HU-EP-14/29

One-loop spectroscopy of semiclassically quantized strings: bosonic sector

Valentina Forini[a,1], Valentina Giangreco M. Puletti[b,c,2],
Michael Pawellek[a,1] and Edoardo Vescovi[a,1]

[a] *Institut für Physik, Humboldt-Universität zu Berlin, IRIS Adlershof, Zum Großen Windkanal 6, 12489 Berlin, Germany*

[b] *University of Iceland, Science Institute, Dunhaga 3, 107 Reykjavik, Iceland*

[c] *Department of Fundamental Physics, Chalmers University of Technology, 412 96 Göteborg, Sweden*

Abstract

We make a further step in the analytically exact quantization of spinning string states in semiclassical approximation, by evaluating the exact one-loop partition function for a class of two-spin string solutions for which quadratic fluctuations form a non-trivial system of coupled modes. This is the case of a folded string in the $SU(2)$ sector, in the limit described by a quantum Landau-Lifshitz model. The same applies to the full bosonic sector of fluctuations over the folded spinning string in AdS_5 with an angular momentum J in S^5. Fluctuations are governed by a special class of fourth-order differential operators, with coefficients being meromorphic functions on the torus, which we are able to solve exactly.

[1] {`valentina.forini,michael.pawellek,edoardo.vescovi`}`@physik.hu-berlin.de`

[2] `vgmp@hi.is`

Contents

1 Introduction

The perturbative approach to string quantization based on semiclassical analysis has proven to be an extremely useful tool for investigating the structure of the AdS/CFT correspondence [1]. Beyond the leading, classical order, direct 2d quantum field theory computations of string energies are - in general - difficult. An exception is the case of rational rigid string solutions, so-called "homogeneous" [2–4] in that derivatives of the background fields are constant, i.e. independent on (τ,σ) [3]. In this case the semiclassical analysis is highly simplified since the quadratic fluctuation Lagrangian turns out to have also constant coefficients. Then, the operator determinants entering the one-loop partition function are expressed in terms of characteristic frequencies which are relatively simple to calculate, and the computation of quantum corrections can be extended to two-loop order [4] by standard diagrammatic methods [8, 9] [5].

Next to simplest cases are "non-homogenous" configurations such as rigid spinning string elliptic solutions, the one-spin folded string solution rotating in AdS_5 [15, 16] - being a well-known example. This is a stationary soliton problem for which the classical equations of motion consists in a one-dimensional sinh-Gordon equation. In a static gauge where fluctuations along the worldsheet directions are set to zero, fluctuations turn out to be governed by differential operators of a single-gap Lamé type [17]. Their determinants can be derived explicitly, leading to an analytically closed integral expression for the full one-loop string partition function. Even if the latter is a complicated integral that is not known how to solve explicitly, a merit of this analysis is to facilitate the investigation of various regimes of interest (BPS or far-from-BPS) furnishing a "spectroscopy" much more precise than the one obtained via a perturbative treatment of the fluctuation interactions [6]. This kind of analysis has been then successfully applied also to the single-spin/parameter case of pulsating string solutions in AdS_5 and S^5 [19], to open string duals of space-like Wilson loops describing quark-antiquark systems [20] or the so-called [21] Bremsstrahlung function [22] and to the case of backgrounds relevant for the $\mathrm{AdS}_4/\mathrm{CFT}_3$ and $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence [23].

In the very general case of non-homogenous solutions with more than one spin, or of single-spin solutions [15, 16] in conformal gauge where bosonic fluctuations couple via the Virasoro constraints [7], the evaluation of the classical energy requires the diagonalization of highly non-trivial second-order *matrix* 2d differential operators whose coefficients have a complicated coordinate-dependence. The same is true for fluctuations over open string solutions for which the corresponding cusped Wilson loops have an expectation value which depends on the cusp angle and on another internal angle [22, 23]. In all these cases the evaluation of the spectrum has been performed setting to zero one of the spins/parameters involved in the problem - thus falling back in the category discussed above - or resorting to perturbation theory in them [24]. In the case of the single-spin string, it has been possible to evaluate the exact one-loop partition function only in static gauge, where mixing is absent, the equivalence with the partition function in conformal gauge being only shown numerically [17].

[3] Non homogenous solutions can become homogeneous in certain limits, as for the folded string with spin S in AdS_5 and momentum J in S^5 in the limit $\mathcal{S}=\frac{S}{\sqrt{\lambda}}\to\infty$ with $\frac{J}{\sqrt{\lambda}\log S}$ fixed [5, 6]. Similarly, in certain cases one can arrange to make the coefficients in the fluctuation Lagrangian constant [7].

[4] Comments on higher-loop calculations in such homogenous case are in [8].

[5] Another way in which sigma-model perturbation theory has been importantly used in the study of the integrable structure underlying the AdS/CFT system is the calculation of the worldsheet S-matrix, where results exists at tree-level [10], one-loop [11–13] and two-loop order [12] (for further references see [14]).

[6] See for example, in the large spin limit, the detection of turning-point contributions for the energy of a single-spin string rotating in AdS_5 [17] which are missed by naive perturbation theory, or the possibility to check at high orders the peculiar reciprocity-respecting structure of subleading corrections [18].

[7] In the single-spin case, fermions are naturally decoupled at quadratic level. In the two-spin case, they couple in a way which mirrors the bosonic sector [23].

Together with the pedagogical motivation of enriching the class of problems that can be solved analytically, the diagonalization of the mixed-modes fluctuation problem for the largest possible set of string configurations is interesting for a number of reasons. First, it provides the natural setup for a detailed comparison between the algebraic curve approach and the direct worldsheet computation of energies for string states, as in some relevant cases passing from one approach to the other implies taking a certain limit which happens to be non-analytic (see discussion in [25, 26]). Also, it should help in the solution of existing caveats for the semiclassical analysis in short string regime [25] as well as in the BPS limit of the ABJM Bremsstrahlung function (see discussion in [23, 27]). In general, the classical and the expected quantum integrability of the $\mathrm{AdS}_5 \times \mathrm{S}^5$ model should be manifest at the level of small fluctuations near a given solution, regardless how complicated is the latter.

In this paper we make a first step into the exact, detailed solution to the mixed-modes fluctuation spectrum in the case of a non-trivial solitonic configuration, the folded string spinning in S^5 with two large angular momenta (J_1, J_2) - as solution of the Landau-Lifshitz (LL) effective action of [28] [8]. In that this effective model only involves a part of the bosonic fluctuations modes (those corresponding to the $SU(2)$ sector), therefore missing bosonic and fermionic contributions crucial for the UV finiteness of the quantum result for the energy, it must be equipped with an appropriate regularization. Calculations for the spectrum of the LL model linearized around the folded $SU(2)$ string solution have been made in [24] using operator methods via perturbative evaluation (in the parameter $J_2/J,\ J = J_1 + J_2$) of characteristic frequencies, with a sum over them cured via ζ-function regularization. We will evaluate here the exact one-loop effective action over the same solution, regularized by referring the determinants to the limit $J_2 = 0$, which ensures the expected vanishing of the partition function, and proceeding with a ζ-function-inspired regularization of the path integral. The "semiclassically exact analysis" explained below provides an efficient and elegant tool to find in one step the needed spectral information. The result obtained in [24] using perturbation theory up to the n-th order means here an n-th order Taylor expansion.

The procedure exploits the integrability of a type of fourth-order linear differential equations with doubly periodic coefficients [9] in terms of which the bosonic fluctuation problem can be usefully re-written, and that emerges as the natural generalization of the Lamé differential equation. Not surprisingly, the same operator appears to govern the mixed-modes bosonic sector of fluctuations for the *full* $\mathrm{AdS}_5 \times \mathrm{S}^5$ action when expanded around the folded string solution with non-vanishing AdS_3 spin and S^1 orbital momentum [16]. As noticed in [23] [10], the mixing of bosonic fluctuations here has its supersymmetric counterpart in a non-trivial fermionic mass matrix. Annoyingly, the differential equations governing the fermionic spectrum do not satisfy the conditions which allowed us to diagonalize the bosonic system, and it is apparently non-trivial to find the necessary generalization of the tools we have developed. We leave the solution of the full string mixed-mode problem for the future [11], but we notice,

[8] See Section 2.1 below, for a detailed review see [29].

[9] A first attempt to study such kind of equations was done by Mittag-Leffler in [30].

[10] See Appendix D there.

[11] Let us underline that the quantum Landau-Lifshitz model should, provided an appropriate regularization

as nice byproduct of our analysis, that the tools developed here allow an analytic proof of equivalence between the full (including fermions [12]) exact one-loop partition function for the one-spin folded string in conformal and static gauge – a non-trivial statement which in [17] has been verified only numerically.

The paper proceeds as follows. In Section 2 we present the mixed fluctuation problems for the folded string both in the LL and in the full bosonic sectors. In Section 3 we discuss and construct the solutions of the fourth order differential operator governing those spectral problems, which we then solve analytically in Section 4. Three Appendices follow, collecting details on Sections 2, 3 and 4 respectively.

2 Bosonic fluctuation spectrum for the folded string

In this section we consider fluctuations over the classical string configuration representing a string solution folded on itself and rotating with two angular momenta. We present the two examples of coupled system of fluctuations which we will able to diagonalize in Section 4, using the tools presented in Section 3.

2.1 Landau-Lifshitz fluctuation spectrum for the SU(2) folded string

The LL effective action, obtained in string theory as a "fast-string" limit of the Polyakov action [28,31], coincides on the gauge theory side with an effective action for the ferromagnetic (bosonic) spin chain in the thermodynamic limit [13]. As such, its role as a bridge between quantum string theory and the spin chain description underlying the AdS/CFT Bethe ansatz has been explored in a number of papers (for a review see [29]). We briefly review now the quantum LL approach for the study of a folded 2-spin (J_1, J_2) string rotating in S^5 [24].

The starting point is the LL action for the SU(2) sector [28, 31], which is obtained considering a string state whose motion with two large spins is restricted to the S^3 part of S^5. The collective "fast" coordinate (β) associated to the total angular momentum is gauged away, while only transverse "slow" coordinates remain to describe the low-energy string motion. This is practically implemented by parameterising the 3-sphere coordinates as $X_1 + iX_2 = U_1 e^{i\beta}$, $X_3 + iX_4 = U_2 e^{i\beta}$, $U_a U_a^\star = 1$, fixing the gauge $t = \tau$, $p_\beta = \text{const} = J$, and rescaling the t coordinate via $\tilde\lambda = \lambda/J^2$, which plays the role of an effective parameter. To first order in the $\tilde\lambda$ expansion, one gets [24]

$$S_{\rm LL} = J \int d\tau \int_0^{2\pi} \frac{d\sigma}{2\pi} \mathcal{L}\,, \qquad \mathcal{L} = -iU_a^\star \partial_\tau U_a - \frac{\tilde\lambda}{2} |D_\sigma U_a|^2 + \mathcal{O}(\tilde\lambda^2)\,, \quad \tilde\lambda \equiv \frac{\lambda}{J^2}\,, \qquad (2.1)$$
$$DU_a = dU_a - iCU_a\,, \qquad DU_a^\star = dU_a^\star + iCU_a^\star\,, \qquad C = -iU_a^\star dU_a\,.$$

prescription is given, still correctly reproduce the full string result [4] while neglecting fermionic fluctuations (as well as a part of the bosonic ones).

[12] This is because fermions are decoupled in conformal gauge.

[13] The SU(2) sector contains operators of the form $\text{Tr}(\Phi_1^{J_1}\Phi_2^{J_2})$, and the thermodynamic limit reads $J = J_1 + J_2 \gg 1$.

For the two-spin folded string, describing a closed folded string at the center of AdS, at fixed angle in S^5, rotating within a $S^3 \subset S^5$ with arbitrary frequencies $w_1\,, w_2$, the non-vanishing part of the metric is

$$ds^2 = -dt^2 + d\psi^2 + \cos^2\psi\, d\varphi_1^2 + \sin^2\psi\, d\varphi_2^2\,, \tag{2.2}$$

and one can write [28]

$$ds^2 = -dt^2 + dX_a dX_a^\star\,, \qquad X_a X_a^\star = 1\,, \qquad X_a = e^{i\beta} U_a\,, \tag{2.3}$$
$$U_1 = \cos\psi\, e^{i\varphi}\,, \qquad U_2 = \sin\psi\, e^{-i\varphi}\,, \qquad \varphi = \frac{\varphi_1 - \varphi_2}{2}\,, \qquad \beta = \frac{\varphi_1 + \varphi_2}{2}\,. \tag{2.4}$$

Hence, the initial Lagrangian (2.1) becomes [24]

$$\mathcal{L} = \cos 2\psi\, \dot{\varphi} - \frac{\tilde{\lambda}}{2}\left(\psi'^2 + \sin^2 2\psi \varphi'^2\right)\,. \tag{2.5}$$

The equations of motion following from (2.5) are in terms of a 1d sine-Gordon equation

$$\psi'' + 2\mathrm{w}\sin 2\psi = 0\,, \qquad \varphi = -w\,t\,, \qquad w = \frac{w_2 - w_1}{2} > 0\,, \qquad \mathrm{w} = \frac{w}{\tilde{\lambda}}\,, \tag{2.6}$$
$$\psi'^2 = 2\mathrm{w}\left(\cos 2\psi - \cos 2\psi_0\right)\,,$$

whose solution can be written as [24]

$$\sin\psi(\sigma) = k\,\mathrm{sn}(C\sigma, k^2), \qquad \cos\psi(\sigma) = \mathrm{dn}(C\sigma, k^2), \qquad k^2 = \sin^2\psi_0\,, \tag{2.7}$$
$$\sqrt{\mathrm{w}} = \frac{1}{\pi}\mathbb{K}(k^2), \qquad C = \frac{2}{\pi}\mathbb{K}(k^2) = 2\sqrt{\mathrm{w}}\,, \qquad \frac{\mathbb{E}(k^2)}{\mathbb{K}(k^2)} = 1 - \frac{J_2}{J}\,.$$

The two non-zero spins (J_1, J_2) are

$$J_1 = w_1\sqrt{\lambda}\int_0^{2\pi}\frac{d\sigma}{2\pi}\cos^2\psi\,, \qquad J_2 = w_2\sqrt{\lambda}\int_0^{2\pi}\frac{d\sigma}{2\pi}\sin^2\psi\,, \qquad \frac{J_1}{w_1} + \frac{J_2}{w_2} = \sqrt{\lambda}\,. \tag{2.8}$$

From the Lagrangian (2.5), one can expand around the classical solution

$$\varphi = \varphi_{cl} + \frac{1}{\sqrt{J}}\delta\varphi(\tau,\sigma)\,, \qquad \psi = \psi_{cl} + \frac{1}{\sqrt{J}}\delta\psi(\tau,\sigma)\,, \tag{2.9}$$

obtaining, with the field redefinition

$$f_1 = -\sin(2\psi_{cl})\delta\varphi\,, \qquad f_2 = \delta\psi\,, \tag{2.10}$$

and after symmetrization, a fluctuation Lagrangian [2, 31, 32] which can be usefully written as follows [24]

$$\mathcal{L}_{LL} = 2\,f_2\,\dot{f}_1 - \frac{1}{2}\tilde{\lambda}\left[\,f_1'^2 + f_2'^2 - V_1(\sigma)\,f_1^2 - V_2(\sigma)\,f_2^2\,\right]. \tag{2.11}$$

Above, in terms of Jacobi functions, one has

$$V_1(\sigma) = 4\mathrm{w}\,\left[1 + 4k^2 - 6k^2\,\mathrm{sn}^2(C\sigma, k^2)\right]\,, \qquad V_2(\sigma) = 4\mathrm{w}\,\left[1 - 2k^2\,\mathrm{sn}^2(C\sigma, k^2)\right]\,. \tag{2.12}$$

The time-independence of the potentials allows the Fourier-transform $\partial_\tau = i\,\omega$, after which the fluctuation equations following from (2.11) form the following non-trivial matrix eigenvalue problem for the characteristic frequencies ω [14]

$$
\begin{aligned}
-\, f_2''(\sigma) - V_2(\sigma) f_2 \,&=\, i\omega f_1\,, &(2.13)\\
f_1''(\sigma) + V_1(\sigma) f_1 \,&=\, i\omega f_2. &(2.14)
\end{aligned}
$$

This system can be solved perturbatively in the elliptic modulus k^2 (or equivalently, in J_2/J), and so has been done in [24]. The main result of this paper is the *analytically exact diagonalization* of this non trivial spectral problem.

To proceed in an analytically exact fashion, we start by decoupling (2.13)-(2.14) into two fourth-order equations

$$
\begin{aligned}
f_2'''' + [V_1(\sigma)+V_2(\sigma)]\, f_2'' + 2V_2'(\sigma) f_2' + \left[V_2''(\sigma) + V_1(\sigma)V_2(\sigma)\right] f_2 \,&=\, 4\omega^2 f_2, &(2.15)\\
f_1'''' + [V_1(\sigma)+V_2(\sigma)]\, f_1'' + 2V_1'(\sigma) f_1' + \left[V_1''(\sigma) + V_1(\sigma)V_2(\sigma)\right] f_1 \,&=\, 4\omega^2 f_1, &(2.16)
\end{aligned}
$$

which, using operator notation

$$
\mathcal{O}_i = -\frac{\mathrm{d}^2}{\mathrm{d}\sigma^2} - V_i(\sigma), \tag{2.17}
$$

can be compactly written as

$$
\mathcal{O}_1\mathcal{O}_2 f_2 = \omega^2 f_2, \qquad\qquad \mathcal{O}_2\mathcal{O}_1 f_1 = \omega^2 f_1\,. \tag{2.18}
$$

The diagonalization of the matrix eigenvalue problem defined by (2.13)-(2.14) is then equivalent to the diagonalization of

$$
\mathcal{O}_{LL} = \begin{pmatrix} \mathcal{O}_1 & -\,2\partial_\tau \\ 2\partial_\tau & \mathcal{O}_2 \end{pmatrix}, \tag{2.19}
$$

where we used the definitions (2.17).

Defining a new coordinate $x = C\sigma = 2\sqrt{\mathrm{w}}\sigma$, the first equation of the system (2.13)-(2.14) can be rewritten as (we define $f_2 \equiv f$, and omit in the Jacobi functions the dependence on the modulus k^2)

$$
\mathcal{O}^{(4)}\, f(x) = 0\,, \qquad \mathcal{O}^{(4)} = \partial_x^4 + 2\left(1+2k^2-4k^2\,\mathrm{sn}^2(x)\right)\partial_x^2 - 8k^2\,\mathrm{sn}(x)\,\mathrm{cn}(x)\,\mathrm{dn}(x)\,\partial_x + 1 - \Omega^2\,, \tag{2.20}
$$

with

$$
\Omega = \frac{\omega}{2\mathrm{w}} \equiv \frac{\omega\,\pi^2}{2\,\mathbb{K}^2}\,. \tag{2.21}
$$

Equation (2.20) is a fourth-order differential equation with doubly-periodic elliptic coefficient functions [15] with period $2L = 4\mathbb{K}$ (following from the 2π-periodicity of the closed string) and only one regular singular pole, in Fuchsian classification. A first (incomplete) attempt

[14] As in [24], the time has been rescaled by $\tilde{\lambda}$, which we will restore in the final expressions.

[15] For a concise review of the relevant properties and identities for Jacobi elliptic functions see for example Appendix A of [17].

to study this kind of equations was done by Mittag-Leffler [16] in [30], and to our knowledge not much else is known in literature. In Section 3 we will present a systematic study of the eigenvalue problem associated to this equation, showing that the corresponding determinant can be computed analytically. Before doing that we show that this specific class of operators is of more general interest, as it appears governing (at least in the bosonic case) the spectrum of fluctuations above the folded string with two angular momenta [16], and thus it can likely be of help for the study of a large variety of problems involving a coupled system of fluctuations above elliptic string solutions [17].

2.2 Folded string in full bosonic sigma-model

Quadratic fluctuations over a folded string solution rotating with two angular momenta (S, J) in AdS_5 and in S^5 [16] are non-trivially coupled both in their bosonic sector [16] and in the fermionic one [23], and regardless of the gauge choice. Notice the different conventions to label frequencies and parameters characterising the classical solution between our work and [16,17].

2.2.1 Bosonic sector

Bosonic fluctuations over the classical closed string solution

$$t = \kappa\,\tau\,, \qquad \phi = \bar{w}\,\tau \qquad \varphi = \nu\,\tau\,, \qquad \kappa, \bar{w}, \nu = \text{const}\,, \tag{2.22}$$

$$\rho = \rho(\sigma) = \rho(\sigma + 2\pi)\,, \qquad \beta_u = 0\,,\ (u = 1, 2)\,, \qquad \psi_s = 0\,,\ (s = 1, 2, 3, 4)\,, \tag{2.23}$$

with (t, ρ, ϕ, β_u) describing AdS_5 and (φ, ψ_s) spanning S^5, are described in conformal gauge by the following Lagrangian [16]

$$\begin{aligned}\mathcal{L}_B^{\text{folded}} = & -\partial_a\tilde{t}\partial^a\tilde{t} - \mu_t^2\tilde{t}^2 + \partial_a\tilde{\phi}\partial^a\tilde{\phi} + \mu_\phi^2\tilde{\phi}^2 + \partial_a\tilde{\rho}\partial^a\tilde{\rho} + \mu_\rho^2\tilde{\rho}^2 + 4\,\tilde{\rho}(\kappa\sinh\rho\;\partial_0\tilde{t} - \bar{w}\,\cosh\rho\;\partial_0\tilde{\phi}) \\ & + \partial_a\tilde{\beta}_u\partial^a\tilde{\beta}_u + \mu_\beta^2\tilde{\beta}_u^2 + \partial_a\tilde{\varphi}\partial^a\tilde{\varphi} + \partial_a\tilde{\psi}_s\partial^a\tilde{\psi}_s + \nu^2\,\tilde{\psi}_s^2\,. \end{aligned}\tag{2.24}$$

Here the fields with tildes are fluctuations over the background (2.22)-(2.23), and have masses

$$\begin{aligned} \mu_t^2 &= 2\rho'^2 - \kappa^2 + \nu^2, & \mu_\phi^2 &= 2\rho'^2 - \bar{w}^2 + \nu^2, \\ \mu_\rho^2 &= 2\rho'^2 - \bar{w}^2 - \kappa^2 + 2\nu^2, & \mu_\beta^2 &= 2\rho'^2 + \nu^2 \end{aligned}\tag{2.25}$$

given in terms of the non-trivial classical field ρ satisfying the equation of motion ($\mathbb{K} \equiv \mathbb{K}(k^2)$)

$$\rho'^2 = \kappa^2\cosh^2\rho - \bar{w}^2\sinh^2\rho - \nu^2\,, \qquad k^2 = \frac{\kappa^2 - \nu^2}{\bar{w}^2 - \nu^2} \tag{2.26}$$

$$\rho'^2(\sigma) = (\kappa^2 - \nu^2)\,\mathrm{sn}^2(\sqrt{\bar{w}^2 - \nu^2}\,\sigma + \mathbb{K}\,|\,k^2)\,. \tag{2.27}$$

The $\tilde{\beta}_u$ fluctuating fields, transverse to the motion of the classical solution and decoupled from the other but with nontrivial mass, give a contribution to the one-loop partition function that has been evaluated exactly in [17].

[16] Or by the student he mentions in a footnote of [30].

[17] This observation is based on the already noticed similarity between the fluctuation spectra over the minimal surfaces corresponding space-like Wilson loops of [22] and the one of [16,17].

The remaining three AdS$_3$ fields (t,ρ,ϕ) and the φ field in S^5 are non-trivially coupled through Virasoro constraints. Their equations of motion read

$$(\partial_\tau^2-\partial_\sigma^2)\,\tilde t+\mu_t^2\,\tilde t+2\,\kappa\,\sinh\rho\,\partial_\tau\tilde\rho=0\,, \tag{2.28}$$
$$(\partial_\tau^2-\partial_\sigma^2)\,\tilde\rho+\mu_\rho^2\,\tilde\rho+2\,(\kappa\,\sinh\rho\,\partial_\tau\tilde t-\bar w\,\cosh\rho\,\partial_\tau\tilde\phi)=0\,, \tag{2.29}$$
$$(\partial_\tau^2-\partial_\sigma^2)\,\tilde\phi+\mu_\phi^2\,\tilde\phi+2\,\bar w\,\cosh\rho\,\partial_\tau\tilde\rho=0\ , \tag{2.30}$$

together with the free field equation for $\tilde\varphi$. From the conformal gauge conditions (Virasoro constraints) it follows

$$-\kappa\,\cosh^2\rho\,\partial_\tau\,\tilde t+(\bar w^2-\kappa^2)\,\sinh\rho\,\cosh\rho\,\tilde\rho+\nu\,\partial_\tau\tilde\varphi+\rho'\,\partial_\sigma\,\tilde\rho+\bar w\,\sinh^2\rho\,\partial_\tau\,\tilde\phi=0\,, \tag{2.31}$$
$$-\kappa\,\cosh^2\rho\,\partial_\sigma\,\tilde t+\bar w\,\sinh^2\rho\,\partial_\sigma\tilde\phi+\nu\,\partial_\sigma\tilde\varphi+\rho'\,\partial_\tau\,\tilde\rho=0\ . \tag{2.32}$$

Since the ρ-background does not depend on τ and since the above equations are linear we may consider to pass at the Fourier mode level, i.e. replacing $\tilde t\to e^{i\,\omega\,\tau}\tilde t$ e $\phi\to e^{i\,\omega\,\tau}\tilde\phi$. Then the Virasoro constraints imply for $\tilde t$ and $\tilde\phi$ (not yet switching to Euclidean)

$$\tilde t=\frac{\nu\,\cosh\rho}{\kappa}\,\tilde\varphi+\frac{i\,\sinh\rho}{2\,\kappa\,\omega}\left[\partial_\sigma^2-2\rho'\,\coth\rho\,\partial_\sigma+(\omega^2+\kappa^2-\bar w^2)\right]\tilde\rho\,, \tag{2.33}$$
$$\tilde\phi=\frac{\nu\,\sinh\rho}{\bar w}\,\tilde\varphi+\frac{i\,\cosh\rho}{2\,\bar w\,\omega}\left[\partial_\sigma^2-2\rho'\,\tanh\rho\,\partial_\sigma+(\omega^2-\kappa^2+\bar w^2)\right]\tilde\rho\,. \tag{2.34}$$

Substituting in (2.28)-(2.30) the expressions (2.33) and (2.34) we get that one of them is satisfied automatically while the other two become equivalent (they differ only up to the free equation of motion for $\tilde\varphi$) to the following equation

$$\mathcal{O}^{(4)}\,\tilde\rho=-4\,i\,\nu\,\omega\,\rho'\,\partial_\sigma\tilde\varphi \tag{2.35}$$

where

$$\mathcal{O}^{(4)}=\frac{1}{\rho'}(\partial_\sigma^2+\omega^2-V(\sigma))\,\rho'^2\,(\partial_\sigma^2+\omega^2)\frac{1}{\rho'}-4\,\nu^2\,\omega^2 \tag{2.36}$$

with

$$V(\sigma)=2\,\rho'^2+2\frac{(\kappa^2-\nu^2)(\bar w^2-\nu^2)}{\rho'^2}\ . \tag{2.37}$$

Being $\tilde\varphi$ a free field one can write (2.35) as [18]

$$(\partial_\sigma^2+\omega^2)\,\frac{1}{\rho'}\,\mathcal{O}^{(4)}\tilde\rho=0. \tag{2.38}$$

Changing to Euclidean signature, $\omega^2\to-\omega^2$ and introducing the new coordinate $x=\sqrt{\bar w^2-\nu^2}\sigma$ the operator gets the canonical form

$$\mathcal{O}^{(4)}=\partial_x^4+2[-\bar\Omega^2+k^2+1-4k^2\mathrm{sn}^2(x)]\partial_x^2-8k^2\mathrm{sn}(x)\mathrm{cn}(x)\mathrm{dn}(x)\partial_x+[(\bar\Omega^2+1+k^2)^2-4k^2]+\frac{4\nu^2\bar\Omega^2}{\bar w^2-\nu^2}, \tag{2.39}$$

[18] This structure has been understood in collaboration with M. Beccaria, G. Dunne and A. A. Tseytlin.

where we used the short notation

$$\bar{\Omega}^2 = \frac{\omega^2}{\bar{w}^2 - \nu^2}. \tag{2.40}$$

The operator above is strikingly similar to the one (2.20) emerging in the LL quantum model, displaying however a significative difference as it cannot be seen as a "traditional" eigenvalue problem. Indeed, $\bar{\Omega}$ does not only appear in the constant term but also in the coefficient of the second-order derivative [19]. In Appendix C.1 we show that the same operator appears in static gauge.

It should be noticed that even in the simpler case of single spin, with the folded string solution only rotating in AdS, bosonic fluctuations are still coupled in conformal gauge with a fourth order operator which can be easily obtained from (2.39) setting $\nu = 0$.

2.2.2 Fermionic sector

As found in [23] (see Appendix D there), in the two-spins case to the coupled system of bosonic modes corresponds a non trivial fermionic mass matrix. After two local boosts and a standard field redefinition [20] the fermionic fluctuation Lagrangian can be put in the form $\mathcal{L}_F = 2\,\bar{\psi}\,D_F\,\psi$, where

$$D_F = i\Big[\,\Gamma^a\,\partial_a + a(\sigma)\,\Gamma_{234} + b(\sigma)\,\Gamma_{129}\,\Big], \tag{2.41}$$

with

$$a(\sigma) = -\sqrt{\rho'^2+\nu^2}, \qquad\qquad b(\sigma) = \frac{\nu\,\kappa\,\bar{w}}{2(\rho'^2+\nu^2)}\ . \tag{2.42}$$

Squaring (2.41), one obtains

$$D_F^2 = -\partial_a\partial^a - \Gamma_{29}\Big(2\,b(\sigma)\,\partial_\sigma + b'(\sigma)\Big) + a^2(\sigma) + b^2(\sigma) - a'(\sigma)\,\Gamma_{1234} + 2a(\sigma)\,b(\sigma)\,\Gamma_{1349} \tag{2.43}$$

Noticing that the matrices appearing in (2.43) satisfy, together with their product, a 2-dimensional Dirac algebra,

$$\{\Gamma_{29},\Gamma_{1234}\} = 0 = \{\Gamma_{29},\Gamma_{1349}\} = \{\Gamma_{1234},\Gamma_{1349}\}\ , \qquad \Gamma_{1234}^2 = \Gamma_{1349}^2 = -\Gamma_{29}^2 = \mathbf{1_{32}}\ , \tag{2.44}$$

one may therefore choose a representation in which

$$\Gamma_{29} = i\,\sigma_2\times\mathbf{1_8}, \qquad \Gamma_{1234} = \sigma_3\times\mathbf{1_8}, \qquad \Gamma_{1349} = \sigma_1\times\mathbf{1_8}\ , \tag{2.45}$$

where σ_i are Pauli matrices. The fermionic fluctuations are then equivalent to 8 copies of coupled fields ψ_1, ψ_2 whose equations of motion, going to Euclidean space and Fourier transforming in τ, read

$$\Big(-\partial_\sigma^2 + \omega^2 + a^2 + b^2 - a'\Big)\,\psi_1 - \Big(2b\,\partial_\sigma + b' - 2a\,b\Big)\,\psi_2 = 0 \tag{2.46}$$

$$\Big(-\partial_\sigma^2 + \omega^2 + a^2 + b^2 + a'\Big)\,\psi_2 + \Big(2b\,\partial_\sigma + b' + 2a\,b\Big)\,\psi_1 = 0\ . \tag{2.47}$$

[19] Such case is apparently called in literature as "polynomial operator pencil" [33].
[20] See Appendix D in [23].

It is possible to decouple the above equations and write a fourth-order differential equation for example for ψ_1. However, we do not report here its lengthy expression, as its coefficient functions are not meromorphic functions on the torus, which is the kind of differential operator we have been able to solve with the method exposed in Section 3. In particular, it is the presence of the function $a(\sigma) = -\sqrt{\rho'^2 + \nu^2}$ which introduces branch-cuts ruining the simple pole structure at the basis of the procedure described below in Section 3.3. The study of a suitable generalization of such procedure does not appear to be trivial [21].

We conclude this section recalling that in the single-spin case ($\nu = 0$, therefore $b = 0$) the system (2.46)-(2.47) decouples giving two second-order Lamé type equations which differ only due to the $\pm a'$ term. The corresponding fermionic functional determinants (which in fact coincide) have been evaluated exactly in [17].

3 Fourth order linear differential equations with doubly periodic coefficients

In this Section we study the properties of fourth order differential equations with doubly periodic coefficients. We first generalize the Floquet analysis of second order linear differential equations with periodic coefficients as done in [34], then find the explicit solution for the specific class of operators of interest in this paper. Because of the striking similarity of the Bloch solutions (3.20) and quasi-momentum finite-gap structure (B.39) found here with the corresponding ones in the second-order decoupled case studied in [17] (see also [19]), we can define the operators here analyzed as a higher-order generalization of the second order finite-gap Lamé equation.

3.1 Floquet theory of determinants of fourth order one-dimensional operators

Consider the fourth order differential operator

$$\mathcal{O}^{(4)} = \partial_x^4 + v_1(x)\partial_x^2 + v_2(x)\partial_x + v_3(x), \tag{3.1}$$

where the coefficient functions $v_i(x)$ have a fundamental period L,

$$v_i(x + L) = v_i(x) \ . \tag{3.2}$$

The solution of the corresponding eigenvalue problem

$$\mathcal{O}^{(4)} f(x) = \Lambda f(x), \qquad f(x + L) = f(x) \tag{3.3}$$

[21] Also, the use of local target space rotations of the type already used, for example, in [16] or [22, 23] does not lead to any simplification of the coefficients.

consists of four independent functions $f_i(x)$, which can be normalized as

$$\begin{aligned}
&f_1(0)=1, \qquad f_1'(0)=0, \qquad f_1''(0)=0, \qquad f_1'''(0)=0,\\
&f_2(0)=0, \qquad f_2'(0)=1, \qquad f_2''(0)=0, \qquad f_2'''(0)=0,\\
&f_3(0)=0, \qquad f_3'(0)=0, \qquad f_3''(0)=1, \qquad f_3'''(0)=0,\\
&f_4(0)=0, \qquad f_4'(0)=0, \qquad f_4''(0)=0, \qquad f_4'''(0)=1\ .
\end{aligned} \tag{3.4}$$

The Wronskian determinant is therefore normalized to

$$W = f_1(0) f_2'(0) f_3''(0) f_4'''(0) = 1\ . \tag{3.5}$$

Given a complete set of solutions $f_i(x)$, the periodicity of (3.3) implies that also $f_i(x+L)$ is a solution, which can be written as a linear combination of the $f_i(x)$:

$$f_i(x+L) = \sum_{j=1}^{4} a_{ij} f_j(x), \qquad i = 1,2,3,4\,. \tag{3.6}$$

Setting $x = 0$ one gets

$$a_{ij} = f_i^{(j-1)}(L)\,, \tag{3.7}$$

with $f_i^{(0)} = f_i$, which defines the monodromy matrix

$$\mathcal{M}(\Lambda) = \begin{pmatrix} f_1(L) & f_1'(L) & f_1''(L) & f_1'''(L) \\ f_2(L) & f_2'(L) & f_2''(L) & f_2'''(L) \\ f_3(L) & f_3'(L) & f_3''(L) & f_3'''(L) \\ f_4(L) & f_4'(L) & f_4''(L) & f_4'''(L) \end{pmatrix}\ . \tag{3.8}$$

By diagonalizing this matrix one obtains a new set of four linear independent solutions $\bar{f}_i(x)$, the Floquet or Bloch solutions with the property $f_i(x+L) = \rho_i f_i(x)$, or

$$\begin{pmatrix} f_1(L)-\rho & f_1'(L) & f_1''(L) & f_1'''(L) \\ f_2(L) & f_2'(L)-\rho & f_2''(L) & f_2'''(L) \\ f_3(L) & f_3'(L) & f_3''(L)-\rho & f_3'''(L) \\ f_4(L) & f_4'(L) & f_4''(L) & f_4'''(L)-\rho \end{pmatrix} \begin{pmatrix} \bar{f}_1(x) \\ \bar{f}_2(x) \\ \bar{f}_3(x) \\ \bar{f}_4(x) \end{pmatrix} = 0. \tag{3.9}$$

As usual, this equation has non-trivial solutions $\bar{f}_i(x)$ provided that $\det(\mathcal{M}-\rho\mathbf{1})=0$, with

$$
\begin{aligned}
\det(\mathcal{M}-\rho\mathbf{1}) &= \rho^4-(f_1(L)+f_2'(L)+f_3''(L)+f_4'''(L))\rho^3+\Big[\det\begin{pmatrix} f_1 & f_1' \\ f_2 & f_2'\end{pmatrix}+\det\begin{pmatrix} f_1 & f_1'' \\ f_3 & f_3''\end{pmatrix}+\\
&+\det\begin{pmatrix} f_1 & f_1''' \\ f_4 & f_4'''\end{pmatrix}+\det\begin{pmatrix} f_2' & f_2'' \\ f_3' & f_3''\end{pmatrix}+\det\begin{pmatrix} f_2' & f_2''' \\ f_4' & f_4'''\end{pmatrix}+\det\begin{pmatrix} f_3'' & f_3''' \\ f_4'' & f_4'''\end{pmatrix}\Big]\rho^2+ \qquad (3.10)\\
&-\left[\det\begin{pmatrix} f_1 & f_1' & f_1'' \\ f_2 & f_2' & f_2'' \\ f_3 & f_3' & f_3'''\end{pmatrix}+\det\begin{pmatrix} f_1 & f_1' & f_1'' \\ f_2 & f_2' & f_2''' \\ f_4 & f_4' & f_4'''\end{pmatrix}+\det\begin{pmatrix} f_1 & f_1'' & f_1''' \\ f_3 & f_3'' & f_3''' \\ f_4 & f_4'' & f_4'''\end{pmatrix}+\det\begin{pmatrix} f_2' & f_2'' & f_2''' \\ f_3' & f_3'' & f_3''' \\ f_4' & f_4'' & f_4'''\end{pmatrix}\right]\rho+1,
\end{aligned}
$$

where we have used (3.5). Let ρ_i, $i=1,2,3,4$ be the roots of this fourth order polynomial equation. Then, we can write

$$
\begin{aligned}
\det(\mathcal{M}(\Lambda)-\rho\mathbf{1}) &= \rho^4-(\rho_1+\rho_2+\rho_3+\rho_4)\rho^3+(\rho_1\rho_2+\rho_1\rho_3+\rho_1\rho_4+\rho_2\rho_3+\rho_2\rho_4+\rho_3\rho_4)\rho^2-\\
&-(\rho_1\rho_2\rho_3+\rho_1\rho_2\rho_4+\rho_1\rho_3\rho_4+\rho_2\rho_3\rho_4)\rho+\rho_1\rho_2\rho_3\rho_4. \qquad (3.11)
\end{aligned}
$$

Comparing the expressions (3.10) and (3.11) gives a condition on the Floquet factors

$$\rho_1\rho_2\rho_3\rho_4=1. \qquad (3.12)$$

A general solution to the equation above would require the introduction of three functions. We can however proceed conveniently introducing just *two* quasi-momenta functions $p_i(\Lambda)$, $i=1,2$ with

$$
\begin{aligned}
\rho_1 &= e^{ip_1(\Lambda)L}, \qquad \rho_2=e^{-ip_1(\Lambda)L},\\
\rho_3 &= e^{ip_2(\Lambda)L}, \qquad \rho_4=e^{-ip_2(\Lambda)L}. \qquad (3.13)
\end{aligned}
$$

Knowing the quasi-momenta allows us to immediately compute the determinants. In particular, we have

i) <u>Functions with period L:</u>

Periodic eigenfunctions $f_i(x+L)=f_i(x)$ exist only for special values of Λ which are determined by setting $\rho=1$ in (3.11) and using (3.13)

$$
\begin{aligned}
\det{}_{P,L}\mathcal{O}^{(4)} &= 4-4\cos(p_1L)-4\cos(p_2L)+4\cos(p_1L)\cos(p_2L)=\\
&= 16\sin^2\left(\frac{L}{2}p_1(\Lambda)\right)\sin^2\left(\frac{L}{2}p_2(\Lambda)\right). \qquad (3.14)
\end{aligned}
$$

ii) <u>Anti-periodic functions by L:</u>

We get the determinant for antiperiodic eigenfunctions $f_i(x+L)=-f_i(x)$ by setting $\rho=-1$ in (3.11) and using (3.13)

$$
\begin{aligned}
\det{}_{AP,L}\mathcal{O}^{(4)} &= 4+4\cos(p_1L)+4\cos(p_2L)+4\cos(p_1L)\cos(p_2L)=\\
&= 16\cos^2\left(\frac{L}{2}p_1(\Lambda)\right)\cos^2\left(\frac{L}{2}p_2(\Lambda)\right). \qquad (3.15)
\end{aligned}
$$

iii) <u>Functions with period $2L$:</u>

In this case one has to take the product of the previous two determinants, which gives

$$\det{}_{P,2L}\mathcal{O}^{(4)}=\det{}_P\mathcal{O}^{(4)}\det{}_{AP}\mathcal{O}^{(4)}=16\sin^2\left(Lp_1(\Lambda)\right)\sin^2\left(Lp_2(\Lambda)\right). \qquad (3.16)$$

3.2 Construction of the solutions: a Hermite-Bethe ansatz

In this section we find the Bloch solutions for a certain class of fourth order periodic differential equations, which we will argue in B.3 to be higher order generalizations of the second order finite-gap Lamé equation. A first attempt to study this kind of equations was done by Mittag-Leffler in [30].

In the following we will use the fact that an elliptic function without any poles in a fundamental period parallelogram of the complex plane is merely a constant [35]. The differential operators of interest are of the type

$$\mathcal{O} = \frac{\mathrm{d}^4 f}{\mathrm{d}x^4} + v_1(x)\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} + v_2(x)\frac{\mathrm{d}f}{\mathrm{d}x} + v_3(x)f(x), \tag{3.17}$$

where the "potentials"

$$\begin{aligned} v_1(x) &= \alpha_0 + \alpha_1 k^2 \mathrm{sn}^2(x), \\ v_2(x) &= \beta_0 + \beta_1 k^2 \mathrm{sn}^2(x) + 2\beta_2 k^2 \mathrm{sn}(x)\mathrm{cn}(x)\mathrm{dn}(x), \\ v_3(x) &= \gamma_0 + 2\gamma_3 k^2 + (\gamma_1 - 4(1+k^2)\gamma_3)k^2\mathrm{sn}^2(x) + 2\gamma_2 k^2 \mathrm{sn}(x)\mathrm{cn}(x)\mathrm{dn}(x) + 6\gamma_3 k^4 \mathrm{sn}^4(x), \end{aligned} \tag{3.18}$$

are given by elliptic functions *with only one regular singular pole* (in Fuchsian classification) at $x = i\mathbb{K}'$. The coefficients $\alpha_0, \alpha_1, \beta_0, \beta_1, \beta_2, \gamma_0, \gamma_1, \gamma_2, \gamma_3$ are so far free parameters. We will now find conditions on these parameters such that the following eigenvalue equation

$$\mathcal{O}f(x) = \Lambda f(x), \qquad f(x+L) = f(x) \tag{3.19}$$

is solved by a Hermite-Bethe-like ansatz [35] [22]

$$f(x) = \prod_{r=1}^{n} \frac{H(x+\bar{\alpha}_r)}{\Theta(x)} e^{x\rho} e^{x\lambda} \ . \tag{3.20}$$

The constants ρ and $\bar{\alpha}_r$ are determined by analyticity constraints on the eigenfunction as follows. Let us introduce the function F

$$F(x) = \frac{1}{f(x)}\mathcal{O}f(x)\,, \tag{3.21}$$

which is an elliptic function with periods $2\mathbb{K}$ and $2i\mathbb{K}'$ and a certain number of poles x_i of order p_i in the period-parallelogram. In terms of F, the eigenvalue equation becomes

$$F(x) = \Lambda\,, \tag{3.22}$$

[22] The ansatz (3.20) provides four linearly independent solutions - see for example (C.24)-(C.34), or Fig. 2 which gives a graphical representation of them. However, at the edges (a finite set of points) where the color lines meet, there can be a problem, since two or all four functions become linearly dependent. This is expected from the second-order case [34, 35], where the ansatz gives all two linear independent solutions, except for a finite number of problematic points (the 'band edge solutions' for Lamé operators [17]). The missing solutions at those points can be found, see the procedure in [36], and this is expected to be generalizable to our fourth-order case. For our purpose of evaluating a partition function, it is sufficient - see discussion below (4.13) - the knowledge of the solutions (in terms of associated quasi-momenta) in the physical region $\Omega^2 < 0$ which is free - see (B.23) - from such problematic "edge points".

and if $f(x)$ in (3.20) is a solution of the differential equation, then the elliptic function $F(x)$ should merely be a constant. Therefore, we have to impose that in the Laurent expansion of $F(x)$

$$F(\varepsilon + x_i) = \frac{A_{i,p_i}}{\varepsilon^{p_i}} + \frac{A_{i,p_i-1}}{\varepsilon^{p_i-1}} + ... + \frac{A_{i,1}}{\varepsilon} + a_{i,0} + a_{i,1}\varepsilon + ... \tag{3.23}$$

all coefficients $A_{i,j}$ of the principal part vanish. This will constrain the free parameters in (3.18) and deliver the corresponding Bethe-ansatz equations for the spectral parameters $\bar{\alpha}_i$.

3.3 Pole structure

In order to proceed we need to collect information about the pole structure of the functions appearing in (3.20)-(3.21). In the study of their analytic properties, it is useful to introduce yet another function

$$\Phi(x) \equiv \frac{1}{f}\frac{\mathrm{d}f}{\mathrm{d}x} = \sum_{r=1}^{n}\left[Z(x+\bar{\alpha}_r+i\mathbb{K}') - Z(x)\right] + \rho + \lambda + \frac{n\pi i}{2\mathbb{K}}\,, \tag{3.24}$$

which has $n+1$ poles at $x = i\mathbb{K}'$ and $x = -\bar{\alpha}_1, -\bar{\alpha}_2, \ldots, -\bar{\alpha}_n$, up to translations by the periods $2\mathbb{K}$ and $2i\mathbb{K}'$. We separately examine these two cases.

Expansion around the pole $x = i\mathbb{K}'$

The expansion of the auxiliary function Φ (3.24) around this singular point provides

$$\begin{aligned}
&\Phi(\varepsilon+i\mathbb{K}') = \frac{A_1}{\varepsilon} + a_0 + a_1\varepsilon + a_2\varepsilon^2 + a_3\varepsilon^3 + \ldots, \quad \Phi'(\varepsilon+i\mathbb{K}') = -\frac{A_1}{\varepsilon^2} + a_1 + 2a_2\varepsilon + 3a_3\varepsilon^2 + \ldots\,,\\
&\Phi''(\varepsilon+i\mathbb{K}') = \frac{2A_1}{\varepsilon^3} + 2a_2 + 6a_3\varepsilon + \ldots\,, \qquad \Phi'''(\varepsilon+i\mathbb{K}') = -\frac{6A_1}{\varepsilon^4} + 6a_3 + \ldots\,,
\end{aligned} \tag{3.25}$$

where we denoted

$$\begin{aligned}
A_1 &= -n, \qquad a_0 = \sum_{r=1}^{n} Z(\bar{\alpha}_r) + \rho + \lambda, \qquad a_1 = \frac{n}{3}(1+k^2) - k^2\sum_{r=1}^{n}\mathrm{sn}^2(\bar{\alpha}_r),\\
a_2 &= -k^2\sum_{r=1}^{n}\mathrm{sn}(\bar{\alpha}_r)\mathrm{cn}(\bar{\alpha}_r)\mathrm{dn}(\bar{\alpha}_r),\\
a_3 &= \frac{n}{45}(1-16k^2+k^4) + \frac{2}{3}(1+k^2)k^2\sum_{r=1}^{n}\mathrm{sn}^2(\bar{\alpha}_r) - k^4\sum_{r=1}^{n}\mathrm{sn}^4(\bar{\alpha}_r).
\end{aligned} \tag{3.26}$$

The same procedure applied on the potentials (3.18) leads to the series

$$\begin{aligned}
v_1(\varepsilon+i\mathbb{K}') &= \frac{\alpha_1}{\varepsilon^2} + \alpha_0 + \frac{\alpha_1}{3}(1+k^2) + \frac{\alpha_1}{15}(1-k^2+k^4)\varepsilon^2 + 0\cdot\varepsilon^3 + ... \\
v_2(\varepsilon+i\mathbb{K}') &= -\frac{2\beta_2}{\varepsilon^3} + \frac{\beta_1}{\varepsilon^2} + \beta_0 + \frac{\beta_1}{3}(1+k^2) + \frac{2\beta_2}{15}(1-k^2+k^4)\varepsilon + \frac{\beta_1}{15}(1-k^2+k^4)\varepsilon^2 + ...\\
v_3(\varepsilon+i\mathbb{K}') &= \frac{6\gamma_3}{\varepsilon^4} - \frac{2\gamma_2}{\varepsilon^3} + \frac{\gamma_1}{\varepsilon^2} + \gamma_0 + \frac{\gamma_1}{3}(1+k^2) + \frac{2\gamma_3}{15}(1-k^2+k^4) + \frac{2\gamma_2}{15}(1-k^2+k^4)\varepsilon + \ldots\,.
\end{aligned} \tag{3.27}$$

Expansion around the poles $x = -\bar{\alpha}_i$, $i = 1, ..., n$

The analysis carried out for the family of poles $-\bar{\alpha}_i$ yields

$$\Phi(\varepsilon - \bar{\alpha}_i) = \frac{1}{\varepsilon} + b_{0,i} + b_{1,i}\varepsilon + b_{2,i}\varepsilon^2 \tag{3.28}$$

where we identify the ε-coefficients with

$$\begin{aligned}
b_{0,i} &= \sum_{r \neq i = 1}^{n} Z(\bar{\alpha}_r - \bar{\alpha}_i + i\mathbb{K}') + nZ(\bar{\alpha}_i) + \frac{i\pi(n-1)}{2\mathbb{K}} + \rho + \lambda, \\
b_{1,i} &= -\sum_{r \neq i = 1}^{n} \mathrm{cs}^2(\bar{\alpha}_i - \bar{\alpha}_r) + \frac{1}{3}(2 - k^2) - n\mathrm{dn}^2(\bar{\alpha}_i), \\
b_{2,i} &= -\sum_{r \neq i = 1}^{n} \frac{\mathrm{cn}(\bar{\alpha}_i - \bar{\alpha}_r)\mathrm{dn}(\bar{\alpha}_i - \bar{\alpha}_r)}{\mathrm{sn}^3(\bar{\alpha}_i - \bar{\alpha}_r)} - nk^2\mathrm{sn}(\bar{\alpha}_i)\mathrm{cn}(\bar{\alpha}_i)\mathrm{dn}(\bar{\alpha}_i).
\end{aligned} \tag{3.29}$$

The potentials are regular functions.

3.4 Consistency equations

From the behaviour of Φ (3.24) and the potentials (3.18) around the singularities, it is now possible to reconstruct the pole structure of F, since the differential operators in (3.21) translate into combinations of the auxiliary function and its derivatives:

$$\begin{aligned}
\frac{1}{f}\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} &= \Phi(x)^2 + \Phi'(x), \\
\frac{1}{f}\frac{\mathrm{d}^3 f}{\mathrm{d}x^3} &= \Phi(x)^3 + 3\Phi(x)\Phi'(x) + \Phi''(x), \\
\frac{1}{f}\frac{\mathrm{d}^4 f}{\mathrm{d}x^4} &= \Phi(x)^4 + 6\Phi(x)^2\Phi'(x) + 4\Phi(x)\Phi''(x) + 3\Phi'(x)^2 + \Phi'''(x).
\end{aligned} \tag{3.30}$$

The condition of vanishing Laurent coefficients of F at $x = i\mathbb{K}'$ gives constraining equations on the numerical parameters α_i, β_i and γ_i, provided we take into account (3.25)-(3.27):

$$\begin{aligned}
\rho &= -\sum_{r=1}^{n} Z(\bar{\alpha}_r), \\
0 &= n(n+1)(n+2)(n+3) + n(n+1)\alpha_1 + 2n\beta_2 + 6\gamma_3, \\
0 &= \lambda[4(n+2)(n+1)n + 2(n\alpha_1 + \beta_2)] + (n\beta_1 + 2\gamma_2)\,, \\
0 &= \lambda^2[6n(n+1) + \alpha_1] + \lambda\beta_1 + a_1[-2n(n+1)(2n+1) + \alpha_1(1-2n) - 2\beta_2] \\
&\quad + n(n+1)(\alpha_0 + \frac{1}{3}\alpha_1(1+k^2)) + \gamma_1\,, \\
0 &= 4\lambda^3 n - 2\lambda[a_1(6n^2 + \alpha_1) - n(\alpha_0 + \frac{\alpha_1}{3}(1+k^2))] - a_1\beta_1 + 2a_2[2n(1+n^2) + \alpha_1(n-1) + \beta_2] + \\
&\quad + n(\beta_0 + \frac{\beta_1}{3}(1+k^2))\,.
\end{aligned} \tag{3.31}$$

In particular, the term of $\mathcal{O}(\varepsilon^0)$ in the Laurent expansion gives the relation between the eigenvalue parameter Λ and the spectral parameters $\bar{\alpha}_i$:

$$\begin{aligned}\Lambda =\ & \lambda^4 + \lambda^2[6a_1(1-2n) + (\alpha_0 + \frac{\alpha_1}{3}(1+k^2))] + \lambda\,[\,a_2(4(2+3n(n-1)) + 2\alpha_1) + (\beta_0 + \frac{\beta_1}{3}(1+k^2))] + \\ & + a_1^2[3(1-2n(1-n)) + \alpha_1] + a_1(1-2n)(\alpha_0 + \frac{\alpha_1}{3}(1+k^2)) + a_2\beta_1 + \\ & + a_3[2(2n-1)(n(1-n)-3) + \alpha_1(3-2n) - 2\beta_2] + \\ & + \frac{1}{15}(1-k^2+k^4)[n(n+1)\alpha_1 - 2n\beta_2 + 2\gamma_3] + \gamma_0 + \frac{\gamma_1}{3}(1+k^2)\ . \end{aligned} \tag{3.32}$$

Finally, imposing that the $1/\varepsilon$-coefficient around the poles $x = -\bar{\alpha}_i$ should vanish gives the Bethe-ansatz equations for the spectral parameters ($i = 1, ..., n$)

$$4b_{0,i}^3 + 2b_{0,i}(6b_{1,i} + \alpha_0 + \alpha_1 k^2 \mathrm{sn}^2(\bar{\alpha}_i)) + 8b_{2,i} + \beta_0 + \beta_1 k^2 \mathrm{sn}^2(\alpha_i) - 2\beta_2 k^2 \mathrm{sn}(\bar{\alpha}_i)\mathrm{cn}(\bar{\alpha}_i)\mathrm{dn}(\bar{\alpha}_i) = 0\ , \tag{3.33}$$

where we used (3.28). In deriving these conditions, we have assumed that $\alpha_i \neq \alpha_j$ for any $i, j = 1, ..., n$.

It is important to mention that in all the examples successfully analyzed in this paper we have made use of the $n = 1$ consistency equations alone - therefore using a single factor in the product defining (3.20) - which we report here separately for reader's convenience.

$n = 1$ consistency equations

$$\begin{aligned} 0 =\ & 12 + \alpha_1 + \beta_2 + 3\gamma_3, \\ 0 =\ & \lambda(24 + 2(\alpha_1 + \beta_2)) + \beta_1 + 2\gamma_2\ , \\ 0 =\ & \lambda^2(12 + \alpha_1) - a_1(\alpha_1 + 12 + 2\beta_2) + \lambda\beta_1 + 2\left(\alpha_0 + \frac{1}{3}\alpha_1(1+k^2)\right) + \gamma_1\ , \\ 0 =\ & -4\lambda^3 - 2\lambda\left(\alpha_0 - a_1(6+\alpha_1) + \frac{1}{3}\alpha_1(1+k^2)\right) + a_1\beta_1 - 2a_2(4+\beta_2) - \beta_0 - \frac{1}{3}\beta_1(1+k^2)\ , \\ \Lambda =\ & \lambda^4 + \lambda^2\,[-6a_1 + \alpha_0 + \frac{1}{3}\alpha_1(1+k^2)\,] + \lambda\,[\,2a_2(\alpha_1+4) + \beta_0 + \frac{1}{3}\beta_1(1+k^2)\,] + \\ & +\ a_1^2(\alpha_1+3) - a_1(\alpha_0 + \frac{1}{3}\alpha_1(1+k^2)) + a_2\beta_1 + a_3(\alpha_1 - 6 - 2\beta_2) + \\ & +\ \frac{2}{15}(1-k^2+k^4)(\alpha_1 - \beta_2 + \gamma_3) + \gamma_0 + \frac{1}{3}\gamma_1(1+k^2)\ . \end{aligned} \tag{3.34}$$

For $n = 1$, the condition for the pole at $x = -\bar{\alpha}$ to vanish turns out to be equivalent to the fourth equation in (3.34) and therefore it does not give any further constraint.

Our result for the consistency equations is in partial disagreement with the study in [30]. However, the examples discussed in the next section and in Appendix A, as well as numerical cross checks of the provided solutions, give strong evidence for the correctness of our procedure.

4 Exact bosonic one-loop partition functions for folded string

We are now ready to use the analysis performed in Section 3 for the computation of determinants of the fluctuation operators discussed in Section 2.

4.1 Exact partition function and one-loop energy for the LL folded string

The fourth order differential operator in (2.20), governing the fluctuations of the LL quantum model defined by (2.11)-(2.19), is easily seen to be of the type (3.17) once the following identification is performed

$$\begin{aligned} \alpha_0 &= 2(1+2k^2), \qquad \alpha_1 = -8, \qquad \beta_0 = \beta_1 = 0, \qquad \beta_2 = -4\,, \\ \gamma_0 &= 1 - \frac{\omega^2}{4\mathrm{w}^2}, \qquad \gamma_1 = \gamma_2 = \gamma_3 = 0\,. \end{aligned} \tag{4.1}$$

Using the $n = 1$ consistency equations (3.34), one finds (here $\bar{\alpha} = \alpha$)

$$\lambda = \pm k\sqrt{\mathrm{sn}^2(\alpha) - 1}\,, \tag{4.2}$$

where the relation between Ω and α is

$$\begin{aligned} \Omega^2_{\mp}(\alpha) &= 4k^2 - 4k^2(1+2k^2)\mathrm{sn}^2(\alpha) + 8k^4\mathrm{sn}^4(\alpha) \mp 8k^3\mathrm{sn}(\alpha)\mathrm{cn}(\alpha)\mathrm{dn}(\alpha)\sqrt{\mathrm{sn}^2(\alpha)-1} \\ &= 4k^2\mathrm{cn}^2(\alpha)\left[ik\,\mathrm{sn}(\alpha) \mp \mathrm{dn}(\alpha)\right]^2\,. \end{aligned} \tag{4.3}$$

It seems advantageous to consider $\alpha \in \mathbb{C}$ as the independent parameter and therefore Ω as a doubly periodic function of α as in (4.3). There should exist four values of α, which correspond to one value of Ω. To be more precise, for the *physical spectrum* we are looking for all values of α, which correspond to a *real* Ω^2. The analysis of Appendix B.1 is devoted to this study, and is nicely summarized in Fig. 1 where in the complex α plane the lines where $\Omega^2(\alpha)$ is real are plotted. The "physical" four linear independent solutions of the fourth order differential operator (2.15) live on these lines, and in the fundamental domain represented in Fig. 2 they correspond to the different colours. In the following let $0 < k < 1/\sqrt{2}$. The case $1/\sqrt{2} < k < 1$ can be obtained by applying the duality transformation of Appendix B.2 on the following results.

For a given real value of Ω^2 the four independent solutions read

$$f_i(x,\Omega,k) = \frac{H(x+\alpha_i)}{\Theta(x)}e^{-x[Z(\alpha_i)-ik\mathrm{cn}(\alpha_i)]}, \qquad i = 1,\ldots,4\,, \tag{4.4}$$

where the α_i as function of Ω have to be chosen according to the range of Ω. For example, for $-\infty < \Omega^2 < 0$, one has

$$\begin{aligned} \alpha_1(\Omega,k) &= u(\Omega,k) - iv(\Omega,k), \qquad (4.5)\\ \alpha_2(\Omega,k) &= 2\mathbb{K} - u(\Omega,k) + iv(\Omega,k), \\ \alpha_3(\Omega,k) &= 2\mathbb{K} + u(\Omega,k) + iv(\Omega,k), \\ \alpha_4(\Omega,k) &= 2\mathbb{K} - u(\Omega,k) + 2i\mathbb{K}' - iv(\Omega,k), \end{aligned}$$

where

$$u(\Omega,k) = \mathrm{sn}^{-1}\left[\sqrt{\tfrac{2}{\Omega^2}(1-\sqrt{1-\Omega^2})},\, k\right], \qquad v(\Omega,k) = \mathrm{sn}^{-1}\left[\sqrt{\tfrac{\Omega^2-2k^2+2k^2\sqrt{1-\Omega^2}}{\Omega^2-4k^2k'^2}},\, k'\right]\,. \tag{4.6}$$

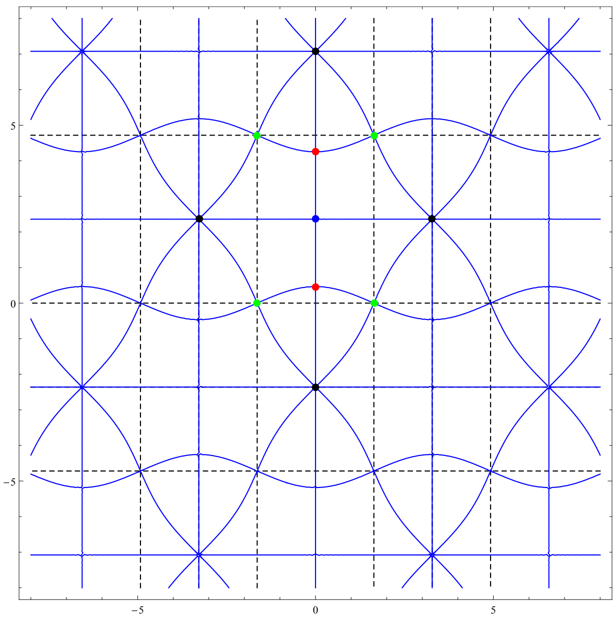

Figure 1: In the complex α plane one can plot the lines where $\Omega^2(\alpha)$ is real. The "physical" four linear independent solution live on these lines. The green dots represent places where $\Omega^2 = 0$, red for $\Omega^2 = 4k^2k'^2$, blue for $\Omega^2 = 1$ and black for poles. We have chosen $k = 0.4$.

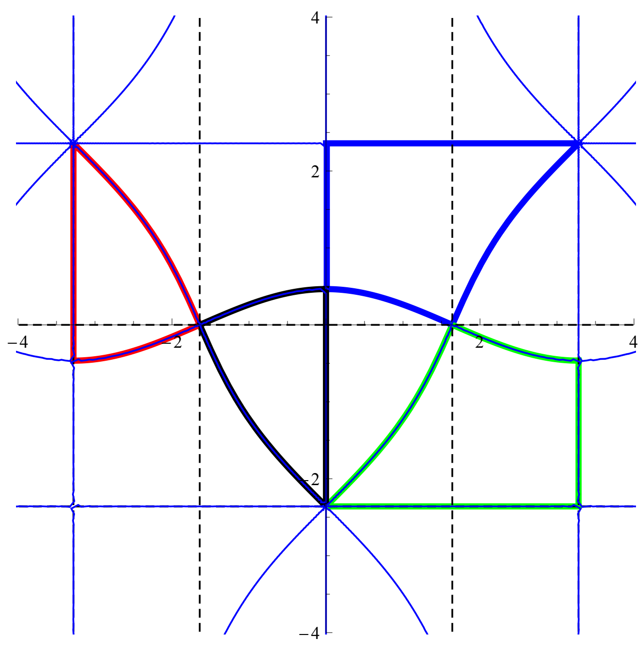

Figure 2: In the fundamental domain the four independent solutions of the fourth order differential operator (2.20) are marked with colors. Walking on such a closed path, Ω^2 runs from $-\infty$ to $+\infty$. We have chosen $k = 0.4$.

The expressions for the α_i's in the other ranges of Ω are collected in (B.25)-(B.30). The quasi-momenta p_i are obtained by $f_i(x+2\mathbb{K}) = e^{2\mathbb{K}ip_i} f_i(x)$ as

$$p_i(\Omega, k) = iZ(\alpha_i, k) + k\mathrm{cn}(\alpha_i, k) - \frac{\pi}{2\mathbb{K}}, \tag{4.7}$$

where the corresponding $\alpha_i(\Omega, k)$ have to be chosen according to the previous list. By construction [23], only two of the quasi-momenta are independent, which we will call p_1 and p_2.

The *exact* determinant of the Landau-Lifshitz model defined by (2.11)-(2.19), using (3.16) with $2L = 4\mathbb{K}$, reads then

$$\det \mathcal{O}_{LL} \; = \; 16 \sin^2 \big(2\mathbb{K}\, p_1(\Omega, k)\big) \, \sin^2 \big(2\mathbb{K}\, p_2(\Omega, k)\big) \; . \tag{4.8}$$

We can immediately recover the characteristic frequencies of the problem, found in [24] using operator methods up to second-order perturbation theory, by simply looking at the zeroes of the determinant (4.8), where the quasi momenta are built with the α's in the branch $\Omega^2 > 0$ and are Taylor-expanded around $k = 0$. It is enough to look to the factor in (4.8) involving p_1, whose expansion re-expressed in terms of the ω is

$$\begin{aligned} p_1 \; = \; & \sqrt{2\omega+1} - \frac{\omega}{2\sqrt{2\omega+1}} k^2 + \frac{\big(-10\omega^4 - 11\omega^3 + 6\omega + 2\big)}{32\omega^2(2\omega+1)^{3/2}} k^4 \\ & + \frac{\big(-22\omega^7 - 39\omega^6 - 19\omega^5 + 2\omega^4 + 10\omega^3 + 16\omega^2 + 10\omega + 2\big)}{64\omega^4(2\omega+1)^{5/2}} k^6 + O(k^8) \; . \end{aligned} \tag{4.9}$$

Inserting it into (4.8) and requiring the vanishing of the expression order by order in small k^2, one finds the (squared) frequencies to be

$$\omega^2 = \frac{1}{4}(n^2-1)^2 + \frac{1}{4}(1-n^2)k^2 + \frac{\big(3n^4 - 2n^2 + 15\big)}{64(1-n^2)} k^4 - \frac{\big(n^8 + n^6 + 7n^4 + 27n^2 + 28\big)}{128\,(n^2-1)^3} k^6 + O\left(k^8\right) \; , \tag{4.10}$$

where we do not report higher orders, but notice that it is straightforward to calculate them. The first three orders of the expansion above coincide with the ones of [24].

The one-loop correction to the $SU(2)$ LL string energy can be of course obtained perturbatively via a regularized sum over the frequencies given above [24], or exactly in terms of the one-loop world-sheet effective action $\Gamma^{(1)}$, and thus in terms of the corresponding partition function Z_{LL}, as follows

$$E_1 = \frac{\Gamma^{(1)}}{\mathcal{T}} = -\frac{\log Z_{LL}}{\mathcal{T}} \, , \qquad\qquad \mathcal{T} = \int_{-\infty}^{\infty} d\tau \, . \tag{4.11}$$

The Euclidean LL partition function is obtained from the functional determinant as

$$Z_{LL} = \det{}^{-1/2} \mathcal{O}_{LL} \, , \tag{4.12}$$

which using (4.8) can be explicitly written as [24]

$$\Gamma^{(1)} = -\log Z_{LL} = \frac{\mathcal{T}}{2} \int_{-\infty}^{\infty} \frac{d\Omega}{2\pi} \log \left[16 \sin^2 \big(2\mathbb{K}\, p_1(\Omega, k)\big) \, \sin^2 \big(2\mathbb{K}\, p_2(\Omega, k)\big)\right] \; . \tag{4.13}$$

[23] See discussion around (3.12)-(3.13).

[24] It is convenient to use as integration variable the rescaled frequency Ω, see (2.21).

Above, the Euclidean setting requires the quasi-momenta p_i to be built out of the α's in the branch $\Omega^2 < 0$, which are given in (4.5). The integral in (4.13) is divergent. A first meaningful choice of regularization of the functional determinant is to refer it to the $k = 0$ case. Indeed this limit, as discussed in [24], represent a nearly point-like string and the correction to the ground-state energy should vanish. Hence, we obtain

$$\Gamma^{(1)}_{reg} = \frac{\mathcal{T}}{2} \int_{-\infty}^{\infty} \frac{d\Omega}{2\pi} \log \left[\frac{\sin^2\left(2\mathbb{K}\, p_1(\Omega, k)\right)\, \sin^2\left(2\mathbb{K}\, p_2(\Omega, k)\right)}{\sin^2\left(\pi\, p_1(\Omega, 0)\right)\, \sin^2\left(\pi\, p_2(\Omega, 0)\right)} \right] \tag{4.14}$$

where, at the denominator, the quasi-momenta $p_i(\Omega, 0)$ are computed at $k = 0$. In order to analytically perform the above integral over Ω, we can resort to the short string expansion $k^2 \simeq 0$. This again means to consider the small k expansion of quasi-momenta p_i (which differ from the ones considered above, as we are in a different branch for the α's), as reported in Appendix B.4, and then after, to integrate over Ω the corresponding expressions computed order by order in k. Each term in the k^2-series for $\Gamma^{(1)}_{reg}$ must be further regularized, which is of course expected as only certain bosonic degrees of freedom and no fermionic ones (crucial for UV finiteness) participate to the effective LL action. In Appendix B.4 we report two different ways of regularizing (one inspired by ζ-function regularization and one with standard cutoff) which lead to the same result. The resulting expression for the k^2-expansion of the one-loop energy is the same as in [24]

$$E_1 = \frac{\Gamma^{(1)}_{reg}}{\mathcal{T}} = \frac{1}{4}k^2 + \frac{1}{16}\left(1 - \frac{\pi^2}{3}\right) k^4 + O(k^5)\,. \tag{4.15}$$

It is interesting to notice that this result follows smoothly by our standard regularization of the 2d LL string effective action, while in [24] it is implied by a ζ-function regularization supplemented by a general prescription for the vacuum energy in terms of characteristic frequencies of a mixed system of oscillators [37].

From equation (2.7), in terms of the physical parameter J_2/J, the short string limit $k^2 \to 0$ reads

$$\frac{J_2}{J} = \frac{k^2}{2} + \frac{k^4}{16} + \mathcal{O}(k^5)\,, \qquad k^2 = \frac{2\,J_2}{J} - \frac{1}{2}\left(\frac{J_2}{J}\right)^2\,, \tag{4.16}$$

and the expression for the energy becomes

$$E_1 = \frac{\tilde{\lambda}}{2}\left(\frac{J_2}{J} + \left(\frac{1}{4} - \frac{\pi^2}{6}\right)\left(\frac{J_2}{J}\right)^2\right) + O\left(\left(\frac{J_2}{J}\right)^2\right)\,, \tag{4.17}$$

where we restored the $\tilde{\lambda}$ dependence. The first three terms in the formula above are in agreement with [24].

For completeness, we mention that the analysis for the LL folded string in the SL(2) sector (where strings rotate in $\mathrm{AdS}_3 \subset \mathrm{AdS}_5$ with center of mass moving along a big circle of S^5) is totally analogous. Using the following analytical continuation [24, 32]

$$\psi \to -i\rho\,, \qquad \varphi \to \eta\,, \qquad w_1 \to \kappa\,, \qquad w_2 \to w_1\,, \tag{4.18}$$

one can easily see that the system of coupled fluctuations is effectively described by the fourth order differential operator in (2.20) where now k^2 is negative. Its solutions are then trivially generalised to the case $k^2 < 0$ as basically in each formula one should substitute $k \to \sqrt{-k^2}$ and omit all imaginary constants i in the exponentials.

4.2 Folded string in full bosonic sigma-model

The fourth order differential operator in (2.39) [25] is again of the type (3.17) with the identification

$$\begin{aligned} \alpha_0 &= 2(\bar{\Omega}^2 + k^2 + 1), \qquad \alpha_1 = -8, \qquad \beta_0 = \beta_1 = 0, \qquad \beta_2 = -4 \\ \gamma_0 &= [(-\bar{\Omega}^2 + 1 + k^2)^2 - 4k^2] - \frac{4\nu^2\bar{\Omega}^2}{\bar{w}^2 - \nu^2}, \qquad \gamma_1 = \gamma_2 = \gamma_3 = 0\,, \end{aligned} \tag{4.19}$$

where $\bar{\Omega}$ is defined in (2.40). Using the consistency equations (3.34) one finds

$$\lambda = \pm\sqrt{k^2\mathrm{sn}^2(\alpha) - \bar{\Omega}^2}\ , \tag{4.20}$$

where the relation between $\bar{\Omega}$ and α is

$$8k^4\mathrm{sn}^4(\alpha) - 4(1+k^2+\bar{\Omega}^2)k^2\mathrm{sn}^2(\alpha) \pm 8k^2\mathrm{sn}(\alpha)\mathrm{cn}(\alpha)\mathrm{dn}(\alpha)\sqrt{k^2\mathrm{sn}^2(\alpha) - \bar{\Omega}^2} - \tfrac{4\nu^2\bar{\Omega}^2}{\bar{w}^2-\nu^2} = 0\,. \tag{4.21}$$

The study reported in Appendix C.2 shows that the "physical" four linear independent solutions of (2.39) live on the straight and ellipse-like lines in Fig. 3.

Using in $f_i(x+2\mathbb{K}) = e^{2\mathbb{K}ip_i}f_i(x)$ their explicit expressions - cf. (C.24) - the quasi-momenta are then obtained as

$$p_n(\bar{\Omega}) = \pm i\left[Z(\alpha_n) + \frac{k\,\mathrm{sn}(\alpha_n)}{(\kappa^2-\nu^2)\mathrm{sn}^2(\alpha_n) + \nu^2}\left(\kappa\bar{w} \pm \sqrt{\kappa^2-\nu^2}\sqrt{\bar{w}^2-\nu^2}\,\mathrm{cn}(\alpha_n)\mathrm{dn}(\alpha_n)\right)\right] + \frac{\pi}{2\mathbb{K}}, \tag{4.22}$$

where α_n as function of $\bar{\Omega}$ has to be chosen from the list in (C.26)-(C.34). The functional determinant is again given by

$$\det \mathcal{O}_\nu = 16\sin^2(L\,p_1)\sin^2(L\,p_2)\ , \tag{4.23}$$

with $2L = 4\mathbb{K}$.

As a first check of the correctness of the procedure, one can take the long string limit $k \to 1$, $\bar{w}^2 \to \kappa^2$ (see Section C.1) and look at the zeroes of the expression above choosing the positive-frequency range (C.32) and (C.34) for the α's in (4.22). We obtain the characteristic frequencies [26]

$$\omega_n = \sqrt{n^2 + 2\kappa^2 \pm 2\sqrt{\kappa^4 + n^2\nu^2}}, \tag{4.25}$$

[25] In this section we are working in Minkowski signature, so that (4.19) are obtained from (3.17) analytically continuing the frequencies.

[26] In this limit, we obtain, from (C.32) and (C.34),

$$\mathrm{sn}(\alpha_{1,2}) \to -\nu^2\omega/\Big[\sqrt{\kappa^2-\nu^2}\sqrt{\left(\kappa^2 \pm \sqrt{(\kappa^2-\nu^2)^2+\nu^2\omega^2}\right)^2 - \nu^4}\Big]\ , \tag{4.24}$$

where we have used (2.40).

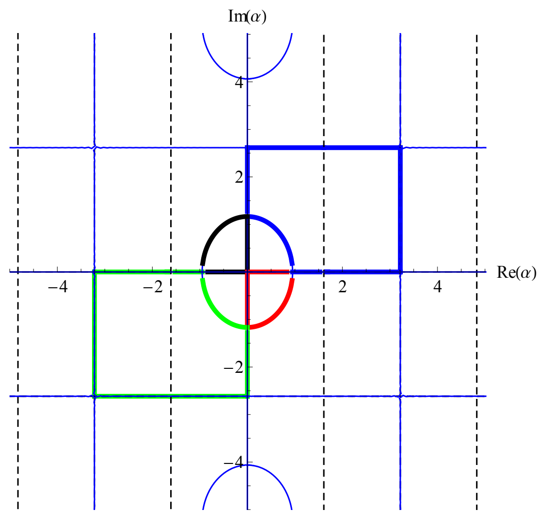

Figure 3: The places, in the fundamental domain of the complex α plane, where the four linear independent solutions (C.24)-(C.25) (marked with different colours) for the fourth order differential operator (2.39) live. Here we have chosen $k = 3, w = 6, \nu = 2.5$. See also Fig. 4.

which is the same result as found in [5] [27], see also (C.12).

Since we are missing (see Introduction and Section 2.2.2) the fermionic counterpart of (4.23), we cannot proceed with the exact evaluation of the full (superstring) one-loop partition function on the folded two-spin solution. However, we observe that a nice consequence of our procedure is the possibility of making a non-trivial, analytical statement on the equivalence of partition functions in conformal and static gauge in the single-spin ($\nu = 0$) case. While here the fermionic determinant can be given exactly for all values of the spin [17], it is only the bosonic partition function in static gauge - where fluctuations are naturally decoupled - which has been written down in an analytically exact closed form, and reads [17]

$$\log Z^{\rm bos}_{\rm static\ gauge} = -\,\frac{\mathcal{T}}{2}\int \frac{d\omega}{2\pi} \log\Big(\det \mathcal{O}_\phi \det{}^2\mathcal{O}_\beta \det{}^5\mathcal{O}_0\Big) \tag{4.27}$$

where

$$\det \mathcal{O}_\phi = 4\sinh^2[2\tilde{\mathbb{K}} Z(\alpha_\phi|\tilde{k}^2)]\,, \quad \det \mathcal{O}_\beta = 4\sinh^2\left[2\mathbb{K}\, Z(\alpha_\beta|k^2)\right]\,, \quad \det\mathcal{O}_0 = 4\sinh^2(\pi\omega) \tag{4.28}$$

and

$$\mathrm{sn}(\alpha_\phi\,|\tilde{k}^2) = \frac{1}{\tilde{k}}\sqrt{1+\left(\frac{\pi\omega}{2\tilde{\mathbb{K}}}\right)^2}\,, \quad \mathrm{sn}(\alpha_\beta|k^2) = \frac{1}{k}\sqrt{1+k^2+\left(\frac{\pi\omega}{2\mathbb{K}}\right)^2}\,, \tag{4.29}$$

with $\tilde{k}^2 = 4k/(1+k)^2$ and $\tilde{\mathbb{K}} = \mathbb{K}(\tilde{k}^2)$. The analysis in Section 2.2.1 shows that, in conformal gauge, the spectral problem associated to the mixed-mode, 3×3 matrix differential

[27] One can obtain this result also from (2.39), which in this limit becomes

$$\lim_{k\to 1} \mathcal{O}^{(4)} = \partial_x^4 + 2[\omega^2 - 2(\kappa^2-\nu^2)]\partial_x^2 + (\omega^4 - 4\omega^2\kappa^2)\;. \tag{4.26}$$

operator corresponding to (2.28)-(2.30) can be evaluated, see (2.38), via the product of a free determinant times the determinant of the fourth order differential operator (2.39), and thus [28]

$$\log Z^{\rm bos}_{\rm conformal\ gauge} = -\frac{\mathcal{T}}{2}\int \frac{d\omega}{2\pi}\log\Big(\det\mathcal{O}_{\nu=0}\det{}^2\mathcal{O}_\beta\det{}^4\mathcal{O}_0\Big)\ , \tag{4.30}$$

where in the counting of massless operators we already have taken into account the two conformal gauge massless ghosts [38], and (see Appendix C.3)

$$\det\mathcal{O}_{\nu=0} = 16\sinh^2\left[2\mathbb{K}\left(Z(\alpha|k^2)+\frac{1+{\rm cn}(\alpha|k^2){\rm dn}(\alpha|k^2)}{{\rm sn}(\alpha|k^2)}\right)\right]\sinh^2(2\mathbb{K}\bar\Omega)\ , \tag{4.31}$$

with (switching to Euclidean signature)

$${\rm sn}^2(\alpha|k^2) = \frac{-4\bar\Omega^2}{(1+k^2-\bar\Omega^2)^2-4k^2}\ . \tag{4.32}$$

One can see that the second factor in (4.31) corresponds to the same massless boson mode of (4.28) (recalling (2.40) and that for $\nu=0$ it is $\bar w = \frac{2\mathbb{K}}{\pi}$), while for the first factor one should use for the Jacobi Zeta function the transformation (C.35) which, writing $\tilde\alpha = \alpha/(1+\tilde k') + i\mathbb{K}'/(1+\tilde k')$, leads to the identity

$$2\mathbb{K}\left[Z(\alpha|k^2)+\frac{1+{\rm cn}(\alpha|k^2){\rm dn}(\alpha|k^2)}{{\rm sn}(\alpha|k^2)}\right] = 2\tilde{\mathbb{K}}Z(\tilde\alpha|\tilde k^2)+i\pi\ . \tag{4.33}$$

This establishes analytically the equivalence of static and conformal gauge bosonic determinants (4.27)-(4.30) [29].

5 Outlook

In this paper we have made a first step into the analytic solution of the matrix fluctuations determinant for nontrivial string configurations relevant for the study of the AdS/CFT integrable systems, evaluating exactly the one-loop partition function for the quantum Landau-Lifshitz model on the $SU(2)$ folded string solution of [24]. The same procedure allows the diagonalization of the bosonic sector of fluctuations of the full $\mathrm{AdS}_5\times\mathrm{S}^5$ excitations over the two-spin folded string solution of [16].

[28] While we worked at the operatorial level with the linearized (near folded string solution) form of the string equations of motion, and did not prove the formal equivalence between the determinant of the 3×3 matrix differential operator corresponding to (2.28)-(2.30) and the product $\det\mathcal{O}_{\nu=0}\det\mathcal{O}_0$, (4.30) should be formally correct. This is not different from the steps (2.11)-(2.19) followed in setting the LL spectral problem, with a new ingredient here consisting in the implementation of Virasoro constraints. As thoroughly discussed in [17], at the level of path integral the step analogous to (2.33)-(2.34) will produce an extra $\det\mathcal{O}_0$ factor as required for balance of degrees of freedom.

[29] At the operator level, it was noticed already in [17] that $\mathcal{O}_{\nu=0}$ manifestly factorizes as a product of two second-order ones

$$\mathcal{O}^{(4)}_{\nu=0} = \mathcal{O}_1\cdot\mathcal{O}_2\ ,\quad \mathcal{O}_1 = (\rho')^{-1}\Big[\partial_\sigma^2+\omega^2-2\,\rho'^2-2\frac{\kappa^2\,w^2}{\rho'^2}\Big]\,\rho'\ ,\quad \mathcal{O}_2 = \rho'\ \ [\partial_\sigma^2+\omega^2]\ (\rho')^{-1}\ , \tag{4.34}$$

where the operators within brackets are those, decoupled, appearing in static gauge [16].

This result calls for the complete (i.e. including fermions) solution of the fluctuation problem for non-homogeneous configurations of elliptic type, which might require a nontrivial field redefinition for the corresponding Lagrangian, or equivalently a modification of the ansatz for the solution of the related differential operator. This class of solutions includes the relevant case of open string configurations corresponding to the space-like Wilson loops of [22] (also in other backgrounds [23]). Completing in this sense the analysis here performed should give an answer to the caveats of the semiclassical analysis mentioned in the Introduction, enlarging the range of applicability of the procedure and opening the way to the detailed understanding of the relation between this quantum field-theoretical approach and the one based on the algebraic curve [39].

Acknowledgments

We are grateful to M. Beccaria, G. Dunne and A. A. Tseytlin for earlier collaboration on the topic of Section 2.2.1 and C.1. It is a pleasure to thank M. Beccaria, G. Dunne, S. Frolov, L. Griguolo, D. Seminara, L. Thorlacius and A. A. Tseytlin for discussions. The work of V.F., M.P. and E.V. is funded by the Emmy Noether Programme "Gauge Field from Strings" funded by DFG. M.P. also acknowledges support from SFB 647 "Space-Time-Matter. Analytic and Geometric Structures". V.G.M.P. acknowledges partial support from Swedish Research Council for funding under the contract 623-2011-1186.

A The squared Lamé operator

As a check of the procedure described in Section 3 and of the involved algebraic manipulations there performed, we consider the fourth order differential operator obtained by squaring the Lamé operator

$$\mathcal{O}_L = -\partial_x^2 + 2k^2\mathrm{sn}^2(x|k^2) + \Omega^2, \tag{A.1}$$

which gives

$$\begin{aligned}\mathcal{O}_L^2 = &\ \partial_x^4 - 2(2k^2\mathrm{sn}^2(x) + \Omega^2)\partial_x^2 - 8k^2\mathrm{sn}(x)\mathrm{cn}(x)\mathrm{dn}(x)\partial_x \\ &-8k^4\mathrm{sn}^4(x) + 4(2(1+k^2) + \Omega^2)k^2\mathrm{sn}^2(x) - 4k^2 + \Omega^4\ .\end{aligned} \tag{A.2}$$

Since the solution of the Lamé equation is well known we can immediately write down the Floquet solutions for

$$\mathcal{O}_L^2 f(x) = \Lambda f(x), \tag{A.3}$$

given by

$$\begin{aligned} f_1(x) &= \frac{H(x+\alpha_+)}{\Theta(x)}e^{-xZ(\alpha_+)}, \qquad & f_2(x) &= \frac{H(x-\alpha_+)}{\Theta(x)}e^{xZ(\alpha_+)}, \\ f_3(x) &= \frac{H(x+\alpha_-)}{\Theta(x)}e^{-xZ(\alpha_-)}, \qquad & f_4(x) &= \frac{H(x-\alpha_-)}{\Theta(x)}e^{-xZ(\alpha_-)}, \end{aligned} \tag{A.4}$$

with

$$\mathrm{sn}(\alpha_\pm|k^2)=\sqrt{\frac{1+k^2\mp\sqrt{\Lambda}+\Omega^2}{k^2}}\,. \tag{A.5}$$

For the squared Lamé operator (A.2) we can read off the coefficients

$$\begin{aligned}\alpha_0 &= -2\Omega^2, \qquad \alpha_1=-4, \quad \beta_0=0, \qquad \beta_1=0, \qquad \beta_2=-4\\ \gamma_0 &= -\frac{4}{3}k^2+\Omega^4, \qquad \gamma_1=\frac{8}{3}(1+k^2)+4\Omega^2, \qquad \gamma_2=0, \qquad \gamma_3=-\frac{4}{3}\,.\end{aligned} \tag{A.6}$$

One can see that the first consistency condition in (3.34) is satisfied. Further one finds $\lambda=0$, which is also consistent with (3.33). Equation (3.32) gives now the relation between Λ and α

$$k^4\mathrm{sn}^4(\alpha)-2[(1+k^2)+\Omega^2]k^2\mathrm{sn}^2(\alpha)+(1+k^2+\Omega^2)^2=\Lambda, \tag{A.7}$$

which can be solved as

$$\mathrm{sn}(\alpha_\pm|k^2)=\sqrt{\frac{1+k^2\mp\sqrt{\Lambda}+\Omega^2}{k^2}}, \tag{A.8}$$

which agrees with the result (A.5) directly obtained using the square property.

B Landau-Lifshitz $SU(2)$ folded string analysis: details

B.1 Spectral domain

Useful properties of $\Omega_\pm$ defined in (4.3) are

$$\begin{aligned}\Omega_\pm(-\alpha,k) &= -\Omega_\mp(\alpha,k),\\ \Omega_\pm(\alpha+2i\mathbb{K}',k) &= -\Omega_\mp(\alpha,k),\\ \Omega_\pm(\alpha+\mathbb{K}+i\mathbb{K}',k) &= \mp\Omega_\mp(i\alpha,k'),\\ \Omega_\pm(\alpha+2\mathbb{K}+2i\mathbb{K}',k) &= -\Omega_\pm(\alpha,k)\;.\end{aligned} \tag{B.1}$$

It is then easy to see that $\Omega_\pm(\alpha)$ is a doubly periodic function

$$\Omega_\pm(\alpha+4\mathbb{K},k)=\Omega_\pm(\alpha,k), \qquad \Omega_\pm(\alpha+4i\mathbb{K}',k)=\Omega_\pm(\alpha,k) \tag{B.2}$$

Important special values are

$$\begin{aligned}&\Omega_+^2(\mathbb{K},k)=0, \qquad \Omega_+^2(i\mathbb{K}',k)=1,\\ &\Omega_+^2\left(i\mathbb{K}'+i\mathrm{cn}^{-1}\left(\frac{k^2}{k'^2}\,,\,k'\right),k\right)=\Omega_+^2\left(i\mathbb{K}'-i\mathrm{cn}^{-1}\left(\frac{k^2}{k'^2}\,,\,k'\right)\right)=4k^2k'^2.\end{aligned} \tag{B.3}$$

For the physical spectrum only those values of the complex parameter $\alpha=u+iv$ corresponding to a real Ω^2 are of interest. We decompose

$$\Omega_+(u+iv)=\mathrm{Re}(\Omega_+)(u,v)+i\,\mathrm{Im}(\Omega_+)(u,v) \tag{B.4}$$

with

$$
\begin{aligned}
\mathrm{Re}(\Omega_+)(u,v) &= \frac{2k\mathrm{dn}(u,k)(\mathrm{dn}(v,k')-k\mathrm{cn}(u,k)\mathrm{sn}(v,k'))}{(1-\mathrm{dn}^2(u,k)\mathrm{sn}^2(v,k'))^2}(\mathrm{cn}(u,k)\mathrm{cn}^2(v,k')+k\mathrm{sn}^2(u,k)\mathrm{sn}(v,k')\mathrm{dn}(v,k')),\\
\mathrm{Im}(\Omega_+)(u,v) &= \frac{2k\mathrm{sn}(u,k)\mathrm{cn}(v,k')(\mathrm{dn}(v,k')-k\mathrm{cn}(u,k)\mathrm{sn}(v,k'))}{(1-\mathrm{dn}^2(u,k)\mathrm{sn}^2(v,k'))^2}(k\mathrm{cn}(u,k)-\mathrm{dn}^2(u,k)\mathrm{sn}(v,k')\mathrm{dn}(v,k')).
\end{aligned}
$$

In order to have Ω_+^2 real, we find the cases

- $v=\mathbb{K}'$, then

$$\Omega_+(u+i\mathbb{K}') = \frac{2\mathrm{dn}(u,k)}{\mathrm{sn}^2(u,k)}(1-\mathrm{cn}(u,k)). \tag{B.5}$$

  Setting $u=2w$ gives

$$\Omega_+(2w+i\mathbb{K}') = \mathrm{dc}^2(w,k)-k^2\mathrm{sn}^2(w,k). \tag{B.6}$$

  Varying w from 0 to $\mathbb{K}$, then $\Omega_+^2(2w+i\mathbb{K}')$ covers the interval $[1,\infty)$. Therefore we can solve for w as

$$\mathrm{sn}^2(w) = 1-\frac{\Omega}{2k^2}+\frac{1}{2k^2}\sqrt{\Omega^2-4k^2k'^2}, \tag{B.7}$$

  with

$$0<\mathrm{sn}^2(w)<1 \qquad \text{for} \qquad 1<\Omega<\infty. \tag{B.8}$$

- $u=0$, then

$$\Omega_+(iv) = \frac{2k}{\mathrm{cn}^2(v,k')}(\mathrm{dn}(v,k')-k\mathrm{sn}(v,k')). \tag{B.9}$$

  Varying v from $\mathbb{K}'-\mathrm{cn}^{-1}(k^2/k'^2,k')$ to $\mathbb{K}'$, then $\Omega_+^2(iv)$ covers the interval $[4k^2k'^2,1]$. Therefore we can solve for v as

$$\mathrm{sn}^2(v,k') = 1+\frac{4k^2}{\Omega^2}\left[-1+2k^2+\sqrt{\Omega^2-4k^2k'^2}\right], \tag{B.10}$$

  with

$$\frac{k^2}{k'^2}<\mathrm{sn}^2(v,k')<1 \qquad \text{for} \qquad 2kk'<\Omega<1 \qquad \text{and} \qquad 0<k^2<\frac{1}{2}. \tag{B.11}$$

- $k\mathrm{cn}(u,k)-\mathrm{dn}^2(u,k)\mathrm{sn}(v,k')\mathrm{dn}(v,k')=0$

  For $0<u<\mathbb{K}$ this can be solved for $v=v(u,k)$ as

$$\mathrm{sn}^2(v,k') = \frac{1-\sqrt{1-4k^2k'^2\frac{\mathrm{cd}^2(u,k)}{\mathrm{dn}^2(u,k)}}}{2k'^2}, \tag{B.12}$$

  then

$$\alpha(u,k) = u+i\,\mathrm{sn}^{-1}\left[\frac{1}{\sqrt{2}k'}\sqrt{1-\sqrt{1-4k^2k'^2\frac{\mathrm{cd}^2(u,k)}{\mathrm{dn}^2(u,k)}}},k'\right]. \tag{B.13}$$

  After using some elliptic function identities one finds

$$\Omega_+^2(\alpha(u,k),k) = 4k^2k'^2\frac{\mathrm{cn}^2(u,k)}{\mathrm{dn}^4(u,k)}, \qquad 0<u<\mathbb{K}. \tag{B.14}$$

Solving for u gives

$$\mathrm{sn}^2(u,k) = \frac{1}{k^2} + \frac{2k'^2}{k^2}\frac{\sqrt{1-\Omega^2}-1}{\Omega^2}, \tag{B.15}$$

with

$$0 < \mathrm{sn}^2(u,k) < 1, \qquad \text{for} \qquad 0 < \Omega < 2kk' \qquad \text{and} \qquad 0 < k^2 < \frac{1}{2}. \tag{B.16}$$

- $\mathrm{cn}(u,k)\mathrm{cn}^2(v,k') + k\mathrm{sn}^2(u,k)\mathrm{sn}(v,k')\mathrm{dn}(v,k') = 0$

  For $\mathbb{K} < u < 2\mathbb{K}$ this can be solved for v as

$$\begin{aligned}\mathrm{sn}^2(v,k') &= \frac{2\mathrm{cn}^2(u,k)}{2\mathrm{cn}^2(u,k) + k^2\mathrm{sn}^4(u,k) + k^2\mathrm{sn}^2(u)\sqrt{4\mathrm{cn}^2(u,k)+\mathrm{sn}^4(u,k)}} = \\ &= \frac{2\mathrm{cn}^2(u,k) + k^2\mathrm{sn}^4(u,k) - k^2\mathrm{sn}^2(u,k)\sqrt{4\mathrm{cn}^2(u,k)+\mathrm{sn}^4(u,k)}}{2(\mathrm{cn}^2(u,k) + k^2k'^2\mathrm{sn}^4(u,k))} = \\ &= \frac{2\frac{\mathrm{cn}^2(u,k)}{\mathrm{sn}^4(u,k)} + k^2 - k^2\sqrt{1+4\frac{\mathrm{cn}^2(u,k)}{\mathrm{sn}^4(u,k)}}}{2\left(k^2k'^2 + \frac{\mathrm{cn}^2(u,k)}{\mathrm{sn}^4(u,k)}\right)},\end{aligned} \tag{B.17}$$

  and then

$$\alpha(u,k) = u + i\,\mathrm{sn}^{-1}\left[\sqrt{\frac{2\frac{\mathrm{cn}^2(u,k)}{\mathrm{sn}^4(u,k)} + k^2 - k^2\sqrt{1+4\frac{\mathrm{cn}^2(u,k)}{\mathrm{sn}^4(u,k)}}}{2\left(k^2k'^2 + \frac{\mathrm{cn}^2(u,k)}{\mathrm{sn}^4(u,k)}\right)}}, k'\right]. \tag{B.18}$$

  After using some elliptic function identities one finds

$$\Omega_+^2(\alpha(u),k) = -\frac{4\mathrm{cn}^2(u,k)}{\mathrm{sn}^4(u,k)}, \qquad \text{for} \qquad \mathbb{K} < u < 2\mathbb{K}, \tag{B.19}$$

  or

$$\Omega_+^2(\alpha(\tilde{u}+\mathbb{K}),k) = -4k'^2\frac{\mathrm{sn}^2(\tilde{u},k)\mathrm{dn}^2(\tilde{u},k)}{\mathrm{cn}^4(\tilde{u},k)}. \tag{B.20}$$

  Solving for u gives

$$\mathrm{sn}^2(u,k) = \frac{2}{\Omega^2}(1-\sqrt{1-\Omega^2}), \tag{B.21}$$

  with

$$0 < \mathrm{sn}^2(u,k) < 1 \qquad \text{for} \qquad -\infty < \Omega^2 < 0. \tag{B.22}$$

The expressions of α_i's in the different branches for Ω^2 read as follows:

- For $-\infty < \Omega^2 < 0$ as

$$\begin{aligned}\alpha_1(\Omega,k) &= u(\Omega,k) - iv(\Omega,k), \\ \alpha_2(\Omega,k) &= 2\mathbb{K} - u(\Omega,k) + iv(\Omega,k), \\ \alpha_3(\Omega,k) &= 2\mathbb{K} + u(\Omega,k) + iv(\Omega,k), \\ \alpha_4(\Omega,k) &= 2\mathbb{K} - u(\Omega,k) + 2i\mathbb{K}' - iv(\Omega,k),\end{aligned} \tag{B.23}$$

where

$$u(\Omega,k)=\mathrm{sn}^{-1}\left[\sqrt{\frac{2}{\Omega^2}(1-\sqrt{1-\Omega^2})},\ k\right],\qquad v(\Omega,k)=\mathrm{sn}^{-1}\left[\sqrt{\frac{\Omega^2-2k^2+2k^2\sqrt{1-\Omega^2}}{\Omega^2-4k^2k'^2}},\ k'\right]. \tag{B.24}$$

- For $0<\Omega^2<4k^2k'^2$ as

$$\begin{aligned}\alpha_1(\Omega,k)&=2\mathbb{K}-u_2(\Omega,k)-iv_2(\Omega,k),\\ \alpha_2(\Omega,k)&=u_2(\Omega,k)+iv_2(\Omega,k),\\ \alpha_3(\Omega,k)&=2\mathbb{K}+u_2(\Omega,k)-iv_2(\Omega,k),\\ \alpha_4(\Omega,k)&=2\mathbb{K}-u_2(\Omega,k)+2i\mathbb{K}'+iv_2(\Omega,k),\end{aligned} \tag{B.25}$$

where

$$u_2(\Omega,k)=\mathrm{sn}^{-1}\left[\frac{1}{k}\sqrt{1-2k'^2\frac{1-\sqrt{1-\Omega^2}}{\Omega^2}}\right],\qquad v_2(\Omega,k)=\mathrm{sn}^{-1}\left[\frac{1}{\sqrt{2}k'}\sqrt{1-\sqrt{1-\Omega^2}},k'\right]. \tag{B.26}$$

- For $4k^2k'^2<\Omega^2<\infty$ as

$$\begin{aligned}\alpha_3(\Omega,k)&=2\mathbb{K}-i\mathbb{K}'+2i\alpha_0(\Omega,k'),\\ \alpha_4(\Omega,k)&=2\mathbb{K}+3i\mathbb{K}'-2i\alpha_0(\Omega,k').\end{aligned} \tag{B.27}$$

- For $4k^2k'^2<\Omega^2<1$

$$\begin{aligned}\alpha_1(\Omega,k)&=2\mathbb{K}-i\,\mathrm{sn}^{-1}\left[\sqrt{1-\frac{4k^2}{\Omega^2}\left(1-2k^2-\sqrt{\Omega^2-4k^2k'^2}\right)},k'\right],\\ \alpha_2(\Omega,k)&=i\,\mathrm{sn}^{-1}\left[\sqrt{1-\frac{4k^2}{\Omega^2}\left(1-2k^2-\sqrt{\Omega^2-4k^2k'^2}\right)},k'\right].\end{aligned} \tag{B.28}$$

- For $1<\Omega^2<\infty$ as

$$\begin{aligned}\alpha_1(\Omega,k)&=2\mathbb{K}-i\mathbb{K}'-2\alpha_0(\Omega,k)\\ \alpha_2(\Omega,k)&=i\mathbb{K}'+2\alpha_0(\Omega,k),\end{aligned} \tag{B.29}$$

where

$$\alpha_0(\Omega,k)=\mathrm{sn}^{-1}\left[\sqrt{1-\frac{\Omega}{2k^2}+\frac{1}{2k^2}\sqrt{\Omega^2-4k^2k'^2}},\ k\right]. \tag{B.30}$$

B.2 A duality property of the LL fourth order differential operator

Defining $z=ix$ we can rewrite (2.20) as

$$\begin{aligned}\mathcal{O}^{(4)}(z,k')f_{1,2}(-iz-\mathbb{K}+i\mathbb{K}',\alpha,k)&=\Omega_-^2(\alpha,k)f_{1,2}(-iz-\mathbb{K}+i\mathbb{K}',\alpha,k),\\ \mathcal{O}^{(4)}(z,k')f_{3,4}(-iz-\mathbb{K}+i\mathbb{K}',\alpha,k)&=\Omega_+^2(\alpha,k)f_{3,4}(-iz-\mathbb{K}+i\mathbb{K}',\alpha,k).\end{aligned} \tag{B.31}$$

Interchanging the role of k and k' and using (B.1) we get

$$\mathcal{O}^{(4)}(x,k) f_{3,4}(-ix - \mathbb{K}' + i\mathbb{K}, i\alpha - i\mathbb{K} + \mathbb{K}', k') = \Omega_-^2(\alpha,k) f_{3,4}(-ix - \mathbb{K}' + i\mathbb{K}, i\alpha - i\mathbb{K} + \mathbb{K}', k')\,. \tag{B.32}$$

Using elliptic function identities one can show that

$$f_3(-ix - \mathbb{K}' + i\mathbb{K}, i\alpha - i\mathbb{K} + \mathbb{K}', k') = c(\alpha,k) f_2(x,\alpha,k), \tag{B.33}$$

with a x-independent constant

$$c(\alpha,k) = \exp\left[\frac{\pi}{4\mathbb{K}\mathbb{K}'}(\mathbb{K}-\alpha)^2 - \frac{i\pi\alpha}{2\mathbb{K}} + (\mathbb{K}' - i\mathbb{K})(-iZ(\alpha,k) + k\,\mathrm{cn}(\alpha,k))\right]. \tag{B.34}$$

The duality implies that an eigenfunction for Ω_-^2 and $0 < k^2 < 1/2$ becomes an eigenfunction for Ω_+^2 and $1/2 < k^2 < 1$. An analogous relation holds for f_1 and f_4.

B.3 Finite-gap structure: a microscopical spectral curve

To uncover the finite-gap structure of the semi-classical fluctuation spectral problem governed by the fourth order differential operator (2.20) one starts by evaluating the differential of the quasi-momentum function (4.7), entering the eigenfunctions of the LL operator (2.20), in two steps according to

$$\frac{\mathrm{d}p}{\mathrm{d}(\Omega^2)} = \frac{\mathrm{d}p}{\mathrm{d}\alpha}\frac{\mathrm{d}\alpha}{\mathrm{d}(\Omega^2)}. \tag{B.35}$$

As an example, we choose $-\infty < \Omega^2 < 0$ and plug (B.23)-(B.24) into (4.7). Choosing the two linearly independent quasi-momenta as in (B.41) we get

$$\frac{\mathrm{d}p_1}{\mathrm{d}(\Omega^2)} = i\frac{-k'^2 - \frac{i}{2}\sqrt{4k^2k'^2 - \Omega^2} + \frac{\mathbb{E}}{\mathbb{K}}}{2\sqrt{-\Omega^2}\sqrt{-1 + 2k^2 - i\sqrt{4k^2k'^2 - \Omega^2}}\sqrt{4k^2 - 4k^4 - \Omega^2}} \tag{B.36}$$

$$\frac{\mathrm{d}p_2}{\mathrm{d}(\Omega^2)} = i\frac{k'^2 - \frac{i}{2}\sqrt{4k^2k'^2 - \Omega^2} - \frac{\mathbb{E}}{\mathbb{K}}}{2\sqrt{-\Omega^2}\sqrt{-1 + 2k^2 + i\sqrt{4k^2k'^2 - \Omega^2}}\sqrt{4k^2 - 4k^4 - \Omega^2}}\,, \tag{B.37}$$

and similarly for the three branches covering the positive-frequency range (see Appendix B.1). Focussing on the first of these formulas, we introduce a new spectral parameter

$$z = \frac{1}{2}\sqrt{\Omega^2 - 4k^2(1-k^2)} \tag{B.38}$$

which results in the following differential of the quasi-momentum p_1

$$\frac{\mathrm{d}p_1}{\mathrm{d}z} = \frac{z + z_0}{\sqrt{2(z-z_1)(z-z_2)(z-z_3)}} \tag{B.39}$$

with

$$z_0 = 1 - k^2 - \frac{\mathbb{E}}{\mathbb{K}}, \qquad z_1 = -ikk', \qquad z_2 = ikk', \qquad z_3 = k^2 - \frac{1}{2}. \tag{B.40}$$

The set of points described by the elliptic curve $y^2 = (z-z_1)(z-z_2)(z-z_3)$ on the complex z plane defines what one could call a "microscopical" spectral curve for the LL string action, in the sense that it encodes the dynamics of the one-loop fluctuations above the classical, "macroscopical" spectral curve emerging in the finite-gap picture of [40]. The differential (B.39) clearly uncovers the one-gap structure of the corresponding spectral curve and justifies to consider the corresponding differential equations as fourth order analogs of the second order, finite-gap, Lamé operators of [17].

B.4 The short string expansion

In this appendix we spell out the expansion of p_1 and p_2 as series of k in the physical branch $\Omega^2 < 0$. The starting point is the expression of the quasi-momentum function (4.7), evaluated on the four functions α_i given in (B.23). As recalled in Section 3, the corresponding values of the momenta are not all linearly independent, and we choose

$$\begin{aligned} p_1 &= iZ\left(\alpha_2, k\right) + k\mathrm{cn}\left(\alpha_2, k\right) - \frac{\pi}{2\mathbb{K}}\,, \\ p_2 &= iZ\left(\alpha_3, k\right) + k\mathrm{cn}\left(\alpha_3, k\right) - \frac{\pi}{2\mathbb{K}}. \end{aligned} \tag{B.41}$$

In the following we will provide the details on the expansion of the former, since the analysis of the latter proceeds in the same fashion, modulo some minus signs.
Using the Jacobi Zeta function the addition formula for complex argument leads to

$$\begin{aligned} p_1 = & -iZ\left(u,k\right) + Z\left(v,k'\right) - k\mathrm{cn}\left(u - iv, k\right) + \frac{v\pi}{2\mathbb{K}\mathbb{K}'} - \frac{\pi}{2\mathbb{K}} \\ & -i\frac{k^2\mathrm{sn}\left(u,k\right)\mathrm{cn}\left(u,k\right)\mathrm{dn}\left(u,k\right)\mathrm{sn}^2\left(v,k'\right)}{1-\mathrm{sn}^2\left(v,k'\right)\mathrm{dn}^2\left(u,k\right)} - \frac{\mathrm{dn}^2\left(u,k\right)\mathrm{cn}\left(v,k'\right)\mathrm{sn}\left(v,k'\right)\mathrm{dn}\left(v,k'\right)}{1-\mathrm{sn}^2\left(v,k'\right)\mathrm{dn}^2\left(u,k\right)}. \end{aligned} \tag{B.42}$$

The functions $u\left(\Omega, k\right)$ and $v\left(\Omega, k\right)$, parametrizing the real and imaginary part of α_2 respectively, were presented in (B.24). The $k \sim 0$ expansion of some terms above can be treated expanding their derivatives with respect to $\sqrt{\Omega^2}$ and integrating back at the end. The derivative of the Jacobi Zeta function with argument $0 < x < \mathbb{K}$, the function x being $u\left(\Omega, k\right)$ or $v\left(\Omega, k\right)$, is conveniently expressed as

$$\frac{\partial Z\left(x,k\right)}{\partial\left(\sqrt{\Omega^2}\right)} = \frac{1}{\sqrt{1-\mathrm{sn}^2\left(x,k\right)}\sqrt{1-k^2\mathrm{sn}^2\left(x,k\right)}}\frac{\partial\mathrm{sn}\left(x,k\right)}{\partial\left(\sqrt{\Omega^2}\right)}\left[1 - k^2\mathrm{sn}^2\left(x,k\right) - \frac{\mathbb{E}}{\mathbb{K}}\right]. \tag{B.43}$$

Up to fourth order, the expansion for the two independent momenta reads

$$p_1 = -i\sqrt{-1-i\sqrt{\Omega^2}} + \frac{\left(2-i\sqrt{\Omega^2}\right)\sqrt{1+i\sqrt{\Omega^2}}}{8\Omega^2}k^4 + \mathcal{O}\left(k^6\right) \tag{B.44}$$

$$p_2 = +i\sqrt{-1+i\sqrt{\Omega^2}} + \frac{\left(2+i\sqrt{\Omega^2}\right)\sqrt{1-i\sqrt{\Omega^2}}}{8\Omega^2}k^4 + \mathcal{O}\left(k^6\right)\,, \tag{B.45}$$

to which corresponds the expansion in the regularized effective action (4.14). The latter can be written as

$$\Gamma^{(1)}_{reg} = \sum_{i=0}^{\infty}\Gamma^{(1)}_{i,reg}k^{2i}\,, \tag{B.46}$$

where the first term is vanishing by construction. The quasi-momenta in (4.14) are computed in the physical region $\Omega^2 < 0$, which corresponds to an Euclidean partition function. However, we find convenient to perform our integrals by analytically-continuing all the expressions to $\Omega^2 \to -\Omega^2$ [30]. This results in the following expressions for the first few terms

$$\Gamma^{(1)}_{0,reg} = 0\,, \tag{B.47}$$

$$\frac{\Gamma^{(1)}_{1,reg}}{\mathcal{T}} = \frac{\pi}{8}\int \frac{d\Omega}{2\pi}\left[\frac{\sqrt{-1-i\sqrt{\Omega^2}}}{\tanh\left(\pi\sqrt{-1-i\sqrt{\Omega^2}}\right)}+\frac{\sqrt{-1+i\sqrt{\Omega^2}}}{\tanh\left(\pi\sqrt{-1+i\sqrt{\Omega^2}}\right)}\right], \tag{B.48}$$

$$\begin{aligned}\frac{\Gamma^{(1)}_{2,reg}}{\mathcal{T}} = \frac{\pi}{32}\int \frac{d\Omega}{2\pi}&\left[-\pi+\frac{\pi\left(1+i\sqrt{\Omega^2}\right)}{2\tanh^2\left(\pi\sqrt{-1-i\sqrt{\Omega^2}}\right)}+\frac{\pi\left(1-i\sqrt{\Omega^2}\right)}{2\tanh^2\left(\pi\sqrt{-1+i\sqrt{\Omega^2}}\right)}\right.\\ &+\frac{\left(-16+8i\sqrt{\Omega^2}+17\Omega^2\right)\sqrt{-1-i\sqrt{\Omega^2}}}{4\Omega^2\tanh\left(\pi\sqrt{-1-i\sqrt{\Omega^2}}\right)}\\ &\left.+\frac{\left(-16-8i\sqrt{\Omega^2}+17\Omega^2\right)\sqrt{-1+i\sqrt{\Omega^2}}}{4\Omega^2\tanh\left(\pi\sqrt{-1+i\sqrt{\Omega^2}}\right)}\right],\end{aligned} \tag{B.49}$$

where we notice that there is no obstruction to go to higher terms. The integrals above are divergent, which is due to the absence in the LL action of fermionic and some bosonic modes which are crucial for UV finiteness. A first form of regularization is realized embedding our real integrals (B.48)-(B.49) in the complex plane, in order to exploit Cauchy's residue theorem. The integrands turn out to be meromorphic functions on $\mathbb{C}\setminus\{0\}$, featuring poles on the imaginary axis at

$$\Omega_n^{\pm} \equiv \pm i\left|n^2-1\right|\,, \qquad n = 2, 3, ... \tag{B.50}$$

By closing the contour of integration on the anti-clockwise upper-half (clockwise lower-half) plane, the finite part is then conventionally defined to be the ζ-regularized sum of the residues, dropping possibly divergent contributions from the arc wrapping the poles Ω_n^+ (resp. Ω_n^-). This prescription brings to the finite answers

$$\frac{\Gamma^{(1)}_{1,reg}}{\mathcal{T}} = 2\pi i\sum_{n=2}^{\infty}\frac{in^2}{8\pi} = \frac{1}{4} \tag{B.51}$$

$$\frac{\Gamma^{(1)}_{2,reg}}{\mathcal{T}} = 2\pi i\sum_{n=2}^{\infty}\frac{in^2\left(35-30n^2+11n^4\right)}{128\pi\left(n^2-1\right)^2} = \left(\frac{1}{16}-\frac{\pi^2}{48}\right), \tag{B.52}$$

which finally lead to the expected one-loop energy (4.15).

Alternatively, we can cut off the frequency domain $\epsilon < |\Omega| < L$ and safely work the real

[30] In this way, see (B.50), the poles of the integrand are on the imaginary axis. Alternatively, one could not perform the analytic continuation and treat the poles appearing on the real axis with an $i\epsilon$ prescription.

integrals out by employing the infinite sum representation for coth and $\coth^2$

$$\coth \pi x = \frac{1}{\pi x} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{x}{n^2 + x^2} \tag{B.53}$$

$$\coth^2 \pi x = 1 + \frac{1}{\pi^2 x^2} - \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{n^2 - x^2}{(n^2 + x^2)^2}. \tag{B.54}$$

The trick consists in performing the integrations first

$$\Gamma^{(1)}_{1,reg} = \lim_{L\to\infty} \left[\frac{3L}{4\pi} + \sum_{n=2}^{\infty} \left(-\frac{n^2}{4} + \frac{L}{2\pi} \right) \right] , \tag{B.55}$$

$$\begin{aligned}\Gamma^{(1)}_{2,reg} = \lim_{L\to\infty} \lim_{\epsilon\to 0} \Bigg\{ \frac{45L}{64\pi} - \frac{3}{8\pi\epsilon} + \sum_{n=2}^{\infty} \Bigg[-\frac{11n^2}{64} + \frac{15L}{32\pi} + \frac{1}{8} \\ - \frac{1}{16\,(n+1)^2} - \frac{1}{16\,(n-1)^2} + \frac{1}{2\pi\,(n^2-1)\,\epsilon} \Bigg] \Bigg\} ,\end{aligned} \tag{B.56}$$

followed by the ζ-regularized sums. Upon ζ-regularization, the ϵ- and 1/L-coefficients are finite and drop out in the limit. On the other hand, UV- and IR-divergencies happen to cancel out and the cut-off regularization reproduces the same one-loop energy contribution (4.15) of the residue prescription.

C Folded string in full sigma-model: details

C.1 Fluctuation Lagrangian in static gauge

A fluctuation Lagrangian derived from Pohlmeyer reduction that agrees with the Nambu action in static gauge (in which fluctuations of t and ρ are set to zero) result is [31]

$$L = \partial_a x \partial^a x - \mu_x^2 x^2 + \partial_a y \partial^a y - \mu_y^2 y^2 + 2q\ y \partial_0 x\ , \tag{C.1}$$

$$q = \frac{2\nu \bar{w} \kappa}{\rho'^2 + \nu^2} \tag{C.2}$$

$$\mu_x^2 = 2\rho'^2 + \nu^2 + \frac{\kappa^2 \bar{w}^2 (2\rho'^2 - \nu^2)}{(\rho'^2 + \nu^2)^2}\ , \qquad \mu_y^2 = \nu^2 \Big[-1 + \frac{2(\bar{w}^2 + \kappa^2)}{\rho'^2 + \nu^2} - \frac{3\bar{w}^2 \kappa^2}{(\rho'^2 + \nu^2)^2} \Big] \tag{C.3}$$

Here x and y are two physical fluctuations in AdS_3 sector. When $\nu \to 0$ we get one massless mode and a mode with $\mu^2 = 2\rho'^2 + \frac{2\kappa^2 \bar{w}^2}{\rho'^2}$ as expected.

All other fluctuations have same mass as in the conformal gauge discussed in [16], e.g., β_u $(u = 1, 2)$ – the fluctuations in AdS_5 that are transverse to AdS_3 – have mass $\mu_\beta^2 = 2\rho'^2 + \nu^2$.

The corresponding equations of motion are $(x \to e^{i\omega\tau} \tilde{x}(\sigma),\quad y \to e^{i\omega\tau} \tilde{y}(\sigma))$ [32]

$$(\partial_\sigma^2 + \omega^2 - \mu_x^2)\tilde{x} - iq\,\omega\,\tilde{y} = 0\ , \qquad (\partial_\sigma^2 + \omega^2 - \mu_y^2)\tilde{y} + iq\,\omega\,\tilde{x} = 0\ . \tag{C.4}$$

[31] We thank I. Iwashita, R. Roiban and A. A. Tseytlin for this information. There are other more complicated forms of the fluctuation action that should be related by field redefinitions.

[32] Here we did not yet switch to Euclidean time $\tau \to i\tau$, i.e. $\omega \to i\omega$.

Here q, μ_x, μ_y depend on σ. Solving the first equation for $\tilde{y}$ and substituting into the second equation, we get fourth order equation

$$\Big[(\partial_\sigma^2 + \omega^2 - \mu_x^2)[\frac{1}{q}(\partial_\sigma^2 + \omega^2 - \mu_y^2)] - q\omega^2\Big]\tilde{y} = 0 \tag{C.5}$$

or explicitly

$$\Big[(\partial_\sigma^2 + \omega^2 - \mu_x^2)[(\rho'^2 + \nu^2)(\partial_\sigma^2 + \omega^2 - \mu_y^2)] - \frac{4\nu^2\kappa^2\bar{w}^2}{\rho'^2 + \nu^2}\omega^2\Big]\tilde{y} = 0\,. \tag{C.6}$$

Redefining

$$\tilde{y} = \frac{1}{\sqrt{\rho'^2 + \nu^2}}\,\tilde{\rho} \tag{C.7}$$

one recovers the fourth-order differential equation

$$\Big[\partial_\sigma^4 + 2(\bar{w}^2 + \kappa^2 - 2\nu^2 + \omega^2 - 4\rho'^2)\;\partial_\sigma^2 - 8\rho'\rho''\;\partial_\sigma + \kappa^4 + (\bar{w}^2 - \omega^2)^2 - 2\kappa^2(\omega^2 + \bar{w}^2)\Big]\tilde{\rho} = 0\,, \tag{C.8}$$

which coincides with the one (2.39) obtained in conformal gauge, once one uses the classical equation of motion for ρ (2.26), and performs a Wick rotation $\Omega^2 \to -\Omega^2$.

In the long string limit the operator (C.5) agrees with the results first found in [5]. This limit (also known as large spin regime) corresponds to

$$\rho = \mu\sigma\ , \qquad \bar{w} = \kappa = \sqrt{\mu^2 + \nu^2}\ , \qquad \mu = \frac{1}{\pi}\log S \gg 1\ , \tag{C.9}$$

and the mass operators (C.2)-(C.3) become

$$\mu_x^2 \to 4\mu^2\ , \qquad \mu_y^2 \to 0\ , \qquad q \to 2\nu\ , \tag{C.10}$$

and thus (C.5) (with $\partial_\sigma \to in$) gives the following characteristic polynomial

$$(\omega^2 - n^2 - 4\mu^2)(\omega^2 - n^2) - 4\nu^2\omega^2 = 0\ , \tag{C.11}$$

with

$$\omega^2 = n^2 + 2(\mu^2 + \nu^2) \pm 2\sqrt{n^2\nu^2 + (\mu^2 + \nu^2)^2}\ , \tag{C.12}$$

which agrees with the expression in [5] where ω and n are rescaled by $\kappa = \sqrt{\mu^2 + \nu^2}$, i.e. ($p = \frac{n}{\kappa}$)

$$\bar{\Omega}^2 = p^2 + 2 \pm 2\sqrt{p^2u^2 + 1}\ , \qquad u \equiv \frac{\nu}{\kappa}\ . \tag{C.13}$$

Taking into account the other masses for the remaining fluctuations in this long string limit, that is

- two transverse fluctuations in AdS_5: $\mu_\beta^2 = 2\rho'^2 + \nu^2 \to 2\mu^2 + \nu^2 = \kappa^2(2 - u^2)$;
- four fluctuations in S^5: $\mu_{sph}^2 = \nu^2 = \kappa^2 u^2$;
- eight fermionic fluctuations $\mu_\psi^2 = \rho'^2 + \nu^2 = \kappa^2$,

the corresponding characteristic frequencies (C.12) produce the well-known one-loop expression for the energy in this scaling limit [5].

C.2 Spectral domain and four linear independent solutions

We can repeat the analysis of the previous section B.1 for the folded string. Here calculations are carried out in Minkowski signature. Again, from the consistency equations we obtain the relations

$$
\begin{aligned}
&\lambda = \pm\sqrt{k^2\mathrm{sn}^2(\alpha) - \bar{\Omega}^2}\,, \\
&\bar{\Omega}^2_{\pm}(\alpha) = -(\kappa^2-\nu^2)\mathrm{sn}^2(\alpha)\left(\frac{\bar{w}k\,\mathrm{cn}(\alpha)\pm\kappa\,\mathrm{dn}(\alpha)}{(\kappa^2-\nu^2)\mathrm{sn}^2(\alpha)+\nu^2}\right)^2 .
\end{aligned}
\tag{C.14}
$$

As before it is useful to work out duality relations for the eigenvalues, that is

$$
\bar{\Omega}^2_{\pm}(\alpha+2\mathbb{K}) = \bar{\Omega}^2_{\mp}(\alpha)\,, \qquad \bar{\Omega}^2_{\pm}(\alpha+2i\mathbb{K}') = \bar{\Omega}^2_{\pm}(\alpha)\,,
\tag{C.15}
$$

which allows us to work only with one kind of rescaled frequency, that is $\bar{\Omega}(\alpha)\equiv\bar{\Omega}_+(\alpha)$. In order to further manipulate the expression for Ω and find the corresponding "physical" spectral curve, it is advantageous to introduce the function

$$
a^2(\alpha) = (\kappa^2-\nu^2)\mathrm{sn}^2(\alpha)+\nu^2\,,
\tag{C.16}
$$

with the property

$$
\frac{\mathrm{d}}{\mathrm{d}\alpha}a^2(\alpha) = \frac{2}{\sqrt{\bar{w}^2-\nu^2}}\sqrt{a^2(\alpha)-\nu^2}\sqrt{\kappa^2-a^2(\alpha)}\sqrt{\bar{w}^2-a^2(\alpha)}\,.
\tag{C.17}
$$

Thus, one can rewrite (C.14) as follows

$$
\bar{\Omega}^2_{\pm}(\alpha) = -\frac{1}{\bar{w}^2-\nu^2}\frac{a^2(\alpha)-\nu^2}{a^4(\alpha)}\left(\bar{w}\sqrt{\kappa^2-a^2(\alpha)}\pm\kappa\sqrt{\bar{w}^2-a^2(\alpha)}\right)^2 ,
\tag{C.18}
$$

and it easily follows now

$$
\frac{\partial\bar{\Omega}^2}{\partial\alpha} = 2\bar{\Omega}^2(\alpha)\frac{1}{(\kappa^2-\nu^2)\mathrm{sn}^2(\alpha)+\nu^2}\left(\nu^2\frac{\mathrm{cn}(\alpha)\mathrm{dn}(\alpha)}{\mathrm{sn}(\alpha)}+k\,\kappa\,\bar{w}\,\mathrm{sn}(\alpha)\right) ,
\tag{C.19}
$$

$$
\lambda(\alpha) = \frac{k\,\mathrm{sn}(\alpha)}{(\kappa^2-\nu^2)\mathrm{sn}^2(\alpha)+\nu^2}\left(\kappa\bar{w}\pm\sqrt{\kappa^2-\nu^2}\sqrt{\bar{w}^2-\nu^2}\,\mathrm{cn}(\alpha)\mathrm{dn}(\alpha)\right) .
\tag{C.20}
$$

We are interested in all values of the complex parameter $\alpha = u+iv$ that correspond to real values of $\bar{\Omega}^2$. The condition $\mathrm{Im}(\bar{\Omega}^2)(u,v)=0$ results in the cases $u=0,\ i\mathbb{K}',\ v=0,\ 2\mathbb{K}$ (modulo periodicity of $\bar{\Omega}^2(\alpha)$), which gives the straight lines in Fig 4. However, this does not exhaust all the possibilities: There are still orbits in the α complex plane where $\bar{\Omega}^2$ is real, which correspond to the ellipse-like lines in Fig. 4. In order to find a parametrization $v(u)$ of such curves, one has to find the real root of the cubic polynomial (setting for short $x\equiv\mathrm{dn}(v,k')$)

$$
\begin{aligned}
P_3(x;u) = {}&\kappa\,\mathrm{cd}(u)\left[k^2\bar{w}^2-\kappa^2\mathrm{ns}^2(u)\right]x^3-\bar{w}\,k\left[\kappa^2\mathrm{cs}^2(u)-\nu^2k'^2\right]x^2+\\
&+\kappa k^2\mathrm{cd}(u)\left[\bar{w}^2\mathrm{ds}^2(u)+\nu^2k'^2\right]x+\bar{w}\,k^3\left[\bar{w}^2\mathrm{ns}^2(u)-\kappa^2\right].
\end{aligned}
\tag{C.21}
$$

Since all the coefficients are elliptic functions with periods $4\mathbb{K}$ and $2i\mathbb{K}$, the roots $x_i(u)$, $i = 1, 2, 3$ will also be elliptic functions of u, with the same periods, such that the polynomial itself $P_3(x(u); u)$ will be an elliptic function. Imposing $P_3(x; u) = 0$ for any value of u, implies that the poles of $x(u)$ have to be canceled by the zeros of the coefficient functions of $P_3(x; u)$. By studying the locus of points where the coefficient functions of $P_3(x(u), u)$ vanish, it allows us to compute

$$x(u) = \frac{k\,\bar{w}}{\kappa}\mathrm{dc}(u, k)\,, \qquad v(u) = \mathrm{dn}^{-1}\left[\frac{k\,\bar{w}}{\kappa}\mathrm{dc}(u, k), k'\right]. \tag{C.22}$$

For completeness, we report other special values of $\bar{\Omega}^2$ which appear in Fig. 4,

$$\bar{\Omega}^2\left(\mathbb{K} - \mathrm{sn}^{-1}\left(\frac{\bar{w}}{\kappa}\sqrt{\frac{\kappa^2 - \nu^2}{\bar{w}^2 - \nu^2}}, k\right)\right) = -\left(\frac{\kappa^2}{\nu^2} - 1\right), \qquad \bar{\Omega}^2(\mathbb{K}) = -\frac{(\kappa^2 - \nu^2)(\bar{w}^2 - \kappa^2)}{\kappa^2(\bar{w}^2 - \nu^2)},$$
$$\bar{\Omega}^2(2\mathbb{K} + i\mathbb{K}') = \frac{(\bar{w} - \kappa)^2}{\bar{w}^2 - \nu^2}, \qquad \bar{\Omega}^2(i\mathbb{K}') = \frac{(\bar{w} + \kappa)^2}{\bar{w}^2 - \nu^2}, \qquad \bar{\Omega}^2(0) = 0, \qquad \bar{\Omega}^2(\mathbb{K} + i\mathbb{K}') = 1 - \frac{\kappa^2}{\bar{w}^2}.$$

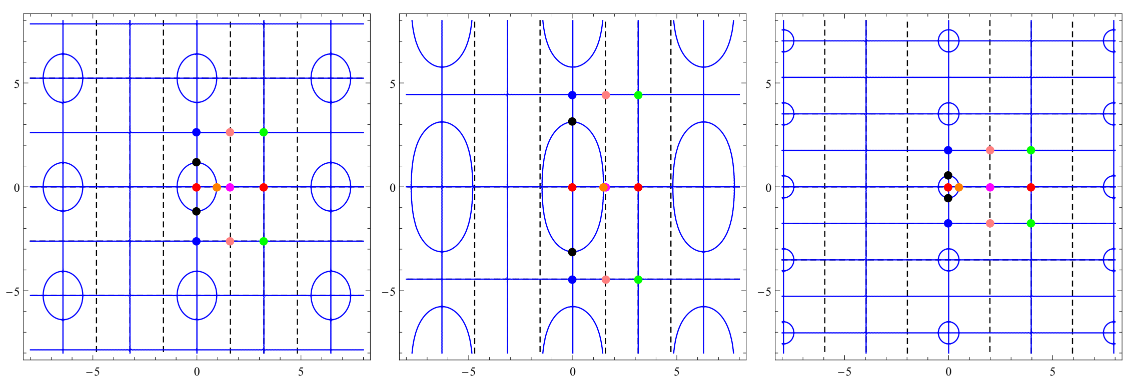

Figure 4: In the complex α plane one can plot the lines where $\bar{\Omega}^2(\alpha)$ is real. The parameters are chosen on the left as $\kappa = 3$, $\omega = 6$, $\nu = 2.5$ ($k \sim 0.1$), in the middle as $\kappa = 3$, $\omega = 6$, $\nu = 2.99$ ($k \sim 0.002$) and on the right as $\kappa = 5$, $\omega = 6$, $\nu = 2.5$ ($k \sim 0.63$). The special points are marked with colors as follows $\bar{\Omega}^2 = -(\frac{\kappa^2}{\nu^2} - 1)$ (orange), $\bar{\Omega}^2 = -\frac{(\kappa^2-\nu^2)(\omega^2-\kappa^2)}{\kappa^2(\omega^2-\nu^2)}$ (magenta), $\bar{\Omega}^2 = 0$ (red), $\bar{\Omega}^2 = \frac{(\omega-\kappa)^2}{\omega^2-\nu^2}$ (green), $\bar{\Omega}^2 = 1 - \frac{\kappa^2}{\omega^2}$ (pink), $\bar{\Omega}^2 = \frac{(\omega+\kappa)^2}{\omega^2-\nu^2}$ (blue).

We are now ready to illustrate the various branches for $\bar{\Omega}^2$. For convenience, we define

$$\chi_\pm(\bar{\Omega}) = \left(\kappa\,\bar{w} \pm \sqrt{(\bar{w}^2 - \nu^2)(\kappa^2 - \nu^2 + \nu^2\bar{\Omega}^2)}\right)^2 - \nu^4. \tag{C.23}$$

For a given real value of $\bar{\Omega}^2$ the linear independent solutions of the fourth order differential equation (2.39) are

$$f_{1,2}(x, \bar{\Omega}) = \frac{H(x \pm \alpha_1)}{\Theta(x)}e^{\mp x[Z(\alpha_1)+\lambda(\alpha_1)]}, \tag{C.24}$$
$$f_{3,4}(x, \bar{\Omega}) = \frac{H(x \pm \alpha_2)}{\Theta(x)}e^{\mp x[Z(\alpha_2)+\lambda(\alpha_2)]}, \tag{C.25}$$

where the α_i's as functions of $\bar{\Omega}$ have to be chosen (see Fig. 4)

- for $-\infty < \bar\Omega^2 < -(\frac{\kappa^2}{\nu^2}-1)$ as

$$\alpha_{1,2}(\Omega) = \mathrm{sn}^{-1}\left[\sqrt{\frac{(\kappa^2-\nu^2)+(\bar w^2-\nu^2)(1-\bar\Omega^2)-\sqrt{((\kappa^2+\bar w^2)-(\bar w^2-\nu^2)\bar\Omega^2)^2-4\kappa^2\bar w^2}}{2(\kappa^2-\nu^2+\nu^2\bar\Omega^2)}}, k^2\right] \pm \tag{C.26}$$
$$\pm i\,\mathrm{dn}^{-1}\left[\frac{k}{\kappa}\sqrt{\frac{(\bar w^2-\kappa^2)^2-(\bar w^2-\nu^2)(\kappa^2+\bar w^2)\bar\Omega^2+(\bar w^2-\kappa^2)\sqrt{((\kappa^2+\bar w^2)-(\bar w^2-\nu^2)\bar\Omega^2)^2-4\kappa^2\bar w^2}}{-2(\bar w^2-\nu^2)\bar\Omega^2}}, k'\right],$$

- for $-(\frac{\kappa^2}{\nu^2}-1) < \bar\Omega^2 < -\frac{(\kappa^2-\nu^2)(\bar w^2-\kappa^2)}{(\bar w^2-\nu^2)\kappa^2}$ as

$$\alpha_1(\bar\Omega) = \mathbb{K} - \mathrm{sn}^{-1}\left[\sqrt{\frac{\chi_-(\bar\Omega)+\nu^4\bar\Omega^2/k^2}{\chi_-(\bar\Omega)+\nu^4\bar\Omega^2}}, k\right], \tag{C.27}$$

- for $-\frac{(\kappa^2-\nu^2)(\bar w^2-\kappa^2)}{(\bar w^2-\nu^2)\kappa^2} < \bar\Omega^2 < 0$ as

$$\alpha_1(\bar\Omega) = 2\mathbb{K} - \mathrm{sn}^{-1}\left[\sqrt{\frac{-\nu^4\bar\Omega^2}{k^2\chi_-(\bar\Omega)}}, k\right], \tag{C.28}$$

- for $0 < \bar\Omega^2 < \frac{(\bar w-\kappa)^2}{\bar w^2-\nu^2}$ as

$$\alpha_1(\bar\Omega) = 2\mathbb{K} + i\,\mathrm{sn}^{-1}\left[\sqrt{\frac{\nu^4\bar\Omega^2}{k^2\chi_-(\bar\Omega)+\nu^4\bar\Omega^2}}, k'\right], \tag{C.29}$$

- for $\frac{(\bar w-\kappa)^2}{\bar w^2-\nu^2} < \bar\Omega^2 < 1-\frac{\kappa^2}{\bar w^2}$ as

$$\alpha_1(\bar\Omega) = 2\mathbb{K} + i\mathbb{K}' - \mathrm{sn}^{-1}\left[\sqrt{\frac{\chi_-(\bar\Omega)}{-\nu^4\bar\Omega^2}}, k\right], \tag{C.30}$$

- for $1-\frac{\kappa^2}{\bar w^2} < \bar\Omega^2 < \frac{(\bar w+\kappa)^2}{\bar w^2-\nu^2}$ as

$$\alpha_1(\bar\Omega) = \mathbb{K} + i\mathbb{K}' - \mathrm{sn}^{-1}\left[\sqrt{\frac{\chi_-(\bar\Omega)+\nu^4\bar\Omega^2}{k^2\chi_-(\bar\Omega)+\nu^4\bar\Omega^2}}, k\right], \tag{C.31}$$

- for $\frac{(\bar w+\kappa)^2}{\bar w^2-\nu^2} < \bar\Omega^2 < \infty$ as

$$\alpha_1(\bar\Omega) = i\,\mathrm{sn}^{-1}\left[\sqrt{\frac{\nu^4\bar\Omega^2}{k^2\chi_-(\bar\Omega)+\nu^4\bar\Omega^2}}, k'\right], \tag{C.32}$$

- for $-(\frac{\kappa^2}{\nu^2}-1) < \bar\Omega^2 < 0$ as

$$\alpha_2(\bar\Omega) = \mathrm{sn}^{-1}\left[\sqrt{-\frac{\nu^4\bar\Omega^2}{k^2\chi_+(\bar\Omega)}}, k\right], \tag{C.33}$$

- for $0 < \bar\Omega^2 < \infty$ one has

$$\alpha_2(\bar\Omega) = i\,\mathrm{sn}^{-1}\left[\sqrt{\frac{\nu^4\bar\Omega^2}{k^2\chi_+(\bar\Omega)+\nu^4\bar\Omega^2}}, k'\right]. \tag{C.34}$$

In the main body we have used the following identity, which does not seem to be tabulated but can be easily checked to be true

$$Z(\alpha,k) = \frac{2}{1+\tilde k'}Z\left(\frac{\alpha}{1+\tilde k'}+\frac{i\mathbb{K}'}{1+\tilde k'}, \tilde k\right) - \frac{1+\mathrm{cn}(\alpha,k)\mathrm{dn}(\alpha,k)}{\mathrm{sn}(\alpha,k)} + \frac{i\pi}{2\mathbb{K}}, \tag{C.35}$$

where $\tilde k$ is the Landen transformed modulus, *i.e.* $\tilde k^2 = 4k/(1+k)^2$.

C.3 The $\nu = 0$ limit

Here we consider (C.26)-(C.34) and outline explicitly the $\nu \to 0$ limit of those α's in the negative-frequency range useful to obtain the determinant (4.31) as a $\nu \to 0$ limit of (4.22)-(4.23). We keep in mind that $k^2 = (\kappa^2 - \nu^2)/(\omega^2 - \nu^2)$. For the first quasi-momentum, we notice that the ellipse segments parameterized by (C.26) shrink for $\nu \to 0$ to a point and (C.26) becomes irrelevant. The interval corresponding to (C.27) extends to $-\infty < \bar{\Omega}^2 < -k'^2$ and (C.27) becomes

$$\alpha_1(\bar{\Omega}) \to \mathbb{K} - \mathrm{sn}^{-1}\left[\sqrt{\frac{(k'^2 + \bar{\Omega}^2)^2}{(k'^2 - \bar{\Omega}^2)^2}}, k\right]. \tag{C.36}$$

The interval corresponding to (C.28) extends to $-k'^2 < \bar{\Omega}^2 < 0$ and (C.28) becomes

$$\alpha_1(\bar{\Omega}) \to 2\mathbb{K} - \mathrm{sn}^{-1}\left[\sqrt{-\frac{4\bar{\Omega}^2}{(1 + k^2 - \bar{\Omega}^2)^2 - 4k^2}}, k\right]. \tag{C.37}$$

Considering the second quasi-momentum, the interval corresponding to (C.33) extends for $\nu \to 0$ to $-\infty < \bar{\Omega}^2 < 0$ and one gets

$$\mathrm{sn}^2(\alpha_2(\bar{\Omega})) \sim \frac{\bar{\Omega}^2}{4\kappa^4}\,\nu^4\,, \quad \mathrm{cn}^2(\alpha_2(\bar{\Omega})) \sim 1 - \frac{\bar{\Omega}^2}{4\kappa^4}\,\nu^4\,, \quad \mathrm{dn}^2(\alpha_2(\bar{\Omega})) \sim 1 - \frac{\bar{\Omega}^2}{4\kappa^2\omega^2}\,\nu^4\,, \tag{C.38}$$

so that

$$p_2 \to i\,\bar{\Omega}\,. \tag{C.39}$$

References

[1] A. A. Tseytlin, *"Review of AdS/CFT Integrability, Chapter II.1: Classical* $\mathrm{AdS}_5 \times \mathrm{S}^5$ *string solutions"*, arXiv:1012.3986. • T. McLoughlin, *"Review of AdS/CFT Integrability, Chapter II.2: Quantum Strings in* $\mathrm{AdS}_5 \times \mathrm{S}^5$*"*, arXiv:1012.3987.

[2] S. Frolov and A. A. Tseytlin, *"Multispin string solutions in AdS(5) x S**5"*, Nucl.Phys. B668, 77 (2003), hep-th/0304255.

[3] S. Frolov and A. A. Tseytlin, *"Quantizing three spin string solution in AdS(5) x S**5"*, JHEP 0307, 016 (2003), hep-th/0306130. • S. Frolov, I. Park and A. A. Tseytlin, *"On one-loop correction to energy of spinning strings in S**5"*, Phys.Rev. D71, 026006 (2005), hep-th/0408187. • I. Park, A. Tirziu and A. A. Tseytlin, *"Spinning strings in AdS(5) x S**5: One-loop correction to energy in SL(2) sector"*, JHEP 0503, 013 (2005), hep-th/0501203.

[4] N. Beisert, A. A. Tseytlin and K. Zarembo, *"Matching quantum strings to quantum spins: One-loop versus finite-size corrections"*, Nucl.Phys. B715, 190 (2005), hep-th/0502173.

[5] S. Frolov, A. Tirziu and A. A. Tseytlin, *"Logarithmic corrections to higher twist scaling at strong coupling from AdS/CFT"*, Nucl. Phys. B766, 232 (2007), hep-th/0611269.

[6] R. Roiban, A. Tirziu and A. A. Tseytlin, *"Two-loop world-sheet corrections in AdS(5) x S**5 superstring"*, JHEP 0707, 056 (2007), arXiv:0704.3638.

[7] B. Hoare, Y. Iwashita and A. A. Tseytlin, *"Pohlmeyer-reduced form of string theory in AdS(5) x S(5): Semiclassical expansion"*, J.Phys. A42, 375204 (2009), arXiv:0906.3800.

[8] S. Giombi, R. Ricci, R. Roiban, A. A. Tseytlin and C. Vergu, *"Generalized scaling function from light-cone gauge* $\mathrm{AdS}_5 \times \mathrm{S}^5$ *superstring"*, JHEP 1006, 060 (2010), arXiv:1002.0018.

[9] L. Bianchi, M. S. Bianchi, A. Bres, V. Forini and E. Vescovi, *"Two-loop cusp anomaly in ABJM at strong coupling"*, arXiv:1407.4788.

[10] T. Klose, T. McLoughlin, R. Roiban and K. Zarembo, *"Worldsheet scattering in* $AdS_5 \times S^5$*"*, JHEP 0703, 094 (2007), hep-th/0611169.

[11] L. Bianchi, V. Forini and B. Hoare, *"Two-dimensional S-matrices from unitarity cuts"*, JHEP 1307, 088 (2013), arXiv:1304.1798.

[12] O. T. Engelund, R. W. McKeown and R. Roiban, *"Generalized unitarity and the worldsheet S matrix in* $AdS_n \times S^n \times M^{10-2n}$*"*, JHEP 1308, 023 (2013), arXiv:1304.4281.

[13] L. Bianchi and B. Hoare, *"*$AdS_3 \times S^3 \times M^4$ *string S-matrices from unitarity cuts"*, arXiv:1405.7947. • R. Roiban, P. Sundin, A. Tseytlin and L. Wulff, *"The one-loop worldsheet S-matrix for the* AdS_n *x* S^n *x* T^{10-2n} *superstring"*, JHEP 1408, 160 (2014), arXiv:1407.7883.

[14] N. Beisert et al., *"Review of AdS/CFT Integrability: An Overview"*, arXiv:1012.3982.

[15] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, *"A semi-classical limit of the gauge/string correspondence"*, Nucl. Phys. B636, 99 (2002), hep-th/0204051.

[16] S. Frolov and A. A. Tseytlin, *"Semiclassical quantization of rotating superstring in* $\mathrm{AdS}_5 \times \mathrm{S}^5$*"*, JHEP 0206, 007 (2002), hep-th/0204226.

[17] M. Beccaria, G. V. Dunne, V. Forini, M. Pawellek and A. A. Tseytlin, *"Exact computation of one-loop correction to energy of spinning folded string in* $\mathrm{AdS}_5 \times \mathrm{S}^5$*"*, J.Phys. A43, 165402 (2010), arXiv:1001.4018.

[18] M. Beccaria, V. Forini, A. Tirziu and A. A. Tseytlin, *"Structure of large spin expansion of anomalous dimensions at strong coupling"*, Nucl.Phys. B812, 144 (2009), arXiv:0809.5234. • M. Beccaria, V. Forini and G. Macorini, *"Generalized Gribov-Lipatov Reciprocity and AdS/CFT"*, Adv.High Energy Phys. 2010, 753248 (2010), arXiv:1002.2363. • V. Forini and M. Beccaria, *"QCD-like properties for anomalous dimensions in N=4 SYM"*, Theor.Math.Phys. 159, 712 (2009), arXiv:0810.0101.

[19] M. Beccaria, G. Dunne, G. Macorini, A. Tirziu and A. Tseytlin, *"Exact computation of one-loop correction to energy of pulsating strings in* $\mathrm{AdS}_5 \times \mathrm{S}^5$*"*, J.Phys.A A44, 015404 (2011), arXiv:1009.2318.

[20] V. Forini, *"Quark-antiquark potential in AdS at one loop"*, JHEP 1011, 079 (2010), arXiv:1009.3939.

[21] D. Correa, J. Henn, J. Maldacena and A. Sever, *"An exact formula for the radiation of a moving quark in* $\mathcal{N} = 4$ *super Yang Mills"*, JHEP 1206, 048 (2012), arXiv:1202.4455.

[22] N. Drukker and V. Forini, *"Generalized quark-antiquark potential at weak and strong coupling"*, JHEP 1106, 131 (2011), arXiv:1105.5144.

[23] V. Forini, V. G. M. Puletti and O. Ohlsson Sax, *"The generalized cusp in* AdS_4 *x* CP^3 *and more one-loop results from semiclassical strings"*, J.Phys. A46, 115402 (2013), arXiv:1204.3302.

[24] J. Minahan, A. Tirziu and A. A. Tseytlin, *"1/J corrections to semiclassical AdS/CFT states from quantum Landau-Lifshitz model"*, Nucl.Phys. B735, 127 (2006), hep-th/0509071.

[25] R. Roiban and A. Tseytlin, *"Semiclassical string computation of strong-coupling corrections to dimensions of operators in Konishi multiplet"*, Nucl.Phys. B848, 251 (2011), arXiv:1102.1209. •

N. Gromov, D. Serban, I. Shenderovich and D. Volin, *"Quantum folded string and integrability: From finite size effects to Konishi dimension"*, JHEP 1108, 046 (2011), arXiv:1102.1040.

[26] D. Fioravanti and M. Rossi, *"TBA-like equations and Casimir effect in (non-)perturbative AdS/CFT"*, JHEP 1212, 013 (2012), arXiv:1112.5668.

[27] A. Lewkowycz and J. Maldacena, *"Exact results for the entanglement entropy and the energy radiated by a quark"*, JHEP 1405, 025 (2014), arXiv:1312.5682. • M. S. Bianchi, L. Griguolo, M. Leoni, S. Penati and D. Seminara, *"BPS Wilson loops and Bremsstrahlung function in ABJ(M): a two loop analysis"*, JHEP 1406, 123 (2014), arXiv:1402.4128.

[28] M. Kruczenski, *"Spin chains and string theory"*, Phys.Rev.Lett. 93, 161602 (2004), hep-th/0311203.

[29] M. Kruczenski, A. Ryzhov and A. A. Tseytlin, *"Large spin limit of AdS(5) x S**5 string theory and low-energy expansion of ferromagnetic spin chains"*, Nucl.Phys. B692, 3 (2004), hep-th/0403120. • J. Stefanski, B. and A. A. Tseytlin, *"Large spin limits of AdS/CFT and generalized Landau-Lifshitz equations"*, JHEP 0405, 042 (2004), hep-th/0404133.

[30] M. G. Mittag-Leffler, *"Ueber die Integration der Hermiteschen Differentialgleichungen der dritten und vierten Ordnung, bei denen die Unendlichkeitsstellen der Integrale von der ersten Ordnung sind"*, Annali di Matematica 11, 65 (1882).

[31] M. Kruczenski and A. A. Tseytlin, *"Semiclassical relativistic strings in S**5 and long coherent operators in N=4 SYM theory"*, JHEP 0409, 038 (2004), hep-th/0406189.

[32] N. Beisert, S. Frolov, M. Staudacher and A. A. Tseytlin, *"Precision spectroscopy of AdS / CFT"*, JHEP 0310, 037 (2003), hep-th/0308117.

[33] D. Fursev and D. Vassilevich, *"Operators, Geometry and Quanta: Methods of Spectral Geometry in Quantum Field Theory"*, Springer Verlag (2011).

[34] W. Magnus and S. Winkler, *"Hill's Equation"*, Dover Phoenix Editions (1966), New York.

[35] G. N. Watson and E. T. Whittaker, *"A Course of Modern Analysis"*, Cambridge University Press (1902).

[36] E. L. Ince, *"Ordinary differential equations"*, Dover (1956).

[37] M. Blau, M. O'Loughlin, G. Papadopoulos and A. A. Tseytlin, *"Solvable models of strings in homogeneous plane wave backgrounds"*, Nucl.Phys. B673, 57 (2003), hep-th/0304198.

[38] N. Drukker, D. J. Gross and A. A. Tseytlin, *"Green-Schwarz string in* $\mathrm{AdS}_5 \times \mathrm{S}^5$*: Semiclassical partition function"*, JHEP 0004, 021 (2000), hep-th/0001204.

[39] N. Gromov, V. Kazakov, S. Leurent and D. Volin, *"Quantum Spectral Curve for Planar* $\mathcal{N} =$ *Super-Yang-Mills Theory"*, Phys.Rev.Lett. 112, 011602 (2014), arXiv:1305.1939. • A. Cavagli, D. Fioravanti, N. Gromov and R. Tateo, *"The Quantum Spectral Curve of the ABJM theory"*, Phys.Rev.Lett. 113, 021601 (2014), arXiv:1403.1859.

[40] V. Kazakov, A. Marshakov, J. Minahan and K. Zarembo, *"Classical/quantum integrability in AdS/CFT"*, JHEP 0405, 024 (2004), hep-th/0402207.