ j\neq k}}^{2L-2} \frac{x^-_{4,k}-x^+_{4,j}}{x^+_{4,k}-x^-_{4,j}}\, \frac{1-\frac{g^2}{2\,x^+_{4,k}x^-_{4,j}}}{1-\frac{g^2}{2\,x^-_{4,k}x^+_{4,j}}}\,\sigma^2(u_{4,k},u_{4,j}) \prod_{j=1}^{2L-4}\frac{x^+_{4,k}-\widetilde{x}_{5,j}}{x^-_{4,k}-\widetilde{x}_{5,j}}, \\
1 &= \prod_{j=1}^{L-2}\frac{\widetilde{u}_{5,k}-u_{6,j}+\frac{i}{2}}{\widetilde{u}_{5,k}-u_{6,j}-\frac{i}{2}} \prod_{j=1}^{2L-2}\frac{\widetilde{x}_{5,k}-x^+_{4,j}}{\widetilde{x}_{5,k}-x^-_{4,j}}, \\
1 &= \prod_{\substack{j=1\\ j\neq k}}^{L-2}\frac{u_{6,k}-u_{6,j}+i}{u_{6,k}-u_{6,j}-i} \prod_{j=1}^{2L-4}\frac{u_{6,k}-\widetilde{u}_{5,j}-\frac{i}{2}}{u_{6,k}-\widetilde{u}_{5,j}+\frac{i}{2}}.
\end{aligned} \tag{3.11}$$

Here, $\widetilde{u}_5$ are the $2L-4$ roots dual to u_5. At one loop, they are the roots of the polynomial

$$\begin{aligned}
Q_5(u) =\ & 3u^{2L}+(-i+u)^L(-2i+u)^L+(2i+u)^L\left((i+u)^L+(-2i+u)^L\right) \\
& -u^L\left((-i+u)^L+(i+u)^L+2(-2i+u)^L+2(2i+u)^L\right).
\end{aligned} \tag{3.12}$$

The roots at nodes 4 and 6 are still given at one-loop by the previous polynomials. At generic g, the anomalous dimension is obtained from the roots $u_{4,k}(g)$ alone and reads

$$\gamma_L(g) = 2\,L + g^2\sum_{k=1}^{K_4}\left(\frac{i}{x^+(u_{4,k})}-\frac{i}{x^-(u_{4,k})}\right). \tag{3.13}$$

Finally, let us consider the dressing phase. It enters the calculation starting from four loops. Its general form is discussed in [24]. The terms relevant for a computation up to five loops are simply

$$\sigma^2(u,u') = e^{i\,\vartheta(u,u')}, \tag{3.14}$$

where

$$\vartheta(u,u') = (\zeta_3\,g^6-5\,\zeta_5\,g^8)\,(q_2(u)\,q_3(u')-q_2(u')\,q_3(u))+\cdots, \tag{3.15}$$

$$q_2(u) = i\left(\frac{1}{x^+(u)}-\frac{1}{x^-(u)}\right),\quad q_3(u)=\frac{i}{2}\left(\frac{1}{x^+(u)^2}-\frac{1}{x^-(u)^2}\right).$$

The coefficients ζ_n are transcendental sums $\zeta_n = \sum_{\ell=1}^{\infty}\ell^{-n}$.

3.2 The one-loop Bethe roots: some numerics at large but finite L

The one-loop Bethe roots are the zeroes of the polynomials $Q_{4,5,6}(u)$. It is instructive to study them at large L comparing with the results of [28] obtained in the $L\to\infty$ limit. First the (dual) roots u_5. They are complex. We show them at $L = 100, 200, 350, 500$ in Fig. (1). As predicted, most of them are distributed along two segments with $\mathrm{Im}\,u_5 = \pm\frac{i}{2}$. Apart from these roots, other ones are scattered in the complex plane according to a nice regular pattern. To understand these roots, we look for a Bethe root admitting the expansion

$$u = L\left(x_0 + \frac{x_1}{L^{1/2}} + \frac{x_2}{L} + \cdots\right). \tag{3.16}$$

A roots with leading behavior $u \sim L$ is called *extremal* in [28]. Replacing this expansion in $Q_6(u)$, we obtain a well defined large L expansion for the ratio

$$R(x_0, x_1, \dots; L) = \frac{Q_6(u)}{u^{2L}}. \tag{3.17}$$

The leading term is

$$R = 64\,\cos^2\frac{1}{2\,x_0}\,\sin^4\frac{1}{2\,x_0} + \mathcal{O}(L^{-1/2}), \tag{3.18}$$

leading to $x_0 = \frac{1}{n\,\pi}$ for any integer n. Considering separately the cases n even/odd and expanding at higher order in $L^{-1/2}$ one finds the solutions (in the first quadrant)

$$\alpha_{\pm,k} = \frac{1}{(2\,k+1)\,\pi}\left(L \pm \sqrt{\frac{L}{2}}\right) + \mathcal{O}\left(L^{-1/2}\right), \qquad k = 0, 1, 2, \dots, \tag{3.19}$$

$$\beta_{\pm,k} = \frac{1}{2\,k\,\pi}\left(L \pm \frac{1}{2}\sqrt{L}\sqrt{3 \pm i\sqrt{15}} \pm \frac{i}{\sqrt{15}}\right) + \mathcal{O}\left(L^{-1/2}\right), \qquad k = 1, 2, \dots. \tag{3.20}$$

The other roots are related by reflection with respect to the coordinate axis. For large L, the α-roots appear in real close pairs. These pairs are closer to the origin as k is increased. In general, for a given L, only a finite number of such pairs is well approximated by the above formula. The β-roots have an imaginary part and also appear in close pairs. In Fig. (1) we draw crosses at the first α and β pairs.

The roots $u_{4,n}$ and $u_{6,n}$ are real. Their density is defined in the $L \to \infty$ continuum limit as $\rho(u) = dn/du$ and the analytical prediction is

$$\rho_4(u) = \frac{1}{2\,\pi}\left(\frac{1}{u^2+\frac{1}{4}} + \frac{3}{u^2+\frac{9}{4}}\right), \qquad \rho_6(u) = \frac{1}{2\,\pi}\frac{3}{u^2+\frac{9}{4}}. \tag{3.21}$$

In the discrete case at finite L, we can plot the points

$$\left(\frac{u_n + u_{n+1}}{2}, \frac{4\,L}{u_{n+1} - u_n}\right). \tag{3.22}$$

The result is shown in Figs. (2,3) for the Bethe roots at $L = 200$. The agreement is quite good in the case of u_4. For u_6, we observe a deviation in the tails of the distribution at large $|u_6|$. It can be understood as in the above discussion of extremal u_5 roots.

4. Perturbative expansion of the long-range Bethe equations

Starting from the exact (*i.e.* known with arbitrarily high precision) one-loop Bethe roots we can make a perturbative expansion in even powers of g

$$u_{a,k} = \sum_{n=0}^{\infty} g^{2\,n}\,u_{a,k}^{(n)}, \qquad a = 4, 5, 6, \qquad k = 1, \dots, K_a, \tag{4.1}$$

where we relabel $\tilde{u}_5 \equiv u_5$, $\tilde{K}_5 \equiv K_5$. The explicit five loop expansion of the anomalous dimension can be compared with the results of [28] up to $L = 8$. We have extended the calculation up to $L = 60$. The results at five loops for $L \leq 20$ are shown in Appendix (A).

The expansion is a rational combination of 1, ζ_3 and ζ_5. As we mentioned, the zero and one loop results are proportional to L. In general, it is convenient to redefine

$$\gamma_L(g) = L\left(2 + 3\,g^2 + \sum_{n\geq 2} c_n(L)\,g^{2\,n}\right). \tag{4.2}$$

We now show that it is possible to provide simple closed expressions for the non-trivial functions $c_n(L)$. As a constraint, we must meet the exact expansion in the $L \to \infty$ limit obtained in [28] and reading at five loops

$$\begin{aligned} c_2(\infty) &= -\frac{51}{8}, \\ c_3(\infty) &= \frac{393}{16}, \\ c_4(\infty) &= -\frac{59487}{512} - \frac{27}{4}\zeta_3, \\ c_5(\infty) &= \frac{632661}{1024} + \frac{1665}{32}\zeta_3 + \frac{135}{4}\zeta_5. \end{aligned} \tag{4.3}$$

As a general remark, it is instructive to plot the numerical values of $c_n(L)$ at the first values of L. Indeed, it is immediately clear that factors $(-1)^L$ can appear in the closed formula for $c_n(L)$. Therefore, we shall analyze the odd and even L cases separately.

4.1 Two loops

For odd $L = 5, 7, 9, \dots$, we subtract the asymptotic value $c_2(\infty)$ and rescale to find

$$\frac{8}{3}(c_2(L) - c_2(\infty)) = -\frac{1}{5}, -\frac{1}{21}, -\frac{1}{85}, -\frac{1}{341}, -\frac{1}{1365}, -\frac{1}{5461}, -\frac{1}{21845}, -\frac{1}{87381}, -\frac{1}{349525}, \dots \tag{4.4}$$

A careful inspection reveals that the denominators are simply related to powers of 2 minus one. The precise formula is easily found and reads

$$c_2(L) = -\frac{51}{8} - \frac{9}{8}\frac{1}{2^{L-1} - 1}, \qquad L \text{ odd}. \tag{4.5}$$

We checked it for all the L that we have explored. Remarkably, it works also for the even L case if the sign of the term $\sim 2^L$ is changed. The final formula is thus

$$c_2(L) = -\frac{51}{8} + \frac{9}{8}\frac{1}{(-1)^L\,2^{L-1} + 1}. \tag{4.6}$$

This simple result is rather remarkable. It holds at finite L and predict exponentially suppressed deviations from the trivial linear scaling of the anomalous dimension $\gamma \sim L$, valid up to the one-loop level. Is it possible to obtain a similar result for the next three loop contribution ?

4.2 Three loops

Following the strategy adopted in the two-loop case, we start again from odd $L = 5, 7, 9, \ldots$ and evaluate

$$c_3(L) - c_3(\infty) = +\frac{111}{1600}, -\frac{425}{3136}, -\frac{3628101}{39304000}, \\ -\frac{9904623}{230701504}, -\frac{7874523}{463736000}, -\frac{63804855621}{10423090379584}, \ldots. \tag{4.7}$$

This sequence appears to be definitely non trivial and much more complicated than the two-loop case. In particular, the signs are not definite and the denominators do not have simple factorization properties. However, the sequence enjoys a remarkable property. If we multiply it by $(2^{L-1} - 1)^3$ and apply a constant scaling, we find

$$\frac{2^6}{3^4}(2^{L-1} - 1)^3 (c_3(L) - c_3(\infty)) = 185, -26775, -1209367, -36316951, -921319191, \\ -21268285207, -461958727447, -9613145655063, -193758643734295, \ldots. \tag{4.8}$$

Indeed, the sequence is integer for all considered L. As a second feature, one can plot the following function of L

$$(2^{L-1} - 1)(c_3(L) - c_3(\infty)), \tag{4.9}$$

and it turns out to be curve quite close to a quadratic parabola. From these two features, it is natural to look for a closed formula of the form

$$c_3(L) - c_3(\infty) = \frac{1}{(2^L - 2)^3} \sum_{p=0}^{2} 2^{p\,L} \sum_{q=0}^{2} c_{p,q}\, L^q. \tag{4.10}$$

Indeed, it turns out that all the three loop results at odd L are reproduced by

$$c_3(L) = \frac{393}{16} + \frac{-9 \cdot 2^{2L}\,(9\,L^2 - 33\,L - 104) - 18 \cdot 2^L\,(9\,L^2 + 15\,L + 202) + 3528}{64\,(-2 + 2^L)^3}, \quad L \text{ odd}. \tag{4.11}$$

Looking back at Eq. (4.6), there is a striking similarity suggesting an all order structure. In particular, the same formula works for even L, if we apply the modification rules

$$2^{2\,p\,L} \to 2^{2\,p\,L}, \qquad 2^{(2\,p+1)\,L} \to -(-1)^L\, 2^{(2\,p+1)\,L}. \tag{4.12}$$

The general formula is then

$$c_3(L) = \frac{393}{16} + \frac{9 \cdot 2^{2L}\,(9\,L^2 - 33\,L - 104) - 18 \cdot (-1)^L\, 2^L\,(9\,L^2 + 15\,L + 202) - 3528}{64 \cdot 8\,[(-1)^L\, 2^{L-1} + 1)]^3}. \tag{4.13}$$

4.3 Four loops

At four loops, we attempt to repeat the game. The only new feature is the transcendental contribution from the dressing phase. This is a piece of $c_4(L)$ proportional to ζ_3. From the numerics, it is independent on L and reads

$$c_4(L) = c_4^{(0)}(L) + c_4^{(3)}(L)\,\zeta_3, \qquad c_4^{(3)}(L) \equiv -\frac{27}{4}. \tag{4.14}$$

The $c_4^{(0)}(L)$ is a rational contribution with properties quite analogous to those of $c_3(L)$. In particular, for odd L

$$(i) \quad \frac{2^8}{3^7}\,(2^{L-1}-1)^5\,(c_4^{(0)}(L)-c_4^{(0)}(\infty)) \in \mathbb{N}, \tag{4.15}$$

$$(ii) \quad (2^{L-1}-1)\,(c_4^{(0)}(L)-c_4^{(0)}(\infty)) \sim L^4, \;\; \text{for } L\to\infty. \tag{4.16}$$

Again, it is natural to postulate from (i) and (ii) the closed formula

$$c_4^{(0)}(L)-c_4^{(0)}(\infty) = \frac{1}{(2^L-2)^5}\sum_{p=0}^{4} 2^{p\,L}\sum_{q=0}^{4} d_{p,q}\,L^q. \tag{4.17}$$

Replacing the explicit anomalous dimensions in this formula we find that indeed it is satisfied by all considered (odd) L with coefficients $d_{p,q}$ giving

$$c_4^{(0)}(L) = -\frac{59487}{512}+\frac{2^{4L}\,Q_{4,4}+2^{3L}\,Q_{4,3}+2^{2L}\,Q_{4,2}+2^{L}\,Q_{4,1}-1335168}{2^{15}\,(2^{L-1}-1)^5}, \quad L \text{ odd}, \tag{4.18}$$

where the Q polynomials are

$$\begin{aligned}
Q_{4,1} &= -72\left(27\,L^4+90\,L^3-1485\,L^2-2004\,L-38456\right), \\
Q_{4,2} &= -108\left(99\,L^4-18\,L^3+513\,L^2+2958\,L+19924\right), \\
Q_{4,3} &= -18\left(297\,L^4-1080\,L^3+1647\,L^2-12060\,L-41264\right), \\
Q_{4,4} &= -9\left(27\,L^4-252\,L^3-1053\,L^2+5190\,L+10676\right).
\end{aligned} \tag{4.19}$$

The case L even is obtained changing the sign of 2^L in the powers $(2^L)^p$ and correcting with a shift in the boundary case $L=4$. The final result is

$$\begin{aligned}
c_4^{(0)}(L) &= -\frac{5}{64}\,\delta_{L,4}-\frac{59487}{512}+ \\
&\quad -\frac{2^{4L}\,Q_{4,4}-(-1)^L\,2^{3L}\,Q_{4,3}+2^{2L}\,Q_{4,2}-(-1)^L\,2^{L}\,Q_{4,1}-1335168}{2^{15}\,[(-1)^L\,2^{L-1}+1]^5}.
\end{aligned} \tag{4.20}$$

The above shift, as well as the corrections appearing in the five loop formula (4.26) below, are possibly related to short wrapping effects - the lack of the asymptotic conditions prevents in the boundary cases the validity of the Bethe equations.

4.4 Five loops

At five loops, we have a more complicated dressing contribution with two different transcendentality terms

$$c_5(L) = c_5^{(0)}(L)+c_5^{(3)}(L)\,\zeta_3+c_5^{(5)}(L)\,\zeta_5, \qquad c_5^{(0,3,5)}(L)\in\mathbb{Q}. \tag{4.21}$$

The maximum transcendentality $c_5^{(5)}$ is independent on L

$$c_5^{(5)}(L) = \frac{135}{4}. \tag{4.22}$$

Repeating the above heuristic analysis for the other terms we find for the transcendentality 3 term

$$c_5^{(3)}(L) = \frac{1665}{32} - \frac{3}{8}\,\delta_{L,4} + \frac{81\cdot(-1)^L\,2^L\,(L-4)-648}{2^7\,[(-1)^L\,2^{L-1}+1]^2}. \tag{4.23}$$

The purely rational term has the representation

$$c_5^{(0)}(L) - c_5^{(0)}(\infty) = \frac{1}{(2^L-2)^7}\sum_{p=0}^{6} 2^{p\,L}\sum_{q=0}^{6} e_{p,q}\,L^q, \tag{4.24}$$

with the explicit final result, holding for even or odd L

$$c_5^{(0)} = \frac{632661}{1024} + \frac{14987}{12288}\,\delta_{L,4} - \frac{333}{4096}\,\delta_{L,5} + \frac{G(L)}{2^{22}\,\left[(-1)^L\,2^{L-1}+1\right]^7}, \tag{4.25}$$

$$G(L) = \sum_{p=0,6} (-1)^{p\,(L+1)}\,2^{p\,L}\,Q_{5,p}, \tag{4.26}$$

where

$$\begin{aligned}
Q_{5,0} &= -2^{11}\cdot 3^2\,(432\,L+56639), \\
Q_{5,1} &= 2^5\cdot 3^4\,\left(9\,L^6+45\,L^5-1581\,L^4-4113\,L^3+39492\,L^2+53316\,L+1253696\right), \\
Q_{5,2} &= 2^4\cdot 3^3\,\left(1539\,L^6+1755\,L^5-37503\,L^4+41409\,L^3-370980\,L^2-961116\,L-9751792\right), \\
Q_{5,3} &= 2^4\cdot 3^2\,\left(12231\,L^6-17172\,L^5+68067\,L^4+158976\,L^3+358722\,L^2+3589416\,L+20219128\right), \\
Q_{5,4} &= 2^3\cdot 3^5\,\left(453\,L^6-1956\,L^5+3769\,L^4-15096\,L^3+13278\,L^2-163616\,L-582008\right), \\
Q_{5,5} &= 2\cdot 3^3\,\left(1539\,L^6-13095\,L^5-10611\,L^4+82683\,L^3-290952\,L^2+1783716\,L+4340656\right), \\
Q_{5,6} &= 9\left(81\,L^6-1377\,L^5-6129\,L^4+103653\,L^3+195912\,L^2-1277388\,L-2247232\right).
\end{aligned} \tag{4.27}$$

The extension to higher loops seems to be a computational issue. One has to generate a large enough number of terms in $c_n(L)$ and must check that an Ansatz similar to the previous ones matches it.

5. Large L expansion of $\gamma_L(g)$

The five-loop results described in the previous sections are valid at finite L. Nevertheless, it is interesting to look at the dominant terms at large L. As remarked in the Introduction, the resulting expression can admit a thermodynamical interpretation. Collecting the formulae for c_n and expanding at large L, we find

$$\frac{\gamma_L(g)}{L} = f_0(g) + f_1(g,L)\,e^{-L\,\log 2} + f_2(g,L)\,e^{-2\,L\,\log 2} + f_3(g,L)\,e^{-3\,L\,\log 2} + \cdots. \tag{5.1}$$

The leading term agrees by construction with the result of [28]

$$f_0(g) = 2+3\,g^2-\frac{51}{8}\,g^4+\frac{393}{16}\,g^6+\left(-\frac{27\zeta_3}{4}-\frac{59487}{512}\right)\,g^8+\left(\frac{1665\zeta_3}{32}+\frac{135\zeta_5}{4}+\frac{632661}{1024}\right)\,g^{10}+\dots \tag{5.2}$$

The first exponentially suppressed term has a prefactor

$$
\begin{aligned}
f_1(g,L) = & -\frac{9}{4}\,g^4 + \left(-\frac{81L^2}{64} + \frac{297L}{64} + \frac{117}{8}\right) g^6 + \\
& + \left(-\frac{243L^4}{1024} + \frac{567L^3}{256} + \frac{9477L^2}{1024} - \frac{23355L}{512} - \frac{24021}{256}\right) g^8 + \\
& + \left(-\frac{729L^6}{32768} + \frac{12393L^5}{32768} + \frac{55161L^4}{32768} - \frac{932877L^3}{32768} - \frac{220401L^2}{4096} + \right. \\
& \left. + \left(\frac{2874123}{8192} - \frac{81\zeta_3}{32}\right) L + \frac{81\zeta_3}{8} + \frac{316017}{512}\right) g^{10} + \ldots .
\end{aligned} \tag{5.3}
$$

At order $\mathcal{O}(g^{2\,n})$, the leading power of the length is $L^{2\,n-4}$ and comes always in transcendentality 0 terms unrelated to dressing. The large L limit of $f_1(g,L)$ can be compactly written as

$$
f_1(g,L) = -\frac{9}{4}\,g^4 \left(1 + z^2 + \frac{z^4}{3} + \frac{z^6}{18} + \cdots\right), \qquad z = \frac{3}{4}\,L\,g, \tag{5.4}
$$

and in particular, given the absence of transcendental contributions, do not depend on the dressing phase. It seems reasonable that this structural properties could persist at all orders.

6. Discussion and Conclusions

In this paper, we have considered the chiral operator Tr $\mathcal{F}^L$ in $\mathcal{N} = 4$ SYM. At one-loop, it scales with a definite anomalous dimension γ_L proportional to L. At two-loops and beyond, it mixes with the other $\mathfrak{psu}(2,2|4)$ fields. The length L is no more a conserved quantity and γ_L/L is not constant. In principle, this ratio is not expected to be expressed by a simple expression at finite L. One would just resort to compute systematically its corrections at large L.

Nevertheless, the main result of this paper shows that some unexpected structure exists at finite L. We have been able to provide a closed form for γ_L/L up to five-loops. Radiative corrections follow a simple pattern order by order in perturbation theory, including transcendental dressing effects. They are sensitive to the parity of L and are exponentially suppressed as $L \to \infty$.

A closed formula for the multi-loop size dependence is a remarkable fact that has no counterpart in existing calculations for other operators in the various subsectors of $\mathcal{N} = 4$ SYM. It can be due to the simplicity of the considered operator or could hint to some hidden relation obeyed by the anomalous dimensions as a function of L. The closed formulae are a mere conjecture, although with a strong empirical basis. It is clear that a (dis)proof would be certainly enlightening.

In the large volume regime our results read

$$
\frac{\gamma_L(g)}{L} = f_0(g) + g^4 h(g\,L)\,e^{-L\,\log\,2} + \cdots . \tag{6.1}
$$

Eq. (6.1) claims that starting at two-loops, exponentially suppressed corrections appear with a g independent correlation length $\xi = 1/\log 2$ and the combination $g\,L$ as a natural scaling variable for the prefactor. It would be interesting to understand such features from the point of view of the spin-chain interpretation of the dilatation operator H. We emphasize that the $\mathcal{O}(2^{-L})$ corrections have nothing to do with much smaller $\mathcal{O}(\lambda^L)$ wrapping effects. A natural explanation for the exponential corrections could take into account length-changing processes as suggested in [28]. An explicit two-loop calculation of H would be important to clarify these issues.

We conclude with a remark concerning the dressing phase ϑ. Currently, this is a well understood ingredient appearing in the S-matrix. However, it would be very nice to classify the special kind of interactions that are associated with it in the dilatation operator. A relevant step in this direction has been recently described in [60] where it is linked to so-called maximal reshuffling interactions. In our investigation, the special feature of dressing effects is that they are subleading at large L and up to five-loops. Transcendental contributions drop out from the function $f_1(L, g)$ being characterized by subdominant powers of the length L.

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A. Five loop anomalous dimensions for $L \leq 20$

$$\begin{aligned}
\gamma_4(g) &= 8 + 12\,g^2 - 25\,g^4 + \frac{1515}{16}\,g^6 + \left(-\frac{513937}{1152} - 27\,\zeta_3\right)\,g^8 \\
&\quad + \left(\frac{22129823}{9216} + \frac{1651}{8}\,\zeta_3 + 135\,\zeta_5\right)\,g^{10} + \cdots\,, \\
\gamma_5(g) &= 10 + 15\,g^2 - \frac{129}{4}\,g^4 + \frac{39411}{320}\,g^6 + \left(-\frac{7346253}{12800} - \frac{135}{4}\,\zeta_3\right)\,g^8 \\
&\quad + \left(\frac{1539949881}{512000} + \frac{8307}{32}\,\zeta_3 + \frac{675}{4}\,\zeta_5\right)\,g^{10} + \cdots\,, \\
\gamma_6(g) &= 12 + 18\,g^2 - \frac{837}{22}\,g^4 + \frac{6278355}{42592}\,g^6 + \left(-\frac{29266837713}{41229056} - \frac{81}{2}\,\zeta_3\right)\,g^8 \\
&\quad + \left(\frac{76857234976107}{19954863104} + \frac{605205}{1936}\,\zeta_3 + \frac{405}{2}\,\zeta_5\right)\,g^{10} + \cdots\,, \\
\gamma_7(g) &= 14 + 21\,g^2 - \frac{179}{4}\,g^4 + \frac{76603}{448}\,g^6 + \left(-\frac{181131695}{225792} - \frac{189}{4}\,\zeta_3\right)\,g^8 \\
&\quad + \left(\frac{8959397257}{2107392} + \frac{11641}{32}\,\zeta_3 + \frac{945}{4}\,\zeta_5\right)\,g^{10} + \cdots\,, \\
\gamma_8(g) &= 16 + 24\,g^2 - \frac{2190}{43}\,g^4 + \frac{125533809}{636056}\,g^6 + \left(-\frac{8840715968859}{9408540352} - 54\,\zeta_3\right)\,g^8 \\
&\quad + \left(\frac{346753221469919673}{69585564443392} + \frac{3080871}{7396}\,\zeta_3 + 270\,\zeta_5\right)\,g^{10} + \cdots\,, \\
\gamma_9(g) &= 18 + 27\,g^2 - \frac{19521}{340}\,g^4 + \frac{8655987591}{39304000}\,g^6 + \left(-\frac{2364798793587021}{2271771200000} - \frac{243}{4}\,\zeta_3\right)\,g^8 \\
&\quad + \left(\frac{73033337654861466627}{13130837536000000} + \frac{108214623}{231200}\,\zeta_3 + \frac{1215}{4}\,\zeta_5\right)\,g^{10} + \cdots\,, \\
\gamma_{10}(g) &= 20 + 30\,g^2 - \frac{7265}{114}\,g^4 + \frac{486455845}{1975392}\,g^6 + \left(-\frac{179336215108445}{154033166592} - \frac{135}{2}\,\zeta_3\right)\,g^8 \\
&\quad + \left(\frac{4103422374381475165}{667271677676544} + \frac{27055595}{51984}\,\zeta_3 + \frac{675}{2}\,\zeta_5\right)\,g^{10} + \cdots\,, \\
\gamma_{11}(g) &= 22 + 33\,g^2 - \frac{8697}{124}\,g^4 + \frac{5656701069}{20972864}\,g^6 + \left(-\frac{206094402320199}{161239378432} - \frac{297}{4}\,\zeta_3\right)\,g^8 \\
&\quad + \left(\frac{5120226507678887973357}{749963046538055568} + \frac{17597781}{30752}\,\zeta_3 + \frac{1485}{4}\,\zeta_5\right)\,g^{10} + \cdots\,, \\
\gamma_{12}(g) &= 24 + 36\,g^2 - \frac{52245}{683}\,g^4 + \frac{1504241745363}{5097791792}\,g^6 + \left(-\frac{26503161491873431953}{19024510362066304} - 81\,\zeta_3\right)\,g^8 \\
&\quad + \left(\frac{262052003573439955673753835}{35498899257159792346624} + \frac{2330333685}{3731912}\,\zeta_3 + 405\,\zeta_5\right)\,g^{10} + \cdots\,, \\
\gamma_{13}(g) &= 26 + 39\,g^2 - \frac{11603}{140}\,g^4 + \frac{11382640977}{35672000}\,g^6 + \left(-\frac{61846229508401447}{40901515200000} - \frac{351}{4}\,\zeta_3\right)\,g^8 \\
&\quad + \left(\frac{4197004793411747026501}{521085303648000000} + \frac{26513707}{39200}\,\zeta_3 + \frac{1755}{4}\,\zeta_5\right)\,g^{10} + \cdots\,, \\
\gamma_{14}(g) &= 28 + 42\,g^2 - \frac{487473}{5462}\,g^4 + \frac{224231872961943}{651801084512}\,g^6 \\
&\quad + \left(-\frac{631952167345694351239 65}{38890942307856038656} - \frac{189}{2}\,\zeta_3\right)\,g^8 \\
&\quad + \left(\frac{10018082258604066826691362529007}{1160250749448653889305611264} + \frac{86929777809}{119333776}\,\zeta_3 + \frac{945}{2}\,\zeta_5\right)\,g^{10} + \cdots\,,
\end{aligned}$$

$$
\begin{aligned}
\gamma_{15}(g) \;=\;& 30 + 45\, g^2 - \frac{2088855}{21844}\, g^4 + \frac{3839300288893665}{10423090379584}\, g^6 \\
&+ \left(-\frac{4337543080186955113152555}{2486742653840334490112} - \frac{405}{4}\, \zeta_3 \right)\, g^8 \\
&+ \left(\frac{2751363092765611488400885340297145}{29664374006299642391355764 9408} + \frac{744805296585}{954320672}\, \zeta_3 + \frac{2025}{4}\, \zeta_5 \right)\, g^{10} + \cdots\,, \\
\gamma_{16}(g) \;=\;& 32 + 48\, g^2 - \frac{371380}{3641}\, g^4 + \frac{75888854970083}{193073214884}\, g^6 \\
&+ \left(-\frac{342362914175507036271937}{184287501648332409888} - 108\, \zeta_3 \right)\, g^8 \\
&+ \left(\frac{10731015336070384155651542820939}{108581221295077626947930636 8} + \frac{22072903471}{26513762}\, \zeta_3 + 540\, \zeta_5 \right)\, g^{10} + \cdots\,, \\
\gamma_{17}(g) \;=\;& 34 + 51\, g^2 - \frac{557049}{5140}\, g^4 + \frac{963880152452127}{2308544648000}\, g^6 \\
&+ \left(-\frac{1024443318894708985760901}{518422022549556800000} - \frac{459}{4}\, \zeta_3 \right)\, g^8 \\
&+ \left(\frac{122265300022822344725447154392091}{1164202709690773020828800000 0} + \frac{46737698103}{52839200}\, \zeta_3 + \frac{2295}{4}\, \zeta_5 \right)\, g^{10} + \cdots\,, \\
\gamma_{18}(g) \;=\;& 36 + 54\, g^2 - \frac{10027071}{87382}\, g^4 + \frac{1180027266090140025}{2668860863627872}\, g^6 \\
&+ \left(-\frac{85208553177354546108032208051}{40756782343071289195379456} - \frac{243}{2}\, \zeta_3 \right)\, g^8 \\
&+ \left(\frac{3461778112225195512629525977023541804641}{311203054756192480704870130697145344} + \frac{28604992980447}{30542455696}\, \zeta_3 \right. \\
&\left. + \frac{1215}{2}\, \zeta_5 \right)\, g^{10} + \cdots\,, \\
\gamma_{19}(g) \;=\;& 38 + 57\, g^2 - \frac{742739}{6132}\, g^4 + \frac{681479838093317}{1460288902464}\, g^6 \\
&+ \left(-\frac{3455324577868750102309487}{1564904852245239998976} - \frac{513}{4}\, \zeta_3 \right)\, g^8 \\
&+ \left(\frac{1093345538309226287514039899518949}{931675305424734835401555578 88} + \frac{74344962037}{75202848}\, \zeta_3 + \frac{2565}{4}\, \zeta_5 \right)\, g^{10} + \cdots\,, \\
\gamma_{20}(g) \;=\;& 40 + 60\, g^2 - \frac{22282275}{174763}\, g^4 + \frac{41954422215573649845}{85402081606607152}\, g^6 \\
&+ \left(-\frac{48481600634126052932187928286535}{20866875547860781826069765504} - 135\, \zeta_3 \right)\, g^8 \\
&+ \left(\frac{31509537567806883433426191930418295740388445}{2549273313592296157452673880096407176704} + \frac{254263222600515}{244336849352}\, \zeta_3 \right. \\
&\left. + 675\, \zeta_5 \right)\, g^{10} + \cdots\,,
\end{aligned}
\tag{A.1}
$$

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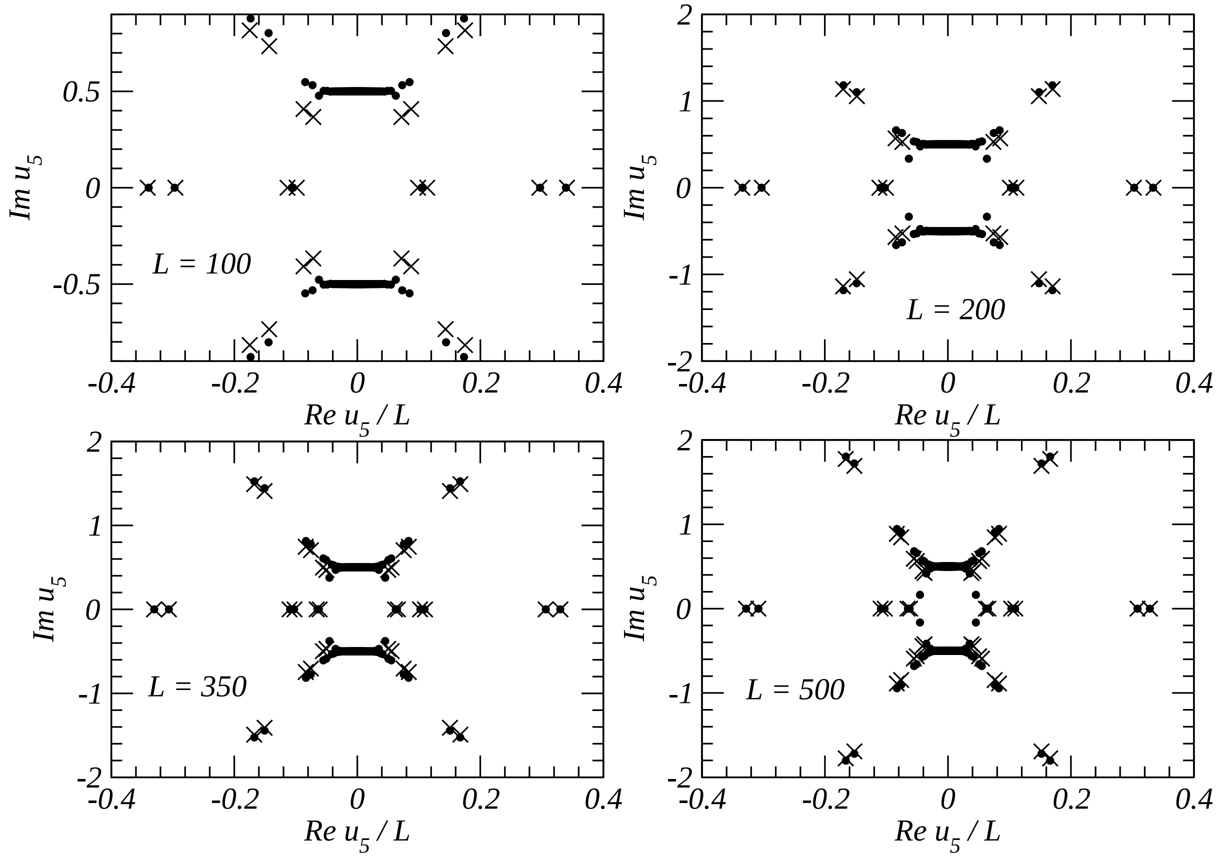

Figure 1: Dual Bethe roots u_5 computed at one-loop with $L = 100, 200, 350, 500$. Crosses on the x axis are pairs of $\alpha_{\pm}$ extremal roots. Crosses with non zero imaginary part are $\beta_{\pm}$ roots.

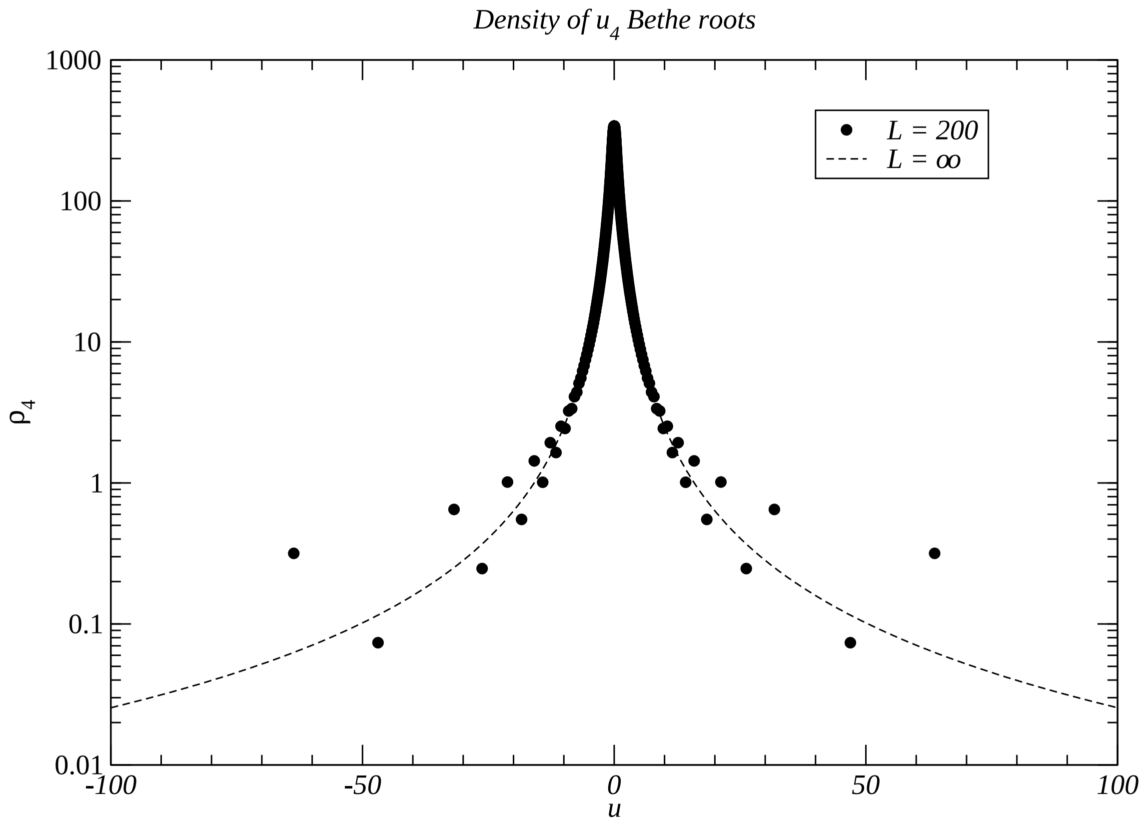

Figure 2: Density of Bethe roots u_4 from the analytical prediction ρ_4 and from the numerical roots at $L = 200$.

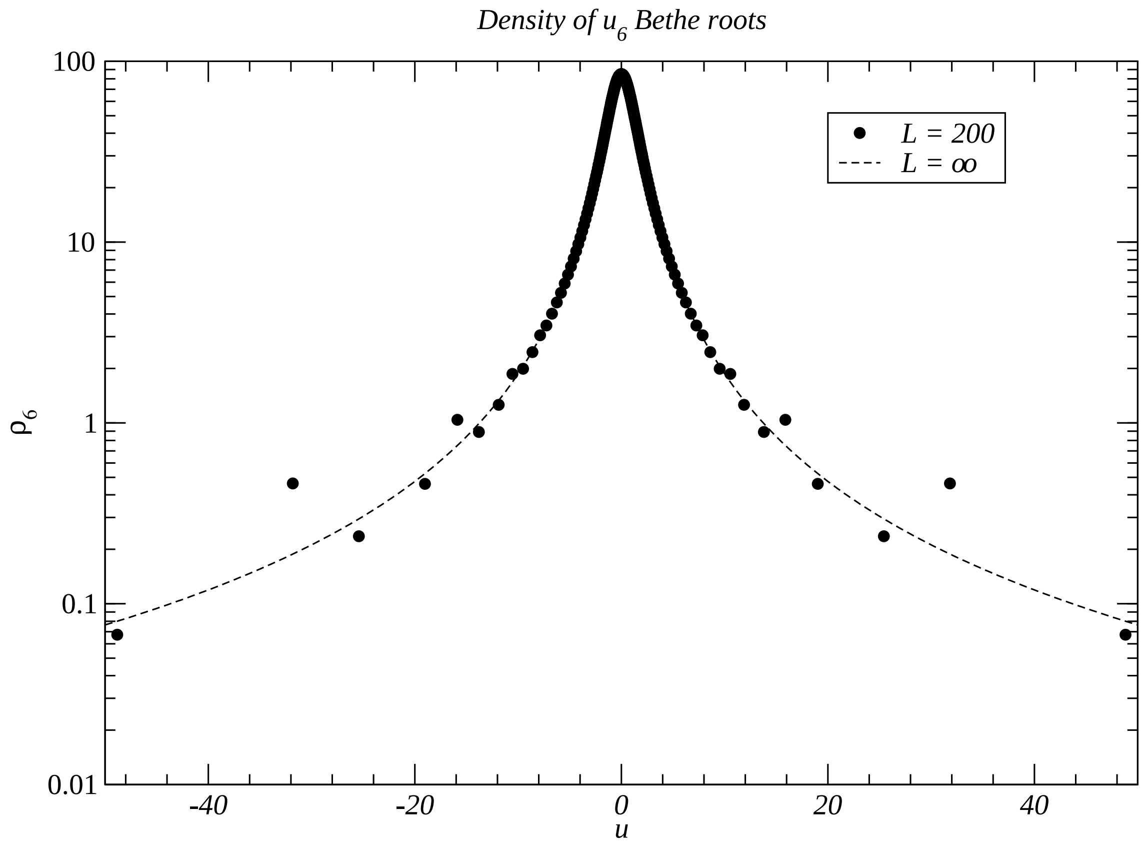

Figure 3: Density of Bethe roots u_6 from the analytical prediction ρ_6 and from the numerical roots at $L = 200$.