Finite size corrections and integrability of $\mathcal{N}=2$ SYM and DLCQ strings on a pp-wave

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Abstract: We compute the planar finite size corrections to the spectrum of the dilatation operator acting on two-impurity states of a certain limit of conformal $\mathcal{N}=2$ quiver gauge field theory which is a Z_M-orbifold of $\mathcal{N}=4$ supersymmetric Yang-Mills theory. We match the result to the string dual, IIB superstrings propagating on a pp-wave background with a periodically identified null coordinate. Up to two loops, we show that the computation of operator dimensions, using an effective Hamiltonian technique derived from renormalized perturbation theory and a twisted Bethe ansatz which is a simple generalization of the Beisert-Dippel-Staudacher [1] long range spin chain, agree with each other and also agree with a computation of the analogous quantity in the string theory. We compute the spectrum at three loop order using the twisted Bethe ansatz and find a disagreement with the string spectrum very similar to the known one in the near BMN limit of $\mathcal{N}=4$ super-Yang-Mills theory. We show that, like in $\mathcal{N}=4$, this disagreement can be resolved by adding a conjectured "dressing factor" to the twisted Bethe ansatz. Our results are consistent with integrability of the $\mathcal{N}=2$ theory within the same framework as that of $\mathcal{N}=4$.

Keywords: AdS-CFT correspondence, pp-wave background.

Contents

1. Introduction

The idea that the planar limit of $\mathcal{N} = 4$ supersymmetric Yang-Mills theory and its string theory dual, the IIB superstrings propagating on the $\mathrm{AdS}_5 \times \mathrm{S}^5$ background, could both be exactly integrable has attracted a good deal of attention [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18]. Both ideas have seen significant development and there is now some hope of an exact solution of one or both theories. This could give a remarkably detailed check of the AdS/CFT correspondence [19, 20, 21] at the level of matching planar Yang-Mills theory to non-interacting strings.

In particular, the gauge theory results have progressed to the point where integrability has been checked explicitly up to three loop order [7] and there are now proposals for integrable systems in various sectors of the theory which would be equivalent to planar Yang-Mills theory to all orders in its loop expansion [7, 1, 22, 23].

If string theory on $AdS_5 \times S^5$ is integrable, the theory on simple orbifolds of that space would also be expected to be integrable. In the Yang-Mills dual, orbifolding reduces the amount of supersymmetry and this gives some hope of finding integrability in theories with less supersymmetry[24, 25, 26, 27]. In this Paper, we shall consider the issue of integrability of an $\mathcal{N} = 2$ supersymmetric $SU(N)^M$ quiver gauge theory [28] which can be obtained as a particular Z_M-orbifold of $\mathcal{N} = 4$ [29]. This system is also conjectured to be integrable using a twisted version of the Bethe ansatz [30]. Its string theory dual is IIB superstrings on the space $\mathrm{AdS}_5 \times \mathrm{S}^5/Z_M$.

Thus far, explicit solutions of string theory on these backgrounds are not known. Quantitative results are limited to the supergravity limit, or to some large quantum number limits [31, 32, 33, 27]. For example, a Penrose limit of $\mathrm{AdS}_5 \times \mathrm{S}^5/Z_M$, together with a large order limit of the orbifold group, $M \to \infty$ can be taken in such a way that it obtains a plane-wave [34] with a periodically identified null coordinate. The IIB superstring can be solved explicitly in this background. Mukhi, Rangamani and Verlinde (MRV) [29] observed that it is possible to find the Yang-Mills dual of this theory by taking an analog of the BMN limit [35, 36, 37] of the $\mathcal{N} = 2$ quiver gauge theory. It is a double-scaling limit where $M \to \infty$ and $N \to \infty$ with the "effective string coupling", $g_2 = \frac{M}{N}$, and light-cone radius[1]

$$R_- = \frac{1}{2}\alpha' \sqrt{g_{YM}^2 \frac{N}{M}} \equiv \frac{1}{2}\alpha' \sqrt{\lambda'} \tag{1.1}$$

held finite.

In that limit, they found a beautiful matching of the discrete light-cone quantized (DLCQ) free string spectrum and planar conformal dimensions of the appropriate Yang-Mills operators. Subsequently, some of the simplifying aspects of DLCQ have been exploited to examine string loop corrections in this model [27].

Our aim in this Paper is to present a computation of the leading finite size correction to the MRV limit. We will concentrate on planar Yang-Mills theory and non-interacting strings. In the course of our work, we will give an explicit demonstration that the twisted integrability ansatz for the $\mathcal{N} = 2$ gauge theory indeed matches the diagrammatic computation of operator dimensions to two loop order.

We will compute the $1/M$ corrections to the spectrum of two-impurity operators to three loop order, λ'^3, in both the gauge theory and the DLCQ string theory. We shall find perfect agreement to two loop order and a disagreement at three-loop order.

A three-loop order disagreement is already well-known to occur in the $\mathcal{N} = 4$ theory [5, 7, 1]. We can check that, in the appropriate limit, our result matches the one for $\mathcal{N} = 4$.

[1] This is similar to the usual definition of λ' in the BMN limit of $\mathcal{N} = 4$ super-Yang-Mills theory,

$$\frac{1}{(\alpha' p^+)^2} = \frac{g_{YM}^2 NM}{(kM)^2} \equiv \frac{\lambda'}{k^2} \quad \text{or} \quad 2p^+ = \frac{k}{R_-}$$

.

We have tested the statement in Ref. [30] that the orbifolding of $\mathcal{N} = 4$ gauge theory results in the modification of the Bethe ansatz by a simple twist. Our conclusion is that it works at least to two-loop order, and we have strong evidence that it also works at three-loop order. [2]

In addition, we construct the dressing factor [10] that must be taken into account to find the factorized S-matrix [12] when the twisted Bethe ansatz is applied to the string sigma model on the orbifolded background in the near-MRV limit.

1.1 Beisert-Dippel-Staudacher ansatz for $\mathcal{N} = 4$

In its most advanced form, the result of integrability of $\mathcal{N} = 4$ super-Yang-Mills theory is a rather simple proposal for computing dimensions of operators. The typical operators are composites of the scalar fields $\Phi^i(x)$, $i = 1, ..., 6$. For simplicity, we shall concentrate on the $\mathfrak{su}(2)$ bosonic sector. In that sector, one restricts attention to four of the scalars in the complex combinations $Z = \Phi^1 + i\Phi^2$ and $\Phi = \Phi^3 + i\Phi^4$ and the composite operators

$$\mathrm{Tr}\,(\Phi ZZZ\Phi\Phi Z\Phi ZZZ...)$$

At the tree level, since scalar fields have dimension one, the dimension of this operator is given by the number of scalars that it contains (we will usually call this L). This spectrum is degenerate, in that it is the same for whatever scalar fields are used to make the composite operator. The problem at hand is to evaluate quantum corrections to the classical dimensions. These corrections should resolve the degeneracy. They are obtained by finding linear combinations of the composite operators which diagonalize the action of the dilatation operator. The analogy of this problem with diagonalizing the Hamiltonian of a spin chain, and the fact that, in the leading order of perturbation theory, it is identical to the integrable Heisenberg spin chain was observed by Minahan and Zarembo [2].

There is a recent proposal which, upon assuming that planar Yang-Mills theory is integrable, gives an elegant presentation of the problem of computing operator dimensions to all orders in the coupling constant [1]. We emphasize at this point, that we shall only use this proposal up to three loop order, where its equivalence to renormalized Yang-Mills perturbation theory has been firmly established. In fact, we shall mainly be interested in a twisted generalization of it, which is conjectured to describe a Z_M-orbifold of $\mathcal{N} = 4$ super-Yang-Mills theory.

In the proposal, the problem for computing eigenvalues of the dilatation operator is summarized in three equations. First, it makes use of the Bethe equation for $\mathcal{M}$

[2] An explicit computation of string energies on orbifolds using twisted Bethe equation was first considered by Ideguchi [38]. He computed the spectrum of infinite length operators of $\mathcal{N} = 0, 1, 2$ planar orbifold field theories to one loop order and showed that they matched the semi-classical spectra of circular string solutions of the strings in $\mathrm{AdS}_5 \times \mathrm{S}^5 / Z_M$.

magnons on a chain of length L:

$$e^{ip_jL} = \prod_{l=1\ ;\ l\neq j}^{\mathcal{M}} \frac{\varphi_j - \varphi_l + i}{\varphi_j - \varphi_l - i} = \prod_{l=1\ ;\ l\neq j}^{\mathcal{M}} S(p_j, p_l) \qquad l = 1, \ldots, \mathcal{M} \tag{1.2}$$

where p_i are the magnon momenta and φ_i are the corresponding rapidities. The factorization to 2-body S-matrices $S(p_i, p_j)$ is also shown. The momenta in (1.2) are constrained by the "level-matching condition"

$$\sum_{i=1}^{\mathcal{M}} p_i = 0 \bmod 2\pi \tag{1.3}$$

which results from the periodicity of the spin chain. Then, there is the BDS "all-loop ansatz" [1], which are the remaining two equations. One relates momenta and rapidities, which depends on the 't hooft coupling λ,

$$\varphi(p_j) = \frac{1}{2} \cot \frac{p_j}{2} \sqrt{1 + \frac{\lambda}{\pi^2} \sin^2 \frac{p_j}{2}}\ . \tag{1.4}$$

The other gives the spectrum of dimensions as a function of the momenta,

$$\Delta = L - \mathcal{M} + \sum_{j=1}^{\mathcal{M}} \sqrt{1 + \frac{\lambda}{\pi^2} \sin^2 \frac{p_j}{2}} \tag{1.5}$$

The program of computing operator dimensions is implemented as follows. Eqs. (1.2) and (1.4) should first be solved to find p_i. The solutions must depend on λ and can in principle be found at least order-by-order in an expansion in λ. Then, the solutions must be inserted into Eq. (1.5) to find the operator dimensions. The statement is that this procedure should yield the dimensions of this class of operators in $\mathcal{N} = 4$ super-Yang-Mills theory. Explicit computations and comparison with diagrammatic perturbation theory have shown that this procedure agrees with renormalized Yang-Mills perturbation theory to at least third order, and is conjectured to do so for higher orders. There is a number of quite non-trivial checks of this fact which are outlined in Ref. [1].

1.2 $\mathcal{N} = 2$ quiver gauge theory as orbifolded $\mathcal{N} = 4$

Before we go on to discuss integrability of the $\mathcal{N} = 2$ theory, we pause to review some facts about the structure of the theory and the procedure for computing operator dimensions there.

The $\mathcal{N} = 2$ quiver gauge theory with gauge group $SU(N)^M$ is obtained from $\mathcal{N} = 4$ with gauge group $SU(MN)$ by a well-known projection. Details of this construction can be found in the literature [28, 24, 39]. The conventions and notation that we use are those of Refs. [29],[27] and details can be found there.

The procedure for obtaining the quiver gauge theory from $\mathcal{N}=4$ begins by embedding the orbifold group Z_M, which is a subgroup of the R-symmetry group, into the gauge group. We will assume that Z_M is in the $\mathfrak{su}(2)$ subgroup of the $\mathfrak{su}(4)$ R-symmetry so that orbifolding preserves $\mathcal{N}=2$ supersymmetry. If γ is an element of Z_M, $R(\gamma)$ is the corresponding element of the R-symmetry group and $U(\gamma)$ is a $U(MN)\times U(MN)$ matrix containing N copies of the regular representation of Z_M, we consider that subset of the $\mathcal{N}=4$ fields which obey the constraint

$$X = U(\gamma)\left[R(\gamma)\circ X\right]U^{\dagger}(\gamma) \tag{1.6}$$

This is accomplished by setting to zero all of those components which do not obey this condition. In the present case, choosing $U(\gamma)$ having the $N\times N$ blocks

$$U(\gamma) = \begin{pmatrix} \bar{1} & 0 & 0 & 0 & ... \\ 0 & \omega & 0 & 0 & ... \\ 0 & 0 & \omega^2 & 0 & ... \\ . & . & . & . & ... \\ 0 & 0 & 0 & ... & \omega^{M-1} \end{pmatrix}$$

where $\omega = e^{\frac{2\pi i}{M}}$ and the action

$$R(\gamma)Z = \omega Z \quad , \quad R(\gamma)\Phi = \Phi$$

we see that the surviving components of the two scalar fields which are of interest to us are $N\times N$ matrices which are embedded in $MN\times MN$ $\mathcal{N}=4$ variables as follows

$$Z = \begin{pmatrix} 0 & 0 & 0 & ... & A_M \\ A_1 & 0 & 0 & 0 & ... \\ 0 & A_2 & 0 & 0 & ... \\ 0 & 0 & A_3 & 0 & ... \\ . & . & . & . & ... \\ 0 & 0 & 0 & 0 & ... \end{pmatrix} \quad , \quad \bar{Z} = \begin{pmatrix} 0 & \bar{A}_1 & 0 & 0 & ... \\ 0 & 0 & \bar{A}_2 & 0 & ... \\ 0 & 0 & 0 & \bar{A}_3 & ... \\ . & . & . & . & ... \\ \bar{A}_M & 0 & 0 & 0 & ... \end{pmatrix} \tag{1.7}$$

$$\Phi = \begin{pmatrix} \Phi_1 & 0 & 0 & 0 & ... \\ 0 & \Phi_2 & 0 & 0 & ... \\ 0 & 0 & \Phi_3 & 0 & ... \\ . & . & . & . & ... \\ 0 & 0 & 0 & ... & \Phi_M \end{pmatrix} \quad , \quad \bar{\Phi} = \begin{pmatrix} \bar{\Phi}_1 & 0 & 0 & 0 & ... \\ 0 & \bar{\Phi}_2 & 0 & 0 & ... \\ 0 & 0 & \bar{\Phi}_3 & 0 & ... \\ . & . & . & . & ... \\ 0 & 0 & 0 & ... & \bar{\Phi}_M \end{pmatrix} \tag{1.8}$$

It is convenient to think of the blocks as being labelled periodically, $A_{M+1}=A_1$, etc. The gauge group is $[SU(N)]^M$ with elements labelled by U_I, $I=1,...,M$ and each field transforms as

$$A_I \to U_I A_I U_{I+1}^{\dagger} \quad , \quad \bar{A}_I \to U_{I+1} A_I U_I^{\dagger} \tag{1.9}$$

$$\Phi_I \to U_I \Phi_I U_I^{\dagger} \quad , \quad \bar{\Phi}_I \to U_I \bar{\Phi}_I U_I^{\dagger} \tag{1.10}$$

States of the $\mathfrak{su}(2)$ sector of $\mathcal{N}=4$ super-Yang-Mills were words made from Z and Φ,

$$\mathrm{Tr}(ZZ\Phi Z\Phi ZZZZ\Phi ZZZ...)$$

Since the remaining gauge transformations (1.9) and (1.10) now commute with $U(\gamma)$, there are additional gauge invariant twisted operators

$$\mathrm{Tr}\ \left[U(\gamma)^{\ell} ZZ\Phi Z\Phi ZZZZ\Phi ZZZ...\right] \quad , \quad \ell = 0,1,...,M-1 \tag{1.11}$$

These are translated into words with (A_I, Φ_I) by substituting (1.7) and (1.8). For example,

$$\mathrm{Tr}Z^J \quad \rightarrow \quad M\mathrm{Tr}\left[(A_1A_2...A_M)^k\right] \tag{1.12}$$

Here, the trace would vanish unless the total number of fields is given by $J = kM$ with k an integer. In the string theory dual, which is DLCQ strings, the integer k is the number of units of light-cone momentum and the operator (1.12) corresponds to the vacuum state of the string sigma model in the sector with discrete light-cone momentum $2p^+ = k/R_-$.

States with impurities are made by inserting Φ_I into the trace. Because of the possible twists of the trace, there are more possible states with these insertions than occurred in the parent $\mathcal{N}=4$ theory. For example, in $\mathcal{N}=4$, the cyclic property of the trace implies that there is only one possible one-impurity state,

$$\mathrm{Tr}\Phi Z^J$$

In the analogous operator of the $\mathcal{N}=2$ theory, there are M inequivalent one-impurity states

$$\mathrm{Tr}\left[A_1...A_{I-1}\Phi_I A_I...A_M(A_1...A_M)^{k-1}\right] \quad , \quad I = 1,...,M \tag{1.13}$$

In the string dual, the extra degrees of freedom that result from this richer structure turns out to be related to the wrapping number of the string world sheet on the compact null direction. A naive Fourier transform of the 1-impurity state, assuming that the are kM positions that the impurity could take up is

$$\sum_{I=1}^{kM} e^{i\frac{2\pi}{kM}nI}\mathrm{Tr}\left[A_1...A_{I-1}\Phi_I A_I...A_M(A_1...A_M)^{k-1}\right] \quad , \quad n = 0,1,...,kM-1$$

The degree of freedom in the dual string theory corresponding to the wave-number n in this Fourier transform is the world-sheet momentum. However, cyclicity of the trace implies that $n = k\cdot\ell$ where ℓ is an integer. This is the level-matching condition and the integer ℓ is dual to the wrapping number of the string around the periodic null direction. Once we realize that $n = k\cdot\ell$, we would recover the twisted expression (1.13), and identify the string wrapping number ℓ with the twist in (1.13).

If the orbifold symmetry group is not spontaneously broken, ℓ is a good quantum number of the states of the theory and operators with different values of ℓ do not mix with each other. In addition it is known that [39], in the planar limit, the correlation functions of un-twisted operators of the $\mathcal{N}=2$ theory are identical to those of their parent operators in $\mathcal{N}=4$ super-Yang-Mills theory once one makes the replacement $\lambda \to \lambda/M$. This means that, for the untwisted operators, with $\ell=0$ in Eq. (1.11), the dimension should be identical to that in $\mathcal{N}=4$ super-Yang-Mills theory. This will give a consistency check for some of our computations in the following.

For the most part, in this Paper we will be interested in two-impurity operators of the form

$$\mathcal{O}_{IJ} = \mathrm{Tr}\left(A_1...A_{I-1}\Phi_I A_I...A_M(A_1....A_M)^p A_1...A_{J-1}\Phi_J A_J...A_M(A_1....A_M)^{k-p-2}\right) \tag{1.14}$$

where we take I and J as running from 1 to kM. Distinct operators are enumerated by taking $I \le J$. The number of scalar fields in this operator is $kM+2$. The cyclic property of the trace implies the conditions

$$\mathcal{O}_{I,kM+1} = O_{1I} \tag{1.15}$$

and

$$\mathcal{O}_{I+M,J+M} = \mathcal{O}_{I,J} \tag{1.16}$$

which will be important to us.

1.2.1 The dilatation operator

Just as in $\mathcal{N}=4$ supersymmetric Yang-Mills theory [40, 3], the computation of dimensions of the operators of interest to us can be elegantly summarized by the action of an effective Hamiltonian. This technique was invented in Ref. [40]. The $\mathcal{N}=4$ dilatation operator is known explicitly in terms of its action on fields up to two loop order, and implicitly to three loop order [3, 41, 42]. That part which is known explicitly can be projected, using the orbifold projection, to obtain a dilatation operator for the $\mathcal{N}=2$ theory. Here, we shall be interested in computing dimensions of operators in the scalar $\mathfrak{su}(2)$ sector, so we only retain the parts of the operator which will contribute there. They can be obtained by simply substituting the matrices in Eqs. (1.7) and (1.8) into the analogous terms of the $\mathcal{N}=4$ operator. The result is

$$D = D_{tree} + D_{1\ loop} + D_{2\ loops} \tag{1.17}$$

where

$$D_{tree} = \sum_{L=1}^{M} \mathrm{Tr}\left(A_L \bar{A}_L + \Phi_L \bar{\Phi}_L\right) \tag{1.18}$$

$$D_{1\ loop} = -\frac{g_{YM}^2 M}{8\pi^2} \sum_{L=1}^{M} \mathrm{Tr}(A_L\Phi_{L+1}\bar{A}_L\bar{\Phi}_L - A_L\Phi_{L+1}\bar{\Phi}_{L+1}\bar{A}_L - \Phi_L A_L \bar{A}_L\bar{\Phi}_L + \Phi_L A_L \bar{\Phi}_{L+1}\bar{A}_L) \tag{1.19}$$

$$
\begin{aligned}
D_{2\ loops} = {} & \frac{g_{YM}^4 N M^2}{64\pi^4} \sum_{L=1}^{M} \mathrm{Tr}(A_L \Phi_{L+1} \bar{A}_L \bar{\Phi}_L - A_L \Phi_{L+1} \bar{\Phi}_{L+1} \bar{A}_L - \Phi_L A_L \bar{A}_L \bar{\Phi}_L + \Phi_L A_L \bar{\Phi}_{L+1} \bar{A}_L) \\
& + \frac{g_{YM}^4 M^2}{128\pi^4} \sum_{L=1}^{M} \mathrm{Tr}(\Phi_L A_L \bar{A}_L A_L \bar{\Phi}_{L+1} \bar{A}_L - A_L \Phi_{L+1} \bar{A}_L A_L \bar{\Phi}_{L+1} \bar{A}_L + A_L \Phi_{L+1} A_{L+1} \bar{\Phi}_{L+2} \bar{A}_{L+1} \bar{A}_L \\
& - \Phi_L A_L A_{L+1} \bar{\Phi}_{L+2} \bar{A}_{L+1} \bar{A}_L + A_L \Phi_{L+1} \bar{A}_L \bar{\Phi}_L \bar{A}_{L-1} A_{L-1} - \Phi_L A_L \bar{A}_L \bar{\Phi}_L \bar{A}_{L-1} A_{L-1} \\
& + \Phi_L A_L \bar{\Phi}_{L+1} \bar{A}_L A_L \bar{A}_L - A_L \Phi_{L+1} \bar{\Phi}_{L+1} \bar{A}_L A_L \bar{A}_L + A_L \Phi_{L+1} \bar{A}_L A_L \bar{A}_L \bar{\Phi}_L - \Phi_L A_L \bar{A}_L \bar{\Phi}_L A_L \bar{A}_L \\
& - \Phi_L A_L \bar{A}_L A_L \bar{A}_L \bar{\Phi}_L + \Phi_L A_L A_{L+1} \bar{A}_{L+1} \bar{\Phi}_{L+1} \bar{A}_L - A_L \Phi_{L+1} A_{L+1} \bar{A}_{L+1} \bar{\Phi}_{L+1} \bar{A}_L \\
& + \Phi_L A_L \bar{A}_L \bar{A}_{L-1} \bar{\Phi}_{L-1} A_{L-1} - A_L \Phi_{L+1} \bar{A}_L \bar{A}_{L-1} \bar{\Phi}_{L-1} A_{L-1} + A_L \Phi_{L+1} \bar{A}_L \bar{\Phi}_L A_L \bar{A}_L) \\
& + \frac{g_{YM}^4 M^2}{128\pi^4} \sum_{L=1}^{M} \mathrm{Tr}(\Phi_L A_L \bar{\Phi}_{L+1} \Phi_{L+1} \bar{\Phi}_{L+1} \bar{A}_L - A_L \Phi_{L+1} \bar{\Phi}_{L+1} \Phi_{L+1} \bar{\Phi}_{L+1} \bar{A}_L + \Phi_L A_L \Phi_{L+1} \bar{A}_L \bar{\Phi}_L \bar{\Phi}_L \\
& - \Phi_L A_L \Phi_{L+1} \bar{\Phi}_{L+1} \bar{A}_L \bar{\Phi}_L + A_L \Phi_{L+1} \bar{\Phi}_{L+1} \bar{\Phi}_{L+1} \bar{A}_L \Phi_L - \Phi_L A_L \bar{\Phi}_{L+1} \bar{\Phi}_{L+1} \bar{A}_L \Phi_L \\
& - A_L \Phi_{L+1} \bar{\Phi}_{L+1} \bar{A}_L \Phi_L \bar{\Phi}_L + A_L \Phi_{L+1} \bar{\Phi}_{L+1} \Phi_{L+1} \bar{A}_L \bar{\Phi}_L + \Phi_L A_L \bar{\Phi}_{L+1} \bar{A}_L \Phi_L \bar{\Phi}_L \\
& - \Phi_L A_L \bar{\Phi}_{L+1} \Phi_{L+1} \bar{A}_L \bar{\Phi}_L + A_L \Phi_{L+1} \Phi_{L+1} \bar{\Phi}_{L+1} \bar{A}_L \bar{\Phi}_L - A_L \Phi_{L+1} \Phi_{L+1} \bar{A}_L \bar{\Phi}_L \bar{\Phi}_L \\
& + \Phi_L A_L \bar{\Phi}_{L+1} \bar{A}_L \bar{\Phi}_L \Phi_L - A_L \Phi_{L+1} \bar{\Phi}_{L+1} \bar{A}_L \bar{\Phi}_L \Phi_L + A_L \Phi_{L+1} \bar{A}_L \bar{\Phi}_L \Phi_L \bar{\Phi}_L - \Phi_L A_L \bar{A}_L \bar{\Phi}_L \Phi_L \bar{\Phi}_L)
\end{aligned}
\tag{1.20}
$$

The number of loops which contribute to each order is exhibited in the power of the Yang-Mills coupling constant g_{YM}^2 which precedes each term. Later we will use either the parent $\mathcal{N}=4$ 't hooft coupling,

$$\lambda \equiv g_{YM}^2 N M$$

which is important for the planar limit, or the modified 't hooft coupling

$$\lambda' \equiv \frac{g_{YM}^2 N}{M} = \frac{\lambda}{M^2}$$

which is held constant in the MRV limit. In the latter limit, N and M are both put to infinity so that λ' and the effective string coupling,

$$g_2 \equiv \frac{M}{N}$$

are held constant. The effective string coupling controls the appearance of non-planar diagrams and, to get the planar limit, which we will for the most part be interested in, it must also be put to zero. Inspection of the 1-loop and 2-loop dilatation operators shows that, in order for this MRV limit to make sense, their action should be suppressed by some powers of $\frac{1}{M}$ further to those exhibited in Eqs. (1.19) and (1.20). We shall see that this is indeed the case.

The action of the operators in Eqs. (1.18), (1.19) and (1.20) on a composite of the form (1.14) is implemented with the following procedure.

We note that each term in the dilatation operators contains a few $\bar{A}_I$'s and $\bar{\Phi}_I$'s. We take a term in D, and we Wick-contract the $\bar{A}_I$'s and $\bar{\Phi}_I$'s which appear in that term with each occurrence of A_I and Φ_I in the trace (1.14) according to the rules

$$\langle [\bar{A}_I]_{ab} [A_J]_{cd} \rangle_0 = \delta_{IJ}\delta_{ad}\delta_{bc} \quad , \quad \langle [\bar{\Phi}_I]_{ab} [\Phi_J]_{cd} \rangle_0 = \delta_{IJ}\delta_{ad}\delta_{bc}$$

Here we are treating the fields as if they are simply matrices in a Gaussian matrix model, ignoring their space-time dependence and simply substituting them with other fields according to the rules of performing the contractions. The space-time dependence, that of course must be taken into account in order to compute dimensions in renormalized perturbation theory, has already been taken care of in formulating the effective Hamiltonian.

In doing these contractions with the first term in (1.17), the tree-level operator, we find the tree level contribution to the conformal dimension. The procedure merely counts the number of scalar fields, giving $kM + 2$ in the case of (1.14).

When we Wick-contract with the 1-loop and 2-loop terms, (1.19) and (1.20), once all possible contractions are done, we find a superposition of operators where the total number of fields in each operator is the same and the number of impurities in each operator is still two, but the positions of the impurities have been shifted.

All of the operators in the superposition have the same tree-level dimensions. It means that, at the outset, we could have began with linear combinations of them. We could then have chosen the coefficients in the linear combinations in such a way as to diagonalize the action of the dilatation operator. Upon doing this, we would find the eigenvalues, i.e. the dimensions, and the linear combinations that we find would be the scaling operators themselves.

Once the Wick contractions are explicitly performed, the action of the one loop dilatation operator on the operators (1.14) is given by two equations, depending on whether the impurities lie next to each other or not

$$D_{1\ loop} \circ O_{IJ} = \frac{\lambda' M^2}{8\pi^2}\big(- O_{I+1,J} - O_{I-1,J} + 4O_{IJ} - O_{I,J+1} - O_{I,J-1}\big), \quad I < J \tag{1.21}$$

$$D_{1\ loop} \circ O_{II} = \frac{\lambda' M^2}{8\pi^2}\big(- O_{I-1,I} - O_{I,I+1} + 2O_{II}\big) \tag{1.22}$$

At two loops, the action of the dilation operator results in three equations,

$$\begin{aligned} D_{2\ loops} \circ O_{IJ} = \frac{\lambda'^2 M^4}{128\pi^4}\big(& - O_{I-2,J} - O_{I+2,J} + 4O_{I-1,J} + 4O_{I+1,J} \\ & - O_{I,J-2} - O_{I,J+2} + 4O_{I,J-1} + 4O_{I,J+1} - 12O_{IJ}\big) \end{aligned} \tag{1.23}$$

for $J - I \geq 2$ and

$$\begin{aligned} D_{2\ loops} \circ O_{II} = \frac{\lambda'^2 M^4}{128\pi^4}\big(& - O_{I-2,I} + 4O_{I-1,I} - O_{I-1,I-1} \\ & - 4O_{I,I} + 4O_{I,I+1} - O_{I+1,I+1} - O_{I,I+2}\big) \end{aligned} \tag{1.24}$$

$$D_{2\ loops} \circ O_{I,I+1} = \frac{\lambda'^2 M^4}{128\pi^4}\big(- O_{I,I+3} + 4O_{I+1,I+1} + 4O_{I,I+2} - 14O_{I,I+1}$$

$$+ 4O_{I,I} + 4O_{I-1,I+1} - O_{I-2,I+1}\Big) \tag{1.25}$$

where the second and the third formulae represent, respectively, the nearest ($I = J$) and the next-to-nearest ($J = I + 1$) neighbor cases. We see explicitly that the dilatation operator is acting like a lattice differential operator on the matrix chains. The result is an effective spin-chain Hamiltonian. The problem of finding the eigenvalues of this Hamiltonian is integrable and can be attacked using the twisted Bethe ansatz, which we summarize in the next subsection.

1.3 Twisted Bethe ansatz for the orbifold

The conjecture [30] is that the spectrum of operator dimensions in the $\mathfrak{su}(2)$ sector of the $\mathcal{N} = 2$ quiver theory which is a Z_M orbifold of $\mathcal{N} = 4$ is found by including a simple twist in the Bethe equation (1.2). The other equations, (1.4) and (1.5) are applied unchanged.

For example, for two magnons, the twisted Bethe equations are

$$e^{ip_1(kM+2)} = \omega^\ell \, \frac{\varphi_1 - \varphi_2 + i}{\varphi_1 - \varphi_2 - i} \quad , \quad e^{ip_2(kM+2)} = \omega^\ell \, \frac{\varphi_2 - \varphi_1 + i}{\varphi_2 - \varphi_1 - i} \tag{1.26}$$

Here, as in (1.2), $L = kM + 2$ is the length of the chain. The twist is the M'th root of unity factor ω^ℓ in front the right-hand-sides of (1.26). $\omega = e^{\frac{2\pi}{M}i}$ and the integer ℓ is the charge of the state under the U(1) symmetry which is used in the orbifold projection. In the dual string theory, it coincides with the wrapping number of the string world-sheet on the compact null direction. Because of (1.30), it is related to the total world-sheet momentum $e^{i(p_1+p_2)} = \omega^\ell$. As in the $\mathcal{N} = 4$ theory, the momenta and rapidities are still related by

$$\varphi_1 = \frac{1}{2}\cot\frac{p_1}{2}\sqrt{1 + \frac{\lambda}{\pi^2}\sin^2\frac{p_1}{2}} \quad , \quad \varphi_2 = \frac{1}{2}\cot\frac{p_2}{2}\sqrt{1 + \frac{\lambda}{\pi^2}\sin^2\frac{p_2}{2}} \ . \tag{1.27}$$

and the spectrum is

$$\Delta = kM + \sqrt{1 + \frac{\lambda}{\pi^2}\sin^2\frac{p_1}{2}} + \sqrt{1 + \frac{\lambda}{\pi^2}\sin^2\frac{p_2}{2}} \tag{1.28}$$

Multiplying the two equations in (1.26) gives the condition on the total momentum

$$e^{i(p_1+p_2)kM} = 1 \quad \to \quad p_1 + p_2 = \frac{2\pi}{kM}s \quad , \quad s = \text{integer} \tag{1.29}$$

The "level-matching condition" (1.3) is replaced by

$$\sum_{i=1}^{\mathcal{M}} p_i = \frac{2\pi}{M}\cdot\ell \quad , \quad \ell = \ \text{integer} \tag{1.30}$$

and it implies

$$s \ = \ k\cdot \ \text{integer} \tag{1.31}$$

It is clear from the form of the equations (1.26) and (1.27) that the momenta, which are their solutions, generally depend on λ and the parameter kM. It is also clear that the momenta which solve them must be small when M is large, $p_i \propto \frac{1}{kM}$. This is also needed for consistency of the MRV limit where $M \to \infty$ and $\lambda \to \infty$ in such a way that $\lambda' = \frac{\lambda}{M^2}$ remains finite. Equation (1.27) also implies that φ_1 and φ_2 are both of order M in that limit. Later in this Paper, we shall consider the leading corrections to this limit in an expansion in $1/M$. In the remainder of this subsection, for a warmup exercise, we will seek the solutions for p_i in the MRV limit, where $M \to \infty$. In this limit, we hold $\lambda' = \frac{\lambda}{M^2}$ finite.

Even in this limit, we shall not be able to solve equations (1.26) and (1.27) for arbitrary values of λ'. We will be limited to considering a Taylor expansion of Eq. (1.27) in λ' and then seeking momenta which are also expressed as expansions in λ'. We begin with the leading order where we simply set λ' to zero in Eq. (1.27). [3] Then, it is easy to see that the momenta must be given by

$$p_1 = \frac{2\pi}{kM} n_1 + \mathcal{O}\left(\frac{1}{M^2}\right) \quad , \quad p_2 = \frac{2\pi}{kM} n_2 + \mathcal{O}\left(\frac{1}{M^2}\right) \tag{1.32}$$

where n_1 and n_2 are integers. Level matching gives the further condition

$$n_1 + n_2 = k \cdot \ell$$

where ℓ is an integer. Then Eq. (1.28) implies

$$\Delta = kM + \sqrt{1 + \lambda' \frac{n_1^2}{k^2}} + \sqrt{1 + \lambda' \frac{n_2^2}{k^2}} \tag{1.33}$$

which agrees beautifully with the spectrum of DLCQ free strings on the plane-wave background.

1.4 Coordinate Bethe ansatz

There is another, equivalent procedure which is sometimes convenient, called the coordinate Bethe ansatz. Since we will make use of it later, we shall review it here for the special case of a two-impurity operator.

Consider the dilatation operator in the form of the difference operators (1.21)-(1.25) which we derived using the effective Hamiltonian. Finding the spectrum of the dilatation operator entails finding the eigenstates and eigenvalues of the combination of difference operators (1.21)-(1.25), operating on the space of two-impurity operators. Here, for illustration, we will review the argument that, to order λ', this is equivalent to the task of solving the twisted Bethe ansatz which was set out in

[3]We do this by setting λ to zero, but we must be careful to see, a posteriori, that indeed $p_i \sim \mathcal{O}\left(\frac{1}{M}\right)$, so that setting $\lambda = 0$ is equivelent to setting $\lambda' = 0$. We shall see this shortly, in Eq. (1.32).

the previous sub-section. Later on in this Paper, we will show that this also holds to order ${\lambda'}^2$ (and then we will assume that it holds to order ${\lambda'}^3$).

To begin, we take the linear super-position of two-impurity operators

$$\mathcal{O} \equiv \sum_{1\le I\le J\le kM} \Psi_{IJ} O_{IJ} \tag{1.34}$$

Our task is to find the coefficients Ψ_{IJ} in this series so that this operator is an eigenstate of the dilation operator. If we impose the same periodicity conditions on Ψ_{IJ} as the operators O_{IJ} obey in (1.15), the action of the dilatation operator as difference operators in (1.21)-(1.25) is self-adjoint[4] and we can recast the problem of diagonalizing dilatations as the problem of finding eigenvalues for the action of the difference operators acting on the wave-functions Ψ_{IJ}.

The coordinate Bethe ansatz was used in refs. [38] and [27] to find the spectrum of the one-loop operator in the large M limit. To introduce the technique, we shall review the essential parts of the argument here. At one-loop order, the eigenvalue equation is

$$E^{(1)}\Psi_{IJ} = g^2\left(-\Psi_{I+1,J} - \Psi_{I-1,J} + 4\Psi_{IJ} - \Psi_{I,J+1} - \Psi_{I,J-1}\right) \qquad I<J \tag{1.35}$$

$$E^{(1)}\Psi_{IJ} = g^2\left(-\Psi_{I-1,I} - \Psi_{I,I+1} + 2\Psi_{II}\right) \qquad I=J \tag{1.36}$$

where $g^2 = g_{YM}^2 NM/(8\pi^2)$. To look for a solution, we make the plane-wave ansatz

$$\Psi_{IJ} = \mu_1^I\mu_2^J + S_0(\mu_2,\mu_1)\mu_2^I\mu_1^J \tag{1.37}$$

where $\mu_1 = e^{ip_1}$ and $\mu_2 = e^{ip_2}$. Then, Eq. (1.35) yields the eigenvalue,

$$E^{(1)} = \frac{\lambda' M^2}{2\pi^2}\left(\sin^2\frac{p_1}{2} + \sin^2\frac{p_2}{2}\right) \tag{1.38}$$

which is the expansion to first order in λ' of the square roots in (1.28). The problem of finding the allowed values of (p_1,p_2) remains.

Then, (1.36) yields the equation

$$S_0(\mu_2,\mu_1) = -\frac{\mu_1}{\mu_2}\frac{\mu_1\mu_2 - 2\mu_2 + 1}{\mu_1\mu_2 - 2\mu_1 + 1} \tag{1.39}$$

where it should be noticed that $S_0(\mu_1,\mu_2)^{-1} = S_0(\mu_2,\mu_1)$.

The boundary condition $\Psi_{I,kM+1} = \Psi_{1,I}$ gives

$$\mu_2^{kM} = S_0(\mu_2,\mu_1) \quad , \quad \mu_1^{kM} = S_0(\mu_2,\mu_1)^{-1} \tag{1.40}$$

Eqs. (1.40) together with (1.39) are identical to the twisted Bethe equations (1.26), together with (1.27) with λ' set to zero. The level-matching condition is obtained by noticing that

$$\Psi_{I+M,J+M} = \Psi_{IJ} \tag{1.41}$$

[4]We note that the detailed form of the contact terms in the difference operators are essential in demonstrating the self-adjoint property.

implies

$$(\mu_1\mu_2)^M = 1 \tag{1.42}$$

1.5 Outline

In the remainder of this Paper, we shall compute the finite size corrections to the spectrum of dimensions of the two-impurity operators in the $\mathfrak{su}(2)$ bosonic sector that we have been discussing so far. We will use the twisted Bethe ansatz, summarized in Eqs. (1.26)-(1.28) and will compute to three-loop order. We also will check explicitly that the coordinate Bethe ansatz technique which used the difference operator form of the dilatation operator exhibited in Eqs. (1.21)-(1.25) indeed produces the same result to two loop order.

Then, we will adopt the string theory computation which was originally used in Ref. [5] for the near pp-wave limit of $AdS_5 \times S^5$ to the present case of the near DLCQ pp-wave limit of $AdS_5 \times S^5/Z_M$. This is the string theory dual of the "near"-MRV limit of the $\mathcal{N} = 2$ theory. We compute the spectrum of the string in this case, expanded to order $1/M$. On the string side, the expression that is obtained is exact to all orders in λ'. When expanded to third order, we find beautiful agreement with the $\mathcal{N} = 2$ gauge theory prediction up to second order in λ', i.e. two loops, and disagreement at third, or three loop order.

This disagreement is similar to the one which is found in the $\mathcal{N} = 4$ theory in Ref. [7, 1]. In fact, in the de-compactified limit, $k \to \infty, R_- \to \infty$ with $p^+ = k/R_-$ fixed, it approaches that result.

In addition, we show that, like in the case of $\mathcal{N} = 4$ super-Yang-Mills theory, the discrepancy can be taken into account by a dressing factor [12].

2. Finite size corrections at one loop

In order to calculate the first finite size corrections to Eq.(1.32) we make the following general ansatz for the magnon momenta

$$\begin{aligned} p_1 &= \frac{2n_1\pi}{kM} + \frac{A\pi}{M^2} \\ p_2 &= \frac{2n_2\pi}{kM} - \frac{A\pi}{M^2} \end{aligned} \tag{2.1}$$

Recall that we solve at one loop order by simply setting $\lambda' \to 0$ in the equation for the rapidity (1.27), so that it is given by

$$\varphi_j = \frac{1}{2}\cot\frac{p_j}{2}. \tag{2.2}$$

By requiring that the Bethe equations (1.26) are satisfied by (2.1) at both leading and next to leading order in $\frac{1}{M}$ one gets the following value for A

$$A = \frac{2\left(n_1^2 + n_2^2\right)}{k^2(n_2 - n_1)} \tag{2.3}$$

We can then insert this solution in the expression (1.38) for the anomalous dimension in terms of p_i and expand in a $\frac{1}{M}$ series. The first finite size correction to the planar anomalous dimension reads

$$\Delta_{1\ loop} = \frac{\lambda'}{2}\left[\frac{n_1^2+n_2^2}{k^2} - \left(\frac{2}{kM}\right)\frac{(n_1^2+n_2^2)}{k^2} + O\left(\frac{1}{M^2}\right)\right] \tag{2.4}$$

As a first consistency check, it is easy to verify that when the $\mathcal{N}=4$ level-matching condition $n_2=-n_1$ is imposed – this gives the result for the unwrapped, $\ell=0$ state – recalling that $J=kM$ and the appropriate re-definition of λ', the $\mathcal{N}=4$ result [7, 1] is recovered.

The zeroth order term in (2.4) equals the one-loop free string spectrum in the plane-wave limit and the first finite size correction, $\frac{1}{M}$ order, will be compared with the corresponding $1/R^2$ correction on the string side of the duality.

3. Two loops

To find the correction to the dimension at two loops, we must expand (1.27) to linear order in λ' and then use it in (1.26) to find the momenta, also to linear order in λ'. The resulting twisted Bethe equation reads

$$e^{ip_2(kM+2)} = e^{i(p_1+p_2)}\frac{\frac{1}{2}\cot\frac{p_2}{2}+\frac{\lambda}{8\pi^2}\sin p_2 - \frac{1}{2}\cot\frac{p_1}{2}+\frac{\lambda}{8\pi^2}\sin p_1 + i}{\frac{1}{2}\cot\frac{p_2}{2}+\frac{\lambda}{8\pi^2}\sin p_2 - \frac{1}{2}\cot\frac{p_1}{2}+\frac{\lambda}{8\pi^2}\sin p_1 - i} \tag{3.1}$$

The simultaneous expansion of the momenta in λ' and $\frac{1}{M}$ will have the form

$$p_1 = \frac{2n_1\pi}{kM} + \frac{A\pi}{M^2} + \lambda'\frac{B\pi}{M^2} + ... \quad , \quad p_2 = \frac{2n_2\pi}{kM} - \frac{A\pi}{M^2} - \lambda'\frac{B\pi}{M^2} + ... \tag{3.2}$$

where A, given in Eq. (2.3), was calculated in the previous section. We could also have included a contribute of order λ'/M to the momenta, but Eq.(3.1), expanded as a power series in λ' and $1/M$, would force it to be zero.

The corrections, indicated by three dots are at least of order $\frac{1}{M^3}$ or $\frac{\lambda'^2}{M^2}$. (In the next Section, we will compute the $\frac{\lambda'^2}{M^2}$ correction.)

B can be fixed by requiring that the Bethe equation (3.1) is satisfied at the first order in the λ' expansion

$$B = \frac{2\ n_1^2 n_2^2}{k^4(n_2-n_1)} \tag{3.3}$$

To calculate the $O(\lambda'^2)$ contribution to the planar anomalous dimension, one plugs the solution of the Bethe equation into the eigenvalue formula (1.28). Performing a double series expansion, in λ' and $\frac{1}{M}$, we obtain the following expression for the two loops planar anomalous dimension, up to the first finite size correction

$$\Delta_{2\ loops} = \frac{\lambda'^2}{8}\left[-\frac{n_1^4+n_2^4}{k^4} + \left(\frac{4}{kM}\right)\frac{n_1^4+n_1^3 n_2+n_1 n_2^3+n_2^4}{k^4} + O\left(\frac{1}{M^2}\right)\right]. \tag{3.4}$$

As a consistency check, we take the case where $\ell=(n_1+n_2)/k=0$ We see that (3.4) agrees with the $\mathcal{N}=4$ solution [7, 1] in that case.

4. Two loops revisited: the perturbative asymptotic Bethe ansatz

In order to diagonalize the two-loop corrected dilatation operator (1.17) the ansatz for the wave-function (1.37) has to be adjusted in a perturbative sense in order to take into account long range interactions. When interactions are included at the next order, the wave-functions are no longer plane waves. The technique which is used, termed as perturbative asymptotic Bethe ansatz (PABA) [43, 12], begins with

$$\Psi_{IJ} = \mu_1^I \mu_2^J \; f(J - I + 1, \mu_1, \mu_2) + \mu_2^I \mu_1^J \; f(kM - J + I + 1, \mu_1, \mu_2) \; S(\mu_2, \mu_1) \quad (4.1)$$

where the S-matrix and the function f have the perturbative expansions

$$\begin{aligned} S(\mu_2, \mu_1) &= S_0(\mu_2, \mu_1) + \sum_{n=1}^{\infty} (g^2)^n \; S_n(\mu_2, \mu_1) \\ f(J - I + 1, \mu_1, \mu_2) &= 1 + \sum_{n=0}^{\infty} (g^2)^{n+|J-I+1|} f_n(\mu_1, \mu_2) \end{aligned} \quad (4.2)$$

where $g^2 = g_{\rm YM}^2 MN/(8\pi^2) = \lambda' M^2/(8\pi^2)$. The number of powers of the coupling in the second of Eqs.(4.2) clearly indicates the interaction range on the lattice.

Note that, once it is determined at the leading order, the wave-function at the next order should be uniquely determined by quantum mechanical perturbation theory. Here, we are postulating that the result of determining it can be put in the form of Eq. (4.1). We will justify this postulate by showing that (3.4) does satisfy the equation to the required order and that the process of finding the solution is encoded in the twisted Bethe ansatz.

To derive the two loop Bethe equations it is sufficient to keep only the following terms in the ansatz (4.1)

$$\begin{aligned} \Psi_{IJ} = &\, \mu_1^I \mu_2^J \left[1 + g^{2|J-I+1|} f_0(\mu_1, \mu_2)\right] \\ &+ \mu_2^I \mu_1^J \left[S_0(\mu_2, \mu_1) + g^2 S_1(\mu_2, \mu_1)\right] \left[1 + g^{2|kM+1-J+I|} f_0(\mu_1, \mu_2)\right] \end{aligned} \quad (4.3)$$

The boundary conditions $\Psi_{I,kM+1} = \Psi_{1,I}$ on (4.3) imply the Bethe equations

$$\begin{aligned} \mu_2^{kM} &= S_0(\mu_2, \mu_1) + g^2 S_1(\mu_2, \mu_1) \\ \mu_1^{kM} &= \left[S_0(\mu_2, \mu_1) + g^2 S_1(\mu_2, \mu_1)\right]^{-1} \end{aligned} \quad (4.4)$$

The Schrödinger equation is obtained, as in Section 1.4, by acting on the wave-function Ψ_{IJ} with the dilatation operator as difference operators according to (1.21)-(1.25). In doing so, the two-loop contributions coming from the action of the 1-loop dilatation operator on the order λ' part of the wave-function have to be kept into account. Note that, since $\mu_i = e^{ip_i}$ and in general the p_i's depend on λ', the wave function has an implicit dependence on λ' through its dependence on μ_i.

The difference equation for $J-I \geq 2$ reads

$$
\begin{aligned}
&(D_{1\ loop} + D_{2\ loop}) \circ \Psi_{IJ} = \\
&g^2 \left(-\Psi_{I+1,J} - \Psi_{I-1,J} + 4\Psi_{IJ} - \Psi_{I,J+1} - \Psi_{I,J-1}\right) \\
&\frac{g^4}{2} \left(-\Psi_{I-2,J} - \Psi_{I+2,J} + 4\Psi_{I-1,J} + 4\Psi_{I+1,J}\right. \\
&\left.-\Psi_{I,J-2} - \Psi_{I,J+2} + 4\Psi_{I,J-1} + 4\Psi_{I,J+1} - 12\Psi_{IJ}\right) \qquad J-I \geq 2
\end{aligned} \tag{4.5}
$$

Using the ansatz (4.3) and keeping only terms up to order g^4 we see that, when $J-I \geq 2$ the dilatation operator acting on the wave-function returns its form times an eigenvalue,

$$
(D_{1\ loop} + D_{2\ loop}) \circ \Psi_{IJ} = \left[4g^2 \left(\sin^2 \frac{p_1}{2} + \sin^2 \frac{p_2}{2}\right) - \frac{g^4}{8} \left(\sin^4 \frac{p_1}{2} + \sin^4 \frac{p_2}{2}\right)\right] \Psi_{IJ} \tag{4.6}
$$

In order for (4.3) to be a eigenstate of the dilatation operator up to two loops, this must also be so for the contact terms in the dilatation operator. For this, the following equations must hold:

$$
\begin{aligned}
&(D_{1\ loop} + D_{2\ loop}) \circ \Psi_{II} = \\
&g^2 \left(-\Psi_{I-1,I} - \Psi_{I,I+1} + 2\Psi_{I,I}\right) \\
&+\frac{g^4}{2} \left(-\Psi_{I-2,I} + 4\Psi_{I-1,I} - \Psi_{I-1,I-1} - 4\Psi_{I,I} + 4\Psi_{I,I+1} - \Psi_{I+1,I+1} - \Psi_{I,I+2}\right) \\
&\equiv \left[4g^2 \left(\sin^2 \frac{p_1}{2} + \sin^2 \frac{p_2}{2}\right) - \frac{g^4}{8} \left(\sin^4 \frac{p_1}{2} + \sin^4 \frac{p_2}{2}\right)\right] \Psi_{II}
\end{aligned} \tag{4.7}
$$

$$
\begin{aligned}
&(D_{1\ loop} + D_{2\ loop}) \circ \Psi_{I,I+1} = \\
&g^2 \left(-\Psi_{I+1,I+1} - \Psi_{I-1,I+1} + 4\Psi_{I,I+1} - \Psi_{I,I+2} - \Psi_{I,I}\right) \\
&+\frac{g^4}{2} \left(-\Psi_{I,I+3} + 4\Psi_{I+1,I+1} + 4\Psi_{I,I+2} - 14\Psi_{I,I+1} + 4\Psi_{I,I} + 4\Psi_{I-1,I+1} - \Psi_{I-2,I+1}\right) \\
&\equiv \left[4g^2 \left(\sin^2 \frac{p_1}{2} + \sin^2 \frac{p_2}{2}\right) - \frac{g^4}{8} \left(\sin^4 \frac{p_1}{2} + \sin^4 \frac{p_2}{2}\right)\right] \Psi_{I,I+1}
\end{aligned} \tag{4.8}
$$

We regard these equations as determining p_i.

Using (4.3) and (1.39) in (4.8) the function $f_0(\mu_1, \mu_2)$ is uniquely derived as

$$
f_0(\mu_1, \mu_2) = -\frac{(\mu_1 - 1)(\mu_2 - 1)(\mu_1 - \mu_2)}{\mu_2(1 + \mu_1(\mu_2 - 2))} \tag{4.9}
$$

Plugging (4.9) in (4.7) one can fix also the function $S_1(\mu_1, \mu_2)$ as

$$
S_1(\mu_2, \mu_1) = -\frac{(\mu_1 - 1)^2(\mu_2 - 1)^2(\mu_1 - \mu_2)(1 + \mu_1\mu_2)}{\mu_2^2(1 + \mu_1(\mu_2 - 2))^2} \tag{4.10}
$$

Using (1.39) and (4.10) the Bethe equation (4.4) becomes

$$e^{ip_2(kM+2)} = e^{i(p_1+p_2)}\left[\frac{\frac{1}{2}\cot\frac{p_2}{2}-\frac{1}{2}\cot\frac{p_1}{2}+i}{\frac{1}{2}\cot\frac{p_2}{2}-\frac{1}{2}\cot\frac{p_1}{2}-i} - \frac{\lambda}{4\pi^2}\frac{\sin p_1-\sin p_2}{\left(\frac{1}{2}\cot\frac{p_2}{2}-\frac{1}{2}\cot\frac{p_1}{2}-i\right)^2}\right] \tag{4.11}$$

This is equivalent to Eq. (3.1) expanded to the first order in λ. We have thus demonstrated that the PABA in Eq. (4.1) solves the eigenvalue equations for the dilatation operator in the form (1.21)-(1.25) and that the process of finding these solutions is equivalent to solving the twisted Bethe equations for the $\mathcal{N}=2$ theory up to two loops.

5. Three loops

The three loop operator dimensions cannot be gotten by direct computation in Yang-Mills perturbation theory, or equivalently, by the perturbative asymptotic Bethe ansatz approach that we used for two loops in the previous Section. The reason is that, so far, no explicit expression for the dilatation operator in terms of fields and their derivatives is available at three loop order. Our approach to computing at three loops will therefore be to assume that the twisted Bethe ansatz, summarized in Eqs. (1.26)-(1.28), correctly describes the spectrum and to derive the three-loop correction to operator dimensions from it.

For this purpose we have to keep $O(\lambda^2)$ terms in Eq.(1.26) so that the twisted Bethe equation now reads

$$e^{ip_2(kM+2)} = e^{i(p_1+p_2)} \frac{\frac{1}{2}\cot\frac{p_2}{2}+\frac{\lambda}{8\pi^2}\sin p_2+\frac{\lambda^2}{64\pi^4}\sin p_2(\cos p_2-1)-\frac{1}{2}\cot\frac{p_1}{2}-\frac{\lambda}{8\pi^2}\sin p_1-\frac{\lambda^2}{64\pi^4}\sin p_1(\cos p_1-1)+i}{\frac{1}{2}\cot\frac{p_2}{2}+\frac{\lambda}{8\pi^2}\sin p_2+\frac{\lambda^2}{64\pi^4}\sin p_2(\cos p_2-1)-\frac{1}{2}\cot\frac{p_1}{2}-\frac{\lambda}{8\pi^2}\sin p_1-\frac{\lambda^2}{64\pi^4}\sin p_1(\cos p_1-1)-i} \tag{5.1}$$

We look for a solution of this equation by means of momenta of the following form

$$\begin{aligned} p_1 &= \frac{2n_1\pi}{kM}+\frac{A\pi}{M^2}+\lambda'\frac{B\pi}{M^2}+\lambda'^2\frac{C\pi}{M^2} \\ p_2 &= \frac{2n_2\pi}{kM}-\frac{A\pi}{M^2}-\lambda'\frac{B\pi}{M^2}-\lambda'^2\frac{C\pi}{M^2}, \end{aligned} \tag{5.2}$$

where A and B have been computed at lower loops, Eqs. (2.3) and (3.3). Recall that $\lambda' = \frac{\lambda}{M^2}$. Requiring that the Bethe equations are satisfied at order λ'^2 we fix C as

$$C = \frac{n_1^2 n_2^2\left(n_1^2-n_1n_2+n_2^2\right)}{2k^6(n_2-n_1)} \tag{5.3}$$

The eigenvalue formula eq.(1.28) expanded up to three loops gives

$$\Delta = kM+2+\frac{\lambda' M^2}{2\pi^2}\left(\sin^2\frac{p_1}{2}+\sin^2\frac{p_2}{2}\right)-\frac{\lambda'^2M^4}{8\pi^4}\left(\sin^4\frac{p_1}{2}+\sin^4\frac{p_1}{2}\right)$$

$$+\frac{\lambda'^3 M^6}{16\pi^6}\left(\sin^6\frac{p_1}{2}+\sin^6\frac{p_2}{2}\right)+O(\lambda'^4) \tag{5.4}$$

Taking into account the λ' dependence of the momenta given in (5.2) and expanding in λ' and $\frac{1}{M}$, we obtain the planar three loop result up to the first finite size correction

$$\Delta_{3\ loops}=\frac{\lambda'^3}{16}\left[\frac{n_1^6+n_2^6}{k^6}-\left(\frac{2}{kM}\right)\frac{3n_1^6+3n_1^5n_2+4n_1^3n_2^3+3n_1n_2^5+3n_2^6}{k^6}+O\left(\frac{1}{M^2}\right)\right]. \tag{5.5}$$

This result has to be compared with the $1/R^2$ corrections to the pp-wave energy spectrum of the corresponding string states.

As a consistency check, we see that when we set the wrapping number to zero to get the $\mathcal{N}=4$ state, i.e. put $n_2=-n_1$, it provides the $\mathcal{N}=4$ result, in beautiful agreement with the one quoted in Refs. [7], [1].

6. On the string side of the duality

In the previous Sections, we discussed the expansion to leading order in $\frac{1}{M}$ about the MRV limit of the $\mathcal{N}=2$ quiver gauge theory. The string dual to the quiver gauge theory is the IIB superstring on the $AdS_5\times S^5/Z_M$ background. The MRV limit of the $\mathcal{N}=2$ theory corresponds to the simultaneous Penrose limit and large M limit of the $AdS_5\times S^5/Z_M$ orbifold where the ratio $R_-=\frac{R^2}{2M}$ is held constant. Here, R is the radius of curvature of $AdS_5\times S^5/Z_M$. The result is the pp-wave background where the null coordinate has been periodically identified with radius R_-. String theory in that background is described by a DLCQ version of the string theory on the maximally symmetric pp-wave. The $\frac{1}{M}$ expansion of Yang-Mills theory about the MRV limit corresponds to an expansion in the ratio $\frac{1}{M}=\frac{2R_-}{R^2}$ about the pp-wave space-time.

Corrections of this kind have already been analyzed in some detail for the case of $\mathcal{N}=4$ super Yang-Mills theory – string on $AdS_5\times S^5$ duality in Ref. [5]. They considered the leading correction to the BMN limit, which was an expansion in the inverse R-charge $\frac{1}{J}$ of Yang-Mills theory or $\frac{\alpha'}{R^2}$ in string theory. In this section, we will generalize their computation to the case of the DLCQ string on the pp-wave background. We will compare the result with our computations of $1/M$-corrections in the quiver gauge theory.

The exact spectrum of states of the string theory on the pp-wave background, as well as the DLCQ of the pp-wave background are well-known. Our goal is to find corrections to the energies of these states to order $\frac{2R_-}{R^2}$. The technique to be used is to first find the correction to the string sigma model which arises from an expansion of the space-time metric and other background fields about the pp-wave. This yields an interaction Hamiltonian. The strategy is then to compute corrections to the energy spectrum by evaluating matrix elements of this interaction Hamiltonian in the pp-wave string theory states. The coefficient of the interaction Hamiltonian contains the factor $\frac{2R_-}{R^2}$.

In the case of $AdS_5 \times S^5$ background, the terms in the interaction Hamiltonian which contain two bosonic creation and two bosonic annihilation operators are expressed in terms of the string oscillators as [5]

$$
\begin{aligned}
H_{BB} = & -\frac{1}{32p^+R^2}\sum\frac{\delta(n+m+l+p)}{\xi}\times \\
& \Big\{2\Big[\xi^2-(1-k_lk_pk_nk_m)+\omega_n\omega_mk_lk_p+\omega_l\omega_pk_nk_m+2\omega_n\omega_lk_mk_p \\
& +2\omega_m\omega_pk_nk_l\Big]a^{\dagger A}_{-n}a^{\dagger A}_{-m}a^B_la^B_p+4\Big[\xi^2-(1-k_lk_pk_nk_m)-2\omega_n\omega_mk_lk_p+\omega_l\omega_mk_nk_p \\
& -\omega_n\omega_lk_mk_p-\omega_m\omega_pk_nk_l+\omega_n\omega_pk_mk_l\Big]a^{\dagger A}_{-n}a^{\dagger B}_{-l}a^A_ma^B_p+4\Big[8k_lk_pa^{\dagger i}_{-n}a^{\dagger j}_{-l}a^i_ma^j_p \\
& +2(k_lk_p+k_nk_m)a^{\dagger i}_{-n}a^{\dagger i}_{-m}a^j_la^j_p+(\omega_l\omega_p+k_lk_p-\omega_n\omega_m-k_nk_m)a^{\dagger i}_{-n}a^{\dagger i}_{-m}a^{j'}_la^{j'}_p \\
& -4(\omega_l\omega_p-k_lk_p)a^{\dagger i}_{-n}a^{\dagger j'}_{-l}a^i_ma^{j'}_p-(i,j\rightleftharpoons i',j')\Big]\Big\}, \qquad (6.1)
\end{aligned}
$$

where p^+ is the space-time momentum conjugate to the light-cone coordinate x^-, $\xi \equiv \sqrt{\omega_n\omega_m\omega_l\omega_p}$, $\omega_n = \sqrt{1+k_n^2}$ and $k_n^2 = \frac{n^2}{\alpha'^2p^{+2}} = \lambda' n^2$, with $\lambda' = g^2_{YM}N/J^2$. The indices l,m,n,p run from $-\infty$ to $+\infty$. The presence of the R-R flux breaks the transverse $SO(8)$ symmetry of the metric to $SO(4)\times SO(4)$. Consequently the notation distinguishes sums over indices of the transverse coordinates in the first $SO(4)$ $(i,j,..)$, the second $SO(4)$ $(i',j',..)$ and over the full $SO(8)$ $(A,B,..)$. The operators in (6.1) are in a normal-ordered form. Since H_{BB} was derived as a classical object, the correct ordering on the operators is not defined and the ambiguity thus arising can be kept into account by introducing a normal ordering function $N_{BB}(k_n^2)$. Such normal-ordering function can however be set to zero following the prescription of Ref.[5].

The DLCQ version of (6.1) can be obtained by taking into account that the light-cone momentum p^+ along the compactified light-cone direction $(x^- \sim x^- + 2\pi R^-)$ is quantized as $p^+ = k/(2R_-)$. R_- is related to R through $R_- = R^2/(2M)$ so that $p^+ = kM/R^2$ and $R^2 = \sqrt{4\pi g_s\alpha'^2NM}$. The Yang-Mills theory coupling constant is then identified with the superstring coupling constant g_s in the usual way $4\pi g_s = g^2_{\rm YM}$ and the double scaling limit is realized by sending both N and M to infinity and keeping the ratio N/M fixed, so that $R_- = \frac{\alpha'}{2}\sqrt{g^2_{YM}\frac{N}{M}} = \frac{\alpha'}{2}\sqrt{\lambda'}$ is also held fixed. As noticed in the introduction, the definition of λ' is in this case related to the YM coupling constant through an analogue of the usual definition $\frac{1}{(\alpha' p^+)^2} = \frac{g^2_{YM}NM}{(kM)^2} \equiv \frac{\lambda'}{k^2}$. This gives for the frequencies ω_n in (6.1) the formula $\omega_n = \sqrt{1+\lambda'\frac{n^2}{k^2}}$.

In the case of the $\mathcal{N}=2$ operator (1.14), the dual string state is the symmetric traceless two-impurity state created by the action of the following combination of bosonic creation operators on the string vacuum[5]

$$|[\mathbf{1},\mathbf{1};\mathbf{3},\mathbf{3}]>= \left[a^{\dagger a}_{n_1}a^{\dagger b}_{n_2}+a^{\dagger b}_{n_1}a^{\dagger a}_{n_2}-\frac{1}{2}\delta^{ab}a^{\dagger g}_{n_1}a^{\dagger g}_{n_2}\right]|0\rangle \tag{6.2}$$

where $n_1+n_2=k\ \ell$.

The general matrix elements of the DLCQ version $H^{Z_M}_{BB}$ of (6.1) between space-time bosons built out of bosonic string oscillators have the following explicit form

$$\begin{aligned}
&\langle 0|\, a^{A}_{-n_2}a^{B}_{-n_1}\, H^{Z_M}_{BB}\, a^{\dagger C}_{n_1}a^{\dagger D}_{n_2}\,|0\rangle = -\frac{1}{2R^2p^+}\frac{1}{\sqrt{1+\lambda'\frac{n_1^2}{k^2}}\sqrt{1+\lambda'\frac{n_2^2}{k^2}}}\\
&\Bigg\{\delta^{AB}\delta^{CD}\lambda'\left[\frac{n_1^2}{k^2}+\frac{n_2^2}{k^2}+2\lambda'\frac{n_1^2n_2^2}{k^4}+2\frac{n_1n_2}{k^2}\sqrt{1+\lambda'\frac{n_1^2}{k^2}}\sqrt{1+\lambda'\frac{n_2^2}{k^2}}\right]\\
&+\delta^{AC}\delta^{BD}\lambda'\left[\frac{n_1^2}{k^2}+\frac{n_2^2}{k^2}+2\lambda'\frac{n_1^2n_2^2}{k^4}-2\frac{n_1n_2}{k^2}\sqrt{1+\lambda'\frac{n_1^2}{k^2}}\sqrt{1+\lambda'\frac{n_2^2}{k^2}}\right]\\
&+\lambda'\left[2\frac{n_1n_2}{k^2}\left(\delta^{ab}\delta^{cd}+\delta^{ac}\delta^{bd}\right)+\frac{(n_1^2+n_2^2)}{k^2}\delta^{ad}\delta^{bc}\right]\\
&-\lambda'\left[2\frac{n_1n_2}{k^2}\left(\delta^{a'b'}\delta^{c'd'}+\delta^{a'c'}\delta^{b'd'}\right)+\frac{(n_1^2+n_2^2)}{k^2}\delta^{a'd'}\delta^{b'c'}\right]\Bigg\}
\end{aligned} \tag{6.3}$$

where lower-case $SO(4)$ indices $a,b,c,d \in 1,\dots,4$ mean that the corresponding $SO(8)$ labels A,B,C,D all lie in the first $SO(4)$, while the indices $a',b',c',d' \in 5,\dots,8$ mean that the $SO(8)$ labels lie in the second $SO(4)$ ($A,B,C,D \in 5,\dots,8$).

Eq. (6.3) can be used to evaluate the first order correction to the energy of the state (6.2), namely the matrix element $<[\mathbf{1},\mathbf{1};\mathbf{3},\mathbf{3}]|H^{Z_M}_{BB}|[\mathbf{1},\mathbf{1};\mathbf{3},\mathbf{3}]>$. Summing all the contributes and dividing the result by the norm of the state

$$<[\mathbf{1},\mathbf{1};\mathbf{3},\mathbf{3}]|[\mathbf{1},\mathbf{1};\mathbf{3},\mathbf{3}]>=2(1+\frac{1}{2}\delta^{ab})$$

one gets the desired first curvature correction to the spectrum of the states (6.2). The final result for the energy levels for a two impurity state with discrete light-cone momentum k, exact to all orders in λ', is

$$\begin{aligned}
E(n_1,n_2) &= \sqrt{1+\lambda'\left(\frac{n_1}{k}\right)^2}+\sqrt{1+\lambda'\left(\frac{n_2}{k}\right)^2}\\
&-\frac{\lambda'}{kM}\left[\frac{\frac{n_1^2}{k^2}+\frac{n_2^2}{k^2}+\lambda'\frac{n_1^2n_2^2}{k^4}+\frac{n_1n_2}{k^2}-\frac{n_1n_2}{k^2}\sqrt{1+\lambda'\left(\frac{n_1}{k}\right)^2}\sqrt{1+\lambda'\left(\frac{n_2}{k}\right)^2}}{\sqrt{1+\lambda'\left(\frac{n_1}{k}\right)^2}\sqrt{1+\lambda'\left(\frac{n_2}{k}\right)^2}}\right]+O\left(\frac{1}{M^2}\right)
\end{aligned} \tag{6.4}$$

where the small parameter governing the strength of the perturbation has been converted from $1/(R^2p^+)$ to $1/(kM)$ in order to make the comparison with the finite

[5] We use the notation of Ref. [5], where the representations of $SO(4)\times SO(4)$ are classified using an $SU(2)$ notation as $SO(4)\approx SU(2)\times SU(2)$.

size corrections of the gauge theory results more clear. Notice that for $n_1 = -n_2$ (6.4) gives back the $\mathcal{N} = 4$ result of Ref.[5], as it should.

A λ' expansion of (6.4) up to $O(\lambda'^2)$ shows perfect agreement with the gauge theory calculations at one and two loops, Eqs.(2.4) and (3.4). As for the parent $\mathcal{N} = 4$ theory [7, 1], the disagreement between the two sides of the duality is manifest at three loops, where the finite size correction to the string energy

$$E_{3\ loops} = \frac{\lambda'^3}{16} \left[\frac{n_1^6+n_2^6}{k^6} - \left(\frac{2}{kM}\right) \frac{3n_1^6+3n_1^5n_2+n_1^4n_2^2+2n_1^3n_2^3+n_1^2n_2^4+3n_1n_2^5+3n_2^6}{k^6} + O\left(\frac{1}{M^2}\right)\right] \tag{6.5}$$

does not match its gauge dual result (5.5).

7. The S-matrix dressing factor

Integrable structures have been found also in the $AdS_5 \times S^5$ string sigma model: from a classical point of view integral Bethe equations were derived in the thermodynamic limit [8], while quantum corrections are believed to yield discrete equations describing a finite number of excitations.

The agreement between the anomalous dimensions of the $\mathcal{N} = 4$ gauge theory operators in the near-BMN limit and the string energies in the near-plane wave limit up to two gauge theory loops suggests that, if we wish to describe the string excitations by the language of a spin chain, the string dynamics should be given by the BDS chain.

The three loop disagreement can actually be encoded by "dressing" the gauge theory S-matrix (*i.e.* the r.h.s. of the Bethe equations for the BDS chain) by a multiplicative factor. From these equations one derives a solution for the momenta of the string excitations which plugged in the BDS dispersion relation (1.28) reproduce the near-plane wave string energies, both in the thermodynamic limit and in the few impurity case [44, 10].

The near-plane wave string energies can therefore be computed in the $AdS_5 \times S^5$ IIB superstring theory by the following Bethe equations:

$$e^{ip_jL} = \prod_{l=1\ ;\ l\neq j}^{\mathcal{M}} S_{string}(p_j, p_l), \tag{7.1}$$

with $L = J + \mathcal{M}$ and

$$S_{string}(p_j, p_l) = \frac{\varphi_j - \varphi_l + i}{\varphi_j - \varphi_l - i} \exp\left\{2i \sum_{r=0}^{\infty} \left(\frac{\lambda}{16\pi^2}\right)^{r+2} [q_{r+2}(p_j)q_{r+3}(p_l) - q_{r+2}(p_l)q_{r+3}(p_j)]\right\} \tag{7.2}$$

where the BDS rapidities are defined in (1.4) and the exponential term is the so called dressing factor, expressed as a function of the BDS conserved charges

$$q_r(p_j) = \frac{2\sin\left(\frac{r-1}{2}p_j\right)}{r-1}\left(\frac{\sqrt{1+\frac{\lambda}{\pi^2}\sin^2\frac{p_j}{2}}-1}{\frac{\lambda}{4\pi^2}\sin\frac{p_j}{2}}\right)^{r-1} \tag{7.3}$$

In particular, the second charge $q_2(p_j)$ is the energy of a single excitation and the energy of a string state with $\mathcal{M}$ excitations is given by

$$E = \frac{\lambda}{8\pi^2}\sum_{j=1}^{\mathcal{M}} q_2(p_j) \tag{7.4}$$

We will now discuss the two magnon case in the orbifolded theory and show that the same dressing factor allows one to compute the DLCQ string energies by means of a Bethe ansatz. The two magnon scattering however is not as trivial as in the parent theory, since the excitations are not forced by the level matching condition to carry opposite momenta.

It is not difficult to check that the string spectrum (6.4) coincides with (7.4) up to $O(\lambda'^3)$ with $\mathcal{M}=2$ if the magnon momenta have the form

$$\begin{aligned} p_1 &= \frac{2n_1\pi}{kM} + \frac{A\pi}{M^2} + \lambda'\frac{B\pi}{M^2} + \lambda'^2\frac{C'\pi}{M^2} \\ p_2 &= \frac{2n_2\pi}{kM} - \frac{A\pi}{M^2} - \lambda'\frac{B\pi}{M^2} - \lambda'^2\frac{C'\pi}{M^2}, \end{aligned} \tag{7.5}$$

with the same A and B found in the gauge theory, Eqs. (2.3) (3.3), and C' given by

$$C' = \frac{n_1^2 n_2^2\,(n_1^2+n_2^2)}{4k^6(n_1-n_2)} \tag{7.6}$$

We conjecture that the string S-matrix for the $\mathrm{AdS}_5\times\mathrm{S}^5/Z_M$ IIB superstring is given by (7.2) with the addition of a twist factor which coincides with the one used in the gauge theory

$$\begin{aligned} S^{orb.}_{string}(p_j,p_l) &= \omega^l\frac{\varphi_j-\varphi_l+i}{\varphi_j-\varphi_l-i} \\ &\exp\left(2i\sum_{r=0}^{\infty}\left(\frac{\lambda}{16\pi^2}\right)^{r+2}[q_{r+2}(p_j)q_{r+3}(p_l)-q_{r+2}(p_l)q_{r+3}(p_j)]\right) \end{aligned} \tag{7.7}$$

with $\omega^l = e^{i(p_1+p_2)}$ for the two magnon case. It is easy to see that the Bethe equations

$$e^{ip_2(kM+2)} = S^{orb.}_{string}(p_2,p_1), \tag{7.8}$$

are in fact satisfied if p_1 and p_2 are exactly (7.5), with the constants A, B and C given in (2.3), (3.3) and (7.6).

Thus we have proved that the dressing factor for the orbifolded theory equals that of the parent theory and therefore, as for the gauge theory, the spectrum can be obtained by just twisting the parent Bethe equations: the three loop disagreement is inherited and does not depend on the orbifold projection.

8. Summary

In this Paper, we have computed the first finite size correction to the anomalous dimension of two-impurity states about the double scaling limit of the $\mathcal{N} = 2$ quiver gauge theory and the analogous quantity in the IIB superstring propagating on the plane-wave background with a periodically identified null coordinate.

In the gauge theory the anomalous dimensions are computed by two independent techniques that agree with each other. We have solved, up to three loops and the first finite size correction, the twisted Bethe equations conjectured in Ref. [30] for the orbifolded theory. Then we have provided an ansatz for the eigenstate of the dilatation operator that up to two loops gives the same spectrum derived with the other procedure. The eigenvalue equation for this wave function reduces to the twisted Bethe equation.

On the string theory side the computation is done by evaluating the first curvature correction to the pp-wave DLCQ spectrum of a bosonic two excitation state.

We have found that the gauge theory and the string theory results agree up to two loop order, but there is a disagreement at three loops. This disagreement is similar to, and a slight generalization of the one which is known to exist at three loop order in the analogous computation in $\mathcal{N} = 4$ super Yang-Mills theory expanded about the BMN limit [7, 1].

In Summary, the results of this Paper are

$$\begin{aligned}\Delta_{YM} =\ & kM + 2 + \frac{\lambda'}{2}\left[\frac{n_1^2+n_2^2}{k^2}\right] - \frac{\lambda'^2}{8}\left[\frac{n_1^4+n_2^4}{k^4}\right] + \frac{\lambda'^3}{16}\left[\frac{n_1^6+n_2^6}{k^6}\right] + ...\\ &+ \frac{\lambda'}{kM}\left[-\frac{(n_1^2+n_2^2)}{k^2} + \frac{\lambda'}{2}\frac{n_1^4+n_1^3n_2+n_1n_2^3+n_2^4}{k^4}\right.\\ &\left.- \frac{\lambda'^2}{8}\frac{3n_1^6+3n_1^5n_2+4n_1^3n_2^3+3n_1n_2^5+3n_2^6}{k^6} + ...\right]\end{aligned} \tag{8.1}$$

$$\begin{aligned}\Delta_{string} =\ & kM + 2 + \frac{\lambda'}{2}\left[\frac{n_1^2+n_2^2}{k^2}\right] - \frac{\lambda'^2}{8}\left[\frac{n_1^4+n_2^4}{k^4}\right] + \frac{\lambda'^3}{16}\left[\frac{n_1^6+n_2^6}{k^6}\right] + ...\\ &+ \frac{\lambda'}{kM}\left[-\frac{(n_1^2+n_2^2)}{k^2} + \frac{\lambda'}{2}\frac{n_1^4+n_1^3n_2+n_1n_2^3+n_2^4}{k^4}\right.\\ &\left.- \frac{\lambda'^2}{8}\frac{3n_1^6+3n_1^5n_2+n_1^4n_2^2+2n_1^3n_2^3+n_1^2n_2^4+3n_1n_2^5+3n_2^6}{k^6} + ...\right]\end{aligned} \tag{8.2}$$

The first two lines of each of the above expressions are identical and they differ in the third line.

We have finally shown that the DLCQ string spectrum is obtained by twisting the string Bethe ansatz proposed in Ref. [10]. The three loop disagreement is encoded in a "dressing factor" added to the gauge theory S-matrix, which coincides with the one of the $\mathcal{N}=4$ theory.

Our computations are consistent with integrability of $\mathcal{N}=2$ quiver gauge theory in the MRV limit and its string theory dual, DLCQ type IIB superstring theory on a plane wave background with a compactified null direction.

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