The dominant dimension of cohomological Mackey functors

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Abstract

We show that a separable equivalence between symmetric algebras preserves the dominant dimensions of certain endomorphism algebras of modules. We apply this to show that the dominant dimension of the category coMack(B) of cohomological Mackey functors of a p-block B of a finite group with a nontrivial defect group is 2.

1 Introduction

Let k be a field. Following Tachikawa [11], the *dominant dimension of a finite-dimensional k-algebra A*, which we will denote by $\mathrm{ddim}(A)$, is the largest nonnegative integer d such that there exists an injective resolution

$$0 \longrightarrow A \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \cdots$$

of A as a right A-module with the property that I^n is projective for $0 \leq n \leq d-1$, provided there is such an integer. If for any injective resolution I of A the term I^0 is not projective, then $\mathrm{ddim}(A) = 0$, and if there exists an injective resolution I of A such that I^n is projective for all $n \geq 0$, then we adopt the convention $\mathrm{ddim}(A) = \infty$. By a result of Müller [9, Theorem 4] the dominant dimension is equal to the obvious left module analogue. In order to calculate $\mathrm{ddim}(A)$ it suffices to consider a minimal injective resolution of A as a right A-module. A finite-dimensional k-algebra A is called *symmetric* if A is isomorphic to its k-dual A^* as an A-A-bimodule. Given two symmetric k-algebras A and B, we say that an A-B-bimodule M *induces a separable equivalence between A and B* if M is finitely generated projective as a left A-module and as a right B-module such that A is isomorphic to a direct summand of $M \otimes_B M^*$ as an A-A-bimodule and B is isomorphic to a direct summand of $M^* \otimes_A M$ as a B-B-bimodule. In that case we say that A and B are *separably equivalent.* This term has been coined by Kadison in [3]; we follow the usage as in [5].

Theorem 1.1. *Let A, B be symmetric k-algebras. Let M be an A-B-bimodule inducing a separable equivalence between A and B. Let U be a finitely generated A-module such that A is isomorphic to a direct summand of U, and let V a finitely generated B-module such that B is isomorphic to a direct summand of V. Suppose that $M^* \otimes_A U \in \mathrm{add}(V)$ and that $M \otimes_B V \in \mathrm{add}(U)$. Then the dominant dimensions of $\mathrm{End}_A(U)$ and of $\mathrm{End}_B(V)$ are equal.*

This will be proved in the next section. If k has prime characteristic p and if A is a source algebra of a block B of a finite group algebra kG with a defect group P, then, by the source algebra version [6, Theorem 1.1] of a result of Yoshida in [13], the category coMack(B) of cohomological Mackey functors of G associated with B is equivalent to the right module category of the endomorphism algebra $E =$

$\mathrm{End}_A(\oplus_{Q\leq P}\, A\otimes_{kQ} k)$. The *dominant dimension of* coMack(B) is defined as the dominant dimension of E.

Theorem 1.2. *Suppose that k has prime characteristic p. Let G be a finite group and B a block of kG with a nontrivial defect group P. The dominant dimension of* coMack(B) *is equal to* 2.

We will present a proof of Theorem 1.2 as an application of Theorem 1.1 (in conjunction with some results from [6], [7], [9]). It is possible to prove Theorem 1.2 more directly; see Remark 3.1. As pointed out by the referee, Theorem 1.2 can also be deduced from results due to Bouc, Stancu and Webb in [2] and standard properties of cohomological Mackey functors from Thévenaz and Webb [12]; see Remark 3.2. Further material on (cohomological) Mackey functors can be found in [10].

2 Proof of Theorem 1.1

Let A be a finite-dimensional k-algebra, and let U, V be finitely generated A-modules. We use without further reference the following standard facts (see e. g. [1, II.2]). If V belongs to add(U), then $\mathrm{Hom}_A(U,V)$ is a projective right $\mathrm{End}_A(U)$-module, and any finitely generated projective right $\mathrm{End}_A(U)$-module is of this form, up to isomorphism. Given two idempotents i, j in A, every homomorphism of right A-modules $iA\to jA$ is induced by left multiplication with an element in jAi. Multiplication by i is exact; in particular, if Z is a complex of A-modules which is exact or which has homology concentrated in a single degree, the same is true for finite direct sums of complexes of the form iZ. Translated to endomorphism algebras this implies that for any two A-modules V, W in add(U), any homomorphism of right $\mathrm{End}_A(U)$-modules $\mathrm{Hom}_A(U,V)\to\mathrm{Hom}_A(U,W)$ is induced by composition with an A-homomorphism $V\to W$. Thus any complex of finitely generated projective right $\mathrm{End}_A(U)$-modules is isomorphic to a complex obtained from applying the functor $\mathrm{Hom}_A(U,-)$ to a complex of A-modules Z whose terms belong to add(U). Moreover, if $\mathrm{Hom}_A(U,Z)$ is exact, then so is any complex of the form $\mathrm{Hom}_A(U',Z)$, where $U'\in$ add(U). We use further the well-known fact that if A, B are symmetric algebras and if M is an A-B-bimodule which is finitely generated projective as a left A-module and as a right B-module, then the functors $M\otimes_B -$ and $M^*\otimes_A -$ between mod(A) and mod(B) are biadjoint.

Proof of Theorem 1.1. The following argument from the proof of [6, Theorem 3.1] shows that we have $\mathrm{add}(M\otimes_B V)=\mathrm{add}(U)$ and $\mathrm{add}(M^*\otimes_A U)=\mathrm{add}(V)$. By the assumptions, we have $\mathrm{add}(M\otimes_B V)\subseteq\mathrm{add}(U)$. Thus $\mathrm{add}(M^*\otimes_A M\otimes_B V)\subseteq\mathrm{add}(M^*\otimes_A U)\subseteq\mathrm{add}(V)$. Since B is isomorphic to a direct summand of the B-B-bimodule $M^*\otimes_A M$, it follows that V is isomorphic to a direct summand of $M^*\otimes_A M\otimes_B V$. Thus the previous inclusions of additive categories are equalities. The same argument with reversed roles shows the second equality.

Set $E=\mathrm{End}_A(U)$ and $F=\mathrm{End}_B(V)$. By the assumptions, add(U) contains all finitely generated projective A-modules; similarly for add(V). In particular, if U' is a projective A-module, then $\mathrm{Hom}_A(U,U')$ is a projective right E-module, and by [7, Proposition 3.2], $\mathrm{Hom}_A(U,U')$ is also an injective right E-module. Let

$$0\longrightarrow E\longrightarrow I^0\longrightarrow I^1\longrightarrow\cdots$$

be an injective resolution of E as a right E-module. Suppose that there is a positive integer d such that I^n is projective for $0 \le n \le d-1$. We will show that there is an injective resolution

$$0 \longrightarrow F \longrightarrow J^0 \longrightarrow J^1 \longrightarrow \cdots$$

of F as a right F-module such that J^n is projective for $0 \le n \le d-1$. Since the right E-modules I^n are projective and injective for $0 \le n \le d-1$, it follows from [7, Proposition 3.2] that there are finitely generated projective A-modules U_n for $0 \le n \le d-1$ such that the sequence

$$0 \longrightarrow E \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \cdots \longrightarrow I^{d-1}$$

is isomorphic to a sequence of the form

$$0 \longrightarrow \mathrm{Hom}_A(U,U) \longrightarrow \mathrm{Hom}_A(U,U_0) \longrightarrow \mathrm{Hom}_A(U,U_1) \longrightarrow \cdots \longrightarrow \mathrm{Hom}_A(U,U_{d-1})$$

which is obtained from applying the functor $\mathrm{Hom}_A(U,-)$ to a sequence of A-modules

$$0 \longrightarrow U \longrightarrow U_0 \longrightarrow U_1 \longrightarrow \cdots \longrightarrow U_{d-1}$$

It follows from the remarks at the beginning of this section that for any A-module U' in $\mathrm{add}(U)$, applying the functor $\mathrm{Hom}_A(U',-)$ to the previous sequence of A-modules yields an exact sequence of right $\mathrm{End}_A(U')$-modules of the form

$$0 \longrightarrow \mathrm{Hom}_A(U',U) \longrightarrow \mathrm{Hom}_A(U',U_0) \longrightarrow \mathrm{Hom}_A(U',U_1) \longrightarrow \cdots \longrightarrow \mathrm{Hom}_A(U',U_{d-1})$$

By the assumptions, the A-module $U' = M \otimes_B V$ belongs to $\mathrm{add}(U)$. Thus we obtain an exact sequence of the form

$$0 \longrightarrow \mathrm{Hom}_A(M \otimes_B V, U) \longrightarrow \mathrm{Hom}_A(M \otimes_B V, U_0) \longrightarrow \cdots \longrightarrow \mathrm{Hom}_A(M \otimes_B V, U_{d-1})$$

Since $M \otimes_B -$ is left adjoint to $M^* \otimes_A -$, it follows that the previous exact sequence is isomorphic to an exact sequence of the form

$$0 \longrightarrow \mathrm{Hom}_B(V, M^* \otimes_A U) \longrightarrow \mathrm{Hom}_B(V, M^* \otimes_A U_0) \longrightarrow \cdots \longrightarrow \mathrm{Hom}_B(V, M^* \otimes_A U_{d-1})$$

Since the A-modules U_n are projective for $0 \le n \le d-1$, it follows that the B-modules $M^* \otimes_A U_n$ are projective as well, hence in $\mathrm{add}(V)$. By [7, Proposition 3.2], the projective right F-modules $\mathrm{Hom}_B(V, M^* \otimes_A U_n)$ are therefore also injective. Thus the preceding sequence is the beginning of an injective resolution of the right F-module $\mathrm{Hom}_B(V, M^* \otimes_A U)$ which has the property that its first d terms are projective. Since V belongs to $\mathrm{add}(M^* \otimes_A U)$, it follows that $F = \mathrm{Hom}_B(V,V)$ is isomorphic, as a right F-module, to a direct summand of a direct sum of finitely many copies of the right F-module $\mathrm{Hom}_B(V, M^* \otimes_A U)$. This implies that F has an injective resolution as a right F-module whose first d terms are projective. This shows that $\mathrm{ddim}(F) \ge \mathrm{ddim}(E)$ (including the case where both are ∞). Exchanging the roles of A, E and B, F, respectively, yields the result. □

3 Proof of Theorem 1.2

We suppose in this section that k is a field of prime characteristic p. Let G be a finite group and B a block of kG with a nontrivial defect group P. Let $i \in B^P$ be a source idempotent of B and set $A = iBi$; that is, A is a source algebra of B. Note that A, B, kP are symmetric algebras. It is well-known that A and kP are separably equivalent via the bimodule A_{kP} and its dual, which is isomorphic to ${}_{kP}A$; see e. g. [4, Proposition 4.2] for a proof (the hypothesis on k being algebraically closed in that paper is not needed for this result). We present first a proof of Theorem 1.2 as an application of Theorem 1.1, and then remark on how to deduce Theorem 1.2 directly from existing results in the literature.

Proof of Theorem 1.2. With the notation above, set $U = \oplus_Q A \otimes_{kQ} k$, where Q runs over the subgroups of P, and set $E = \mathrm{End}_A(U)$. By [6, Theorem 1.1] we have $\mathrm{coMack}(B) \cong \mathrm{mod}(E^{\mathrm{op}})$. Set $V = \oplus_Q kP \otimes_{kQ} k$, where Q runs as before over the subgroups of P, and set $F = \mathrm{End}_{kP}(V)$. As in the proof of [6, Theorem 1.6], the functor $A \otimes_{kP} -$ sends V to $\mathrm{add}(U)$ and the functor ${}_{kP}A \otimes_A -$ sends U to $\mathrm{add}(V)$, because A has a $P \times P$-stable k-basis, hence preserves the classes of p-permutation modules. By definition, the dominant dimension of $\mathrm{coMack}(B)$ is equal to $\mathrm{ddim}(E)$. It follows from Theorem 1.1 that $\mathrm{ddim}(E) = \mathrm{ddim}(F)$. We have $\mathrm{ddim}(F) \geq 2$ by the general Morita-Tachikawa correspondence. Since the argument is very short, we sketch it (the dual argument for left modules is in the proof of [9, Theorem 2], for instance). Let $0 \to V \to I^0 \to I^1 \to \cdots$ be an injective resolution of V as a kP-module, with I^n finitely generated for all $n \geq 0$. The modules I^n are also projective since kP is symmetric. Applying $\mathrm{Hom}_{kP}(V, -)$ yields an exact sequence

$$0 \longrightarrow \mathrm{Hom}_{kP}(V,V) = F \longrightarrow \mathrm{Hom}_{kP}(V, I^0) \longrightarrow \mathrm{Hom}_{kP}(V, I^1)$$

of right F-modules. The last two terms are projective as right F-modules because the finitely generated injective kP-modules I^0 and I^1 are in $\mathrm{add}(V)$, and the last two terms are also injective right F-modules by [7, Proposition 3.2] or by [8, (17.2)].

By Müller's Lemma 3 in [9], in order to show that $\mathrm{ddim}(F) = 2$, it suffices to show that $\mathrm{Ext}^1_{kP}(V,V)$ is nonzero. The summand of V indexed by P is the trivial kP-module. Since P is nontrivial, it follows that $\mathrm{Ext}^1_{kP}(k,k) \neq \{0\}$, and hence $\mathrm{Ext}^1_{kP}(V,V) \neq \{0\}$. Theorem 1.2 follows. □

Remark 3.1. One can prove Theorem 1.2 also without using Theorem 1.1, by showing directly that $\mathrm{Ext}^1_A(U,U)$ is nonzero, and then applying [9, Lemma 3] as in the proof above. Indeed, as mentioned above, kP is isomorphic to a direct summand of A as a kP-kP-bimodule. Thus $A \otimes_{kP} k$ has a trivial summand as a left kP-module. With U, V as above, we have $U \cong A \otimes_{kP} V$. An Eckmann-Shapiro adjunction implies that we have an isomorphism $\mathrm{Ext}^*_A(U,U) \cong \mathrm{Ext}^*_{kP}(V, \mathrm{Res}^A_{kP}(U))$. Both V and $\mathrm{Res}^A_{kP}(U)$ have a trivial summand, and hence $\mathrm{Ext}^*_{kP}(k,k)$ is a summand of $\mathrm{Ext}^*_A(U,U)$ as a graded k-vector space. In particular, $\mathrm{Ext}^1_A(U,U)$ is nonzero since $\mathrm{Ext}^1_{kP}(k,k)$ is nonzero.

Remark 3.2. As pointed out by the referee, Theorem 1.2 follows also from [2, Proposition 5.2]. We sketch the referee's argument. As before, the Tachikawa-Morita correspondence implies that the dominant dimension of $\mathrm{coMack}(B)$ is at least 2 (this follows also from the proof of [2, Proposition 5.2]). The fact that it is at most 2 can be deduced from [2, Proposition 5.2] as follows (using without further comment notation and results on cohomological Mackey functors from [12], such as the fact that restriction and induction of cohomological Mackey functors to/from a subgroup preserves projectives and injectives). Dualising the statement of [2, Proposition 5.2] implies that for a nontrivial finite cyclic p-group, the projective Mackey functor FP_k has injective dimension 2. That is, its minimal injective resolution (in the category of cohomological Mackey functors) has three nonzero terms, the last of which is necessarily nonprojective (since otherwise it would split off the resolution). Thus, reverting to cohomological Mackey functors of G belonging to B, if U is a trivial source module belonging to B with a nontrivial vertex, then the restriction of an injective resolution of the projective cohomological Mackey functor FP_U to a nontrivial cyclic subgroup of a vertex of U has an injective resolution of FP_k for that cyclic subgroup as a direct summand, and hence the third term of any injective resolution of FP_U is nonprojective. This shows that $\mathrm{coMack}(B)$ has dominant dimension equal to 2.

References

[1] M. Auslander, I. Reiten, and S. O. Smalø, *Representation Theory of Artin Algebras*, Cambridge studies in advanced mathematics **36**, Cambridge University Press (1995).

[2] S. Bouc, R. Stancu, and P. Webb, *On the Projective Dimensions of Mackey Functors.* Algebras Representation Theory (2017). https://doi.org/10.1007/s10468-017-9695-y

[3] L. Kadison, *On split, separable subalgebras with counitality condition.* Hokkaido Mathematical Joumal **24** (1995), 527–549.

[4] M. Linckelmann, *Trivial source bimodule rings for blocks and p-permutation equivalences*, Trans. Amer. Math. Soc. **361** (2009), 1279–1316.

[5] M. Linckelmann, *Finite generation of Hochschild cohomology of Hecke algebras of finite classical type in characteristic zero*, Bull. London Math. Soc. **43** (2011) 871–885.

[6] M. Linckelmann, *On equivalences for cohomological Mackey functors*, Represent. Theory **20** (2016), 162–171.

[7] M. Linckelmann and B. Rognerud, *On Morita and derived equivalences for cohomological Mackey functors.* preprint (2016); Math. Z. (to appear).

[8] K. Morita, *Duality for modules and its applications to the theory of rings with minimum condition.* Sci. Repts. Tokyo Kyoiku Daigaku, Sect. A **6** (1958), 83–142.

[9] B. J. Müller, *The classification of algebras by dominant dimensions*, Canad. J. Math. **20** (1968), 398409.

[10] B. Rognerud, *Equivalences between blocks of cohomological Mackey algebras.* Math. Z. **280** (2015), 421–449.

[11] H. Tachikawa, *On dominant dimensions of QF-3 algebras.* Proc. Amer. Math. Soc. **112** (1964), 249–266.

[12] J. Thévenaz and P. Webb *The structure of Mackey functors*, Trans. Amer. Math. Soc. **347** (1995), 1865–1961.

[13] T. Yoshida, *On G-functors (II): Hecke operators and G-functors*, J. Math. Soc. Japan **35** (1983), 179–190.