ALGEBRAIC AND GEOMETRIC METHODS AND PROBLEMS FOR IMPLICIT LINEAR SYSTEMS

BY

DIMITRIS VAFIADIS

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CONTROL ENGINEERING CENTRE
DEPARTMENT OF ELECTRICAL, ELECTRONIC AND
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CITY UNIVERSITY
LONDON EC1V 0HB

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Στη μνήμη του παππού μου Στυλιανού Φανού.

To the memory of my grandfather Stylianos Fanos.

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Contents

ABSTRACT

This thesis investigates a number of problems of the implicit linear systems framework.

First, the problem of realisations of nonproper transfer functions is considered. The main result obtained here is the generalisation of the realisation method from MFDs to the case of the nonproper transfer functions. The obtained realisations are singular systems. The method treats both finite and infinite frequency behaviour in a unified way and generalises the results related to the minimality of the realisation and coprimeness and column reducedness of the MFD. Furthermore, it displays transparently the relation between the extended MacMillan degree of the transfer function and the minimal realisation.

The next problem considered is the problem of canonical forms of minimal singular systems under restricted system equivalence transformations. For systems with outputs a canonical form is obtained and it is shown that it is directly related to the echelon form of the composite matrix of an MFD of the transfer function of the system. This result is a direct generalisation of the results of Popov and Forney for strictly proper systems. The canonical form obtained is of Popov type and may be considered as a direct generalisation of the well known form for strictly proper systems. The second canonical form is for systems without outputs. A Popov type canonical form for a class of these systems is obtained. This class is that of systems with equal reachability indices. For both canonical forms, the sequence of the transformations yielding the canonical description is described in detail. In the general case of systems without outputs a semi canonical Popov type form is obtained.

Another problem considered in the thesis is the problem of first order realisations of autoregressive equations within the external equivalence framework. An alternative to the existing methods is provided; in fact, the proposed method is simpler than the existing ones and allows the derivation of the realisation by inspection of the autoregressive equations. A generalisation of the observability indices is proposed for nonsquare descriptor systems and their connection to the autoregressive equations is established.

The problem of model matching for implicit systems is considered next. This is a generalisation of the model matching problem for systems described by transfer functions. Here a controller is interconnected to the given plant such that the overall system has a desired external behaviour. The problem is studied within the framework of external and $\mathcal{A}$-external (input–output) equivalence. Necessary as well as sufficient conditions

for the solvability of the problem are derived and the equations of the controllers are found in a constructive way.

The last problem considered here is the generalised dynamic cover problem of geometric theory i.e. the problem of finding the family of (A, B)–invariant subspaces covering a given subspace. This problem is formulated here by using the matrix pencil approach of the geometric theory. This approach allows the unification of the problem for state–space and nonsquare descriptor systems. An extension of the problem to the case of infinite spectrum spaces is also obtained. The solution of the above problems is reduced to the solution of appropriately defined systems of linear equations. Finally, an alternative method for the solution involving systems of multilinear equations is proposed using the mathematical tool of Groebner bases.

Chapter 1

INTRODUCTION

Introduction

In the classical system theory, a system is considered as an entity interacting with the environment through the external signals, the inputs and the outputs. The inputs "excite" the system which operates as a "processor" and gives the outputs as the result of this processing. In the framework of this approach a complete theory for linear systems was built during the last four decades. Many methods are developed for the study of linear systems with predominant the transfer function and state–space methods. These methods are based on the external and internal descriptions of the system and each one has certain advantages over the other. Although initially the above methods were developed independently, it was soon realised that they are strongly related and that combination of them would provide powerful tools for analysis and design of systems.

The first and most commonly used type of systems encountered in the literature is the class of proper and strictly proper systems. This terminology is used within the transfer function framework. The corresponding systems within the state–space framework are the regular state–space systems. Although strictly proper systems cover a wide range of systems that we may find in practice, there are many examples where these models are insufficient to describe a physical system. Such models are electrical networks [New., 1981], large scale and interconnected systems [Ros. & Pugh, 1974], economic models [Lue.& Arb., 1977] etc. With these observations in hand, a generalised state model was proposed. This new model was the model of singular or descriptor or generalised state–space system model [Luen., 1977], [Ros., 1974b], [Ver., V.-D. & Kail., 1979]. This is an extended type of state–space system describing the situation where, in addition to the dynamical differential equation involving the states and the external signals, we have algebraic constraints on the state vector. This type of system corresponds to a system with nonproper transfer function [Ver., V.-D. & Kail., 1979].

Singular systems were the first step towards the generalisation of the state–space or strictly proper systems. A further generalisation was motivated by the observation that if we try to obtain the model of a system starting from the elementary differential equations describing the evolution in time, it is not always guaranteed that we can obtain a system were the variables labeled as outputs are expressed as functions of the inputs in an explicit way. This observation applies mainly to multivariable systems i.e. systems with many inputs and outputs. The external (input–output) models describing this type of systems are implicit in the output variable. In this case the most convenient way to study the system is to consider the set of inputs–outputs as the set of the external variables without making any distinction between them. Due to this unification of the roles of inputs and outputs we may term the systems described by implicit differential equations as *nonoriented* systems.

The introduction of implicit systems implies that we may no longer consider the system as the processor of the inputs. Willems [Wil., 1983], [Wil., 1986], [Wil., 1991] proposed a different approach where the system is considered as a constraint in the set of external signals. The signals satisfying the constraint imposed by the system equations are defined as the *behaviour* of the system. According to Willems' approach the system is defined by the evolution in time (trajectories) of the external signals.

In order to study a system we need to have a mathematical representation in hand. For classical systems such representations are transfer function, polynomial matrix description (PMD) [Ros., 1970] and state–space models. For the implicit systems similar descriptions exist. These systems may be described by differential operators containing higher order derivatives [Wil., 1983]. These operators are directly related to polynomial matrices having as indeterminate the differentiation operator σ. The use of these operators may be considered as a generalisation of the matrix fraction descriptions (MFD) although some attention is needed to make this generalisation.

A different type of representation of an implicit system is the first order representation. This may be considered as a generalisation of the state–space model. As it was mentioned above the first extension of state–space models was the singular system possessing transfer function. This model may be extended to the implicit descriptor models i.e. descriptor type equations implicit in the state variable [Karc. & Hay., 1981], [Sch., 1989], [Kar. & Kal., 1989], [Kui. & Sch., 1991], [Kuij. & Sch., 1990], [Lew., 1982]. It may be shown that starting from a system described by differential equations involving only the external variables we may find a descriptor type system having the same external behaviour with the given differential system [Kuij. & Sch., 1990], [Bon., 1991]. The process of going from an external model to a first order model involving auxiliary variables, is called realisation and may be considered as an extension of the classical realisation theory for strictly proper systems. For implicit systems we may have first order realisations other than the descriptor type. [Kuij. & Sch., 1990]. These realisations are also considered in this thesis.

At this point it must be mentioned that in the implicit systems framework there are two main interpretations of the term system. The first is the one based on Willems' approach mentioned above. According to this, the system is directly related to the solution of a set of differential equations. A different approach which is a direct extension of the transfer function approach is the approach that associates the system with certain rational vector spaces [Apl., 1981], [Apl., 1985], [Grimm, 1988]. Roughly speaking, we may say that Willems' approach is a time domain approach while the latter is a frequency domain approach. The different definitions of the system lead to different answers to the question "when two representations describe the same system?". This question led to the definitions of several notions of equivalence and transformations between system representations [Ros., 1970], [Ros., 1974b], [Ver., Lev. & Kail., 1981], [Pugh, Hay. & Fret., 1987], [Sch., 1988].

An important approach to the study of state–space systems is the so called geometric approach or geometric theory [Bas. & Mar., 1969], [Wonh., 1979]. According to this theory a system is defined as a set of mappings between real or complex vector spaces. The use of descriptor models for implicit systems allowed the extension of geometric theory to this type of systems. The notions of the fundamental subspaces related to a linear system were extended, and the characterisation of the behaviour of implicit system was given in geometric terms [Ozc., 1986], [Ozc., Lew., 1989], [Mal., 1989], [Lew., 1982] [Kar. & Kal., 1989] etc.. Many problems in linear systems theory may be formulated as geometric problems. In many cases different problems in the state–space framework may be reduced to a common geometric problem. Thus, instead of solving each one separately we may consider the geometric version and take a solution to all the problems. Such type of problems are the so called generalised cover problems [Mor., 1976], [Em., Sil. & Gl., 1977], [Ant., 1983] etc.. These problems are studied in the present thesis and are extended to the implicit systems framework.

A mathematical tool directly related to descriptor models and geometric theory, is matrix pencil theory [Gant., 1959], . Matrix pencils are related to first order differential equations and allow the algebraic interpretation of their properties. This is done via the theory of Kronecker for the invariants of matrix pencils [Kro., 1890], . Kronecker's theory played an important role in the study of state–space and descriptor systems. Many notions such as controllability, zeros and transmission properties of a system may be defined in terms of appropriate matrix pencils related to the system [Pop., 1973], [Hau., 1969], [Kar. & Kouv., 1979] [Karc. & Hay., 1981], [Jaf. & Kar., 1981]. On the other hand, all the notions of geometric theory may be translated into Kronecker invariants terms [Kar., 1979], [Kar., 1978], [Kar. & Kal., 1989]. This provides an algebraic way of dealing with problems defined in the geometric approach framework.

In the present dissertation several problems from the implicit systems framework are considered. These problems involve descriptor systems as well as external (autoregressive) models. The problems considered are listed below:

- Realisation of nonproper transfer functions.
- Canonical forms under restricted system equivalence.
- Realisations of autoregressive equations.
- Model matching for implicit systems.
- Generalised dynamic cover problem.

The structure of the thesis is the following: In Chapter 2 the basic background and definitions from the theory of polynomial matrices and matrix pencils is given. This is necessary, since most of the problems in this thesis are treated algebraically. The basic results related to system theory are given without proofs and aim to provide a review for the reader.

Chapter 3 is a brief survey of the main results and definitions from implicit systems theory, related to the problems considered in this thesis. First, the most common types of representations of linear systems are given. These are of external or first order type. Next, the several notions of equivalence of systems described by the same type of representation are discussed. It is shown that the term system (and thus, the equivalence of systems) has received several interpretations. The notions of strict system equivalence [Ros., 1970], restricted system equivalence [Ros., 1974b], strong equivalence [Ver., Lev. & Kail., 1981], complete equivalence [Pugh, Hay. & Fret., 1987], fundamental equivalence [Hay., Fret. & Pugh, 1986], external equivalence [Wil., 1986] and $\mathcal{A}$–external equivalence [Apl., 1991] are discussed and the relation between them is clarified. The definitions and criteria for minimality under the above types of equivalence are discussed next. The chapter closes with the basic results of the geometric approach and matrix pencil theory for linear systems.

In the fourth chapter the first problem of this thesis is considered. This is the problem of realising a nonproper transfer function in descriptor form. First, some realisation procedures are given in order to prove formally that a nonproper transfer function may always admit a generalised state–space realisation. Next, the minimality of such a realisation is related to the MacMillan degree of a given coprime and column reduced MFD of the given transfer function. The proof of this result comes as an alternative to other existing proofs [Jan., 1988]. The main result of this chapter is the derivation of a descriptor type realisation directly from a given MFD. This is a direct generalisation of the existing methods for strictly proper transfer functions. Note that the realisation is obtained without resorting to decomposition of the system into fast and slow parts and thus, the finite and infinite behaviour are treated in a unified manner. The form of the proposed realisation gives us some hints about the construction of canonical forms of descriptor systems.

The problem of canonical forms of regular and minimal descriptor systems is considered in Chapter 5. Canonical forms for this type of systems have been examined; however the existing forms, were obtained under a quite rich transformation group, the Brunovsky group [Ros. & Hay.,1974], and the use of feedback of the derivatives of the states [Gl.-Luer., 1990], [Lois., Ozc. et al., 1991], [Leb. & Lois., 1994]. In this chapter we consider the problem of canonical forms under restricted system equivalence transformations [Ros., 1974b]. The set of these transformations is restricted in comparison to the Brunovsky group and therefore, we have restricted freedom of modifying the system. Our effort was to produce canonical forms of Popov type [Pop., 1972], i.e. forms where the controllability/ reachability properties of the system are displayed transparently and the continuous invariants of the system are in complete analogy to the Popov form for state–space systems.

Two subproblems are considered here. First, a canonical form of Popov type is produced for systems with outputs. In this case it is shown that the canonical form is directly related to the echelon canonical form of the composite matrix of a minimal MFD of the transfer function of the system. This result is a generalisation of the well known works of Popov [Pop., 1969] and Forney [For., 1975] for state–space systems. The restricted system equivalence transformations leading to the canonical form are described in detail. The second canonical form considered in this chapter is the canonical form of a singular system without outputs. This problem is solved for a special type of singular systems, namely the systems with all their reachability indices equal. The difficulties for solving the problem in the general case are identified. However, a semi canonical form for the general case is obtained.

In Chapter 6 we are considering the framework of behavioural systems, where the notion of transfer equivalence is replaced by external equivalence. The problem considered here is the realisation of a set of autoregressive equations in first order form, namely the descriptor and pencil form. This problem has already been examined and solved in [Kui. & Sch., 1991]; the contribution of this chapter is that it provides much simpler procedures for obtaining the realisations. The matrices of the realisations are obtained directly from the coefficients of the polynomial entries of the matrix of the autoregressive system. The simplicity of the realisations allows us to propose an extension of the observability indices to the case of nonsquare descriptor systems and to relate them to the row degrees of the polynomial matrix of the autoregressive representation. Another result that is proved useful for the development of Chapter 7, is that in order to derive the realisations a special first order autoregressive–moving average (ARMA) realisation was produced, as a byproduct of the overall methodology.

The topic of Chapter 7 is the study of the model matching problem for implicit systems. This problem belongs to the general family of control problems where we have to find a system such that when it is interconnected to a given system, a desired property of the overall system is obtained. In the case of model matching, it is desired to obtain a final system with a prespecified external behaviour. This problem is an extension of the classical model matching problem for systems described by transfer functions. In this chapter necessary conditions for model matching under external and $\mathcal{A}$–external equivalence are produced and for a class of systems, these conditions are proved to be also sufficient. For the case of model matching under external equivalence a parametrisation of the family of solutions (when exist) is given. For the case of $\mathcal{A}$–external equivalence, it is shown that the problem is a direct extension of model matching under transfer function equivalence. In the cases where sufficient conditions are derived, constructive solutions of both types of model matching problem are developed and thus, the controllers solving the problem are easily found.

Chapter 8 is introductory to Chapter 9. In this chapter a definition of the cover problems of geometric theory and a brief survey of some important control problems which are formulated as cover problems is given. The problems considered in this chapter are the disturbance decoupling [Wonh., 1979], model matching [Em. & Haut., 1980], deterministic identification [Em., Sil. & Gl., 1977] and observer of linear functionals problems [Wonh. & Mor., 1972]. For each one of these, the formulation as cover problem is described. The observer problem is extended to the case of implicit systems and it is shown that it may be formulated as an extended cover problem. Finally the Model Projection Problems [Kar., 1994] are discussed and it is shown that the standard cover problem arises as a special case of this family of problems.

In Chapter 9 the dynamic cover problem is considered. This problem was originally defined as the problem of finding the family of the (A,B)–invariant subspaces containing a given subspace of the state space and are contained in another subspace [Wonh., 1979], [Gl.-Luer. & Hin., 1987]. In this chapter the problem is approached by using the matrix pencil characterisation of the (A,B)–invariant subspaces [Jaf. & Kar., 1981]. It is shown that every cover problem may be formulated as the problem of augmenting by columns an appropriately defined matrix pencil (restriction pencil [Kar., 1979],) such that the final pencil has certain types of Kronecker invariants. By the matrix pencil formulation of the cover problem two advantages are obtained. First, the problem becomes algebraic and second it is easy to extend the solution to the framework of implicit descriptor systems where (A,B)–invariance is replaced by (A,E,B)–invariance. Furthermore, we may enrich the family of cover problems by considering subspaces with infinite spectrum (almost (A,B)–invariant) subspaces [Wil., 1981], [Jaf. & Kar., 1981]. In this chapter it is shown that the matrix pencil formulated cover problem is essentially a two–fold problem: The *Kronecker Invariant Transformation by Matrix Pencil Augmentation* and the *Matrix Pencil Realisation* problems. These two problems may be reduced to the solution of linear systems of equations. It is shown that the parametric solutions of these equations provide the parametrisation of the bases of the covering spaces. The extended cover problems (i.e. the problems concerning infinite spectrum spaces) are tackled by using a slightly modified version of the standard cover problems. Finally, an alternative technique is proposed for the solution of the problem. This is the Groebner basis technique, where the matrix pencil augmentation–realisation problem is reduced to the problem of appropriately defined sets of multilinear equations.

Summarising, the contribution of the present thesis in the area of implicit systems is the following: First, the classical realisation technique via MFDs is extended to the case of singular systems in a way similar to that of the strictly proper systems case. Then, the problem of canonical forms under restricted system equivalence comes naturally and is related to the realisation theory for the case of systems with outputs. The problem of Popov type canonical forms for reachable systems without outputs is solved for a special type of systems, while for the general case a semi–canonical Popov form is provided. Next, alternative simple realisation methods for autoregressive systems are developed and a generalisation of the observability indices to the case of implicit descriptor systems is proposed. The model matching problem under external and $\mathcal{A}$–external equivalence are tackled and necessary conditions for the solvability are provided. These conditions are also necessary in certain cases. Finally, the generalised dynamic cover problems are extended and solved by formulating them as matrix pencil Kronecker invariant transformation problems.

Chapter 2

POLYNOMIAL MATRICES AND MATRIX PENCILS

2.1 Introduction

This Chapter aims at providing the basic background on polynomial matrices and matrix pencils. Polynomial matrices theory is the basic tool in algebraic control theory. The need of application of polynomial matrix theory in control was necessary in order to extend the basic transfer function approach to multivariable systems. Many significant developments in linear system theory were achieved by the use of polynomial matrices [Ros., 1970], [For., 1975], [Wol., 1974], [Wil., 1991] etc..

With the introduction of state–space theory the need of a special type of polynomial matrices emerged. This type is the matrix pencils [Gant., 1959]. Matrix pencil is the basic tool for translating linear systems properties into algebraic terms. By the use of matrix pencils the fundamental notions of linear systems theory such as controllability were related to the theory of Kronecker [Kro., 1890].

In the present thesis, the treatment of the several problems is mainly developed by using polynomial matrix and matrix pencil theory. This Chapter provides a brief presentation of the basic properties and results in polynomial matrix and matrix pencil theory. The material of this Chapter may be found in classical algebra books such as [Wed., 1934], [MacD., 1950], [MacL. & Bir., 1967], [Gant., 1959] etc..

2.2 Polynomial matrices

Polynomial matrices are matrices whose elements are polynomials. If we see polynomial matrices as a subset of the rational matrices we may readily define the rank of a polynomial matrix since it is a matrix with its elements over a field. Consider now a square polynomial matrix of full rank. Clearly the inverse of this matrix is not necessarily a polynomial matrix.

Definition 2.2.1 *A polynomial matrix $U(s) \in \Re^{n\times n}[s]$ is called* **unimodular** *if it has full rank and $U^{-1}(s) \in \Re^{n\times m}[s]$.* □

Notice that the set of square polynomial matrices endowed with the operations of the usual matrix multiplication and addition has the algebraic structure of a ring. Thus, from the definition of the unimodular matrices we have the equivalent statement that a unimodular matrix $U(s) \in \Re^{n\times n}[s]$ is a unit of the ring of polynomial matrices in $\Re^{n\times n}[s]$. An immediate consequence of definition 2.2.1 is the following:

Lemma 2.2.1 *A polynomial matrix $U(s) \in \Re^{n\times n}[s]$ is unimodular if and only if*

$$\det\{U(s)\} = k \in C \tag{2.1}$$

□

The above may be used as an alternative definition of a unimodular matrix. Unimodular matrices are very useful tools in polynomial matrix theory since they represent closed forms of the *elementary operations* on polynomial matrices defined below.

Definition 2.2.2 *Let $P(s)$ be a polynomial matrix. By the term* **elementary column operations** *we define the following transformations on the columns of $P(s)$*

(i) Permutation of any columns of $P(s)$

(ii) Addition of a polynomial multiple of a column of $P(s)$ to another column of $P(s)$

(iii) Multiplication of a column of $P(s)$ by a scalar in C. □

Each one of the above operations may be obtained by post-multiplication by an appropriate unimodular matrix. These matrices corresponding to elementary operations are called *elementary matrices.* Note that we have the analogous definition for the row operations.

We continue with some basic definitions form the theory of polynomial matrices.

Definition 2.2.3 *Let $\underline{p}(s) = [p_1(s), \ldots, p_n(s)]^T$ be a polynomial vector. The degree of $\underline{p}(s)$ is defined as*

$$\deg\{\underline{p}(s)\} = \max_i\{\deg\{p_i(s)\}\} \tag{2.2}$$

□

Definition 2.2.4 *If $P(s)$ is a polynomial matrix of dimensions $m \times n$, the i–th* **index** *of $P(s)$ is defined as $\lambda_i = \deg\{\underline{p}_i(s)\}$ where $\underline{p}_i(s)$ are the columns of $P(s)$.* □

Definition 2.2.5 *[For., 1975] Let $P(s)$ be a $m \times n$ polynomial matrix. Then the order λ of $P(s)$ is defined as*

$$\lambda = \sum_{i=1}^{n} \lambda_i \tag{2.3}$$

□

The above definition will be used later in this Chapter in the discussion of minimal bases of rational vector spaces.

In the case of scalar polynomials we say that $a(s)$ divides $b(s)$ if $b(s) = q(s)a(s)$. This definition may be extended to polynomial matrices as follows.

Definition 2.2.6 *A polynomial matrix $D(s)$ is called a* **right divisor** *of the polynomial matrix $P(s)$ if there exists another polynomial matrix $P'(s)$ such that*

$$P(s) = P'(s)D(s) \tag{2.4}$$

□

Left divisors of $P(s)$ may be defined analogously. The notion of common divisors may be extended to the matrix case as follows.

Definition 2.2.7 *Let $P_1(s)$, $P_2(s)$ be two polynomial matrices in $\Re^{m \times n}[s]$. Then, the polynomial matrix $D(s)$ is said to be a* **common right divisor** *of $P_1(s)$, $P_2(s)$ if there exist polynomial matrices $P_1'(s)$, $P_2'(s)$ such that*

$$P_1(s) = P_1'(s)D(s), \;\; P_2(s) = P_2'(s)D(s) \tag{2.5}$$

□

Similarly, we may define the left common divisors of two matrices.

Definition 2.2.8 *A polynomial matrix $D(s)$ is a* **greatest common right divisor** *of two polynomial matrices $P_1(s)$ and $P_2(s)$ if*

(i) $D(s)$ is a common right divisor of $P_1(s)$, $P_2(s)$

(ii) If $D_1(s)$ is another common right divisor of $P_1(s)$, $P_2(s)$ then, $D_1(s)$ is a right divisor of $D(s)$. □

Notice that the definition of the greatest common divisor is a straightforward extension of the scalar case. After the definition of divisors we may define the coprimeness of polynomial matrices.

Definition 2.2.9 *Two polynomial matrices are* **right coprime** *if their greatest common right divisors are unimodular matrices.* □

Similarly we may define left coprimeness.

The following results provide criteria for the coprimeness of two polynomial matrices [Kail., 1980].

Lemma 2.2.2 *The polynomial matrices $P_1(s)$ and $P_2(s)$ are right coprime if and only if there exist polynomial matrices $X(s)$ and $Y(s)$ such that*

$$X(s)P_1(s) + Y(s)P_2(s) = I \tag{2.6}$$

□

Lemma 2.2.3 *The polynomial matrices $P_1(s)$ and $P_2(s)$ are right coprime if and only if the composite matrix*

$$T(s) = \begin{bmatrix} P_1(s) \\ P_2(s) \end{bmatrix} \tag{2.7}$$

has full rank $\forall\; s \;\in\; C$. □

Consider now a polynomial matrix $P(s) \in \Re^{m\times n}[s]$, $m \geq n$, of full rank and write it in the form

$$P(s) = P_{hc}H(s) + P_{\ell c}L(s) \tag{2.8}$$

where

$$H(s) = \begin{bmatrix} s^{\lambda_1} & & & \\ & s^{\lambda_2} & & \\ & & \ddots & \\ & & & s^{\lambda_n} \end{bmatrix} \tag{2.9}$$

$$L(s) = \text{block} - \text{diag}\{\ldots, [1\ s \ldots s^{\lambda_i - 1}]^T, \ldots\} \tag{2.10}$$

with λ_i the degrees of the columns of $P(s)$. We have now the following.

Definition 2.2.10 *A polynomial matrix $P(s)$ is called* **column reduced** *if the corresponding matrix P_{hc} in* (2.8) *has full rank.* □

Matrix P_{hc} is called the *high order coefficient matrix of* $P(s)$.

Coprimeness and column reducedness are related to the finite and infinite structure of the matrix $T(s)$ in (2.7). This will be discussed after the definition of poles and zeros later in this Chapter.

2.3 Unimodular transformations on polynomial matrices

In the previous section we saw that unimodular matrices correspond to column, row elementary operations on a polynomial matrix. Consider now two matrices $P_1(s)$, $P_2(s)$ such that we may derive $P_2(s)$ from $P_1(s)$ by unimodular column and row operations. We proceed to the following definition.

Definition 2.3.1 *Let $P_1(s)$ and $P_2(s)$ be two polynomial matrices of the same dimensions. We say that $P_1(s)$, $P_2(s)$ are* **unimodularly equivalent** *if there exist unimodular matrices $U(s)$, $R(s)$ such that*

$$P_2(s) = U(s)P_1(s)R(s) \tag{2.11}$$

□

The transformation defined by (2.11) is an equivalence transformation and thus we may say that the set of the polynomial matrices of the same dimensions $m \times n$ is partitioned into equivalence classes by this transformation [MacL. & Bir., 1967]. Then the question of the existence of canonical forms for these equivalence classes arises. Before we proceed, we give the definition of the canonical form [MacL. & Bir., 1967].

Definition 2.3.2 *Given a set $\mathcal{X}$ and an equivalence relation $\sim$, a subset C of $\mathcal{X}$ will be said to be a set of* **canonical forms for $\mathcal{X}$ under $\sim$** *if for every $x \in \mathcal{X}$ there exists one and only one $c \in C$ such that $x \sim c$.* □

Definition 2.3.3 *Consider the map $f : \mathcal{X} \to \mathcal{Y}$. This map is called an* **invariant under the equivalence relation $\sim$** *if*

$$x_1 \sim x_2 \Longrightarrow f(x_1) = f(x_2) \tag{2.12}$$

and **complete invariant under the equivalence relation $\sim$** *if*

$$x_1 \sim x_2 \Longleftrightarrow f(x_1) = f(x_2) \tag{2.13}$$

□

Consider now the map which associates each $x \in \mathcal{X}$ to the unique canonical element $c \in C$ which corresponds to the equivalence class of x. We have the following.

Theorem 2.3.1 *If C is a set of canonical forms for $\mathcal{X}$ under the equivalence relation $\sim$, then the map $f : \mathcal{X} \to C$ that associates to each $x \in \mathcal{X}$ a unique $c \in C$ such that $c \sim x$ is a complete invariant.* □

We continue now with the equivalence transformations of definition 2.3.1.

Theorem 2.3.2 *Consider the polynomial matrix $P(s) \in \Re^{m\times n}[s]$. There exist unimodular matrices $U(s)$, $R(s)$ such that*

$$U(s)P(s)R(s) = S_P(s) = \left[\begin{array}{cccc|c} p_1(s) & & & & \\ & p_2(s) & & & \\ & & \ddots & & \\ & & & p_r(s) & \\ \hline & & & & 0 \end{array}\right] \tag{2.14}$$

where $r = \text{rank}\{P(s)\}$ and $p_i(s)$ are monic polynomials with the division property

$$p_i(s)|p_{i+1}(s), \quad i = 1, \cdots, r-1 \tag{2.15}$$

The polynomials $p_i(s)$ are uniquely defined and (2.14) is a canonical form for the equivalence class of $P(s)$. This form is called **Smith form of $P(s)$**. □

Note that the polynomials $p_i(s)$, called the *invariant polynomials of* $P(s)$, are complete invariants under unimodular equivalence.

The construction of the Smith form is performed by elementary column and row operations on $P(s)$ and is similar to the Gauss elimination method for constant matrices [Gant., 1959].

A final remark to the Smith canonical form is that the invariant polynomials $p_i(s)$ in (2.14) are given as follows

$$p_i(s) = \frac{\Delta_i(s)}{\Delta_{i-1}(s)}, \ \Delta_0 = 1 \tag{2.16}$$

where $\Delta_i(s)$ is the greatest common divisor of all the $i \times i$ minors of $P(s)$. The roots of the polynomials $p_i(s)$ are called *Smith zeros* of $P(s)$. The polynomial $z_p(s) = \prod_{i=1}^{r} p_i(s)$ is called the *zero polynomial* of $P(s)$. If $z_p(s)$ is factorised into irreducible factors over C as $z_p(s) = (s - z_1)^{\tau_1} \cdots (s - z_\mu)^{\tau_\mu}$ then the integer τ_i is called the *algebraic multiplicity* of z_i. For $s = z_i$ the matrix $P(s)$ looses rank. The rank defficiency of $P(s)$ at $s = z_i$ is called *geometric multiplicity* of z_i. By factorising each of the $p_i(s)$ into irreducible factors over C and collecting all terms corresponding to the zero z_i we define the set of *elementary divisor* for z_i, $\mathcal{D}_{P,z_i} \equiv \{(s - z_i)^{q_{ik}}, k = 1, \cdots, v_i\}$ where v_i is the geometric multiplicity and $\sum_{k=1}^{v_i} q_{ik} = \tau_i$.

Theorem 2.3.3 [Kail., 1980] *Given a polynomial matrix $P(s)$ not necessarily column reduced, we may reduce it to a column reduced matrix by elementary column transformations, or equivalently by post-multiplication by an appropriate unimodular matrix.* □

Another result related to column reduced matrices is the following.

Theorem 2.3.4 [Wed., 1934] *Let $D_1(s)$, $D_2(s)$ be two column reduced matrices such that $D_1(s) = D_2(s)U(s)$ where $U(s)$ is unimodular. Then $D_1(s)$ and $D_2(s)$ have the same column degrees.* □

The above property is called *invariance of column degrees* of column reduced matrices.

A result very useful in system theory [Kail., 1980] is the following.

Lemma 2.3.1 [MacD., 1950], [Kail., 1980] *Let $P_1(s) \in \Re^{m\times n}[s]$, $P_2(s) \in \Re^{\ell\times n}[s]$, $m + \ell \geq n$ and consider the composite matrix $T(s) = [P_1^T(s), P_2^T(s)]^T$. Assume that $T(s)$ has full column rank and consider the unimodular matrix $U(s)$ such that*

$$U(s)T(s) = \begin{bmatrix} U_{11}(s) & U_{12}(s) \\ U_{21}(s) & U_{22}(s) \end{bmatrix} \begin{bmatrix} P_1(s) \\ P_2(s) \end{bmatrix} = \begin{bmatrix} H(s) \\ 0 \end{bmatrix} \tag{2.17}$$

where $H(s)$ has full rank. Then $H(s)$ above is a greatest common right divisor of $P_1(s)$, $P_2(s)$. □

The above result is a constructive way to find a greatest common divisor of a pair of matrices. This result may be used in system theory when we want to obtain a coprime MFD from a given MFD [Kail., 1980], [Cal. & Des., 1982]. The extraction of the greatest common divisor from $P_1(s)$, $P_2(s)$ is performed by simply taking $P'(s) = P_1(s)H^{-1}(s)$ and $P_2'(s) = P_2(s)H^{-1}(s)$ and $P'(s)$ and $P_2'(s)$ are right coprime. A criterion for the coprimeness of two matrices is given below.

Lemma 2.3.2 [Kail., 1980] *The polynomial matrices $P_1(s)$ and $P_2(s)$ are right coprime if and only if the Smith form of the composite matrix $T(s) = [P_1^T(s), P_2^T(s)]^T$ is $[I\ 0]^T$.* □

2.4 The Smith–MacMillan form

A canonical form for matrices whose elements are from the field of rational functions is the *Smith–McMillan* form. This form is immediate consequence of the Smith form for polynomial matrices.

A rational matrix $G(s) \in \Re^{m \times n}(s)$ of rank r may be written in the from

$$G(s) = \frac{1}{\psi(s)} P(s) \tag{2.18}$$

where $P(s)$ is a polynomial matrix and $\psi(s)$ is the least common multiple of the denominators of all the entries of $G(s)$. If $S_P(s)$ is the Smith form $P(s)$, then there exist unimodular matrices $U(s)$ and $R(s)$ such that

$$G(s) = \frac{1}{\psi(s)} U(s) S_P(s) R(s) = U(s) [\frac{1}{\psi(s)} S_P(s)] R(s) \tag{2.19}$$

Definition 2.4.1 *The Smith–MacMillan form of $G(s)$ is the matrix*

$$M_G(s) = \frac{1}{\psi(s)} S_P = \left[\begin{array}{cccc|c} \frac{\varepsilon_r(s)}{\psi_r(s)} & & & & \\ & \ddots & & & \\ & & \frac{\varepsilon_2(s)}{\psi_2(s)} & & \\ & & & \frac{\varepsilon_1(s)}{\psi_1(s)} & \\ \hline & & & & 0 \end{array}\right] \tag{2.20}$$

where the polynomials $\varepsilon_i(s)$, $\psi_i(s)$ are relatively prime for $i = 1, \ldots, r$. □

From the above definition it readily follows that all the possible cancellations between the factors of the entries of the numerators and denominators of the entries of $M_G(s)$ must be carried out. From the divisibility properties of the elements of $S_P(s)$, we have the following theorem.

Theorem 2.4.1 *The polynomials $\varepsilon_i(s)$, $\psi_i(s)$ in (2.20) are uniquely defined and satisfy the following divisibility conditions*

(i) $\varepsilon_i(s)$ divides $\varepsilon_{i-1}(s)$, for $i = 1, \ldots, r$

(ii) $\psi_1(s)$ divides $\psi_{i+1}(s)$, for $i = 1, \ldots, r$. □

The polynomials $z_p(s) = \prod_{i=1}^{r} \varepsilon_i(s)$, $p_p(s) = \prod_{i=1}^{r} \psi_i(s)$ are defined as the *zero, pole polynomials* of $G(s)$ and $\deg\{p_p(s)\} \equiv \delta_M$ is defined as the *MacMillan degree* of $G(s)$.

After the definition of the Smith–MacMillan form we are ready to define the poles and zeros of a rational matrix.

Definition 2.4.2 [Ros., 1970] *Let $G(s)$ be a rational matrix with Smith-MacMillan form as in* (2.20). *Then*

(i) The finite zeros of $G(s)$ are the roots of $\varepsilon_i(s)$, $i = 1, \ldots, r$

(ii) The finite poles of $G(s)$ are the roots of $\psi_i(s)$, $i = 1, \ldots, r$. □

The notions of Smith, Smith–MacMillan forms and the zeros of a rational matrix play important role in algebraic theory of linear systems. Many issues such as minimality of systems [Ros., 1970], transmission properties [MacF. & Kar., 1976], controllability, observability [Kal., 1969], [Ros. 1968], matrix fraction descriptions etc. are related to these notions.

So far in this Chapter only the finite structure of poles and zeros of polynomial and rational matrices has been discussed. Next, we give the definitions of poles and zeros of a rational matrix $M(s)$ at infinity.

Definition 2.4.3 [Ver., 1978], [Pug. & Rat., 1979] *The rational matrix $G(s)$ is said to have a pole (zero) at infinity if the matrix $G(1/w)$ has a pole (zero) at $w = 0$.* □

From the above definition we see that in order to obtain the structure at infinity of a rational matrix we may use a bilinear transformation in order to map ∞ at a finite point (zero) and then we follow the classical method to determine the structure of the resulting rational matrix at $s = 0$.

The approach for the determination of the infinite structure of a rational matrix suggested by 2.4.3 is somewhat indirect since we have to perform first the bilinear transformation $s \to \frac{1}{w}$. In [Var., Lim. & Kar., 1982] an alternative way of finding the Smith–MacMillan form at infinity is proposed. The main result of the above paper is the following.

Theorem 2.4.2 [Var., Lim. & Kar., 1982] *Consider the rational matrix $G(s)$ with rank r. There exist biproper matrices $U(s)$ and $R(s)$ such that*

$$U(s)G(s)R(s) = M_G^\infty = \left[\begin{array}{cccc|c} s^{q_1} & & & & \\ & s^{q_2} & & & \\ & & \ddots & & \\ & & & s^{q_r} & \\ \hline & & & & 0 \end{array}\right] \tag{2.21}$$

The matrix M_G^∞ is called **Smith–MacMillan form of $G(s)$ at infinity.** □

The structure at infinity of $G(s)$ is given by the following result.

Theorem 2.4.3 *The structure at infinity of $G(s)$ may be determined form* (2.21) *as follows*

(i) If $q_i > 0$ then $G(s)$ has pole at infinity of order q_i

(ii) If $q_i < 0$ then $G(s)$ has zero at infinity of order q_i

The number $\delta_M^\infty(G(s)) = \sum_{i=1}^{p_\infty} q_i$, $q_i > 0$, p_∞ the number of q_i's with $q_i > 0$, is called the **MacMillan degree of $G(s)$ at infinity.** □

Note that in theorem 2.4.2 the matrices transforming $G(s)$ to the Smith–MacMillan form at infinity are biproper i.e. units of the ring of the proper and stable rational matrices. The reason for the use of biproper matrices instead of unimodular is that unimodular matrices although do not have pole, zero structure at finite s, they may have pole, zero structure at infinity and thus, they introduce pole–zero cancellations at infinity. From the above we see that when we are interested in the finite structure of a rational matrix we use unimodular matrices while in the case of infinite structure we use biproper matrices. A transformation that leaves invariant both finite and infinite zero structure of a polynomial matrix is the *full equivalence transformation* introduced by Hayton, Pugh and Fretwell in [Hay., Pug., & Fre., 1988]. Full equivalence is defined below.

Definition 2.4.4 [Hay., Pug., & Fre., 1988] *Let $P_1(s)$ and $P_2(s)$ be polynomial matrices in $\Re^{m\times n}[s]$. These matrices are said to be fully equivalent if there exist polynomial matrices $M(s)$ and $N(s)$ such that*

$$[M(s)\ P_2(s)]\begin{bmatrix} P_1(s) \\ -N(s) \end{bmatrix} = 0 \tag{2.22}$$

and the composite matrices

$$[M(s)\ P_2(s)], \begin{bmatrix} P_1(s) \\ -N(s) \end{bmatrix} \tag{2.23}$$

such that

(i) they have full rank

(ii) they have neither finite nor infinite zeros

(iii) the following MacMillan degree conditions hold

$$\delta_M([M(s)\ P_2(s)]) = \delta_M(P_2(s)),\ \delta_M(\begin{bmatrix} P_1(s) \\ -N(s) \end{bmatrix}) = \delta_M(P_1(s)) \tag{2.24}$$

□

The relation of the zero structures of two fully equivalent matrices is given below.

Theorem 2.4.4 *If $P_1(s)$, $P_2(s) \in \Re^{m\times n}[s]$ are related by full equivalence, then they possess identical finite and infinite zero structure.* □

The notion of full equivalence was applied to system theory by introducing several notions of system equivalence [Pugh, Hay. & Fret., 1987], [Hay., Fret. & Pugh, 1986].

2.5 Minimal bases

In this section we consider the minimal bases of rational vector spaces. The notion of the minimal basis plays an important role in system theory. Realisation theory [Wol., 1974a], [For., 1975], [Kail., 1980], canonical forms [Pop., 1969], [Pop., 1972], design of minimal controllers [Sc. & And., 1978] are some of the many applications of the minimal basis in control and system theory.

We start our discussion by considering the rational matrix $\tilde{G}(s) \in \Re^{m\times\ell}(s)$. Without loss of generality we may assume that $\tilde{G}(s)$ has full column rank. Then, the columns of $\tilde{G}(s)$ span a rational vector space and they are a basis of this space. Clearly, we may always find a polynomial matrix $G(s)$ of full rank whose columns are a basis of col–span $\{\tilde{G}(s)\}$, by multiplying $\tilde{G}(s)$ with the least common multiple of the denominators of the entries of the latter matrix. Let $\mathcal{V}$ be the rational vector space spanned by the columns of $\tilde{G}(s)$. This vector space does not have a unique polynomial basis. Among the polynomial bases of $\mathcal{V}$ we distinguish a special class of bases, the minimal bases. The definition of a *minimal basis of* $\mathcal{V}$ is given below.

Definition 2.5.1 [For., 1975] *If $\mathcal{V}$ is an $\ell-$dimensional vector space of $m-$tuples the field $\Re(s)$, a* **minimal basis** *of $\mathcal{V}$ is an $m \times \ell$ polynomial matrix $G(s)$ such that the columns of $G(s)$ form a basis for $\mathcal{V}$ and $G(s)$ has least order among all polynomial bases of $\mathcal{V}$.* □

The following theorem gives a criterion for a polynomial matrix $G(s)$ of full rank to be a minimal basis of its column span.

Theorem 2.5.1 *Let $\mathcal{V}$ be an $\ell-$dimensional vector space over $\Re(s)$ and let $G(s)$ be a polynomial basis of $\mathcal{V}$. Then, $G(s)$ is a minimal basis of $\mathcal{V}$ if and only if*

(i) $G(s)$ does not have Smith zeros

(ii) $G(s)$ is column reduced □

Another important result stated in [For., 1975] is that if c_i are the (ordered) column degrees of a minimal basis of $\mathcal{V}$, then any other minimal basis of $\mathcal{V}$ has the same (ordered) column degrees. We have the following definition.

Definition 2.5.2 *The* **invariant indices** *of c_i of a vector space $\mathcal{V}$ are the column degrees (indices) of any minimal basis of $\mathcal{V}$. Its* **invariant dynamical order** *c is the sum of the c_i.* □

Notice that the invariant dynamical order c is often referred to as *Forney order* of the vector space $\mathcal{V}$.

In [For., 1975] it was shown that any polynomial basis $G'(s)$ of $\mathcal{V}$ may be reduced to a minimal basis. This procedure consists essentially in the extraction of the greatest common divisor of the polynomial rows of $G'(s)$ followed by a reduction to a column reduced polynomial matrix.

Another set of integers related to the minimal bases of $\mathcal{V}$ is the set of *pivot indices.* These are defined below.

Definition 2.5.3 [For., 1975] *Let $G(s)$ be a minimal basis with ordered indices $c_1 \leq c_2 \leq \ldots \leq c_\ell$. The* **pivot indices** *p_i of $G(s)$ are the degrees obtained by the following procedure: Let there be n_1 columns of index c_1. Find the first (lowest index) n_1 rows of the high order coefficient matrix G_{hc} of $G(s)$, such that the $n_1 \times n_1$ submatrix so defined is nonsingular. The indices of these rows are the first n_1 pivot indices of $G(s)$. Delete these n_1 rows and columns from $G(s)$ and repeat the above for all the distinct values of c_i.* □

The pivot indices have the following invariance property.

Lemma 2.5.1 *The pivot indices of all the minimal bases of a vector space $\mathcal{V}$ are the same.* □

Consider now two minimal bases $G_1(s)$ and $G_2(s)$ of $\mathcal{V}$. The following result shows how these minimal bases are related.

Theorem 2.5.2 [Wol., 1974a] *Let $G_1(s)$ and $G_2(s)$ be two ordered minimal bases of the same vector space $\mathcal{V}$. Let $c_1 \leq c_2 \leq \ldots \leq c_\ell$ be the column indices of $G_1(s)$ and $G_2(s)$. If $\gamma_1, \ldots, \gamma_\nu$ are the distinct values of the column indices, the above bases are related as follows:*

$$G_2(s) = G_1(s)U(s) \tag{2.25}$$

where

$$U(s) = \begin{bmatrix} U_1 & U_{12}(s) & U_{13}(s) & \cdots & U_{1\nu}(s) \\ 0 & U_2 & U_{23}(s) & \cdots & U_{2\nu}(s) \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & & & & U_{\nu-1,\nu}(s) \\ 0 & 0 & \cdots & 0 & U_\nu \end{bmatrix} \tag{2.26}$$

where $U_i \in \Re^{\gamma_i \times \gamma_i}$, $det\{U_i\} \neq 0$ and $U_{ij}(s) \in \Re^{\gamma_i \times \gamma_j}[s]$ and $deg\{U_{ij}(s)\} \leq c_j - c_i$. The matrix $U(s)$ is unimodular and is called **structured unimodular.** □

We continue with the definition of a special form of polynomial matrices.

Definition 2.5.4 [For., 1975], [Pop., 1969] *A minimal basis $G(s)$ is said to be in* **echelon form** *if*

(i) its indices are ordered $c_1 \leq c_2 \leq \ldots \leq c_\ell$

(ii) its entries $g_{p_i,i}$ are monic polynomials of degree c_i

(iii) for any i and j such that $c_i \leq c_j$ we have $deg\{g_{p_i,i}\} < c_i$. □

In [For., 1975], [Pop., 1969] it was shown that every minimal basis of $\mathcal{V}$ may be transformed to the echelon form by an appropriate post–multiplication by an unimodular matrix. The post–multiplication by unimodular matrices defined an equivalence relation. Forney in [For., 1975] and Popov in [Pop., 1969] showed that the echelon form defined above, is the canonical form related to this transformation.

The canonicity of the echelon form is of great importance in linear systems theory, since it leads to canonical forms of state–space systems [Pop., 1969], [Pop., 1972], [For., 1975].

2.6 Matrix pencils

In this section we consider a special type of polynomial matrices, the *matrix pencils* [Gant., 1959]. Matrix pencils are polynomial matrices of degree one, i.e. they have the form $sF-G$ where F, G are real matrices. The role of matrix pencils in system theory is important, since they are directly related to first order differential systems [Kar., 1979].

The notion of strict equivalence for matrix pencils is defined below.

Definition 2.6.1 *Two matrix pencils sF_1-G_1 and sF_2-G_2 are called* **strictly equivalent** *if there exist constant nonsingular matrices P, Q such that*

$$sF_2 - G_2 = P(sF_1 - G_1)Q \tag{2.27}$$

□

If the pencil $sF - G$ is square and $\det\{sF - G\} \not\equiv 0$then the pencil is called *regular*, otherwise it is called *singular.* If $sF - \hat{s}G$ id the homogeneous pencil obtained from $sF - G$ [Gant., 1959] and $f_i(s,\hat{s})$, $i = 1,\cdots,r$, r =rank$\{sF - G\}$, are the homogeneous invariant polynomials (obtained by reduction to Smith form), then elementary divisors (e.d.) of the type $\hat{s}^q$ are referred to as *infinite elementary divisors* (i.e.d) and those of the type $(s-\alpha\hat{s})^p)$ as *finite elementary divisors* (f.e.d). If the pencil is singular, at least one of the following equations has a solution for polynomial vectors $x(s)$, $y^T(s)$

$$(sF - G)x(s) = 0 \text{ and/or } y(s)^T(sF - G) = 0^T \tag{2.28}$$

If $[x_1(s),\cdots,x_\mu(s)]$ and $[y_1^T(s),\cdots,y_v^T]^T$ are minimal polynomial bases for the right and left null space of $sF - G$ respectively and ε_i, $i = 1,\cdots,\mu$, η_j, $j = 1,\cdots,v$ denote the corresponding degrees, then ε_i are known as *column minimal indices* (c.m.i.) and η_j as *row minimal indices* (r.m.i.) of the pencil. The sets of f.e.d., i.e.d., c.m.i and r.m.i uniquely characterises the strict equivalence class of $sF - G$ and there exists a canonical form obtained by some appropriate transformation pair (P,Q) and defined by $P(sF - G)Q = sF_K - G_K$ where

$$sF_K - G_K = \left[\begin{array}{c|cccc} 0_{q,h} & & & & \\ \hline & L_\eta(s) & & & \\ & & L_\varepsilon(s) & & \\ & & & sH - I & \\ & & & & sI - J \end{array}\right] \tag{2.29}$$

where $0_{g,h}$ is a zero block defined by the g r.m.i., h zero c.m.i., $L_\varepsilon(s)$, $L_\eta(s)$ are blocks associated with nonzero c.m.i. and r.m.i. respectively, $sH - I$ a block associated with the i.e.d. and $sI - J$ a block associated with the f.e.d.. The structure of these blocks is defined below.

$$L_\epsilon(s) = \text{block–diag}[\cdots, L_{\epsilon_i}(s), \cdots], \quad L_{\epsilon_i}(s) = s[I_{\epsilon_i}, \underline{0}] - [\underline{0}, I_{\epsilon_i}] \tag{2.30}$$

$$L_\eta(s) = \text{block–diag}[\cdots, L_{\eta_i}(s), \cdots], \quad L_{\eta_i}(s) = s\left[\begin{array}{c} I_{\eta_i} \\ \hline \underline{0}^T \end{array}\right] - \left[\begin{array}{c} \underline{0}^T \\ \hline I_{\eta_i} \end{array}\right] \tag{2.31}$$

$$sH - I = \text{block–diag}[\cdots, sH_{q_i} - I_{q_i}, \cdots], \quad H_{q_i} = \left[\begin{array}{c|c} 0 & I_{q_i-1} \\ \hline 0 & 0 \end{array}\right] \tag{2.32}$$

$$sI - J = \text{block–diag}[\cdots, sI_{p_i} - J_{p_i}(\alpha), \cdots], \quad J_{p_i}(\alpha) = \alpha I_{p_i} - H_{p_i} \tag{2.33}$$

The above canonical form is called *Kronecker canonical form* of $sF - G$. In the case where the pencil is regular it is characterised only by i.e.d. and f.e.d. and the canonical form has only the blocks $sI - J$ and $sH - I$. In this case the canonical form is called *Weierstrass canonical form.* The computation of the canonical form is a quite involved procedure. In [Gant., 1959] an algorithm for the reduction of a given pencil to the canonical form is given. This algorithm is rather tutorial and it is difficult to be applied in practice. Van Dooren in [VanD., 1979] has proposed stable numerical algorithms for the determination of the Kronecker canonical form.

2.7 Conclusions

In this chapter the basic theory of polynomial, rational matrices and matrix pencils has been discussed briefly. The basic definitions and results related to linear systems theory have been reviewed. In the present thesis, the main tools for the treatment of the several problems are polynomial matrices and matrix pencils. The specialised results and properties about polynomial matrices and matrix pencils are considered in the appropriate chapters.

Chapter 3

A SURVEY ON IMPLICIT LINEAR SYSTEMS

3.1 Introduction

The purpose of this chapter is to provide an overview of the implicit systems theory, the need of such representations, and the approaches adopted from several researchers for the study of the most general form of linear systems.

In the classical system theory we consider systems where the inputs and the outputs are related by an operator $G(s)$, the transfer function. This means that the output $y(s)$ comes as a result of the processing of the input $u(s)$ by the system described by $G(s)$ and $y(s)$ may be described in an explicit way in terms of $u(s)$.

In the theory of implicit systems a different point of view is followed. The system is not considered as a processor of $u(t)$ but it is viewed as a set of constraints on $y(t)$ and $u(t)$. Thus, the pair $u(t)$, $y(t)$ must be such that they satisfy an equation of the form $N(\sigma)u(t) + D(\sigma)y(t) = 0$ where no assumption for invertibility of any of the matrices $N(\sigma)$ and $D(\sigma)$ is made. Clearly, this approach is closer to the theory of linear differential equations, than the transfer function approach. A characteristic of the above approach is that in general we do not treat the inputs and outputs separately but we consider the overall vector $w(t) = [u^T(t), y^T(t)]^T$. This vector is called external behaviour vector. Thus, in the implicit systems theory the distinction between inputs and outputs is not necessary since all external signals are considered in a unified way. For this reason implicit systems are of *non oriented* nature.

The simplest type of implicit systems is the so called regular singular systems i.e. systems described by state equations of the type $E\dot{x} = Ax + Bu$, $y = Cx$, $\det E = 0$ where the matrix pencil $sE - A$ is regular. This type of system possess a transfer function, but has a behaviour different than that of the classical state–space systems. The behaviour of this system depends on the initial conditions of the state vector and in general it is impulsive unless the initial conditions satisfy certain conditions. The transfer function of the singular systems is, in general, a nonproper transfer function.

In the classical theory two systems are said to be transfer equivalent if they have the same transfer function. In the theory of implicit systems the term transfer equivalence is replaced by the term *external equivalence.* There are several definitions of external equivalence in the literature. The issue of external equivalence and the existing definitions is discussed in this Chapter.

The state–space models in classical system theory were introduced in order to allow the study of the internal mechanism of the system and, in some cases, they are very useful for the solution of standard control problems. On the other hand when one has a state–space model in hand, may use entirely numerical methods instead of polynomial methods required when transfer function is used. The relationship between state–space models and transfer function of PMD models was extensively studied during the last

25 years. When we deal with implicit systems we may use models involving internal variables called states, or auxiliary variables or latent variables. These models are obtained from the external descriptions of the system in a way similar to the realisation theory of classical systems and they are models of first order differential equations on the internal variables having the external variables vector as the "exciting" variable.

Since internal descriptions of implicit systems are introduced, it is expected to have the definitions of the notions of observability and controllability considered. These notions were introduced in different ways by several authors. According to Willems' theory [Wil., 1989], [Wil., 1991], controllability is not dependent on the system description but it is an intrinsic property of the system. This aspect is justified from the consideration that the system models are produced by the observation of the external signals related to the system i.e. the system is an entity imposing constraints on the space of signals. Other researchers define controllability in a way entirely conformable to the classical definition for standard state–space systems.

The issue of representations and transformations, as well as the minimality of the representations under several definitions of system equivalence is a topic that has received much attention [Kui. & Sch., 1991], [Wil., 1986], [Wil., 1983], [Ros., 1970], [Pugh, Hay. & Fret., 1987], [Ver., Lev. & Kail., 1981]. The problem transformation between the equivalent representations is important since a type of representations may be useful for the treatment of one problem but there is need for another type of representation to tackle a different problem. The several types of representations and transformations are discussed in this Chapter.

With the introduction of first order internal models, the geometric theory of implicit systems emerged. As in the case of state–space systems the fundamental subspaces were defined and geometric characterisations of the properties of implicit systems were given. The approaches adopted here are either pure geometric in the fashion of Wonham's theory for state–space systems or a mixture of algebraic and geometric methods that is made possible by using the matrix pencil theory as the major tool. The matrix pencil approach is very convenient when we deal with first order differential equations because the dynamic behaviour of the variables may be characterised and studied in algebraic terms such as generalised eigenvalues–eigenvectors, finite and infinite elementary divisors, minimal indices of rational vector spaces etc..

Implicit descriptions are not of theoretical interest only. Many problems encountered in practice may be formulated as implicit systems problems and treated in a convenient way. Implicit equations may be used for modeling of circuits, economic phenomena, large scale systems and systems with differentiators such as PID controllers If we consider internal descriptions of implicit systems (e.g. descriptor models) it may be seen that we have to deal with systems governed by a set of dynamical equations where the variables are constrained by algebraic constraints (nondynamic equations). This is a very common situation in practical systems.

In the study of large scale systems, descriptor equations may be used when a system is strongly coupled and it is simplified by the singular perturbations method. On the other hand, even in the case where a system may be modeled as a standard state–space system, it is sometimes more convenient to consider the state equation $E\dot{x} = Ax + Bu$, where E is invertible instead of producing a standard state–space model because this way we avoid matrix inversions, which is not recommended for ill conditioned problems.

Other cases where we may end up with implicit equations are the cases where our models arise as a result of linearisation procedures.

3.2 Representations of linear systems

The problem of representations and transformations between equivalent representations is important in control and system theory for two reasons: First, one representation may be more preferable than another mathematically equivalent one because the problem under study may be handled more easily using that representation; furthermore, even if a problem is solved theoretically by using one representation, there may be another equivalent representation, which is more convenient for computations. Second, when we are modeling a system, we are usually led to a set of differential equations which usually describe the external behaviour of the system; it may be desirable to have the description of the system in another form and thus we need a way of transforming the system representation to an equivalent representation of different type.

The most common way of describing a linear system, is the description by transfer functions. The systems that can be described by transfer functions are usually obtained from differential equations of the type

$$N(\sigma)u(t) = D(\sigma)y(t) \tag{3.1}$$

where σ denotes the derivative operator $\sigma = \frac{d}{dt}$ and $N(s)$ and $D(s)$ are polynomial matrices with $\det D(s) \neq 0$. Then, the relation between the input $u(t)$ and the output $y(t)$ may be expressed in the frequency domain (where s is the Laplace variable) as

$$y(s) = D^{-1}(s)N(s)u(s) = G(s)u(s) \tag{3.2}$$

and $G(s)$ is the transfer function of the system. In the case where the transfer function is proper, we may find a state–space system

$$\dot{x} = Ax + Bu, \; y = Cx + Du \tag{3.3}$$

which has as transfer function the matrix $G(s)$ defined in (3.2) and $G(s)$ is given in terms of (3.3) as

$$G(s) = C(sI - A)^{-1}B + D \tag{3.4}$$

In the case where $G(s)$ is nonproper, the system described by the equations

$$E\dot{x} = Ax + Bu, \; y = Cx + Du, \; \det\{E\} = 0 \tag{3.5}$$

has $G(s)$ as transfer function and $G(s) = C(sE - A)^{-1}B + D$. The transformation of the system (3.2) to the form (3.3) or (3.4) is called realisation of the transfer function $G(s)$. For proper systems there are many methods developed for the realisation of $G(s)$ [Wol., 1974], [Kalm., 1963], [Kail., 1980], [Ho & Kal., 1966]. For the case of nonproper transfer functions these methods are either based on those of the proper case, or are entirely independent. A new method for realisation in the nonproper case is proposed in Chapter 4 of this thesis, which also generalised a previous classical result.

Although the transfer function description is very convenient for applications of control, it has the disadvantage that it is not always equivalent to the set of differential equations of (3.1) describing the modeled system.

Another description which was proven to be very important in system theory is the polynomial matrix description (PMD) which was introduced by Rosenbrock [Ros., 1970] and is described below

$$T(s)\xi = U(s)u \tag{3.6}$$

$$y = V(s)\xi + W(s)u \tag{3.7}$$

This representation involves the inputs $u(s)$, the outputs $y(s)$ and the auxiliary variables $\xi(s)$. It was shown that the above representation may be transformed to the transfer function representation

$$G(s) = V(s)T^{-1}V(s) + W(s) \tag{3.8}$$

Note that the above may be readily reduced to the matrix fraction description $G(s) = D^{-1}(s)N(s)$. Rosenbrock showed that (3.6), (3.7) may be transformed to the state–space description (3.3) by means of *strict system equivalence* transformations. He has also shown that in the case where $G(s)$ is nonproper then (3.7), (3.8) may be transformed to a model of the type (3.5) with $D = 0$; however this problem had a more refined treatment by Verghese et al. in [Ver., Lev. & Kail., 1981].

Representations of the type (3.5) are important in linear systems theory since they provide the means for the description of a much wider range of systems than the standard state–space representations. This type of representation is called descriptor form representation. The introduction of the descriptor representations has motivated the study of the behaviour of the systems at infinity and in particular the definition of the poles and zeros at infinity, as well as the notions of reachability and observability at infinity [Ver., Lev. & Kail., 1981], [Cobb, 1984] etc..

The representation that fits the implicit systems framework is the external form of description proposed by Willems [Wil., 1979], [Wil., 1983], [Wil., 1991]. This type of description comes directly from the differential equation descriptions of the dynamical systems. According to this approach the dynamical system is defined as the family of the solutions $w(t)$ satisfying differential equations of the types

$$T(\sigma)w(t) = 0 \tag{3.9}$$

or

$$P(\sigma)\xi = 0,\ w(t) = Q(\sigma)\xi \tag{3.10}$$

or

$$P(\sigma)\xi = Q(\sigma)w(t) \tag{3.11}$$

Note that in the above system representations there are no inputs and outputs but instead we have the external signals $w(t)$. The set of vector functions $w(t)$ which satisfy the above equations is called the *external behaviour* of the system. As is the case of the transfer function descriptions we may obtain equivalent (in a proper sense of equivalence) descriptions of the following forms [Kuij. & Sch., 1990], [Kuij., 1992], [Bon., 1991]

$$E\dot{x} = Ax + Bu,\ y = Cx + Du \tag{3.12}$$

and

$$F\dot{\xi} = G\xi,\ H\xi = w \tag{3.13}$$

The first of the above is the descriptor form representation, where the matrices E, A are not necessarily square. Note that in (3.12) we have inputs and outputs. The role of these signals is not necessarily interpreted as in the case of transfer function descriptions where the output is an explicit function of the input. In the case of the above representation $y(t)$ is a subset of the external variables $w(t)$ named as outputs and $u(t)$ is a subset of $w(t)$ named as inputs. The external variables vector is $w(t) = [u^T(t), y^T(t)]^T$. In the present thesis we propose an alternative to the existing method for transforming representations of the type (3.9) to the forms (3.12) and (3.13).

Another type of representation where no distinction between inputs and outputs is made is the representation of the form

$$E\dot{x} = Fx + Gw \tag{3.14}$$

where $x(t)$ is the vector of internal variables and $w(t)$ is the behaviour. Willems in [Wil., 1991], [Wil., 1983] considered this type of representation and showed that behavioural equations of the form (3.9), (3.10) or (3.11), may take a first order realisation of this type. Representations of the type (3.14) have been considered and by Aplevich [Apl., 1985],; however, that approach is different to that of Willems as it will be shown in the following section where the several notions of equivalence of representations are discussed.

3.3 Equivalence of representations

The topic of this section is a brief presentation of the several notions of equivalence defined on the different representations of a system. Of course the notion of equivalence depends on the interpretation we give to the term "system". If a system is identified only by its transfer function, then a controllable and an uncontrollable state–space realisation of this transfer function are equivalent. If we are interested on the overall external behaviour of this system, then the above state–space representations are not equivalent.

Rosenbrock [Ros., 1970] considered models of the type (3.6), (3.7) and defined the strict system equivalence by using the system matrix defined below.

Definition 3.3.1 [Ros., 1970] *Consider the system described by* (3.6), (3.7). *Then, the system matrix $P(s)$ of this system is defined as follows*

$$P(s) = \begin{bmatrix} T(s) & U(s) \\ -V(s) & W(s) \end{bmatrix} \tag{3.15}$$

□

Then the definition of the strict system equivalence of two systems with system matrices $P_1(s)$ and $P_2(s)$ is the following.

Definition 3.3.2 *Consider two systems with system matrices $P_1(s)$ and $P_2(s)$. These systems are called strictly equivalent if*

$$\begin{bmatrix} M(s) & 0 \\ X(s) & I \end{bmatrix} \begin{bmatrix} T_1(s) & U_1(s) \\ -V_1(s) & W_1(s) \end{bmatrix} \begin{bmatrix} N(s) & Y(s) \\ 0 & I \end{bmatrix} = \begin{bmatrix} T_2(s) & U_2(s) \\ -V_2(s) & W_2(s) \end{bmatrix} \tag{3.16}$$

where M(s) and N(s) are unimodular matrices and X(s), Y(s) are polynomial matrices.

□

Note that strict system equivalence leaves invariant the zeros of the system (decoupling, invariant, system zeros) and the transfer function. Note that if the systems considered (3.16) are described by state–space equations then the definition of strict system equivalence coincides with that of similarity equivalence.

In the framework of descriptor systems Rosenbrock gave the definition of *restricted system equivalence* as follows [Ros., 1974b]:

Definition 3.3.3 *Two systems described by the descriptor equations* (3.5) *with feedthrough term $D = 0$, are called restricted system equivalent, if their associated system matrices are related as follows*

$$\begin{bmatrix} M & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} sE_1 - A_1 & -B_1 \\ C_1 & 0 \end{bmatrix} \begin{bmatrix} N & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} sE_2 - A_2 & -B_2 \\ C_2 & 0 \end{bmatrix} \tag{3.17}$$

where M, N are nonsingular constant matrices. □

Note that the systems considered above do not have matrix D and the nondynamic part (corresponding to D) is incorporated into the matrix pencils $sE_1 - A_1$ and $sE_2 - A_2$, in such a way, that the dynamic and nondynamic variables of the system are treated in the same way by the restricted equivalence transformations [Ver., Lev. & Kail., 1981]. As it was pointed out in this paper this is a drawback of the restricted system equivalence since two systems differing only in trivial nondynamic variables are not considered as equivalent under this type of equivalence. In order to overcome this drawback of Rosenbrock's strict system equivalence, Verghese and his co–workers extended the notions of equivalence in such way that two systems differing trivially may be considered as equivalent. This new type of equivalence was termed strong equivalence and was introduced as follows [Ver., Lev. & Kail., 1981].

Definition 3.3.4 *Consider two descriptor systems described by the equations* (3.5). *Then, if the corresponding system matrices are related as*

$$\begin{bmatrix} M & 0 \\ Q & I \end{bmatrix} \begin{bmatrix} sE_1 - A_1 & -B_1 \\ C_1 & D_1 \end{bmatrix} \begin{bmatrix} N & R \\ 0 & I \end{bmatrix} = \begin{bmatrix} sE_2 - A_2 & -B_2 \\ C_2 & D_2 \end{bmatrix} \tag{3.18}$$

where M, N are nonsingular constant matrices and

$$QE_1 = 0, \; E_1 R = 0 \tag{3.19}$$

the two descriptor systems are called **strong equivalent** *and the operations induced by* (3.18), (3.19) *are called* **operations of strong equivalence.** □

Note that strong equivalence operations allow operations of restricted system equivalence and, in addition, elimination or introduction of nondynamic variables provided that the constant feedthrough term does not change. This is guaranteed by conditions (3.19). The strong equivalence transformations in the work of Verghese were not given in a closed form but in the form of catalogue of elementary transformations. The difficulty in giving a closed form of the elementary operations arises from the fact that the strict equivalence transformations allow addition/deletion of trivial dynamics. The closed form of strong equivalence transformations was developed by Pugh, Hayton and Fretwell in [Pugh, Hay. & Fret., 1987]. In this paper the notion of *complete equivalence* of descriptor systems was introduced. Complete equivalence is defined as follows.

Definition 3.3.5 *Consider two descriptor systems of the form* (3.5). *Then, if the corresponding system matrices are related as*

$$\begin{bmatrix} M & 0 \\ X & I \end{bmatrix} \begin{bmatrix} sE_1 - A_1 & -B_1 \\ C_1 & D_1 \end{bmatrix} = \begin{bmatrix} sE_2 - A_2 & -B_2 \\ C_2 & D_2 \end{bmatrix} \begin{bmatrix} N & Y \\ 0 & I \end{bmatrix} \tag{3.20}$$

where the matrices

$$[sE_2 - A_2, M], \begin{bmatrix} -N \\ sE_1 - A_1 \end{bmatrix} \tag{3.21}$$

have neither finite nor infinite zeros, the systems are called **completely equivalent.**

□

The following result shows that complete equivalence provides a closed form of strong equivalence transformations.

Lemma 3.3.1 [Pugh, Hay. & Fret., 1987] *Two system matrices $P_1(s)$ and $P_2(s)$ corresponding to descriptor systems are completely system equivalent, if and only if they can be obtained from each other by strong equivalence transformations.* □

The relation of equivalence and the solutions of the descriptor equations of completely equivalent systems were explored in [Hay., Fret. & Pugh, 1986]. In this paper the notion of *fundamental equivalence* was introduced and defined as follows.

Definition 3.3.6 *Consider two descriptor systems with system matrices $P_1(s)$ and $P_2(s)$ respectively. These systems are fundamentally equivalent if there exist* *(i) a constant, injective map*

$$\begin{bmatrix} x_2(s) \\ u_2(s) \end{bmatrix} = \begin{bmatrix} N & Y \\ 0 & I \end{bmatrix} \begin{bmatrix} x_1(s) \\ -u(s) \end{bmatrix} \tag{3.22}$$

(ii) a constant, surjective map

$$\begin{bmatrix} E_2 x_2(0-) \\ -y(s) \end{bmatrix} = \begin{bmatrix} M & 0 \\ X & I \end{bmatrix} \begin{bmatrix} E_1 x_1(0-) \\ -y(s) \end{bmatrix}, \ XE_1 = 0 \tag{3.23}$$

□

In this paper it was shown that two systems are fundamentally equivalent, if and only if they are completely equivalent. From the definition of fundamental equivalence it is clear that this transformation preserves the input–output behaviour of the system. This is important in the framework of implicit systems, since we are not interested only in transfer equivalence but in external equivalence which will be defined later in this section. The notion of fundamental equivalence was extended to the case of systems described by PMD's in [And., Cop. & Cul., 1985], [Hay., Wal., & Pug., 1990], [Pugh, Kar., Var. & Hay., 1994].

We move now to the framework referred to as behaviour of systems [Wil., 1983], [Wil., 1991]. In this framework the system is identified by its external behaviour $w(t)$ as it was mentioned in the previous section. Here we have the notion of *external equivalence* which is defined as follows.

Definition 3.3.7 [Wil., 1983] *Two representations are called* **externally equivalent,** *if and only if they induce the same external behaviour.* □

In the case of autoregressive representations we have the following criterion for external equivalence [Wil., 1991], [Wil., 1983].

Proposition 3.3.1 *Consider the autoregressive descriptions*

$$R_1(\sigma)w(t) = 0 \text{ and } R_2(\sigma)w(t) = 0$$

where $R_1(s)$, $R_2(s)$ are polynomial matrices of full row rank and have the same dimensions. Then the above systems are externally equivalent, if and only if

$$R_1(s) = U(s)R_2(s) \tag{3.24}$$

where $U(s)$ is unimodular matrix. □

If the representation of the system is of the type $\Sigma(P,Q) : P(\sigma)\xi = 0,\ Q(\sigma)\xi = w$, then we have the following [Sch., 1988], [Wil., 1979]:

Proposition 3.3.2 *Two behavioural systems $\Sigma(P_1, Q_1)$ and $\Sigma(P_2, Q_2)$ are externally equivalent if*

$$Q_1(\sigma)\mathrm{Ker}\{P_1(\sigma)\} = Q_2(\sigma)\mathrm{Ker}\{P_2(\sigma)\} \tag{3.25}$$

□

Note that Q_i, P_i are considered as differential operators and therefore $\mathrm{Ker}\{P_i(\sigma)\}$ is not interpreted as a rational vector space, but it is the set of functions $f(t)$ satisfying $P_i(\sigma)f(t) = 0$. In the framework of external equivalence the system is associated with a set of solutions of differential equations. In [Apl., 1985] first order representations of the type (3.14) were considered. The equivalence criterion in this approach is the following:

Definition 3.3.8 *Two systems represented by* (3.14) *are equivalent if $\mathcal{W}_1 = \mathcal{W}_2$ where*

$$\mathcal{W}_i = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \mathrm{Ker}\{P_i(s)\} \text{ and } P_i(s) = [sE_i - F_i, G_1] \tag{3.26}$$

and $\mathrm{Ker}\{P_i(s)\}$ *is interpreted as the rational vector space annihilating $P_i(s)$.* □

From the above definition we see that in this approach the system is associated with a rational vector space and not to the solution of a differential equation. The equivalence notion of the above definition was termed as external equivalence in [Apl., 1985]. In order to distinguish the external equivalence in the sense of Willems, from the above we give the following definition.

Definition 3.3.9 *If two systems are equivalent, in the sense of definition* 3.3.8, *then they will be called $\mathcal{A}$–externally equivalent.* □

In [Apl., 1991] the above type of equivalence is termed external equivalence. Here, we introduce the term $\mathcal{A}$–external equivalence in order to make the distinction from external equivalence in the sense of Willems.

It is easy to see that if we have system representations as in proposition 3.3.2 then the notion of $\mathcal{A}$–external equivalence may be extended as follows.

Definition 3.3.10 *Two systems of the type $\Sigma(P_1, Q_1)$, $\Sigma(P_2, Q_2)$ are $\mathcal{A}$-externally equivalent if*

$$Q_1(s)\mathrm{Ker}\{P_1(s)\} = Q_2(s)\mathrm{Ker}\{P_2(s)\} \tag{3.27}$$

where $Q_i(s)$, $P_i(s)$ are interpreted as matrices over the field of rational functions and thus $\mathrm{Ker}\{P_i(s)\}$ *is a rational vector space.* □

Note that external equivalence is stronger than $\mathcal{A}$–external equivalence in the sense that if two systems are externally equivalent, then they are $\mathcal{A}$–externally equivalent without the reverse being necessarily true. If we wish to identify the stronger notion of equivalence then it is clear that the strong equivalence notion is the stronger one.

By the introduction of the several types of equivalence, the problem of finding canonical forms under the eqivalence transformations corresponding to each type of equivalence emerges. In linear systems theory the most commonly used type of canonical form is the canonical form under similarity equivalence [Pop., 1972], [Dick., Kail. & Morf, 1974], [Den,. 1974], [Ros. & Hay.,1974], [Luen., 1967]. For the case of regular descriptor systems the problem of canonical forms has been considered under restricted system equivalence for systems without outputs in [Gl.-Luer. & Hin., 1987], [Hel. & Shay., 1989]. These forms are extensions to the controllable canonical forms for state–space systems. However, this problem remains open since a Popov type canonical form is not available yet. In the present thesis the problem of canonical forms under restricted system equivalence transformations is considered for systems with and without outputs. In the first case the problem is entirely solved, while in the latter, a solution for a special case of regular descriptor systems is obtained. There are other types of canonical forms under transformations different than the transformations corresponding to the types of equivalence presented in this section [Gl.-Luer., 1990], [Kar., & MacB, 1981], [Lois., Ozc., et al., 1991], [Leb. & Lois., 1994]. These canonical forms are not discussed here.

The transformations between several types of system representations is an important issue in system theory. The most classical type of such transformations is the realisation of a given transfer function either proper or nonproper mentioned earlier in this section. In the present dissertation the realisation in descriptor form under transfer and external equivalence are considered. We may consider transformations between any type of representations mentioned in this section under a given type of equivalence. A detailed review of system representations and transformations may be found in [Sch., 1989].

3.4 Descriptor systems

The descriptor type representation $E\dot{x} = Ax + Bu$, $y = Cx + Du$ is the main tool for the analysis of implicit systems when they are represented by first order representations. Originally, the use of descriptor type models, was motivated from the observation that state–space models were not able to describe some types of systems such as economic models [Luen., 1978] composite systems [Ros. & Pugh, 1974], electrical circuits [Lew., 1986] etc.. The reason for this is that some systems are described by a combination of dynamic and algebraic equations (algebraic constraints of the state vector).

On the other hand the family of nonproper systems is a typical example of descriptor models of the first order differential representation type. The first class of descriptor systems which was considered was that of the regular type i.e.

$$\det\{sE - A\} \neq 0 \tag{3.28}$$

For such systems a rich theory has been developed. The most important state–space system properties were extended to the case of singular systems and some basic results about this type of systems are given below.

The main difference between state–space and descriptor systems is that the latter may have "behaviour at infinity". Indeed, if the pencil $sE - A$ is decomposed to its Weierstrass canonical form, we have the equivalent descriptor representation (after Laplace transform) [Cobb, 1984], [Ver., Lev. & Kail., 1981], [Ver., V.-D. & Kail., 1979], [Lew., 1986].

$$\begin{bmatrix} sI - A & 0 \\ 0 & sJ - I \end{bmatrix} \begin{bmatrix} x_s(t) \\ x_f(t) \end{bmatrix} = \begin{bmatrix} B_s \\ B_f \end{bmatrix} u(s) \tag{3.29}$$

$$y(s) = [C_s\ C_f]\,[x_s(t)x_f(t)] \tag{3.30}$$

We readily see that the descriptor system is decomposed into two subsystems the "slow" [Cobb, 1984] which is a state–space system and the "fast" which is a system of descriptor type where $E = J$ and J is a Jordan matrix with all its eigenvalues equal to zero. The fast part corresponds to a system with polynomial transfer function i.e. it may be interpreted as a chain of differentiations. The type of the unforced solution of $x(s)$ of the descriptor equations is dependent on the initial conditions since

$$x(s) = (sE - A)^{-1}Ex(0-) \tag{3.31}$$

If we take the inverse Laplace transform of the above expression it is easy to see [Lew., 1986] that if $Ex(0-) \neq 0$ then the time response $x(t)$ is impulsive.

Thus in order to have smooth solutions we must have initial conditions satisfying

$$x(0) \in \text{Ker}\{E\} \tag{3.32}$$

Initial conditions satisfying the above, are called *admissible initial conditions for zero output* [Lew., 1986]. If we allow the use of the inputs to eliminate the impulsive behaviour of the state, we have [Lew., 1986] that the set of the initial conditions not resulting in impulsive behaviour is the maximal (A, E, B)–invariant subspace defined by

$$\mathcal{V}^* = \sup\{\mathcal{V} \subset \Re^n | A\mathcal{V} \subset E\mathcal{V} + \mathcal{B}\} \tag{3.33}$$

where $\mathcal{B}$ denotes the image of B. The subspace $\mathcal{V}^*$ may be computed by the following recursive algorithm [Ozc., 1985]

$$\mathcal{V}_{k+1} = A^{-1}(E\mathcal{V}_k + \mathcal{B}) \tag{3.34}$$

Two notions playing a key role in linear systems theory are the notions of controllability and observability. In the case of descriptor systems reachability is also an important property; note that in the case of continuous time state–space systems, reachability and controllability coincide.

Definition 3.4.1 [Lew., 1986] *A point x_r in the state–space is called* **reachable,** *if there exists input $u(t)$ such that the state vector $x(t)$ is driven from $x(0) = 0$ to x_r in a finite time interval and the state trajectory $x(t)$ is continuously differentiable.* □

Definition 3.4.2 [Lew., 1986] *A point x_c in the state–space is called* **controllable,** *if there exists control input $u(t)$ such that if $x(0) = x_c$, the state may be driven to the origin in finite time and the trajectory $x(t)$ is continuously differentiable.* □

In the case of state–space systems the criteria for the controllability are well known and they are related to the structure of the controllability pencil $[sI - A, -B]$. For the case of descriptor systems we have the following criterion [Ver., Lev. & Kail., 1981].

Theorem 3.4.1 *The descriptor system $E\dot{x} = Ax + Bu$ is controllable, if the following conditions hold*

(i) The pencil $[sE - A, -B]$ does not have finite Smith zeros

(ii) $\mathrm{Im}\{E\}+\mathrm{Im}\{B\} + A\mathrm{Ker}\{E\} = \mathcal{X}_c$

where $\mathcal{X}_c$ is the codomain of E, A. □

Theorem 3.4.2 [Cobb, 1984] *The descriptor system $E\dot{x} = Ax + Bu$ is reachable, if the following conditions hold*

(i) The pencil $[sE - A, -B]$ does not have finite Smith zeros

(ii) $\mathrm{Im}\{E\}+\mathrm{Im}\{B\} = \mathcal{X}_c$ □

Note that in the above theorems the codomain of E is used. This is, in order to include the nonregular descriptor systems i.e. systems where $sE - A$ is singular.

The criteria for reachability and controllability differ only in conditions *(ii)* in the above theorems. These conditions are related to the behaviour of the system at infinity. In the case of state–space systems where $E = I$ the notions of controllability and reachability coincide.

The dual results for observability are the following:

Theorem 3.4.3 [Ver., Lev. & Kail., 1981] *The descriptor system $E\dot{x} = Ax$, $y = Cx$ is called* **observable in the sense of Verghese,** *if the following conditions hold*

(i) The pencil $\begin{bmatrix} sE - A \\ C \end{bmatrix}$ *does not have finite Smith zeros*

(ii) $\mathrm{Ker}\{E\} \cap \mathrm{Ker}\{C\} \cap A^{-1}\mathrm{Im}\,\{E\} = \{0\}$ □

Theorem 3.4.4 [Sch., 1989] *The descriptor system $E\dot{x} = Ax$, $y = Cx$ is called* **observable in the sense of Rosenbrock,** *if*

(i) The pencil $\begin{bmatrix} sE - A \\ C \end{bmatrix}$ *does not have finite Smith zeros*

(ii) $\mathrm{Ker}\{E\} \cap \mathrm{Ker}\{C\} = \{0\}$ □

The reachability and observability properties of the descriptor systems may be expressed in geometric terms. This approach has been developed in [Ozc., Lew., 1989], [Ozc., 1986], [Ozc., 1985], [Mal., 1987], [Mal., 1989], [Lew., 1986], [Lew., 1982].

3.5 Minimality of implicit systems

In the classical theory of state–space systems a system is said to be minimal when it has the least possible dimension among the systems giving rise to the same transfer function. From this definition we see that minimality is defined in the context of the type of equivalence we consider. In this section we will discuss the different definitions of minimality in the context of alternative types of equivalence.

In the case of first order implicit representations, minimality is considered with respect to three numbers: The dimension of the costate–space (the number of the equations), the dimension of the state–space and the rank defect of the matrix that multiplies the vector of the derivatives of the states. For the classical case of regular state–space systems we have the standard result:

Theorem 3.5.1 *A state–space system $\dot{x} = Ax + Bu$, $y = Cx + Du$ is minimal, if*

(i) $[sI - A, -B]$ *does not have Smith zeros*

(ii) $\begin{bmatrix} sI - A \\ C \end{bmatrix}$ *does not have Smith zeros* □

The minimality in the above theorem is directly related to the joint controllability and observability of the system. The following theorem gives the relationship of minimal state–space systems having the same transfer function:

Theorem 3.5.2 [Kail., 1980] *Two minimal state- space systems have the same transfer function if and only if they are related by similarity transformations.* □

When, instead of transfer equivalence we consider external equivalence we have the following definition of the minimality of a state–space system [Wil., 1991].

Theorem 3.5.3 [Wil., 1983] *A state-space system* $\dot{x} = Ax + Bu$, $y = Cx + Du$ *is minimal under external equivalence, if and only if the pencil*

$$\begin{bmatrix} sI - A \\ C \end{bmatrix}$$

does not have Smith zeros. □

In the above theorem we see that controllability is not required for the minimality. The reason for this, is that the controllability properties of the system are invariant under external equivalence [Wil., 1991], [Wil., 1983]. Willems in [Wil., 1983] proved the following:

Theorem 3.5.4 *Two minimal state–space systems are externally equivalent if and only if they are related by similarity transformations.* □

In [Ver., Lev. & Kail., 1981] criteria for the minimality of a singular system $E\dot{x} = Ax + Bu$, $y = Cx + Du$ were given by the following result:

Theorem 3.5.5 *A singular system* $E\dot{x} = Ax + Bu$, $y = Cx + Du$ *is minimal, if and only if the following conditions hold*

(i) $[sE - A, -B]$ *has no finite zeros*

(ii) $\begin{bmatrix} sE - A \\ C \end{bmatrix}$ *has no finite zeros*

(iii) $\mathrm{Im}\{E\} + \mathrm{Im}\{B\} + A\mathrm{Ker}\{E\} = \mathcal{X}_c$

(iv) $\mathrm{Ker}\{E\} \cap \mathrm{Ker}\{C\} \cap A^{-1}\mathrm{Im}\{E\}$

□

The above criterion for minimality was developed under transfer equivalence requirements, i.e. it was assumed that $sE - A$ is invertible and two systems are equivalent if they have the same transfer function. For completeness, it is mentioned that minimality was referred to as irreducibility in that paper [Ver., Lev. & Kail., 1981]. This definition of minimality corresponds to joint observability and controllability, as they are defined in theorem 3.4.3 and theorem 3.4.1. These definitions allow the pencils $[sE - A, -B]$ and $[sE^T - A^T, C^T]^T$ to have linear infinite elementary divisors.

A theorem analogous to theorem (3.5.2) for the case of strong equivalence of descriptor systems is the following:

Theorem 3.5.6 [Ver., Lev. & Kail., 1981] *Two minimal (in the sense of Verghese) descriptor systems are strongly equivalent if and only if they have the same transfer function.* □

If we consider reachability and observability in the sense of Rosenbrock then we have the following [Ros., 1974b], [Sch., 1989]:

Theorem 3.5.7 *A descriptor system $E\dot{x} = Ax + Bu$, $y = Cx + Du$ is minimal under transfer function equivalence if and only if the following conditions hold:*

(i) $[sE - A, -B]$ *has no finite zeros*

(ii) $\begin{bmatrix} sE - A \\ C \end{bmatrix}$ *has no finite zeros*

(iii) $[E\ B]$ *has full row rank*

(iv) $\begin{bmatrix} E \\ C \end{bmatrix}$ *has full column rank* □

In [Grimm, 1988], the case where the matrices E, A of a descriptor representation are rectangular was considered and minimality was defined as follows:

Theorem 3.5.8 *A descriptor system $E\dot{x} = Ax + Bu$, $y = Cx + Du$ where E, A are not necessarily square, is minimal if*

(i) It is minimal in the sense of theorem 3.5.5

(ii) $A\mathrm{Ker}\{E\} \subseteq \mathrm{Im}E$ □

The second condition in the above theorem expresses the requirement that the state equation contains no nondynamic variables [Ver., Lev. & Kail., 1981]. Minimality in the above theorem was considered in the context of $\mathcal{A}$-external equivalence. When external equivalence is considered we have the following [Kui. & Sch., 1991] result.

Theorem 3.5.9 *The descriptor representation $E\dot{x} = Ax+Bu$, $y = Cx+Du$. This representation is minimal under external equivalence, if and only if the following conditions hold* *(i)* $[E\ B]$ *has full row rank*

(ii) $\begin{bmatrix} E \\ C \end{bmatrix}$ *has full column rank*

(iii) $A\mathrm{Ker}\{E\} \subseteq \mathrm{Im}\{E\}$

(iv) $\begin{bmatrix} sE - A \\ C \end{bmatrix}$ *does not have Smith zeros* □

The above result does not take into account the controllability of the system. This is expected, since the result is given in the context of external equivalence. Note, that nondynamic variables are not present.

The relation between two minimal (under external equivalence) descriptor systems having the same external behaviour is considered in the following theorem:

Theorem 3.5.10 [Kui. & Sch., 1991] *Two minimal (under external equivalence) descriptor systems have the same external behaviour if and only if they are related by strong equivalence transformations.* □

If we restrict ourselves to systems without feedthrough matrix D, then the nondynamic variables corresponding to this matrix have to be incorporated in the pencil $sE - A$. In this case, minimality under external equivalence may be inspected by the following criteria [Kuij., 1992].

Theorem 3.5.11 *The representation $E\dot{x} - Ax + Bu$, $y = Cx$ is minimal under external equivalence, if and only if the following conditions hold:*

(i) $[E\ B]$ *has full row rank*

(ii) $\begin{bmatrix} E \\ C \end{bmatrix}$ *has full column rank*

(ii) $\begin{bmatrix} sE - A \\ C \end{bmatrix}$ *does not have finite Smith zeros.* □

The only difference between the above and theorem 3.5.9 is the criterion for the absence of nondynamic variables.

The relationship between two minimal descriptor representations without feedthrough term, with the same behaviour is considered below:

Theorem 3.5.12 [Kuij., 1992] *Two minimal (under external equivalence) descriptor representations without feedthrough term are externally equivalent if and only if they are related by restricted system equivalence transformations.* □

We proceed now to the definitions of minimality for other representations of implicit systems. Consider first systems of the type [Wil., 1991]

$$E\dot{x} + Fx + Gw = 0 \tag{3.35}$$

This system is non oriented and $w(t)$ is the external behaviour vector; the criteria for the minimality of this system, are given below [Wil., 1991].

Theorem 3.5.13 *The system* (3.35) *is minimal under external equivalence, if and only if the following conditions hold*

(i) $\lambda E + \mu F$ *has full column* rank $\forall\ \lambda,\mu\ \in \mathcal{C}$, $|\lambda^2| + |\mu^2| \neq 0$

(ii) $\mathrm{Im}\{E\}\ \subseteq \mathrm{Im}[F, G]$

(iii) $\mathrm{Im}\{F\}\ \subseteq \mathrm{Im}[E, G]$ □

Willems in [Wil., 1991] considered the relationship of two minimal representations of the type $E\dot{x} + Fx + Gw = 0$ and provided the following:

Theorem 3.5.14 *Two minimal (under external equivalence) representations* $E_1\dot{x}_1 + F_1x_1 + G_1w = 0$ *and* $E_2\dot{x}_2 + F_2x_2 + G_2w = 0$ *are externally equivalent if and only if there exist invertible matrices* P *and* Q *such that*

$$E_2 = PE_1Q, \quad F_1 = PF_1Q, \quad G_2 = PG_1 \tag{3.36}$$

□

If instead of external equivalence we consider $\mathcal{A}$–external equivalence, the minimality of (3.35) is defined as follows [Apl., 1985].

Theorem 3.5.15 *The system* (3.35) *is minimal under* $\mathcal{A}$*-external equivalence, if and only if the following conditions hold*

(i) F *has full column rank*

(ii) $[F, G]$ *has full row rank*

(iii) $[E - \lambda F, G]$ *has full row rank* $\forall\ \lambda\ \in\ \mathcal{C}$

(iv) $E - \lambda F]$ *has full row rank* □

We close this section by giving the criteria for minimality of a pencil representation of the form $F\dot{\xi} = G\xi$, $w = H\xi$ under external equivalence [Kuij. & Sch., 1990].

Theorem 3.5.16 *A pencil representation $F\dot{\xi} = G\xi$, $w = H\xi$ is minimal under external equivalence, if and only if the following conditions hold*

(i) F has full row rank

(ii) $\begin{bmatrix} F \\ H \end{bmatrix}$ *has full column rank*

(iii) $\begin{bmatrix} sF - G \\ H \end{bmatrix}$ *does not have finite Smith zeros.* □

The following theorem relates two minimal pencil representations with the same external behaviour

Theorem 3.5.17 [Kui. & Sch., 1991] *Two minimal pencil representations have the same external behaviour if and only if there exist invertible matrices S and T such that*

$$\begin{bmatrix} S & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} sF_1 - G_1 \\ H_1 \end{bmatrix} = \begin{bmatrix} sF_2 - G_2 \\ H_2 \end{bmatrix} T \tag{3.37}$$

□

3.6 Geometric theory and implicit systems

Geometric theory for state–space systems was proven to be an elegant tool for analysis and design. In this approach linear systems described by state–space models are associated to fundamental subspaces expressed in terms of the system matrices A, B, C [Wonh., 1979], [Bas. & Mar., 1969] etc.. In this way the basic structural characteristics of the system may be translated into geometric terms and a classification of the systems according to these properties may be given. Several important problems in system theory have been solved by using geometric theory [Won. & Mor.,1970], [Wonh., 1979], [Wonh. & Mor., 1972]. With the introduction of implicit descriptor models, geometric theory was extended to this type of systems [Lew., 1986], [Lew., 1982], [Ozc., 1986], [Ozc., Lew., 1989], [Mal., 1989]. In this section we give some definitions of the fundamental spaces and algorithms of the geometric approach and we start from the classical results of geometric theory of state–space systems [Wonh., 1979], [Bas. & Mar., 1969], [Wil., 1981]. We restrict ourselves only to the geometric definitions of the several notions without discussing the dynamical aspects, which may be found in the references.

Definition 3.6.1 [Wonh., 1979] *A subspace $\mathcal{V} \subset \mathcal{X}$, ($\mathcal{X}$ being the state–space) is (A, B)–invariant if*

$$A\mathcal{V} \subset \mathcal{V} + \mathcal{B} \tag{3.38}$$

where $\mathcal{B} = Im\{B\}$. □

Given now a subspace $\mathcal{K} \subset \mathcal{X}$ we may define the maximal (A, B)–invariant subspace $\mathcal{V}^{max}$ contained in $\mathcal{K}$ as follows [Wonh., 1979]:

$$\mathcal{V}^{max} = \sup\{\mathcal{V} \subset \mathcal{K} | A\mathcal{V} \subset \mathcal{V} + \mathcal{B}\} \tag{3.39}$$

The notion of a maximal invariant subspace contained in another subspace of the state–space is useful for the solution of several problems such as disturbance decoupling [Wonh., 1979], input–output decoupling [Won. & Mor.,1970], model matching [Mor., 1973], [Mor., 1976], [Em. & Haut., 1980] etc.. A very important algorithm for the determination of $\mathcal{V}^{max}$ was given in [Won. & Mor.,1970], [Bas. & Mar., 1969].

Theorem 3.6.1 *Let $\mathcal{K} \subset \mathcal{X}$. Then $\mathcal{V}^{max}$ is given by the following nonincreasing algorithm:*

$$\mathcal{V}^0 = \mathcal{K} \tag{3.40}$$

$$\mathcal{V}^{\nu+1} = \mathcal{K} \cap A^{-1}(\mathcal{V}^{\nu} + \mathcal{B});\ \nu \geq 0 \tag{3.41}$$

The recursion stops when $\mathcal{V}^{\nu+1} = \mathcal{V}^{\nu}$. □

Definition 3.6.2 [Bas. & Mar., 1969] *A subspace $\mathcal{S} \subset \mathcal{X}$ is called (C, A)–invariant if*

$$A(\mathcal{S} \cap \mathcal{C}) \subseteq \mathcal{S} \text{ where } \mathcal{C} = \mathrm{Ker}C \tag{3.42}$$

□

Let $\mathcal{S}^{min}$ be the minimal (C, A)–invariant subspace containing a given subspace $\mathcal{K} \subset \mathcal{X}$ i.e.

$$\mathcal{S}^{min} = \inf\{\mathcal{S} \subset \mathcal{X} | \mathcal{S} \supseteq A(\mathcal{S} \cap \mathcal{C}), \text{ and } \mathcal{S} \supset \mathcal{K}\} \tag{3.43}$$

The dual of the recursive algorithm of theorem 3.6.1 is the following:

Theorem 3.6.2 *The minimal (C, A)–invariant space containing a given subspace $\mathcal{K} \subset \mathcal{X}$ is given by the following nondecreasing algorithm*

$$\mathcal{S}_0 = \mathcal{K} \tag{3.44}$$

$$\mathcal{S}^{\nu+1} = \mathcal{K} + A(\mathcal{S}^{\nu} \cap \mathcal{C}) \tag{3.45}$$

and the recursion stops when $\mathcal{S}^{\nu} = \mathcal{S}^{\nu+1}$. □

Another fundamental subspace associated with a state–space system is the controllability subspace. This is defined as follows [Wonh., 1979].

Definition 3.6.3 *A subspace $\mathcal{V} \subset \mathcal{X}$ is a controllability subspace of the pair (A, B) if there exists maps $F : \mathcal{X} \rightarrow \mathcal{U}$ ($\mathcal{U}$ is the input space) such that*

$$\mathcal{V} = \langle A + BF | \mathcal{B} \cap \mathcal{V} \rangle \tag{3.46}$$

□

As in the case of the invariant subspaces when there is a given subspace $\mathcal{K} \subset \mathcal{X}$ we may define the maximal controllability subspace $\mathcal{V}^{max}$ contained in $\mathcal{K}$ by using a recursive algorithm [Wonh., 1979]. This algorithm is given below.

Theorem 3.6.3 *Let $\mathcal{K} \subset \mathcal{X}$. Then $\mathcal{V}_c^{max}$ is given by the following nondecreasing algorithm:*

$$\mathcal{V}^0 = 0 \tag{3.47}$$

$$\mathcal{V}^{\nu+1} = \mathcal{V}^{max} \cap (A\mathcal{V}^{\nu} + \mathcal{B}) \tag{3.48}$$

where $\mathcal{V}^{max}$ is the maximal (A, B)–invariant subspace contained in $\mathcal{K}$. □

The almost invariant subspaces (almost controllability) were defined by Willems in [Wil., 1981]. The dynamical definitions of these subspaces are given in Chapter 9. Here we give only the algorithm for finding the maximal almost controllability subspace contained in a given subspace $\mathcal{K} \cap \mathcal{X}$.

Theorem 3.6.4 *Let $\mathcal{K} \subset \mathcal{X}$. Then the maximal almost controllability subspace $\mathcal{V}_a^{max}$ contained in $\mathcal{K}$ is given by the following nondecreasing algorithm.*

$$\mathcal{V}_a^0 = 0 \tag{3.49}$$

$$\mathcal{V}_a^{\nu+1} = \mathcal{K} \cap (A\mathcal{V}_a^{\nu} + \mathcal{B}) \tag{3.50}$$

□

In the context of descriptor systems we have the following definition of the (A, E, B)–invariant subspace [Ozc., 1986].

Definition 3.6.4 *A subspace $\mathcal{V} \subset \mathcal{X}$ is called (A, E, B)–invariant subspace if*

$$A\mathcal{V} \subset E\mathcal{V} + \mathcal{B} \tag{3.51}$$

□

In a analogous fashion to the state–space we have the maximal (A, E, B)–invariant subspace $\mathcal{V}^{max}$ contained in a given subspace $\mathcal{K}$.

$$\mathcal{V}^{max} = \sup\{\mathcal{V} \subset \mathcal{K} | A\mathcal{V} \subset E\mathcal{V} + \mathcal{B}\} \tag{3.52}$$

The algorithmic computation of this subspace is given by the following [Ozc., 1985] result.

Theorem 3.6.5 *Let $\mathcal{K} \subset \mathcal{X}$. Then the maximal (A, E, B)–invariant subspace contained in $\mathcal{K}$ is given as the limit of the following recursion of subspaces*

$$\mathcal{V}^0 = \mathcal{K} \tag{3.53}$$

$$\mathcal{V}^{\nu+1} = \mathcal{K} \cap A^{-1}(E\mathcal{V}^{\nu} + \mathcal{B}) \tag{3.54}$$

□

In the case of descriptor systems the notions of reachability and controllability do not coincide. First we give the geometric definitions of (A, E, B)–invariant reachability and controllability subspaces of a descriptor system [Ozc., Lew., 1989]. Let α be the index of nilpotency of E:

Definition 3.6.5 [Ozc., Lew., 1989] *A subspace $\mathcal{V}_R \subset \mathcal{X}$ is called a reachability subspace, if there exist matrices F and G such that $\mathcal{V}_R$ is the reachable subspace of the triple $(E, A + BF, BG)$.* □

Definition 3.6.6 [Ozc., Lew., 1989] *A subspace $\mathcal{V}_c \subset \mathcal{X}$ is called a controllability subspace if $\mathcal{V}_c = E\mathcal{V}_R$, for some reachability subspace $\mathcal{V}_R \subset \mathcal{X}$.* □

The analogous result to theorem 3.6.3 are the following [Mal., 1989].

Theorem 3.6.6 *Let $\mathcal{K} \subset \mathcal{X}$. Then the maximal reachability subspace $\mathcal{V}_R^{max}$ contained in $\mathcal{K}$ is given by the following subspace recursion*

$$\mathcal{V}_c^0 = \mathcal{V}^{max} \cap \text{Ker}\{E\} \tag{3.55}$$

$$\mathcal{V}_c^{\nu+1} = \mathcal{V}^{max} \cap E^{-1}(A\mathcal{V}_c^{\nu} + \mathcal{B}) \tag{3.56}$$

□

The following result concerns the almost controllability subspaces [Mal., 1987].

Theorem 3.6.7 *Let $\mathcal{K} \subset \mathcal{X}$. Then the maximal almost controllability subspace $\mathcal{V}_{ac}^{max}$ contained in $\mathcal{K}$ is given by the following subspace recursion*

$$\mathcal{V}_a^0 = \mathcal{K} \cap \operatorname{Ker}\{E\} \tag{3.57}$$

$$\mathcal{V}_a^{\nu+1} = \mathcal{K} \cap E^{-1}(A\mathcal{V}_a^{\nu} + \mathcal{B}) \tag{3.58}$$

□

Detailed analysis on the geometric theory for descriptor systems may be found in [Lew., 1986], [Lew., 1982], [Mal., 1989], [Ozc., Lew., 1989], [Ozc., 1986].

3.7 The matrix pencil approach

The matrix pencil approach to linear systems comes as an alternative to the classical geometric theory [Kar., 1979], [Jaf. & Kar., 1981], [Kar. & Kal., 1989]. According to this approach, the fundamental subspace associated to a linear system may be characterised in terms of Kronecker invariants of appropriately defined matrix pencils. The main tool of this approach is the feedback free description of the systems which is defined below:

Consider the descriptor system $E\dot{x} = Ax + Bu$. If $(N, B^{\dagger})$ is a pair of a left annihilator, inverse of $B(NB = 0$, $\operatorname{rank}\{N\} = n - \ell$, $B^{\dagger}B = I_{\ell})$ we may readily see [Kar., 1979] that the descriptor representation is equivalent to

$$NE\dot{x} = NAx \tag{3.59}$$

$$u = B^{\dagger}\{E\dot{x} - Ax\} \tag{3.60}$$

The above differential system (3.59) is known as *input–space restricted state mechanism model.* For every solution of (3.59) the input that generates this solution is given by (3.60). The pencil $R(s) = sNE - NA$ is referred to as *input–state restriction pencil.* Notice that this description is not affected by the introduction of state, state–derivative feedback and thus, it is called *feedback free description.* The controllability and reachability properties of the system $E\dot{x} = Ax + Bu$ may be expressed in terms of the invariants of restriction pencil $R(s)$ as follows [Karc. & Hay., 1981]:

Theorem 3.7.1 *Consider the system $E\dot{x} = Ax + Bu$ and its corresponding restriction pencil $sNE - NA$. Then*

(i) The finite zeros of $sNE - NA$ correspond to the finite decoupling zeros of the system

(ii) The infinte zeros of $sNE - NA$ correspond to decoupling zeros at infinity

(iii) If the system is reachable then, $sNE - NA$ has only column minimal indices

(iv) If the system is controllable, then the restriction pencil may have linear infinite elementary divisors. □

Consider now a subspace $\mathcal{V} \subset \mathcal{X}$. Then we define as *$\mathcal{V}$-restricted pencil* the pencil $R_{\mathcal{V}}(s) = sNEV - NAV$ where V is a basis matrix of $\mathcal{V}$. If $F = NE$, $G = NA$ we have the following [Kar. & Kal., 1989].

Definition 3.7.1 *Let $F, G \in \Re^{m\times n}$, $\mathcal{V} \subset \Re^{n}$ be a subspace with* $\dim\{\mathcal{V}\} = d$

*(i) $\mathcal{V}$ is called (G,F)-***invariant subspace** *if*

$$G\mathcal{V} \subseteq F\mathcal{V} \tag{3.61}$$

or equivalently, for any basis V of $\mathcal{V}$, $\exists\, \overline{A}_V \in \Re^{d\times d}$ such that

$$GV = FV\overline{A}_V \tag{3.62}$$

*(ii) $\mathcal{V}$ is called an (F,G)-***invariant subspace** *if*

$$F\mathcal{V} \subseteq G\mathcal{V} \tag{3.63}$$

or, equivalently, for any basis V of $\mathcal{V}$, $\exists\, \underline{A}_V \in \Re^{d\times d}$ such that

$$FV = GV\underline{A}_V \tag{3.64}$$

(iii) $\mathcal{V}$ is called a **complete-(F,G)-invariant subspace** *if*

$$F\mathcal{V} = G\mathcal{V} \tag{3.65}$$

or, equivalently, for any basis V of $\mathcal{V}$, $\exists\, \overline{A}_V, \underline{A}_V \in \Re^{d\times d}$ such that

$$GV = FV\overline{A}_V \text{ and } FV = GV\underline{A}_V \tag{3.66}$$

□

Theorem 3.7.2 *The type of the Kronecker invariants of $R_{\mathcal{V}}(s)$ is as follows:*

(i) If $\mathcal{V}$ is (G,F)-invariant then $R_{\mathcal{V}}(s)$ may have finite elementary divisors (f.e.d.), column minimal indices (c.m.i.) and possibly zero row minimal indices (z.r.m.i.)

(ii) If $\mathcal{V}$ is (F,G)-invariant then $R_{\mathcal{V}}(s)$ may have i.e.d., c.m.i. and possibly z.r.m.i.

(iii) If $\mathcal{V}$ is complete-(F,G)-invariant subspace, then $R_{\mathcal{V}}(s)$ may have f.e.d., c.m.i. and possible z.r.m.i. but it may not have 0–e.d.. □

Observe the following: The definition of (G, F)-invariant subspaces corresponds to the definition of (A, E, B)-invariant subspaces (see [Kar., 1979]). Thus, we have the following.

Theorem 3.7.3 [Jaf. & Kar., 1981] *A subspace $\mathcal{V} \subset \mathcal{X}$ is (A, E, B)-invariant, if the restriction pencil $R_{\mathcal{V}}(s) = sNEV - NAV$ has f.e.d, c.m.i. and possibly z.r.m.i..* □

A complete characterisation of the fundamental subspaces of a linear system in terms of the invariants of the restriction pencil has been given (see [Jaf. & Kar., 1981], [Kar. & Kal., 1989], [Kar., 1979]). This characterisation will be discussed later in Chapter 9 of this thesis, where the matrix pencil approach is used for the solution of the cover problem.

As a final comment in this Chapter it is mentioned that the recursive algorithms for the maximal elements of several types of invariant subspaces contained in a given subspace $\mathcal{K}$ may be given in terms of the matrices N, E, A appearing in the restriction pencil $R_{\mathcal{V}}(s) = sNEV - NAV$ and thus, a complete translation of the geometric theory is obtained in terms of matrix pencils. Furthermore the matrix pencil characterisation of the subspaces may provide a more refined classification in terms of Kronecker invariants.

3.8 Conclusions

In this Chapter a brief background of the basic definitions and results related to implicit systems was given. First the several types of representations of linear implicit systems were discussed. These representations cover the external, as well as internal descriptions of the systems. The second topic was the several notions of equivalence of systems found in the literature. The equivalence framework we consider, is crucial when we want to inspect when two types of representations describe the same dynamical system. Another important topic of this Chapter was the minimality of systems under the several types of equivalence. Minimality is always desirable since it reduces the computational cost in the analysis and design and allows reduction of the cost when a system is built in practice. The most used type of first order representation of implicit systems was examined next. This type is the descriptor representation. The use of this representation allows the development of geometric theory for implicit systems and thus, a generalisation of Wonham's geometric approach. Finally, the matrix pencil approach was briefly discussed. This approach comes as an alternative to geometric theory and provides the means for relating the several geometric concepts to the Kronecker theory of invariants which is a natural tool for the study of first order representations.

Chapter 4

GENERALIZED STATE-SPACE REALISATIONS OF NONPROPER TRANSFER FUNCTIONS

4.1 Introduction

The purpose of this chapter is to present a realisation method for nonproper transfer functions. It is known that a nonproper transfer function may be transformed to a system of first order differential equations in descriptor form. There are several approaches towards the derivation of the descriptor form. The first is the decomposition of the nonproper transfer function, to the strictly proper and polynomial part and then realise the two parts independently.The realisation of the strictly proper part is obtained using any of the well known techniques, such as realisation from Markov parameters, matrix fraction description method etc. (see [Wol. & Fal., 1969], [Wol., 1973],[Chen, 1984]). The realisation of the polynomial part was studied in [Var., Lim. & Kar., 1982], [Con. & Per., 1982] etc.. In [Var., Lim. & Kar., 1982] the polynomial part realisation is obtained reducing the problem to the problem of the realisation of the "dual" strictly proper system. The minimality of the above realisation is proved to be equivalent to the minimality of the "dual" realisation. Conte and Perdon in [Con. & Per., 1982] follow a module theoretic approach for the realisation of the polynomial part. In [Chr. & Mer., 1986], [Tan & Van., 1987] a generalised state–space realisation is obtained by inspection from a given MFD of the transfer function. Both approaches follow along the lines of the realisations of MFD of strictly proper transfer functions, but there is no discussion on the minimality of the resulting systems (except for the SISO case in [Chr. & Mer., 1986]) and no relationship between the Forney indices of the MFD and the reachability/controllability properties of the state equations is established.

The main result of the present Chapter is a generalisation of the classical realisation technique based on MFDs [Wol., 1973] to the case of nonproper transfer functions. The issues examined here are the following: First, the construction of the generalised state–space description is obtained from a given MFD, by inspection. Then, necessary and sufficient conditions for the minimality of the realisation are produced and the dimension of the minimal state–space description is expressed in terms of the generalised McMillan degree of the transfer function. The relation between the column minimal indices of the realisation and Forney indices of the MFD, first established by Rosenbrock in [Ros., 1974], is verified in a straightforward way. It must be mentioned that the realisation procedure covers the case of the strictly proper transfer functions, i.e. when the transfer matrix is strictly proper, one gets the classical results for the strictly proper systems [Wol. & Fal., 1969], [Chen, 1984], [Kail., 1980] etc.. The proper systems are considered here in a singular system representation by expanding the proper dynamics by nondynamic variables.

The structure of the Chapter is as follows: In section 4.2, the problem of realisation is posed and it is shown that it can be considered as the problem of realisation of a strictly proper transfer function and a polynomial transfer function separately. In section 4.3 it is shown that any polynomial transfer function may have a generalised state–space realisation. Two realisation procedures are given in this section. The minimality of a realisation of a polynomial is considered in section 4.4. Criteria for the minimality are given in terms of the controllability and observability at infinity. In section 4.5 the relationship between the minimality and the extended McMillan degree is discussed. In section 4.6, the realisation procedure of $G(s)$ using MFDs is described, and some related topics are discussed. Finally in section 4.7 the properties of the proposed realisation are discussed.

4.2 Statement of the problem

Let $G(s) \in \Re^{m\times\ell}(s)$ be a rational transfer matrix and let $y(s)$ and $u(s)$ be the Laplace transform of the output and input vector respectively i.e.

$$y(s) = G(s)u(s) \tag{4.1}$$

The problem of the realisation of $G(s)$ in generalised state–space form is finding a quadruple (E, A, B, C) such that the system $\mathcal{S}_e$ described by the equations

$$\mathcal{S}_e : E\dot{x} = Ax + Bu \tag{4.2}$$

$$y = Cx \tag{4.3}$$

gives rise to the transfer function $G(s)$ i.e.

$$G(s) = C(sE - A)^{-1}B \tag{4.4}$$

We are going to investigate the existence and the conditions for a realisation to be minimal. Since $G(s)$ is a nonproper rational matrix it may be written as

$$G(s) = G_{sp}(s) + P(s) \tag{4.5}$$

where $G_{sp}(s)$ is a strictly proper rational matrix and $P(s)$ is a polynomial matrix. It is well known that $G_{sp}(s)$ can be realised minimally in state–space form. In the next section it is proved that the polynomial part $P(s)$ may have a descriptor form realisation. Then the generalised state–space of the realisation of $G(s)$ is taken as the direct sum of the state–space $\mathcal{X}_r$ of the strictly proper part and the generalised $\mathcal{X}_e$ state–space of the polynomial part of $G(s)$. If

$$\mathcal{S}_r : \dot{x}_r = A_r x_r + B_r u_r \tag{4.6}$$

$$y_r = C_r x_r \tag{4.7}$$

is a realisation of the strictly proper part of $G(s)$ and

$$\mathcal{S}_e : E_e \dot{x} = A_e x_e + B_e u_e \tag{4.8}$$

$$y_e = C_e x_e \tag{4.9}$$

then the generalised state-space realisation of $G(s)$ is

$$\begin{bmatrix} I & 0 \\ 0 & E_e \end{bmatrix} \begin{bmatrix} \dot{x}_r \\ \dot{x}_e \end{bmatrix} = \begin{bmatrix} A_r & 0 \\ 0 & A_e \end{bmatrix} \begin{bmatrix} x_r \\ x_e \end{bmatrix} + \begin{bmatrix} B_r \\ B_e \end{bmatrix} u \tag{4.10}$$

$$y = [C_r \ C_e] \begin{bmatrix} x_r \\ x_e \end{bmatrix} \tag{4.11}$$

The above analysis shows the existence of realisation of $G(s)$, as long as we can realise as in (4.8), (4.9) the polynomial part in (4.5).

4.3 Realisation of the polynomial matrix $P(s)$

First we prove that there always exist a generalised state-space realisation of $P(s)$ of the form

$$\mathcal{S}_c : H_c \dot{x}_c = x_c + B_c u \tag{4.12}$$

$$y_c = C_c x_c \tag{4.13}$$

To establish the existence of the above realisation let $p_i(s)$ be the columns of $P(s)$, i.e.

$$P(s) = [p_1(s), p_2(s), \ldots, p_\ell(s)] \in \Re^{m\times\ell}[s] \tag{4.14}$$

and

$$\delta_i^c = \partial\{p_i(s)\} \tag{4.15}$$

Proposition 4.3.1 *The column vector $p_i(s)$ with $\delta_i^c = \partial\{p_i(s)\}$ admits a realisation of the following type:*

$$\mathcal{S}_i : H_i \dot{x} = x + b_i u \tag{4.16}$$

$$y = C_i x \tag{4.17}$$

where

$$H_i = \underbrace{\begin{bmatrix} 0 & 1 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & \cdot & \cdot & \cdot & & \cdot & \\ \cdot & & & \cdot & \cdot & \cdot & 0 \\ \cdot & & & & \cdot & \cdot & \cdot \\ 0 & & & & & \cdot & 1 \\ 0 & \cdot & \cdot & \cdot & \cdot & 0 & 0 \end{bmatrix}}_{\delta_i^c+1}, \quad b_i = \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix} \updownarrow \delta_i^c + 1 \;, \quad p_i(s) = C_i e_{\delta_i^c}(s) \tag{4.18}$$

$e_{\delta_i^c}(s) = [1, s, \ldots, s^{\delta_i^c}]^T$ *and* C_i *are defined by the coefficient matrix of the polynomial vector* $p_i(s)$.

Proof: The transfer function corresponding to the system (4.16), (4.17) is

$$p_i(s) = C_i(sH_i - I)^{-1} b_i \tag{4.19}$$

Note that

$$\begin{bmatrix} -1 & s & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & s \\ 0 & \cdots & \cdots & 0 & -1 \end{bmatrix}^{-1} = - \begin{bmatrix} 1 & s & \cdots & \cdots & s^{\delta_i^c} \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & s \\ 0 & \cdots & \cdots & 0 & 1 \end{bmatrix} \tag{4.20}$$

and thus

$$(sH_i - I)^{-1} b_i = [s^{\delta_i^c}, s^{\delta_i^c - 1}, \ldots, s, 1]^T \tag{4.21}$$

If we now write

$$C_i = \begin{bmatrix} c_i^1 \\ c_i^2 \\ \vdots \\ c_i^m \end{bmatrix}, \quad p_i(s) = C_i e_{\delta_i^c}(s) \tag{4.22}$$

the result follows immediately. □

Proposition 4.3.2 *Every polynomial matrix* $P(s) = [p_1(s), \ldots, p_\ell(s)]$ *admits a realisation of the following type*

$$H_c \dot{x}_c = x_c + B_c u \tag{4.23}$$

$$y = C_c x_x \tag{4.24}$$

where

$$H_c = \text{diag}\{\ldots H_i \ldots\} \tag{4.25}$$

$$B_c = \text{block–diag}\{\ldots, e_{\delta_i^c+1}, \ldots\} \tag{4.26}$$

where e_i is the unit vector with length i and C_c is defined by

$$C_c \text{block–diag}\{\ldots, [s^{\delta_1^c+1}, \ldots s, 1]^T, \ldots\} = P(s) \tag{4.27}$$

The proof is obvious from proposition 4.3.1. □

Note that the structure of the above realisation is defined as the aggregate of the realisations of $p_i(s)$. We can obtain similar results taking the rows of $P(s)$ instead of columns. The corresponding realisation is of the type:

$$H_o \dot{x}_o = x_0 + B_o u \tag{4.28}$$

$$y_o = C_0 x_o \tag{4.29}$$

where

$$H_o = \text{diag}\{\ldots H_i^o \ldots\} \tag{4.30}$$

$$C_o = \text{block–diag}\{\ldots, \hat{e}_{\delta_i^o+1}, \ldots\} \tag{4.31}$$

where $\hat{e}_i$ is the vector $[1\ 0 \ldots 0]$ with length i and B_0 is defined by

$$\text{block–diag}\{\ldots, [s^{\delta_i^o+1}, \ldots s, 1], \ldots\} B_0 = P(s) \tag{4.32}$$

The superscripts "c" and "o" stand for the columns and rows respectively. From the structure of the matrices H_c and H_0 we can see that the pencils $sH_c - I$ and $sH_0 - I$ are characterised by i.e.d. only. The corresponding sets of the i.e.d. are determined from the column and row degrees of $P(s)$. The realisations (4.23), (4.24) and (4.28), (4.29) will be called *canonical column* and *canonical row realisation* respectively. The sets of i.e.d. of the two canonical realisations are as follows $\{s^{\delta_1^c+1}, s^{\delta_2^c+1}, \ldots, s^{\delta_\ell^c+1}\}$ for the column case and $\{s^{\delta_1^o+1}, s^{\delta_2^o+1}, \ldots, s^{\delta_m^o+1}\}$ for the row case and the dimensions of the realisations are

$$n_c = \sum_{i=1}^{\ell} \delta_i^c + \ell, \quad n_o = \sum_{i=1}^{m} \delta_i^o + m$$

Note that the above realisations are not necessarily minimal, i.e. we may in general find g.s.s. realisations of $P(s)$ with number of states less than the states of the above realisations. Generalised state–space systems of the form (4.10), (4.11) can be obtained from (4.23),(4.24) or (4.28),(4.29) by restricted system equivalence transformations [Ver., V.-D. & Kail., 1979].

4.4 Minimal Realisations of $P(s)$

In this section, the definition and the characterisation of the minimality of the generalised state–space realisation of a polynomial transfer function is given.

Definition 4.4.1 *A generalised state-space realisation is called minimal when there is no other realisation of the same transfer function with less number of states.* □

Lemma 4.4.1 *If a realisation of $P(s)$ of the form (4.10),(4.11) is minimal, then the pencil $sE_e - A_e$ is characterised only by infinite elementary divisors.*

Proof: Since $sE_e - A_e$ is a regular pencil, it is not characterised by any type of minimal indices. Let $sE_e - A_e$ be characterised by finite and infinite elementary divisors and let P, Q be the transformation matrices that bring $sE_e - A_e$ to the Weierstrass canonical form i.e.

$$P(sE_e - A_e)Q = \text{diag}\{\ldots, sH_{q_i} - I_{qi}, \ldots, sI_{r_i} - J_{r_i}(\lambda_i), \ldots\} \tag{4.33}$$

where q_i and r_i are the orders, degrees of the infinite and finite elementary divisors, respectively. The generalised state–space description of the transformed system is

$$E_W \dot{x}_W = x_W + B_W u \tag{4.34}$$

$$y = C_W x_W \tag{4.35}$$

where

$$E_W = \text{diag}\{\ldots, H_{q_i}, \ldots, I_{r_i}, \ldots\} \tag{4.36}$$

$$A_W = \text{diag}\{\ldots, I_{q_i}, \ldots, J_{r_i}(\lambda_i), \ldots\} \tag{4.37}$$

$$B_W = PB,\ C_W = CQ \tag{4.38}$$

The transfer function of the original system remains invariant under the transformation (P, Q), that is

$$P(s) = C_e(sE_e - A_e)^{-1}B_e = C_eQQ^{-1}(sE_e - A_e)^{-1}P^{-1}PB_e = C_W(sE_W - A_W)^{-1}B_W =$$

$$= [\ldots, C_{q_i}(sH_{q_i} - I_{q_i})^{-1}, \ldots, C_{r_i}(sI_{r_i} - J_{r_i}(\lambda_i))^{-1}] \begin{bmatrix} \vdots \\ B_{q_i} \\ \vdots \\ B_{r_i} \\ \vdots \end{bmatrix} =$$

$$= \sum_{i=1}^{\mu} C_{q_i}(sH_{q_i} - I_{q_i})^{-1}B_{q_i} + \sum_{i=1}^{f} C_{r_i}(sI_{r_i} - J_{r_i}(\lambda_i))^{-1}B_{r_i} \tag{4.39}$$

From the above we see that for $P(s)$ to be polynomial we must have

$$\sum_{i=1}^{f} C_{r_i}(sI_{r_i} - J_{r_i}(\lambda_i))^{-1}B_{r_i} \equiv 0 \tag{4.40}$$

which implies that there exists a realisation of $P(s)$ of smaller dimension. □

Note that the reverse of the result is not always true. From the above lemma we have that the minimal realisations of $P(s)$ must be sought amongst the realisations of the form

$$\begin{aligned} H\dot{x} &= x + Bu \\ y &= Cx \end{aligned}$$

The structure at infinity (i.e. the existence of i.e.d. of the pencil $sH - I$) of the realisation of polynomials imposes a dynamical behaviour different than the behaviour of the normal state–space systems. The generalised state–space systems are characterised by impulsive behaviour at $t = 0$ excited by the initial conditions $x(0-)$ of the state vector [Ver., Lev. & Kail., 1981]. The impulsive behaviour can be easily verified from the solution of the equation (4.1) taking into account the initial conditions. Next we give the definitions and several tests for the reachability/observability at infinity as they are defined in the work of Cobb [Cobb, 1984]. Let ε be the index of nilpotency of H, C^i the i times continuously differentiable functions on $\Re$ and $\mathcal{D}^+$ the space of distributions with support in $[0, \infty)$.

Definition 4.4.2 [Cobb, 1984] *A generalised state–space realisation is called reachable, if* $\forall t_f > 0$, $x(0-)$ *admissible and* $W \in \Re^n$ *there exists* $u \in C^{\varepsilon-1}$ *such that* $x(t_f) = 0$. □

Definition 4.4.3 [Cobb, 1984] *A generalised state–space realisation is called observable, if knowledge of* $u \in C_p^{\varepsilon-1}$, $y \in \mathcal{D}^+$ *and* $y(0-)$ *is sufficient to determine* $x(0-)$. □

Let $H = \text{block–diag}\{\ldots, H_{q_i}, \ldots\}$, $i = 1, \cdots, t$. The following propositions provide ways of testing the reachability and observability at infinity.

Proposition 4.4.1 [Cobb, 1984], [Yip & Sin., 1981], *A realisation (H, B, C) of $P(s)$ is reachable, if and only if*

$$\operatorname{rank}(R_c) = \sum_{i=1}^{1} \kappa_i \tag{4.41}$$

where

$$R_c = [B\ HB\ H^2B \ldots H^{\kappa-1}B] \tag{4.42}$$

$$\kappa = \sum_{i=1}^{t} \kappa_i \tag{4.43}$$

where κ_i are the dimensions of the Jordan blocks of H. The above is equivalent to the statement that the matrix $[H\ B]$ has full row rank. □

Proposition 4.4.2 [Cobb, 1984] *A realisation (H, B, C) of $P(s)$ is observable, iff*

$$\operatorname{rank}(R_0) = \sum_{i=1}^{m} \kappa_i \tag{4.44}$$

where

$$R_0 = \begin{bmatrix} C \\ CH \\ \vdots \\ CH^{\kappa-1} \end{bmatrix} \tag{4.45}$$

$$\kappa = \sum_{i=1}^{t} \kappa_i \tag{4.46}$$

the above is equivalent to the statement the matrix $[H^T\ C^T]^T$ has full column rank. □

Proposition 4.4.3 [Ros., 1974] *A realisation (H, B, C) of $P(s)$ is reachable, if and only if the matrix pencil*

$$L_c = [sH - I, -B] \tag{4.47}$$

has no infinite zeros. □

The dual result for the observability of a pair (H, C) is the following.

Proposition 4.4.4 [Ros., 1974] *A realisation (H, B, C) of $P(s)$ is observable, if and only if the matrix pencil*

$$L_0(s) = \begin{bmatrix} sH - I \\ -C \end{bmatrix} \tag{4.48}$$

has no infinite zeros. □

In the sequel we give a condition for the reachability (observability) of a realisation, using the Jordan form of the matrix H. Consider a realisation (H, I, B, C) of a polynomial transfer function. The matrix H has all its eigenvalues at $s = 0$ and its Jordan form is

$$H = \mathrm{diag}\{\ldots H_{\kappa_i} \ldots\}, \quad i = 1, \cdots, t \tag{4.49}$$

Let the matrices B and C be as follows

$$B = \begin{bmatrix} B_1 \\ \vdots \\ B_t \end{bmatrix} \tag{4.50}$$

$$B_i = \begin{bmatrix} b_1^i \\ \vdots \\ b_{\kappa_i}^i \end{bmatrix} \tag{4.51}$$

$$C = [C_1 \; C_2 \ldots C_t] \tag{4.52}$$

$$C_i = [c_1^i \; c_2^i \ldots c_{\kappa_i}] \tag{4.53}$$

where κ_i are the orders of the i.e.d. of H.

Lemma 4.4.2 *The controllability (observability) matrix $R_c(R_0)$ has full row (column) rank, if and only if the following condition holds true: The rows (columns) of $B^\star(C^\star)$ where*

$$B^\star = \begin{bmatrix} b_{\kappa_1}^1 \\ \vdots \\ b_{\kappa_t}^t \end{bmatrix} \quad (C^\star = [c_{\kappa_1}^1 \ldots c_{\kappa_t}^t]) \tag{4.54}$$

are linearly independent. □

The following theorem relates the structure at infinity of $P(s)$ and of the pencil $sH - I$.

Theorem 4.4.1 [Ver., V.-D. & Kail., 1979] *Let $P(s)$ be a polynomial transfer function and (H,B,C) be the matrix triple of a reachable and observable realisation of $P(s)$. Then the pole structure at infinity of $P(s)$ is isomorphic to the zero structure of the pencil $sH-I$ at infinity.* □

We can now state the following result:

Theorem 4.4.2 *Let $P(s)$ be a polynomial transfer matrix and (H,B,C) a generalised state–space realisation. This realisation is minimal, if and only if the triple (H,B,C) is reachable and observable.*

Proof: From the nilpotency of the matrix H it follows that the controllability and observability matrices have the form:

$$R_c = [B \; HB \; H^2B \ldots H^{\varepsilon-1}B \;\; 0 \ldots 0] \tag{4.55}$$

$$R_0 = \begin{bmatrix} C \\ CH \\ \vdots \\ CH^{\varepsilon-1} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{4.56}$$

where ε is the index of nilpotency of H.

Let the triple (H,B,C) be an unreachable and/or unobservable realisation of $P(s)$. Then using a similarity transformation we can bring the matrix H to the Jordan form and let $(H^\star,B^\star,C^\star)$ be the transformed system. Then, since the realisation is either unreachable or unobservable, we may reduce the dimensions of this realisation by following the method described in [Ros., 1974b].

Conversely, let (H,B,C) be reachable and observable. From theorem 4.4.1 it follows that the pole structure at infinity of $P(s)$ is isomorphic to the structure of the pencil $sH-I$. Let now $(\overline{H},\overline{B},\overline{C})$ be a generalised state–space realisation of $P(s)$ of order less than the order of the realisation (H,B,C). Then the set of i.e.d. of $s\overline{H}-I$ is different than the set of i.e.d. of $sH-I$. This set of i.e.d. defines the pole structure at infinity of $P(s)$. This is a contradiction since the pole structure of $P(s)$ is defined from the i.e.d. of $sH-I$. Thus, the order of the two systems must be the same and it is minimal. □

4.5 The McMillan degree and MFDs

In this section the *extended McMillan degree* of $G(s)$ is defined and its relationship to the minimality of the generalised state–space representation is derived. The results obtained in this section provide a complete generalisation of the strictly proper systems case result to that of nonproper systems.

Definition 4.5.1 [Ros., 1970], [Kail., 1980] *The extended McMillan degree of $G(s)$ is defined as*

$$\delta^*_M(G(s)) = \delta_M(G_{sp}(s)) + \delta_M(P(1/s)) \tag{4.57}$$

where δ_M denotes the McMillan degree in the usual sense. □

It is known [Ros., 1970], [Kail., 1980] that $\delta^*_M(G(s))$ expresses the total number of finite and infinite poles of $G(s)$ with the orders accounted. We proceed with some results that will be used on the derivation of the main result of this section.

Definition 4.5.2 [Kail., 1980] *Let $t(s) = n(s)/d(s) \in \Re(s)$. The map*

$$\delta_\infty : \Re(s) \longrightarrow Z \cup \{\infty\}$$

is defined by

$$\delta_\infty(t(s)) = \begin{cases} \partial[n(s)] - \partial[d(s)], & t(s) \neq 0 \\ \infty, & t(s) \equiv 0 \end{cases} \tag{4.58}$$

□

The above function is a discrete valuation since it satisfies the properties

$$\delta_\infty(t_1(s) + t_2(s)) \geq \min\{\delta_\infty(t_1(s)), \delta_\infty(t_2(s)) \tag{4.59}$$

$$\delta_\infty(t_1(s) \cdot t_2(s)) = \delta_\infty(t_1(s)) + \delta_\infty(t_2(s)) \tag{4.60}$$

The definition of the valuation at infinity may be extended to rational matrices as follows [Kail., 1980], [Var. & Kar., 1983a]:

Definition 4.5.3 *The i-th valuation $\zeta_i(G(s))$ of a rational matrix $G(s)$ at infinity is defined as*

$$\zeta_i = \begin{cases} \min\limits_{i} \ \{\delta_\infty(G^i(s))\} \\ +\infty \quad if G(s) = 0 \end{cases} \tag{4.61}$$

where $G^i(s)$ is an $i \times i$ minor of $G(s)$. □

Let $\mathcal{P}_G^\infty = \{\tilde{q}_i,\ i \in \underline{\mu},\ \tilde{q}_i > 0\}$, $\mathcal{Z}_G^\infty = \{\overline{q}_i,\ i \in \underline{\nu},\ \overline{q}_i > 0\}$ denote the orders of infinite poles and zeros respectively of $G(s)$. Then [Var., Lim. & Kar., 1982] we have

$$\sum_{i=1}^{r} q_i = \zeta_0(G) - \zeta_r(G) = -\zeta_r(G)$$

where q_i are defined in (2.21) and r =rank$\{G(s)\}$. The above may be written as follows:

$$\sum_{i=1}^{\mu} \tilde{q}_i - \sum_{i=1}^{\nu} \overline{q}_i = -\zeta_r(G)$$

or

$$\boxed{\delta_\infty(G) = \zeta_r(G) = \sum_{i=1}^{\mu} \tilde{q}_i - \sum_{i=1}^{\nu} \overline{q}_i = -\zeta_r(G)} \tag{4.62}$$

{# inf. zeros}-{# inf. poles}

Remark 4.5.1 $\delta_\infty(G) = \zeta_r(G)$ *and expresses the difference between total number of infinite zeros and total number of infinite poles of* $G(s)$. □

The following propositions are important for the derivation of the main result of this section.

Theorem 4.5.1 [Var. & Kar., 1983a] *Let* $G(s) = B(s)A^{-1}(s)$ *be an* $\Re_{pr}(s)$ *MFD. If*

$$M_G^\infty(s) = \left[\begin{array}{ccccccccc|c} s^{q_1} & & & & & & & & & \\ & \ddots & & & & & & & & \\ & & s^{q_\mu} & & & & & & & \\ & & & 1 & & & & & & \\ & & & & \ddots & & & & & \\ & & & & & 1 & & & & \\ & & & & & & s^{-\hat{q}_1} & & & \\ & & & & & & & \ddots & & \\ & & & & & & & & s^{-\hat{q}_\nu} & \\ \hline & & & & & & & & & 0 \end{array}\right], \quad q_i > 0,\ \hat{q}_j > 0 \tag{4.63}$$

then

$$M_B^{\infty}(s) = \left[\begin{array}{cccccc|c} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & s^{-\hat{q}_1} & & & \\ & & & & \ddots & & \\ & & & & & s^{-\hat{q}_\nu} & \\ \hline & & & & & & 0 \end{array}\right] \tag{4.64}$$

$$M_A^{\infty}(s) = \begin{bmatrix} s^{-q_1} & & & & & \\ & \ddots & & & & \\ & & s^{-q_\mu} & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{bmatrix} \tag{4.65}$$

□

Proposition 4.5.1 *Let $G(s) \in \Re^{m\times\ell}(s)$, represented by an $\Re_{pr}(s)-$coprime MFD, $G(s) = B(s)A^{-1}(s)$, $A(s) \in \Re_{pr}^{\ell\times\ell}$, $B(s) \in \Re_{pr}^{m\times\ell}(s)$, then*

(i) $T_G(s) = \begin{bmatrix} A(s) \\ B(s) \end{bmatrix} A^{-1}(s)$ *is and $\Re_{pr}(s)-$coprime proper MFD*

(ii) $T_G(s)$ has no infinite zeros, the same poles as $G(s)$ and

$$\delta_M^{\infty}(T_G(s)) = \delta_{\infty}(\det A(s)) \tag{4.66}$$

Proof: Since $(A(s),\ B(s))$ are right $\Re_{pr}(s)-$coprime

$$\begin{bmatrix} A(s) \\ B(s) \\ A(s) \end{bmatrix} \tilde{} \begin{bmatrix} A(s) \\ B(s) \\ 0 \end{bmatrix}$$

and thus the MFD is coprime which proves part *(i)*. For the part *(ii)*, the coprimeness at $s = \infty$ of the numerator $\begin{bmatrix} A(s) \\ B(s) \end{bmatrix}$ implies that we have no infinite zeros, whereas the infinite poles are determined by $A(s)$ and thus

$$\delta_M^{\infty}(T_G(s)) = \delta_M^{\infty}(G(s)) = \delta_{\infty}(\det A(s))$$

□

Some important properties of the valuation are summarised below [Var. & Kar., 1983a].

Proposition 4.5.2 *Let* $G(s) \in \Re^{m\times\ell}(s)$, $m \geq \ell$, $\mathrm{rank}\{G(s)\} = r$ *and let* $Q(s) \in \Re^{\ell\times\ell}(s)$, $\mathrm{rank}\{Q(s)\} = \ell$ *and let*

$$R(s) = G(s)Q(s) \tag{4.67}$$

Then

$$\delta_\infty(R) = \delta_\infty(G) + \delta_\infty(Q) \tag{4.68}$$

□

In order to examine some further properties of the valuation we also need the concept of the *degree* of a polynomial matrix.

Definition 4.5.4 *Let* $G(s) \in \Re^{m\times\ell}[s]$. *We define as the degree of* $G(s)$ *the function:* $\partial : \Re^{m\times\ell} \longrightarrow Z_{\geq 0} \cup \{-\infty\}$ *defined by:*

$$\partial[G(s)] = \begin{cases} \left\{ \begin{array}{c} \textit{max. deg among the degrees} \\ \textit{of all max. order minors } (r\times r) \textit{ minors} \end{array} \right\} & if G(s) \neq 0 \\ -\infty & if G(s) = 0 \end{cases} \tag{4.69}$$

□

Proposition 4.5.3 *If* $G(s) \in \Re^{m\times\ell}[s]$, $m \geq \ell$, $\mathrm{rank}\{G(s)\} = r$, $Q(s) \in \Re^{\ell\times\ell}[s]$, $\mathrm{rank}\{Q(s)\} = \ell$ *and* $R(s) = G(s)Q(s)$, *then*

$$\partial[R(s)] = \partial[G(s)] + \partial[Q(s)] \tag{4.70}$$

□

From the definition of the valuation and degree we have [Var. & Kar., 1983a]:

Corollary 4.5.1 *If* $G(s) \in \Re^{m\times\ell}[s]$ *then*

$$\delta_\infty(G) = -\partial[G] \tag{4.71}$$

and if $G(s) \in \Re^{\ell\times\ell}[s]$ *is* $\Re[s]-$*unimodular, then*

$$\delta_\infty(G) = -\partial[G] = \partial[\det(G)] = 0 \tag{4.72}$$

□

The next proposition generalises to the matrix case the definition of reduction at $s = \infty$ of a scalar transfer function [Var. & Kar., 1983a].

Proposition 4.5.4 *Let $G(s) \in \Re^{\ell\times\ell}[s]$, $m \geq \ell$ and $G(s) = N(s)D^{-1}(s)$ be an $\Re[s]-$ MFD not necessarily coprime. Then,*

$$\delta_{\infty}(G) = \partial[D] - \partial[N] \tag{4.73}$$

□

With these preliminary results we may state the main result.

Theorem 4.5.2 *Let $G(s) \in \Re^{m\times\ell}(s)$, $m \geq \ell$ and $G(s) = N(s)D^{-1}(s)$ be any right coprime MFD over $\Re[s]$ and let*

$$T(s) = \begin{bmatrix} D(s) \\ N(s) \end{bmatrix} \tag{4.74}$$

Then,

$$\delta_M^*(G(s)) = \delta_M(G(s)) + \delta_M^{\infty}(G(s)) = \partial[T(s)] \tag{4.75}$$

Proof: Consider the matrix

$$T_G(s) = \begin{bmatrix} G(s) \\ I_\ell \end{bmatrix} \tag{4.76}$$

and any $\Re[s]-$coprime polynomial MFD $G(s) = N(s)D^{-1}(s)$. Then,

$$T_G = \begin{bmatrix} D(s) \\ N(s) \end{bmatrix} D^{-1}(s) = T(s)D^{-1}(s) \tag{4.77}$$

is a coprime MFD. Note that $T_G(s)$ has neither finite nor infinite zeros. By the definition

$$\delta_{\infty}(T_G(s)) = \{\#\ inf.\ zeros\}-\{\#\ inf.\ poles\} \tag{4.78}$$

and since $T_G(s)$ has no infinite zeros we have

$$\delta_{\infty}(T_G(s)) = -\{\#\ inf.\ poles\} = -\delta_M^{\infty}(T_G(s)) \tag{4.79}$$

which by proposition 4.5.1 leads to

$$\delta_{\infty}(T_G(s)) = -\delta_M^{\infty}(G(s)) \tag{4.80}$$

By proposition 4.5.4 we also have that

$$\delta_{\infty}(T_G(s)) = \partial[D] - \partial[T] \tag{4.81}$$

given that $\partial[D] = \delta_M(G(s))$ then (4.79) and (4.80) lead to

$$-\delta_M^{\infty}(G(s)) = \delta_M(G(s)) - \partial[T(s)] \tag{4.82}$$

or

$$\partial[T(s)] = \delta_M(G(s)) + \delta_M^{\infty}(G(s)) = \delta_M^*(G(s)) \tag{4.83}$$

□

Note that the degree of the matrix $T(s)$ defined by an irreducible MFD is invariant of the transfer function and known as the Forney dynamical order of the vector space

$$\mathcal{F} \equiv \text{col} - \text{span}_{\Re(s)}\{T(s)\} \tag{4.84}$$

Thus, we have:

Corollary 4.5.2 *Let $\delta_F(\cdot)$ be the Forney order of $\mathcal{F}$, associated with right coprime MFDs of $G(s)$. Then*

$$\delta_F(G(s)) = \delta_M^*(G(s)) \tag{4.85}$$

□

This gives an interpretation of the extended MacMillan degree as a Forney order and vice–versa. This result provides alternative means for the computation of $\delta_M^*(G(s))$ without resorting to the computation of the Smith–MacMillan forms.

Remark 4.5.2 *The above result was proven by Janssen* in [Jan., 1988] *by following different approach.* □

The above results may be used for the determination of the relation of a minimal realisation of $P(s)$) with the MacMillan degree. Let (H, B, C) be a reachable and observable realisation of $P(s)$. From theorem 4.4.1 we have that if $P(s)$ has μ poles at infinity of orders $\tilde{q}_i = \kappa_i - 1$, where κ_i are the multiplicities of the corresponding i.e.d. of the pencil $sH - I$, the MacMillan degree at infinity of $P(s)$ is defined as the total number of the infinite poles of $P(s)$. Then, taking into account and the i.e.d. of $sH - I$ with multiplicity 1 (linear i.e.d.) we have the following result.

Theorem 4.5.3 *Let (H, B, C) be a controllable and observable realisation of $P(s)$. If $P(s)$ has μ infinite poles of orders $\tilde{q}_i$ then the dimension ν of the minimal realisation is*

$$\nu = \sum_{i=1}^{\mu} (\tilde{q}_i + 1) = \sum_{i=1}^{\mu} \kappa_i + p_\ell \tag{4.86}$$

where p_ℓ is the number of linear i.e.d. of $sH - I$. □

Now going back to the overall nonproper transfer function we have that a minimal realisation of the form (4.2) (4.3) has dimension equal to the extended McMillan degree of $G(s)$ plus the number of the infinite poles, plus the number of trivial entries (1's) of M_G^∞ . It must be noted that our discussion is restricted to systems without feedforward term. If such systems are considered, then the dimension of the realisation may be less than that in (4.86). This is because in our approach, the nondynamic variables [Ver., Lev. & Kail., 1981], are incorporated in the dynamic equation (4.1). This inclusion of the nondynamic variables may be obtained as follows: If we write $G(s)$ in the form $G(s) = \tilde{G}(s) + D$, where D is constant and $\tilde{G}(s)$ does not have constant term, we may find a minimal realisation of $G(s)$ of the form

$$E\dot{x} = Ax + Bu, \quad y = Cx + Du \tag{4.87}$$

Now, taking into account that a finite pole of $G(s)$ of order κ corresponds to a finite elementary divisor (f.e.d.) of $sE - A$ of order κ and an infinite pole of $G(s)$ of order κ corresponds to an infinite elementary divisor (i.e.d.) of $sE - A$ of order $\kappa + 1$ [MacF. & Kar., 1976], [Ver., V.-D. & Kail., 1979] we have that a minimal realisation of the form (4.87) $G(s)$ has order

$$n_{min}^d = \delta_M(G(s)) + \#\text{ of infinite poles of } G(s)) \tag{4.88}$$

The matrix D may be always written as $D = \tilde{C}\tilde{B}$ where $\tilde{C} \in \Re^{m\times d}$, $\tilde{B} \in \Re^{d\times \ell}$, d =rank$D$ and thus (4.87) may be expressed as:

$$\begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} \dot{\tilde{x}} = \begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix} + \begin{bmatrix} B \\ \tilde{B} \end{bmatrix} u, \quad y = [C, \tilde{C}]\tilde{x} \tag{4.89}$$

which gives rise to the same transfer function $G(s)$. In (4.89), the feedthrough term D of (4.87) is incorporated in the state equations by introducing nondynamic state variables [Ver., Lev. & Kail., 1981]. Clearly, if (4.87) is minimal, then (4.89) is minimal among the realisations of the form (4.2) giving rise to $G(s)$.

Remark 4.5.3 *The minimal dimension of a realisation of $G(s)$ of the form (4.2) is*

$$n_{min} = \delta_M(G(s)) + \mathcal{P}_\infty(G(s)) \tag{4.90}$$

□

where $\mathcal{P}_\infty(G(s))$ is the number of poles at infinity of $G(s)$ plus the rank of D. The number $\mathcal{P}_\infty(G(s))$ will be referred to as the *index at infinity* of $G(s)$. From (4.89) it is clear that $\mathcal{P}_\infty(G(s))$ is the total number of the i.e.d. of the state pencil of the realisation.

4.6 Realisations of nonproper transfer functions via MFDs

In this section our aim is to find a generalised state–space realisation of $G(s)$ when it is given in MFD (Matrix Fraction Description) form i.e.

$$G(s) = N(s)D(s)^{-1} \tag{4.91}$$

Our approach is the generalisation of the realisation procedure of the MFDs of strictly proper transfer function [Wol., 1973], [Wol. & Fal., 1969], [Chen, 1984]. The generalised state–space realisations are based on the parameters of the MFD and can be obtained by inspection.

For the given systems with MFD as in (4.91), the equations relating the input and output variables (external variables) of the system are

$$D(s)v(s) = u(s) \tag{4.92}$$

$$N(s)v(s) = y(s) \tag{4.93}$$

where $v(s)$ is a vector of internal variables. Let $T(s)$ be the composite matrix of the MFD, i.e.

$$T(s) = \begin{bmatrix} D(s) \\ N(s) \end{bmatrix} \tag{4.94}$$

which can be written in the form

$$T(s) = T_{hc}H(s) + T_{\ell c}L(s) = \begin{bmatrix} D_{hc} \\ N_{hc} \end{bmatrix} H(s) + \begin{bmatrix} D_{\ell c} \\ N_{\ell c} \end{bmatrix} L(s) \tag{4.95}$$

where $T_{hc} = \begin{bmatrix} D_{hc} \\ N_{hc} \end{bmatrix}$ and $T_{\ell c} = \begin{bmatrix} D_{\ell c} \\ N_{\ell c} \end{bmatrix}$ are the highest order and lower order coefficient matrices of $T(s)$ respectively. If $\kappa_1, \ldots, \kappa_\ell$ are the column degrees of $T(s)$ then

$$H(s) = \mathrm{diag}\{s^{\kappa_i}\} \tag{4.96}$$

$$L(s) = \text{block–diag}\{\ldots[1\ s \ldots s^{\kappa_i - 1}]^T \ldots\} \tag{4.97}$$

The equations (4.92), (4.93) may be written in the form

$$D_{hc}H(s)v(s) = -D_{\ell c}L(s)v(s) + u(s) \tag{4.98}$$

$$N_{hc}H(s)v(s) = -N_{\ell c}L(s)v(s) + y(s) \tag{4.99}$$

Now define the new variable vector $\xi(s)$ by

$$\xi(s) = L(s)v(s) \tag{4.100}$$

Then, in the time domain, we have

$$\begin{bmatrix} \xi_{11} \\ \vdots \\ \xi_{1\kappa_1} \\ \hline \vdots \\ \hline \xi_{\ell 1} \\ \vdots \\ \xi_{\ell\kappa_\ell} \end{bmatrix} = \begin{bmatrix} \nu_1 \\ \vdots \\ \nu_1^{(\kappa_1-1)} \\ \hline \vdots \\ \hline \nu_\ell \\ \vdots \\ \nu_\ell^{(\kappa_\ell-1)} \end{bmatrix} \tag{4.101}$$

and from the above

$$\dot{\xi}_{ij} = \xi_{i(j+1)}, \quad i = 1, \ldots, \ell, \quad j = 1, \ldots, \kappa_i - 1 \tag{4.102}$$

Now (4.98) and (4.99) can be written as follows

$$D_{hc} \begin{bmatrix} \dot{\xi}_{1\kappa_1} \\ \vdots \\ \dot{\xi}_{\ell\kappa_\ell} \end{bmatrix} = -D_{\ell c}\xi(t) + u(t) \tag{4.103}$$

and

$$N_{hc} \begin{bmatrix} \dot{\xi}_{1\kappa_1} \\ \vdots \\ \dot{\xi}_{\ell\kappa_\ell} \end{bmatrix} = -N_{\ell c}\xi(t) + y(t) \tag{4.104}$$

If Q is a square invertible matrix such that

$$N_{hc}Q = [N^* \; 0] \tag{4.105}$$

where N^* has full column rank, then (4.104) is equivalent to

$$N_{hc}QQ^{-1} \begin{bmatrix} \dot{\xi}_{\kappa_1} \\ \vdots \\ \dot{\xi}_{\ell\kappa_\ell} \end{bmatrix} = -N_{\ell c}\xi(t) + y(t) \tag{4.106}$$

or

$$[N^* \ 0]Q^{-1}\begin{bmatrix} \dot{\xi}_{\kappa_1} \\ \vdots \\ \dot{\xi}_{\ell\kappa_\ell} \end{bmatrix} = -N_{\ell c}\xi(t) + y(t) \tag{4.107}$$

By partitioning the matrix Q^{-1} as shown below we define a matrix Q

$$Q^{-1} = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} \begin{matrix} \}p \\ \}\ell - p \end{matrix} \tag{4.108}$$

and

$$Q^{-1}\begin{bmatrix} \dot{\xi}_{\kappa_1} \\ \vdots \\ \dot{\xi}_{\ell\kappa_\ell} \end{bmatrix} = \begin{bmatrix} w(t) \\ \hat{\xi}(t) \end{bmatrix} \tag{4.109}$$

Proposition 4.6.1 *The matrix Q_1 is uniquely defined by the given MFD .*

Proof: $NQ = [N^* \ 0]$ or $N\begin{bmatrix} Q_1 \\ \hline Q_2 \end{bmatrix}^{-1} = [N^* \ 0]$ or $N = [N^* \ 0]\begin{bmatrix} Q_1 \\ \hline Q_2 \end{bmatrix} = N^*Q_1$ where N^* has full column rank. Since N^* has full column rank, it is injective and the result follows from the uniqueness of the echelon form of N. □

We may now define p new state variables ($p = \text{rank}\{N_{hc}\}$) as follows

$$w(t) = Q_1\begin{bmatrix} \dot{\xi}_{\kappa_1} \\ \vdots \\ \dot{\xi}_{\ell\kappa_\ell} \end{bmatrix} \tag{4.110}$$

then, from (4.107) we have

$$N^*w(t) + N_{\ell c}\xi(t) = y(t) \tag{4.111}$$

and equations (4.102), (4.103) and (4.111) can be written in matrix form as follows:

$$E\dot{\xi} = Ax + Bu \tag{4.112}$$

$$y = Cx \tag{4.113}$$

where

$$E = \left[\begin{array}{c|c|c|c|c}
\begin{matrix} I_{\kappa_1-1} & \begin{matrix}0\\ \vdots\\ 0\end{matrix} \\ 0\cdots 0 & d^h_{11}\end{matrix} & \begin{matrix} 0 \\ 0\cdots 0 \quad d^h_{12}\end{matrix} & \cdots & \begin{matrix} 0 \\ 0\cdots 0 \quad d^h_{1\ell}\end{matrix} & 0 \\ \hline
\begin{matrix} 0 & \begin{matrix}0\\ \vdots\\ 0\end{matrix} \\ 0\cdots 0 & d^h_{21}\end{matrix} & \begin{matrix} I_{\kappa_2-1} & \begin{matrix}0\\ \vdots\\ 0\end{matrix} \\ 0\cdots 0 & d^h_{22}\end{matrix} & \cdots & \begin{matrix} 0 \\ 0\cdots 0 \quad d^h_{21}\end{matrix} & 0 \\ \hline
 & & \vdots & & \\ \hline
\begin{matrix} 0 \\ 0\cdots 0 \quad d^h_{\ell 1}\end{matrix} & \begin{matrix} 0 \\ 0\cdots 0 \quad d^h_{\ell 2}\end{matrix} & \cdots & \begin{matrix} I_{\kappa_\ell-1} & \begin{matrix}0\\ \vdots\\ 0\end{matrix} \\ 0\cdots 0 & d^h_{\ell\ell}\end{matrix} & 0 \\ \hline
\begin{matrix} 0 & \begin{matrix}q_{11}\\ \vdots\\ q_{p1}\end{matrix}\end{matrix} & \begin{matrix} 0 & \begin{matrix}q_{12}\\ \vdots\\ q_{p2}\end{matrix}\end{matrix} & \cdots & \begin{matrix} 0 & \begin{matrix}q_{1\ell}\\ \vdots\\ q_{p\ell}\end{matrix}\end{matrix} & 0
\end{array}\right] \tag{4.114}$$

$$A = \left[\begin{array}{c|c|c|c|c}
\begin{matrix} \begin{matrix}0\\ \vdots\\ 0\end{matrix} & I_{\kappa_1-1} \\ d^\ell_{11} \cdots & d^\ell_{1\tau_1}\end{matrix} & \begin{matrix} 0 \\ d^\ell_{1\tau_1+1} \cdots d^\ell_{2\tau_2}\end{matrix} & \cdots & \begin{matrix} 0 \\ d^\ell_{1\tau_{\ell-1}+1} \cdots d^\ell_{1\tau_\ell}\end{matrix} & 0 \\ \hline
\begin{matrix} 0 \\ d^\ell_{21} \cdots d^\ell_{2\tau_1}\end{matrix} & \begin{matrix} \begin{matrix}0\\ \vdots\\ 0\end{matrix} & I_{\kappa_2-1} \\ d^\ell_{2\tau_1+1} \cdots & d^\ell_{2\tau_2}\end{matrix} & \cdots & \begin{matrix} 0 \\ d^\ell_{1\tau_{\ell-1}+1} \cdots d^\ell_{2\tau_\ell}\end{matrix} & 0 \\ \hline
 & & \vdots & & \\ \hline
\begin{matrix} 0 \\ d^\ell_{\ell 1} \cdots d^\ell_{\ell\tau_1}\end{matrix} & \begin{matrix} 0 \\ d^\ell_{\ell\tau_1+1} \cdots d^\ell_{\ell\tau_2}\end{matrix} & \cdots & \begin{matrix} \begin{matrix}0\\ \vdots\\ 0\end{matrix} & I_{\kappa_\ell-1} \\ d^\ell_{\ell\tau_{\ell-1}+1} \cdots & \end{matrix} & 0 \\ \hline
0 & 0 & \cdots & 0 & I_p
\end{array}\right] \tag{4.115}$$

$$\tau_i = \sum_{j=1}^{i} \kappa_i, \quad \tau_0 = 0 \tag{4.116}$$

$$B = \left[\begin{array}{c} 0 \\ 1\,0\,\dots\,0 \\ \hline 0 \\ 0\,1\,0\,\dots\,0 \\ \hline \vdots \\ \hline 0 \\ 0\,\dots\,0\,1 \\ \hline 0 \end{array}\right] \begin{array}{l} \Big\updownarrow \kappa_1 \\ \Big\updownarrow \kappa_2 \\ \\ \Big\updownarrow \kappa_\ell \\ \Big\updownarrow p \end{array} \tag{4.117}$$

$$C = [N_{\ell c}\ N^*] \tag{4.118}$$

$$D_{hc} = \begin{bmatrix} d^h_{11} & \dots & d^h_{1\ell} \\ \vdots & & \vdots \\ d^h_{\ell 1} & \dots & d^h_{\ell 1} \end{bmatrix} \tag{4.119}$$

$$D_{\ell c} = -\begin{bmatrix} d^\ell_{11} & \dots & d^\ell_{1,\sum \kappa_i} \\ \vdots & & \vdots \\ d^\ell_{\ell 1} & \dots & d^\ell_{\ell,\sum \kappa_i} \end{bmatrix} \tag{4.120}$$

$$Q_1 = \begin{bmatrix} q_{11} & \dots & q_{1\ell} \\ \vdots & & \vdots \\ q_{p1} & \dots & q_{p\ell} \end{bmatrix} \tag{4.121}$$

Equations (4.112), (4.113) are clearly a state–space representation of the transfer function $G(s)$.

Note that the entries of the above matrices can be found by inspection from the coefficients of the entries of the matrices of the MFD.

Remark 4.6.1 *The above realisation is uniquely defined from the given MFD since Q_1 is unique and E_{11}, A_{11}, B are directly determined from the coefficients of the polynomials of the MFD. Thus, if we start from a given MFD in echelon form, which is unique among the MFDs of $G(s)$,* [For., 1975], *then the resulting realisation is uniquely defined by $G(s)$.* □

4.7 Properties of the MFD based realisation

The properties of the MFD based realisation introduced above are discussed next.

Proposition 4.7.1 *The realisation* (4.112), (4.113) *is reachable.*

Proof: A generalised state–space realisation of the form (4.2), (4.3) is called reachable if

$$\text{rank}[sE - A, -B] = n \ \ \forall\, s \in \mathcal{C} \tag{4.122}$$

and

$$\text{rank}[E\ B] = n \tag{4.123}$$

where n is the dimension of the matrix E. Consider now the matrices E, A, B appearing in (4.114), (4.115), (4.117). From the form of these matrices it is easy to verify that the Smith form of the pencil in (4.124) is

$$S_{E,A,B} = [I\ 0] \tag{4.124}$$

and therefore the reachability pencil does not have any finite zeros. Consider now the matrix $[E\ B]$. By elementary transformations we can bring it to the form

$$\left[\begin{array}{c|c} I_{\tau_\ell} & 0 \\ \hline 0 & Q_1 \end{array}\right] \tag{4.125}$$

Since $n = \sum_{i=1}^{\ell} \kappa_i + p$ it follows that $\text{rank}[E\ B] = n$. □

Going back to the realisation of $G(s)$ we can write the state equations as follows

$$\left[\begin{array}{c|c} E_{11} & 0 \\ E_{12} & 0 \end{array}\right] \dot{x}(t) = \left[\begin{array}{c|c} A_{11} & 0 \\ 0 & I \end{array}\right] x(t) + \left[\begin{array}{c} B_1 \\ 0 \end{array}\right] u(t) \tag{4.126}$$

where

$$E_{11} = \left[\begin{array}{cc|cc|c|cc}
\mathrm{I}_{\kappa_1-1} & \begin{matrix}0\\ \vdots\\ 0\end{matrix} & & 0 & \cdots & & 0 \\
0\ \ldots\ 0 & d^h_{11} & 0\ \ldots\ 0 & d^h_{12} & & 0\ \ldots\ 0 & d^h_{2,\ell} \\ \hline
0 & \begin{matrix}0\\ \vdots\\ 0\end{matrix} & \mathrm{I}_{\kappa_\ell-1} & \begin{matrix}0\\ \vdots\\ 0\end{matrix} & \cdots & & 0 \\
0\ \ldots\ 0 & d^h_{21} & 0\ \ldots\ 0 & d^h_{22} & & 0\ \ldots\ 0 & d^h_{2,\ell} \\ \hline
 & & & & \vdots & & \\ \hline
 & 0 & & 0 & \cdots & \mathrm{I}_{\kappa_\ell-1} & \begin{matrix}0\\ \vdots\\ 0\end{matrix} \\
0\ \ldots\ 0 & d^h_{\ell 1} & 0\ \ldots\ 0 & d^h_{\ell 2} & & 0\ \ldots\ 0 & d^h_{\ell\ell}
\end{array}\right] \tag{4.127}$$

$$E_{11} = \left[\begin{array}{cc|cc|c|cc} & q_{11} & & q_{12} & & & q_{1\ell} \\ 0 & \vdots & 0 & \vdots & \cdots & 0 & \vdots \\ & q_{p1} & & q_{p2} & & & q_{p\ell} \end{array}\right] \tag{4.128}$$

$$A_{11} = \left[\begin{array}{c|c|c|c} \begin{matrix} 0 & \\ \vdots & I_{\kappa_1 - 1} \\ 0 & \end{matrix} & 0 & & 0 \\ d^{\ell}_{11} \dots d^{\ell}_{1\tau_1} & d^{\ell}_{1\tau_1+1} \cdot\cdot d^{\ell}_{2\tau_2} & \cdots & d^{\ell}_{1\tau_{\ell-1}+1} \cdot\cdot d^{\ell}_{1\tau_\ell} \\ \hline 0 & \begin{matrix} 0 & \\ \vdots & I_{\kappa_2 - 1} \\ 0 & \end{matrix} & & 0 \\ d^{\ell}_{21} \dots d^{\ell}_{2\tau_1} & d^{\ell}_{2\tau_1+1} \cdot\cdot d^{\ell}_{2\tau_2} & \cdots & d^{\ell}_{1\tau_{\ell-1}+1} \cdot\cdot d^{\ell}_{2\tau_\ell} \\ \hline & & \vdots & \\ \hline 0 & 0 & & \begin{matrix} 0 & \\ \vdots & I_{\kappa_\ell - 1} \\ 0 & \end{matrix} \\ d^{\ell}_{\ell 1} \dots d^{\ell}_{\ell\tau_1} & d^{\ell}_{\ell\tau_1+1} \cdot\cdot d^{\ell}_{\ell\tau_2} & \cdots & d^{\ell}_{\ell\tau_{\ell-1}+1} \cdot\cdot d^{\ell}_{\ell\tau_\ell} \end{array}\right] \tag{4.129}$$

$$B = \begin{array}{c} \left[\begin{array}{c} 0 \\ 1\,0 \dots 0 \\ \hline 0 \\ 0\,1\,0 \dots 0 \\ \hline \vdots \\ \hline 0 \\ 0 \dots 0\,1 \end{array}\right] \end{array} \begin{array}{l} \\ \kappa_1 \\ \\ \kappa_2 \\ \\ \\ \kappa_\ell \\ \\ \end{array} \tag{4.130}$$

From the form of the matrices E_{11}, A_{11}, *and* B_1 it is clear that

$$(sE_{11} - A_{11})L(s) = B_1 D(s) \tag{4.131}$$

Remark 4.7.1 *Following along similar lines as in* [Kail., 1980], (lemma 6.4-2) *it can be easily proved that*

$$\det(sE - A) = \det(sE_{11} - A_{11}) = \kappa \cdot detD(s) \tag{4.132}$$

where κ is constant. □

From the above remark and (4.131) it follows, that if s_0 is an eigenvalue of $sE - A$ then it is a zero of $D(s)$ and there exists vector q such that

$$(s_0 E_{11} - A_{11}) L(s_0) q = B_1 D(s_0) q = 0 \tag{4.133}$$

Since s_0 is an eigenvalue of $sE_{11} - A_{11}$ we conclude that the eigenvectors of that pencil have the form

$$p = L(s_0) q \tag{4.134}$$

We proceed now to the following main result.

Theorem 4.7.1 *Let the pair* $(N(s), D(s))$ *be column reduced. Then the reachable realisation* (4.112), (4.113) *is observable, if and only if the pair* $(N(s), D(s))$ *is coprime.*

Proof: Let the realisation have finite unobservable modes. Then, $\exists$ nonzero vector $t = [t_1^T, t_2^T]^T$ such that

$$\begin{bmatrix} s_0 E - A \\ C \end{bmatrix} t = 0 \tag{4.135}$$

where s_0 is an eigenvalue of the pencil $s_0 E - A$. The above means

$$(s_0 E - A) t = 0 \tag{4.136}$$

$$Ct = 0 \tag{4.137}$$

or equivalently

$$\begin{bmatrix} s_0 E_{11} - A_{11} & 0 \\ s_0 E_{21} & -I \\ N_{\ell c} & N^* \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} = 0 \tag{4.138}$$

From remark 4.7.1 and (4.134) the above yields

$$t_1 = L(s_0) q \tag{4.139}$$

$$s_0 E_{21} L(s_0) q - t_2 = 0 \tag{4.140}$$

Observe now that

$$s E_{21} L(s) = Q_1 H(s) \tag{4.141}$$

and consider the numerator matrix $N(s)$, i.e.

$$N(s) = N_{hc}H(s) + N_{\ell c}L(s) =$$

$$N_{hc}QQ^{-1}H(s) + N_{\ell c}L(s) =$$

$$[N^* \ 0]\begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} H(s) + N_{\ell c}L(s) =$$

$$(sN^*E_{21} + N_{\ell c}L(s) =$$

$$[N_{\ell c} \ N^*]\begin{bmatrix} I \\ sE_{21} \end{bmatrix} L(s) \tag{4.142}$$

then from (4.139), (4.140) we have

$$N(s_0)q = 0 \tag{4.143}$$

where

$$D(s_0)q = 0 \tag{4.144}$$

or

$$\begin{bmatrix} N(s_0) \\ D(s_0) \end{bmatrix} q = 0 \tag{4.145}$$

i.e. the MFD is not coprime. This is a contradiction, since the MFD is assumed to be coprime. Thus, the realisation does not have unobservable finite modes.

Now, let the realisation be unobservable at infinity. Then the matrix $[E^T \ C^T]^T$ is rank defficient, or equivalently the matrix

$$\begin{bmatrix} E_{11} & 0 \\ E_{21} & 0 \\ N_{\ell c} & N^* \end{bmatrix} \tag{4.146}$$

is rank defficient. Examine first the matrix

$$\begin{bmatrix} E_{11} \\ E_{21} \end{bmatrix} \tag{4.147}$$

From (4.127) and (4.128) we can see that the rank of the above matrix is determined by its nontrivial columns i.e. from the columns $\kappa_1, \kappa_1+\kappa_2, \ldots, \kappa_1+\ldots+\kappa_\ell$, or equivalently from the matrix

$$\begin{bmatrix} D_{hc} \\ Q_1 \end{bmatrix} \tag{4.148}$$

The above matrix may be written as follows:

$$\begin{bmatrix} D_1Q_1 + D_2Q_2 \\ Q_1 \end{bmatrix} = \begin{bmatrix} D_1 & D_2 \\ I & 0 \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} \tag{4.149}$$

where D_1 has full column rank and therefore, the matrix (4.147) has full column rank which yields that matrix (4.146) has full column rank i.e. the system is observable at infinity.

Conversely, consider the case where the MFD is not coprime i.e.

$$\begin{bmatrix} N(s_0) \\ D(s_0) \end{bmatrix} q = 0 \tag{4.150}$$

for $s_0 \in C$ and $q \neq 0$. Then from (4.135) and (4.138)-(4.141) we have that

$$\begin{bmatrix} s_0E_{11} - A_{11} & 0 \\ s_0E_{21} & -I \\ N_{\ell c} & N^* \end{bmatrix} \begin{bmatrix} L(s_0) \\ Q_1H(s_0) \end{bmatrix} q = 0 \tag{4.151}$$

The above means that the realisation is unobservable. □

The following proposition relates the c.m.i. of the controllability pencil of a realisation to the Forney indices of the composite matrix $T(s)$ and it is a generalisation of the strictly proper systems case result.

Proposition 4.7.2 *The c.m.i. of the pencil $[sE - A, B]$ are equal to the Forney indices of the composite matrix $T(s)$, of the MFD $G(s) = N(s)D^{-1}(s)$.*

Proof: From (4.131) and (4.141) we have that

$$\begin{bmatrix} sE_{11} - A_{11} & 0 \\ sE_{21} - 0 & I \end{bmatrix} \begin{bmatrix} L(s) \\ -Q_1H(s) \end{bmatrix} = \begin{bmatrix} B_1D(s) \\ 0 \end{bmatrix} = \begin{bmatrix} B_1 \\ 0 \end{bmatrix} D(s) \tag{4.152}$$

or

$$\left[\begin{array}{cc|c} sE_{11} - A_{11} & 0 & -B_1 \\ sE_{21} - 0 & I & 0 \end{array}\right] \begin{bmatrix} L(s) \\ -Q_1H(s) \\ D(s) \end{bmatrix} = 0 \tag{4.153}$$

The matrix

$$\begin{bmatrix} L(s) \\ -Q_1 H(s) \\ D(s) \end{bmatrix} \tag{4.154}$$

does not have finite zeros, since $L(s)$ contains a square submatrix of dimension ℓ with nonzero determinant. Our aim is to prove that the above matrix is also column reduced. Consider the matrix T_{hc}. This may be written as follows:

$$T_{hc} = \begin{bmatrix} D_{hc} \\ N_{hc} \end{bmatrix} QQ^{-1} = \begin{bmatrix} D_1 & D_2 \\ N^* & 0 \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = \begin{bmatrix} D_1 Q_1 + D_2 Q_2 \\ N^* Q_1 \end{bmatrix} = \begin{bmatrix} D_{hc} \\ N Q_1 \end{bmatrix} =$$

$$\begin{bmatrix} I & 0 \\ 0 & N^* \end{bmatrix} \begin{bmatrix} D_{hc} \\ Q_1 \end{bmatrix} \tag{4.155}$$

Since the above matrix has full column rank ℓ it follows that the matrix

$$\begin{bmatrix} D_{hc} \\ Q_1 \end{bmatrix} \tag{4.156}$$

has full column rank. Therefore the high order coefficient matrix

$$\begin{bmatrix} 0 \\ -Q_1 \\ D_{hc} \end{bmatrix} \tag{4.157}$$

has full column rank i.e. the matrix in (4.154) is column reduced with column degrees equal to the column degrees of $T(s)$. From this and the fact that (4.154) does not have finite zeros, the result follows. □

Corollary 4.7.1 *If the pair $(N(s), D(s))$ is coprime and column reduced, then $\mathcal{P}_\infty = p =$ $\text{rank}\{N_{hc}\}$ is the number of the trivial and nontrivial infinite elementary divisors of $sE - A$.*

Proof: Since the reachable realisation obtained above, is also observable, it is minimal. The dimension of the realisation is

$$n = \sum_{i=1}^{\ell} \kappa_i + p \tag{4.158}$$

From section 4.5 we have that the dimension of the minimal realisation is equal to the extended McMillan degree plus the number of the i.e.d. of $sE - A$. Now, the extended McMillan degree is equal to $\sum_{i=1}^{\ell} \kappa_i$ and the result follows. □

Remark 4.7.2 *Dual results for observable realisation may be obtained using left MFDs of $G(s)$.* □

Remark 4.7.3 *In the case where the transfer function $G(s)$ is strictly proper and the MFD is column reduced, the resulting matrix E of the generalised state–space description is invertible, since $p = 0$ and D_{hc} has full rank and therefore, the realisation is a state–space realisation.* □

Remark 4.7.4 *From the above realisation and proposition* 4.7.1 *we can easily identify the reachability and controllability indices of the singular system* [Gl.-Luer., 1990], [Kar. & Hel., 1990], [Mal., Kuc. & Zag., 1990] *as follows :*

(i) The controllability indices are the column degrees of the coprime and column reduced MFD of $G(s)$. Furthermore, the nonproper controllability indices are the indices corresponding to the columns of $T(s)$ with $\deg(\hat{n}_i(s)) > \deg(d_i(s))$.

(ii) The reachability indices r_i are given by

$$r_i = \deg(\hat{n}_i((s)) + 1 \tag{4.159}$$

where

$$\hat{n}_i = i - th \text{ col}\{\begin{bmatrix} L(s) \\ -Q_1 H(s) \end{bmatrix}\} \tag{4.160}$$

□

Next, we give an example to illustrate the realisation method.

Example 4.7.1 Let

$$G(s) = \begin{bmatrix} \frac{2s^3+4s^2+11s+2}{-s^4-2s^2+s+1} & \frac{s^5+6s^3+5s+1}{-s^4-2s^2+s+1} \\ \frac{3s^5+4s^4+20s^3+3s^2+12s-6}{s^6+5s^4-s^3+5s^2-3s-3} & \frac{s^7+s^5-s^4-5s^3-15s+3}{s^6+5s^4-s^3+5s^2-3s-3} \end{bmatrix}$$

a coprime and column reduced MFD of the above transfer function be

$$T(s) = \begin{bmatrix} D(s) \\ N(s) \end{bmatrix} = \left[\begin{array}{cc} s^2+1 & s^2+3 \\ s+2 & s^4+4s^2+3 \\ \hline s^2 & s^5+8s^3+s^2+15s+3 \\ s^2+5s & s^5+3 \end{array}\right]$$

Then

$$N_{hc}=\begin{bmatrix}1&1\\1&1\end{bmatrix},\ Q=\begin{bmatrix}1&1\\0&-1\end{bmatrix},\ Q_1=[1\ 1],\ D_{hc}=\begin{bmatrix}1&0\\0&0\end{bmatrix},\ N^*=\begin{bmatrix}1\\1\end{bmatrix}$$

$$N_{\ell c}=\begin{bmatrix}0&0&3&15&1&8&0\\0&5&3&0&0&0&0\end{bmatrix},\ D_{\ell c}=\begin{bmatrix}1&0&3&0&1&0&0\\2&1&3&0&4&1&0\end{bmatrix}$$

and the matrices of the generalised state–space realisation are

$$E=\left[\begin{array}{cc|cccccc|c}1&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\\\hline 0&0&1&0&0&0&0&0\\0&0&0&1&0&0&0&0\\0&0&0&0&1&0&0&0\\0&0&0&0&0&1&0&0\\0&0&0&0&0&0&0&0\\\hline 0&1&0&0&0&0&1&0\end{array}\right],\ A=\left[\begin{array}{cc|ccccc|c}0&1&0\ 0&0\ 0&0&0\\-1&0&-3\ 0&-1\ 0&0&0\\\hline 0&0&0\ 1&0\ 0&0&0\\0&0&0\ 0&1\ 0&0&0\\0&0&0\ 0&0\ 1&0&0\\0&0&0\ 0&0\ 0&1&0\\-2&-1&-3\ 0&-4\ 0&-1&0\\\hline 0&0&0\ 0&0\ 0&0&1\end{array}\right],\ B=\left[\begin{array}{cc}0&0\\1&0\\\hline 0&0\\0&0\\0&0\\0&0\\0&1\\\hline 0&0\end{array}\right],$$

$$C=\left[\begin{array}{ccccccc|c}0&0&3&15&1&8&0&1\\0&5&3&0&0&0&0&1\end{array}\right]$$

It can be easily verified that the above realisation is minimal.

4.8 Conclusions

In this chapter the problem of the realisation of nonproper transfer functions has been considered. First, two realisation procedures for the polynomial part of $G(s)$ were given and the overall system was realised by taking the direct sum of the realisation of the strictly proper and polynomial part of $G(s)$. These procedures were used in order to prove formally that we may always take a descriptor realisation of a nonproper transfer function. The minimality of a descriptor realisation was related to the McMillan degree of the composite matrix of a given coprime and column reduced MFD of $G(s)$. The main result of the chapter is the realisation procedure of $G(s)$ via MFDs.

If the MFD is coprime and column reduced, the resulting singular system is minimal. The presented method is a natural generalisation of the realisation of strictly proper transfer functions. The form of the g.s.s. equations reveals the controllability indices of the system as well as the number of the infinite elementary divisors of the pencil $sE-A$, when the realisation is minimal.

Chapter 5

CANONICAL FORMS AND INVARIANTS FOR SINGULAR SYSTEMS

5.1 Introduction

The issue of canonical forms of linear systems under certain transformations has received attention from many researchers for the past 20 years [Luen., 1966], [Pop., 1972], [Den,. 1974], [Brun., 1970] etc.. There are many reasons that make the knowledge of the canonical elements desirable. First, the representation of a canonical element of a family of equivalent systems (equivalence class) usually contains a small number of parameters which characterise the system. Thus, the study of an equivalence class is reduced to the study of a single element (the canonical element) of simple structure. In the case of linear systems, the invariants of the equivalence classes characterise the behaviour of the system and provide criteria for the design and classification of the controllable state–space systems in terms of two types of invariants (discrete and continuous) under similarity transformations of the state–space model. There are several canonical forms that may be defined for regular state–space systems.

When we are interested only in the free response of a given system, the Jordan canonical form of the matrix A of a state–space system provides all the necessary information in a simple way (eigenvalues–eigenvectors of A). When the system is forced (the inputs are nonzero) the need of a canonical form for the pair (A, B) emerges [Luen., 1966], [Pop., 1972], [Den,. 1974], [Brun., 1970].

The canonical form is defined according to a given type of transformation (transformation group) on the system. When the transformation group is richer than the similarity transformation e.g. state–feedback, output injection e.t.c. we end up with different canonical forms than in the case of similarity transformations [Kar., & MacB, 1981]. It is plausible to say that the "larger" the transformation group, the simpler the canonical form and its derivation.

The canonical forms of the state–space model are related to the canonical forms of the MFD descriptions via the echelon canonical forms for polynomial matrices under unimodular transformations [Pop., 1969], [Pop., 1972], [For., 1975]. This relation shows that the input–output and internal variable (state) descriptions are related in a straightforward manner and that some results of state–space theory may be derived directly via the transfer function approach [For., 1975].

The theory of the canonical forms is extended to the case of descriptor systems by using the restricted system equivalence transformations [Ros., 1974b] instead of similarity transformations. Although this extension seems to be straightforward, this is not always the case as it is shown in this chapter.

When we use rich groups of transformations such as the Brunovsky group it is easy to extend the state–space results to the descriptor system. However, if we restrict ourselves to the restricted system equivalence transformations, which means that the freedom of changing the parameters of the system is restricted, there are some cases where the extension is not complete yet.

The problem of canonical forms of $\mathcal{S}(E, A, B)$ was considered by [Dou.& Fen., 1987], [Gl.-Luer. & Hin., 1987], [Hel. & Shay., 1989]. These canonical forms cover the general case of singular systems (the case where the reachability indices are equal). However, these forms are based in the Weierstrass form of the pencil $sE - A$. In the above works the system $\mathcal{S}(E, A, B)$ is decomposed into the slow and fast systems. However these forms are not related to the Popov canonical form and the reachability indices of the system are not displayed through the above canonical forms.

It is the purpose of this Chapter to generalise some of the canonical forms to descriptor systems under strict equivalence transformations.

First, systems with outputs are considered. For those systems, the problem of canonical forms under restricted system equivalence transformations is solved entirely and it is shown that the invariants of the canonical form may be obtained directly by the parameters of the echelon form of the MFD of the input–output transfer function of the system.

Next, the problem of Popov type canonical forms for reachable systems is tackled. For the case where all the reachability indices of the regular descriptor system are equal, the problem is solved completely, while in general case, a semi canonical form is obtained.

5.2 Preliminaries and statement of the problem

Consider the reachable singular system $\mathcal{S}(E, A, B)$ described by the equation

$$\mathcal{S} : E\dot{x} = Ax + Bu \tag{5.1}$$

$$y = Cx \tag{5.2}$$

where $x \in \mathcal{Z} \approx \Re^n$, $u \in \mathcal{U} \approx \Re^\ell$, $y \in \mathcal{Y} \approx \Re^m$, $(E, A, B, C) \in \Re^{n\times n} \times \Re^{n\times n} \times \Re^{n\times \ell} \times \Re^{m\times n}$ rank $\{B\} = \ell$ and rank $\{C\} = m$. System (5.1), (5.2) is assumed to be minimal, i.e. $[sE - A, -B]$, $[sE^T - A^T, C^T]^T$ do not have finite Smith zeros and rank$[E, B]$=rank$[E^T, C^T]^T = n$ and regular i.e. $\det\{sE - A\} \neq 0$. For this system we define the system matrix $P(s)$ [Ros., 1974b].

$$P(s) = \begin{bmatrix} sE - A & -B \\ C & 0 \end{bmatrix} \tag{5.3}$$

The action of restricted system equivalence transformations on the system $\mathcal{S}(E, A, B, C)$

is defined as follows

$$(P,Q) \circ (E,A,B,C) = (PEQ,PAQ,PB,CQ) \tag{5.4}$$

where $(P,Q) \in \Re^{n\times n} \times \Re^{n\times n}$ and $\det\{P\} \neq 0$, $\det\{Q\} \neq 0$.

The result of the above transformation is the equivalent system

$$\mathcal{S}'{:}PEQ\dot{x}' + PBu$$

$$y = CQx'$$

The action of the transformations (5.4) on the system matrix $P(s)$ is the following:

$$P'(s) = \begin{bmatrix} P & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} sE-A & -B \\ C & 0 \end{bmatrix} \begin{bmatrix} Q & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} sPEQ - PAQ & -PB \\ CQ & 0 \end{bmatrix} \tag{5.5}$$

It is well known that the restricted system equivalence transformations define an equivalence relation which partitions the set of all quadruples $(E,A,B,C) \in \Re^{n\times n} \times \Re^{n\times n} \times \Re^{n\times \ell} \times \Re^{m\times n}$ to equivalence classes or orbits. Our aim is to find the canonical element of each orbit and the invariants characterising the orbit.

The problem of finding a canonical form for the singular system (5.1), (5.2) under the transformations (5.4) will be referred to as the problem of *canonical form with outputs.* Clearly, this problem is equivalent to the problem of canonical form of the pencil $P(s)$ under the transformation shown in (5.5). It is mentioned that the problem of canonical forms for descriptor systems with outputs was solved entirely for the case where the transformation group is the Brunovsky group [Brun., 1970] and derivative feedback is allowed, by Lebret and Loiseau in [Leb. & Lois., 1994]. The case where the allowed transformations are strict equivalence and state feedback was considered in [Lois., Ozc., et al., 1991]. The problem considered here, has the difficulty that the transformation group, which is a subgroup of the Brunovsky group, reduces the freedom of the elementary transformations allowed on the pencil $P(s)$. It is expected to end up with a different set of invariants than that of the general (Brunovsky) case.

The second problem considered in this chapter is the problem of finding a canonical form of the triple (E,A,B) i.e. we consider only the state equation $E\dot{x} = Ax + Bu$. This problem will be referred to as *the problem of canonical form.* In this case the action of the transformation is defined as follows

$$(P,Q) \circ (E,A,B) = (PEQ,PAQ,PB) \tag{5.6}$$

The key tool for the development of the canonical form of (5.1) is the *reachability pencil*

$$T(s) = [sE - A, -B] \tag{5.7}$$

The reachability pencil $T'(s)$ of a system in the same orbit to $\mathcal{S}$ is

$$T'(s) = [sPEQ - PAQ, -PB] \tag{5.8}$$

or

$$T'(s) = P[sE - A, -B]\begin{bmatrix} Q & 0 \\ 0 & I \end{bmatrix} = PT(s)\begin{bmatrix} Q & 0 \\ 0 & I \end{bmatrix} \tag{5.9}$$

From the above it is clear that $T(s)$ and $T'(s)$ are related by strict equivalence transformations and thus, they have the same Kronecker invariants.

Since restricted system equivalence transformations on $\mathcal{S}$ induce strict equivalence transformations on $T(s)$, the canonical form of $T(s)$ yields a canonical form of $\mathcal{S}$ in a straightforward manner.

As a preliminary step towards the derivation of the canonical forms described above we consider the following.

Let $N \in \Re^{(n-\ell)\times n}$ be a left annihilator of B and $B^\dagger \in \Re^{\ell\times n}$ a left inverse of B, i.e $NB = 0$ and $B^\dagger B = I_\ell$. Then the matrix $\begin{bmatrix} N \\ B^\dagger \end{bmatrix}$ is invertible. Consider the pencil

$$\begin{bmatrix} N \\ B^\dagger \end{bmatrix}[sE - A, -B] = \begin{bmatrix} sNE - NA, & 0 \\ sB^\dagger E - B^\dagger A, & -I \end{bmatrix} \tag{5.10}$$

which is clearly strictly equivalent to $T(s)$. Furthermore, it is known that the pencil $sNE - NA$ [Kar., 1990] has only column minimal indices c.m.i. which coincide with the reachability indices of the system $\mathcal{S}$. Thus, the pencil (5.10) is strictly equivalent to a pencil of the form:

$$\begin{bmatrix} L(s) & 0 \\ sK - \Lambda & -I \end{bmatrix} \tag{5.11}$$

where

$$L(s) = \text{block} - \text{diag}\{\ldots, L_{\varepsilon_i}(s), \ldots\} \tag{5.12}$$

and

$$L_{\varepsilon_i}(s) = \begin{bmatrix} s & -1 & 0 & \cdot & \cdot & \cdot \\ & \cdot & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & s & -1 \end{bmatrix}_{\varepsilon_i \times (\varepsilon_i + 1)} \tag{5.13}$$

and $sK - \Lambda$ is unspecified.

Similarly, the pencil $P(s)$ may be transformed to the form

$$P(s) \sim \left[\begin{array}{c|c} L(s) & 0 \\ sK - \Lambda & -I \\ \hline \tilde{C} & 0 \end{array}\right] \tag{5.14}$$

As far as the canonical form without outputs is concerned, we are going to investigate only the case where the reachability indices of the system $\mathcal{S}(E, A, B)$ are equal.

The matrix pencil in (5.10) will be referred to as pseudo canonical form of $T(s)$ and (5.14) as the pseudo canonical form of $P(s)$. Forms (5.11) and (5.14) are not canonical since $sK - \Lambda$ is not uniquely defined.

The pseudo canonical form will be used as an intermediate step for the construction of the canonical forms of $T(s)$ and $P(s)$ under transformations (5.6) and (5.4) respectively.

5.3 The stabilizer of the canonical element $L(s)$

In this section we consider the matrix pencil $L(s)$ (see 5.12) and we find its stabilizer. The stabilizer plays an important role to the determination of the canonical form of $T(s)$ since it leads to the transformations that bring $T(s)$ to the canonical form.

Definition 5.3.1 [MacL. & Bir., 1967] *Let $\mathcal{X}$ be a set and G be a transformation group acting on $\mathcal{X}$ with the action $g : G \times \mathcal{X} \to \mathcal{X}$, $gx \mapsto y$, $g \in G$ $x, y \in \mathcal{X}$. Consider a fixed $x_0 \in \mathcal{X}$. Then the set $G_{x_0} \subset G$ with the property $gx_0 \mapsto x_0$, $g \in G_{x_0}$ is defined as the stabilizer of x_0. The set G_{x_0} is a subgroup of G and it is denoted by $Stab(x_0)$.* □

The above definition may be translated as follows. The stabilizer is the subgroup of the transformation group for which the action on x_0, leaves x_0 unchanged.

The following results are related to the derivation of the stabilizer of $L(s)$.

Lemma 5.3.1 *Let P, Q be such that $PL_{\varepsilon_i}(s) = L_{\varepsilon_i}(s)Q$ where $L_{\varepsilon_i}(s)$ is the standard c.m.i. block. Then the forms of P, Q are the following:*

$$P = \lambda I_{\varepsilon_i},\ Q = \lambda I_{\varepsilon_i+1}, \tag{5.15}$$

where λ is a non zero constant.

Proof: Let $\varepsilon_i = 2$ and

$$P = \begin{bmatrix} p_1 & p_2 \\ p_3 & p_4 \end{bmatrix}, Q = \begin{bmatrix} q_1 & q_2 & q_3 \\ q_4 & q_5 & q_6 \\ q_7 & q_8 & q_9 \end{bmatrix}$$

then from $PL_2(s) = L_2(s)Q$ it follows:

$$\begin{array}{ccc} p_1 = q_1 & p_2 = q_2 & q_3 = 0 \\ p_3 = q_4 & p_4 = q_5 & q_6 = 0 \\ q_4 = 0 & p_1 = q_5 & p_2 = q_6 \\ q_7 = 0 & p_3 = q_8 & p_4 = q_9 \end{array}$$

Therefore,

$$Q = p_1 I_3,\ P = p_1 I_2$$

where p_1 is arbitrary. Now, let (5.15) hold true for $\varepsilon_i = \kappa$. We are going to prove that it holds for $\varepsilon_i = \kappa + 1$. The pencil $L_{\kappa+1}(s)$ is as follows

$$L_{\kappa+1}(s) = \left[\begin{array}{c|c} L_\kappa(s) & \underline{0} \\ \hline \underline{0} \quad s & -1 \end{array}\right] \tag{5.16}$$

Let

$$P = \begin{bmatrix} p_1 & \underline{p}_2^T & p_3 \\ \underline{p}_4 & \tilde{P} & \underline{p}_5 \\ p_6 & \underline{p}_7^T & p_8 \end{bmatrix}, Q = \begin{bmatrix} q_1 & \underline{q}_2^T & q_3 \\ \underline{q}_4 & \tilde{Q} & \underline{q}_5 \\ q_6 & \underline{q}_7^t & q_8 \end{bmatrix} \tag{5.17}$$

then from $P[I_{\kappa+1}|0] = [I_{\kappa+1}|0]Q$ (equate the coefficients of s) we have

$$\left[\begin{array}{c|cc|c} p_1 & \underline{p}_2^T & p_3 & 0 \\ \hline \underline{p}_4 & \tilde{P} & \underline{p}_5 & 0 \\ p_6 & \underline{p}_7^T & p_8 & 0 \end{array}\right] = \left[\begin{array}{c|c|c} q_1 & \underline{q}_2^T & q_3 \\ \hline \underline{q}_4 & \tilde{Q} & \underline{q}_5 \end{array}\right] \tag{5.18}$$

and thus,

$$p_1 = q_1,\ [\underline{p}_2^T, p_3] = \underline{q}_2^t,\ q_3 = 0,\ \begin{bmatrix} \underline{p}_4 \\ p_6 \end{bmatrix} = \underline{q}_4,\ \begin{bmatrix} \tilde{P} & \underline{p}_5 \\ \underline{p}_7^t & p_8 \end{bmatrix} = \tilde{Q},\ \underline{q}_5 = 0 \tag{5.19}$$

Now, equating the constant terms we take

$$\left[\begin{array}{c|cc|c} 0 & p_1 & \underline{p}_2^T & p_3 \\ 0 & \underline{p}_4 & \tilde{P} & \underline{p}_5 \\ \hline 0 & p_6 & \underline{p}_7^T & p_8 \end{array}\right] = \left[\begin{array}{c|c|c} \underline{q}_4 & \tilde{Q} & \underline{q}_5 \\ \hline q_6 & \underline{q}_7^T & q_8 \end{array}\right] \tag{5.20}$$

or

$$\begin{bmatrix} \underline{q}_4 \\ q_6 \end{bmatrix} = 0,\ \begin{bmatrix} p_3 \\ \underline{p}_5 \end{bmatrix} = 0,\ \begin{bmatrix} p_1 & \underline{p}_2^T \\ \underline{p}_4 & \tilde{p} \end{bmatrix} = \tilde{Q},\ [p_6\ \underline{p}_7^T] = \underline{q}_7^T,\ p_8 = q_8 \tag{5.21}$$

From (5.19) and (5.21) it follows

$$Q = \left[\begin{array}{cc|c} p_1 & \underline{q}_2^T & 0 \\ 0 & \tilde{Q} & 0 \\ \hline 0 & \underline{q}_7^t & p_8 \end{array}\right] = \left[\begin{array}{c|c} Q' & 0 \\ \hline \underline{q}' & p_8 \end{array}\right] \tag{5.22}$$

$$P = \left[\begin{array}{cc|c} p_1 & \underline{p}_2^T & 0 \\ 0 & \tilde{P} & 0 \\ \hline 0 & \underline{p}_7^T & p_8 \end{array}\right] = \left[\begin{array}{c|c} P' & 0 \\ \hline \underline{p}' & p_8 \end{array}\right] \tag{5.23}$$

Using (5.22), (5.23) and

$$PL_{\kappa+1}(s) = L_{\kappa+1}(s)Q \tag{5.24}$$

we take

$$\left[\begin{array}{c|c} P' & 0 \\ \hline \underline{p}' & p_8 \end{array}\right] \left[\begin{array}{c|c} L_\kappa(s) & 0 \\ \hline e_{\kappa+1}(s) & -1 \end{array}\right] = \left[\begin{array}{c|c} L_\kappa(s) & 0 \\ \hline e_{\kappa+1}(s) & -1 \end{array}\right] \left[\begin{array}{c|c} Q' & 0 \\ \hline \underline{q}' & p_8 \end{array}\right] \tag{5.25}$$

where $e_i(s) = [0, \dots, 0, s, 0, \dots, 0]$ (s in the i–th position).
The equation defined from the top block of (5.25) is

$$P' L_\kappa(s) = L_\kappa(s) Q' \tag{5.26}$$

which means that

$$P' = p_1 I_\kappa, \; Q' = p_1 I_{\kappa+1} \tag{5.27}$$

Therefore from (5.19) and (5.21)

$$\underline{q}_2^t = 0, \; \underline{p}_2^t = 0, \; \underline{p}_7^t = 0, \; \underline{q}_7^t = 0, \; p_1 = p_8 \tag{5.28}$$

and $P = p_8 I_{\kappa+1}$, $Q = p_8 I_{\kappa+2}$ and since p_8 is arbitrary, the result follows. □

The following lemmas may be proven along similar lines.

Lemma 5.3.2 *Let P, Q be such that $PL_{\varepsilon_i}(s) = L_{\varepsilon_j}(s)Q$ where $\varepsilon_i > \varepsilon_j$. Then P, Q have the following forms*

$$P = \begin{bmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_{\varepsilon_i-\varepsilon_j+1} & & & \\ \cdot & \cdot & & \cdot & & & \\ & \cdot & \cdot & & \cdot & & \\ & & \cdot & \cdot & & \cdot & \\ & & \lambda_1 & \lambda_2 & \cdots & \lambda_{\varepsilon_i-\varepsilon_j+1} & \end{bmatrix}_{\varepsilon_j \times \varepsilon_i} \tag{5.29}$$

$$Q = \begin{bmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_{\varepsilon_i-\varepsilon_j+1} & & & \\ \cdot & \cdot & & \cdot & & & \\ & \cdot & \cdot & & \cdot & & \\ & & \cdot & \cdot & & \cdot & \\ & & \lambda_1 & \lambda_2 & \cdots & \lambda_{\varepsilon_i-\varepsilon_j+1} \end{bmatrix}_{(\varepsilon_j+1)\times(\varepsilon_i+1)} \tag{5.30}$$

□

Lemma 5.3.3 *Let P, Q be such that $PL_{\varepsilon_i}(s) = L_{\varepsilon_j}(s)Q$ where $\varepsilon_i < \varepsilon_j$. Then $P = 0$ and $Q = 0$.* □

Consider now the pencil $L(s)$ defined in (5.12). Let P, Q be a pair of constant matrices such that

$$PL(s) = L(s)Q \tag{5.31}$$

without loss of generality we assume that $\varepsilon_1 \leq \ldots \leq \varepsilon_\ell$. Partitioning P and Q according to $L(s)$ we have

$$\begin{bmatrix} P_{11} & P_{12} & \cdots & P_{1\ell} \\ P_{21} & P_{22} & \cdots & P_{2\ell} \\ \vdots & \vdots & & \vdots \\ P_{\ell 1} & P_{\ell 2} & \cdots & P_{\ell\ell} \end{bmatrix} \begin{bmatrix} L_{\varepsilon_1}(s) & & & \\ & L_{\varepsilon_2}(s) & & \\ & & \ddots & \\ & & & L_{\varepsilon_\ell}(s) \end{bmatrix} =$$

$$\begin{bmatrix} L_{\varepsilon_1}(s) & & & \\ & L_{\varepsilon_2}(s) & & \\ & & \ddots & \\ & & & L_{\varepsilon_\ell}(s) \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{12} & \cdots & Q_{1\ell} \\ Q_{21} & Q_{22} & \cdots & Q_{2\ell} \\ \vdots & \vdots & & \vdots \\ Q_{\ell 1} & Q_{\ell 2} & \cdots & Q_{\ell\ell} \end{bmatrix} \tag{5.32}$$

or

$$\left.\begin{array}{llll} P_{11}L_{\varepsilon_1}(s) = L_{\varepsilon_1}(s)Q_{11}, & P_{12}L_{\varepsilon_2}(s) = L_{\varepsilon_1}(s)Q_{12}, & \ldots & , P_{1\ell}L_{\varepsilon_\ell}(s) = L_{\varepsilon_1}(s)Q_{1\ell} \\ P_{21}L_{\varepsilon_1}(s) = L_{\varepsilon_2}(s)Q_{21}, & P_{22}L_{\varepsilon_2}(s) = L_{\varepsilon_2}(s)Q_{22}, & \ldots & , P_{2\ell}L_{\varepsilon_\ell}(s) = L_{\varepsilon_2}(s)Q_{2\ell} \\ \vdots & \vdots & & \vdots \\ P_{\ell 1}L_{\varepsilon_1}(s) = L_{\varepsilon_\ell}(s)Q_{\ell 1}, & P_{\ell 2}L_{\varepsilon_2}(s) = L_{\varepsilon_\ell}(s)Q_{\ell 2}, & \ldots & , P_{\ell\ell}L_{\varepsilon_\ell}(s) = L_{\varepsilon_\ell}(s)Q_{\ell\ell} \end{array}\right\} \tag{5.33}$$

From the above lemmas it follows that

$$P_{ii}=\begin{bmatrix}\lambda_i & & \\ & \ddots & \\ & & \lambda_i\end{bmatrix}_{\varepsilon_i},\ Q_{ii}=\begin{bmatrix}\lambda_i & & \\ & \ddots & \\ & & \lambda_i\end{bmatrix}_{\varepsilon_{i+1}} \tag{5.34}$$

$$P_{ij}=0,\ Q_{ij}=0\ if\ \varepsilon_i<\varepsilon_j \tag{5.35}$$

$$\begin{aligned}P_{ij}&=\begin{bmatrix}\kappa_1^{ij} & \cdots & \kappa_{\varepsilon_i-\varepsilon_j+1}^{ij} & & \\ & \ddots & & \ddots & \\ & & \kappa_1^{ij} & \cdots & \kappa_{\varepsilon_i-\varepsilon_j+1}^{ij}\end{bmatrix}_{\varepsilon_j\times\varepsilon_i},\\ Q_{ij}&=\begin{bmatrix}\kappa_1^{ij} & \cdots & \kappa_{\varepsilon_i-\varepsilon_j+1}^{ij} & & \\ & \ddots & & \ddots & \\ & & \kappa_1^{ij} & \cdots & \kappa_{\varepsilon_i-\varepsilon_j+1}^{ij}\end{bmatrix}_{(\varepsilon_j+1)\times(\varepsilon_i+1)}\end{aligned} \tag{5.36}$$

Or P and Q have the form (for the case of $\ell=4$ and $\varepsilon_1<\varepsilon_2=\varepsilon_3<\varepsilon_4$)

$$P=\begin{array}{c}\begin{array}{cccc}\overbrace{\qquad\qquad}^{\varepsilon_1} & \overbrace{\qquad\qquad}^{\varepsilon_2} & \overbrace{\qquad\qquad}^{\varepsilon_3} & \overbrace{\qquad\qquad}^{\varepsilon_4}\end{array}\\ \left[\begin{array}{c|c|c|c}
\begin{matrix}\star & & & \\ & \star & & \\ & & \ddots & \\ & & & \star\end{matrix} & \begin{matrix}\star & \cdots & \star & \\ \vdots & \ddots & & \vdots \\ & \star & \cdots & \star\end{matrix} & \begin{matrix}\star & \cdots & \star & \\ \vdots & \ddots & & \vdots \\ & \star & \cdots & \star\end{matrix} & \begin{matrix}\star & \cdots & \star & \\ \vdots & \ddots & & \vdots \\ & \star & \cdots & \star\end{matrix} \\ \hline
 & \begin{matrix}\star & & & \\ & \star & & \\ & & \ddots & \\ & & & \star\end{matrix} & \begin{matrix}\star & & & \\ & \star & & \\ & & \ddots & \\ & & & \star\end{matrix} & \begin{matrix}\star & \cdots & \star & \\ \vdots & \ddots & & \vdots \\ & \star & \cdots & \star\end{matrix} \\ \hline
 & \begin{matrix}\star & & & \\ & \star & & \\ & & \ddots & \\ & & & \star\end{matrix} & \begin{matrix}\star & & & \\ & \star & & \\ & & \ddots & \\ & & & \star\end{matrix} & \begin{matrix}\star & \cdots & \star & \\ \vdots & \ddots & & \vdots \\ & \star & \cdots & \star\end{matrix} \\ \hline
 & & & \begin{matrix}\star & & & \\ & \star & & \\ & & \ddots & \\ & & & \star\end{matrix}
\end{array}\right]\begin{array}{l}\}\varepsilon_1\\ \}\varepsilon_2\\ \}\varepsilon_3\\ \}\varepsilon_4\end{array}\end{array} \tag{5.37}$$

$$
Q = \begin{array}{c}
\begin{array}{cccc} \overbrace{\qquad\qquad}^{\varepsilon_1+1} & \overbrace{\qquad\qquad}^{\varepsilon_2+1} & \overbrace{\qquad\qquad}^{\varepsilon_3+1} & \overbrace{\qquad\qquad}^{\varepsilon_4+1} \end{array} \\
\left[\begin{array}{c|c|c|c}
\begin{matrix}\star & & & \\ & \star & & \\ & & \ddots & \\ & & & \star\end{matrix} &
\begin{matrix}\star & \cdots & \star \\ \vdots & & \vdots \\ \star & \cdots & \star\end{matrix} &
\begin{matrix}\star & \cdots & \star \\ \vdots & & \vdots \\ \star & \cdots & \star\end{matrix} &
\begin{matrix}\star & \cdots & \star \\ \vdots & & \vdots \\ \star & \cdots & \star\end{matrix} \\ \hline
 &
\begin{matrix}\star & & & \\ & \star & & \\ & & \ddots & \\ & & & \star\end{matrix} &
\begin{matrix}\star & & & \\ & \star & & \\ & & \ddots & \\ & & & \star\end{matrix} &
\begin{matrix}\star & \cdots & \star \\ \vdots & & \vdots \\ \star & \cdots & \star\end{matrix} \\ \hline
 &
\begin{matrix}\star & & & \\ & \star & & \\ & & \ddots & \\ & & & \star\end{matrix} &
\begin{matrix}\star & & & \\ & \star & & \\ & & \ddots & \\ & & & \star\end{matrix} &
\begin{matrix}\star & \cdots & \star \\ \vdots & & \vdots \\ \star & \cdots & \star\end{matrix} \\ \hline
 & & &
\begin{matrix}\star & & & \\ & \star & & \\ & & \ddots & \\ & & & \star\end{matrix}
\end{array}\right]
\end{array}
\begin{array}{l}
\left.\vphantom{\begin{matrix}a\\a\\a\\a\end{matrix}}\right\}\varepsilon_1+1 \\
\left.\vphantom{\begin{matrix}a\\a\\a\\a\end{matrix}}\right\}\varepsilon_2+1 \\
\left.\vphantom{\begin{matrix}a\\a\\a\\a\end{matrix}}\right\}\varepsilon_3+1 \\
\left.\vphantom{\begin{matrix}a\\a\\a\\a\end{matrix}}\right\}\varepsilon_4+1
\end{array}
\tag{5.38}
$$

The blocks below the diagonal are nonzero only in the case where $\varepsilon_i = \varepsilon_{i+1}$. In the above matrices we take $\varepsilon_2 = \varepsilon_3$ and therefore P_{23} and Q_{23} are diagonal matrices. Note that Q^{-1} has the same block structure as Q. We have thus established the following:

Theorem 5.3.1 *The stabilizer of the canonical element $L(s)$ has the form: P is as in (5.34)–(5.36) and Q^{-1} is the matrix obtained by inverting the matrix Q in (5.34)–(5.36).* □

Definition 5.3.2 *Let M_i and M_j be two matrices with equal number of rows and m_i, m_j, $m_i < m_j$ columns respectively. By $c(\alpha, i, j, k)$ we define the operation of the addition of the columns of M_i with indices $m_i, m_i - 1, \ldots, 1$ (multiplied by the scalar α) to the columns of M_j with indices $m_j - k$, $m_j - k - 1, \ldots, m_j - k - m_i + 1$, $k \leq m_j - m_i$ respectively.* □

Write $L(s) = [\hat{L}_1(s), \ldots, \hat{L}_\ell(s)]$ where

$$
\hat{L}_i(s) = \begin{bmatrix} 0 \\ \vdots \\ L_{\varepsilon_i}(s) \\ \vdots \\ 0 \end{bmatrix}, \; i = 1, \ldots, \ell
$$

In this way $\hat{L}_i(s)$ is partitioned to column blocks where each of the $\hat{L}_i(s)$ has equal number of columns to $L_{\varepsilon_i}(s)$. Then we may state the following proposition.

Proposition 5.3.1 *The column operations on $L(s)$ corresponding to Q, where*

$$(P,Q) \in \mathrm{Stab}(L(s))$$

are the following:

(i) Multiplication of the columns of $\hat{L}_i(s)$ by a scalar

(ii) Addition of $\hat{L}_i(s)$ to $\hat{L}_j(s)$ as it is defined by definition 5.3.2

(iii) Permutation of $\hat{L}_i(s)$, $\hat{L}_j(s)$ if $\varepsilon_i = \varepsilon_j$ □

As an example, let

$$L(s) = \left[\begin{array}{ccc|cccc} s & -1 & 0 & & & & \\ 0 & s & -1 & & & & \\ \hline & & & s & -1 & 0 & 0 \\ & & & 0 & s & -1 & 0 \\ & & & 0 & 0 & s & -1 \end{array}\right]$$

and

$$Q = \left[\begin{array}{ccc|cccc} 2 & 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 4 \\ \hline 0 & 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{array}\right], \quad P = \left[\begin{array}{cc|ccc} \frac{1}{2} & 0 & 0 & -4 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & -4 \\ \hline 0 & 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{3} \end{array}\right]$$

Clearly $(Q,P) \in$ Stab $(L(s))$. Then

$$L(s)Q = \left[\begin{array}{ccc|cccc} 2s & -2 & 0 & 0 & 4s & -4 & 0 \\ 0 & 2s & -2 & 0 & 0 & 4s & -4 \\ \hline 0 & 0 & 0 & 3s & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3s & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3s & -3 \end{array}\right]$$

Now premultiplying the above by P we take

$$PL(s)Q = \left[\begin{array}{ccc|cccc} s & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & s & -1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & s & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & s & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & s & -1 \end{array}\right]$$

The action of Q is the addition of 4 times the first block to the three last columns of the second block i.e. it corresponds to the operation $c(4,1,2,0)$.

5.4 A canonical form for constant matrices

In this section a canonical form under a special transformation group for constant matrices is discussed. This canonical form is important for the derivation of both canonical forms with and without outputs.

Let A be a constant matrix and consider the following elementary operations on the columns of A

(i) Multiplication of a column by a non zero constant

(ii) Addition of a multiple of the i-th column of A to the j-th column, where j>i

The above transformations are a subset of the transformations corresponding to postmultiplication of A by a general invertible matrix. The matrix that corresponds to the above transformations is an upper triangular invertible matrix.

Consider now the following **reduction procedure** on A:

1. Multiply the first nonzero column by an appropriate constant such that the upper nonzero element of this column is 1.

2. By elementary column transformations eliminate all the entries to the right of the first (upper) nonzero element of the first column. Then A is transformed to the following matrix.

$$\begin{bmatrix} 0 & 0 & 0 & x & x & \cdots & x \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & 0 & x & x & \cdots & x \\ \vdots & \vdots & 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & x & x & x & \cdots & x \\ \vdots & \vdots & x & x & x & \cdots & x \\ 0 & 0 & x & x & x & \cdots & x \end{bmatrix} \tag{5.39}$$

3. Repeat the above procedure for all the other columns

The resulting matrix has the following form

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & 0 & \boxed{1} & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \boxed{1} & 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & \mathrm{x} & 0 & 0 & \cdots & \cdots & \mathrm{x} \\ \vdots & \vdots & \mathrm{x} & \mathrm{x} & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \mathrm{x} & \vdots & \boxed{1} & 0 & \cdots \\ 0 & 0 & \mathrm{x} & \mathrm{x} & \boxed{1} & 0 & \cdots & \cdots \end{bmatrix} \tag{5.40}$$

Observe that all the entries to the right of the 1's marked by boxes are zero and all the entries above the 1's are zero as well. The 1's in the boxes will be referred to as *pivot elements.*

Theorem 5.4.1 *Consider the set of matrices A_i of the same dimension which are related by the transformations described above. Then application of the elimination procedure to any of A_i leads to a unique matrix A_c.*

Proof: For the sake of simplicity we consider the case where the dimensions of A_i are 5×3. Let

$$A_1 = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \\ \alpha_{41} & \alpha_{42} & \alpha_{43} \\ \alpha_{51} & \alpha_{52} & \alpha_{53} \end{bmatrix}, \quad A_2 = \begin{bmatrix} \beta_{11} & \beta_{12} & \beta_{13} \\ \beta_{21} & \beta_{22} & \beta_{23} \\ \beta_{31} & \beta_{32} & \beta_{33} \\ \beta_{41} & \beta_{42} & \beta_{43} \\ \beta_{51} & \beta_{52} & \beta_{53} \end{bmatrix} \tag{5.41}$$

Since A_1 and A_2 are related by the transformations 1., 2. it follows that

$$\begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \\ \alpha_{41} & \alpha_{42} & \alpha_{43} \\ \alpha_{51} & \alpha_{52} & \alpha_{53} \end{bmatrix} \begin{bmatrix} \kappa_1 & \kappa_2 & \kappa_3 \\ & \kappa_4 & \kappa_5 \\ & & \kappa_6 \end{bmatrix} = \begin{bmatrix} \beta_{11} & \beta_{12} & \beta_{13} \\ \beta_{21} & \beta_{22} & \beta_{23} \\ \beta_{31} & \beta_{32} & \beta_{33} \\ \beta_{41} & \beta_{42} & \beta_{43} \end{bmatrix} \tag{5.42}$$

which means that the first column of A_2 is a multiple of the first column of A_1. Now, multiplying the first columns of A_1 and A_2 by appropriate constants we have (here it is assumed that $\alpha_{11} \neq 0,\ \beta_{11} \neq 0$) without loss of generality

$$A_1^{(2)} = \begin{bmatrix} 1 & \alpha_{12} & \alpha_{13} \\ c_{21} & \alpha_{22} & \alpha_{23} \\ c_{31} & \alpha_{32} & \alpha_{33} \\ c_{41} & \alpha_{42} & \alpha_{43} \\ c_{51} & \alpha_{52} & \alpha_{53} \end{bmatrix}, \quad A_2^{(2)} = \begin{bmatrix} 1 & \beta_{12} & \beta_{13} \\ c_{21} & \beta_{22} & \beta_{23} \\ c_{31} & \beta_{32} & \beta_{33} \\ c_{41} & \beta_{42} & \beta_{43} \\ c_{51} & \beta_{52} & \beta_{53} \end{bmatrix} \tag{5.43}$$

Now, eliminate the entries of the first row, to the right of 1's

$$A_1^{(3)} = \begin{bmatrix} 1 & 0 & 0 \\ c_{21} & \alpha_{22} & \alpha_{23} \\ c_{31} & \alpha_{32} & \alpha_{33} \\ c_{41} & \alpha_{42} & \alpha_{43} \\ c_{51} & \alpha_{52} & \alpha_{53} \end{bmatrix}, \quad A_2^{(3)} = \begin{bmatrix} 1 & 0 & 0 \\ c_{21} & \beta_{22} & \beta_{23} \\ c_{31} & \beta_{32} & \beta_{33} \\ c_{41} & \beta_{42} & \beta_{43} \\ c_{51} & \beta_{52} & \beta_{53} \end{bmatrix} \tag{5.44}$$

The matrices $A_1^{(3)}$ and $A_2^{(3)}$ are related by transformations of the form 1., 2. since

$$A_1^{(3)} = A_1 T_1^{(3)}, \; A_2^{(3)} = A_2 T_2^{(3)} \tag{5.45}$$

where $T_1^{(3)}$ and $T_2^{(3)}$ are upper triangular invertible matrices. Thus,

$$A_2^{(3)} = A_1^{(3)} M_3 \tag{5.46}$$

where

$$M_3 = \begin{bmatrix} 1 & \lambda_2 & \lambda_3 \\ & \lambda_4 & \lambda_5 \\ & & \lambda_6 \end{bmatrix} \tag{5.47}$$

Observe that (5.46) holds true only if $\lambda_2 = \lambda_3 = 0$. Thus, the second column of $A_2^{(3)}$ is a multiple of the second column of $A_1^{(3)}$ which means that we may transform $A_1^{(3)}$ and $A_2^{(3)}$ to the form:

$$A_1^{(4)} = \begin{bmatrix} 1 & 0 & 0 \\ c_{21} & 0 & 0 \\ c_{31} & 1 & \alpha_{33} \\ c_{41} & d_{41} & \alpha_{43} \\ c_{51} & d_{51} & \alpha_{53} \end{bmatrix}, \quad A_2^{(4)} = \begin{bmatrix} 1 & 0 & 0 \\ c_{21} & 0 & 0 \\ c_{31} & 1 & \beta_{33} \\ c_{41} & d_{41} & \beta_{43} \\ c_{51} & d_{51} & \beta_{53} \end{bmatrix}$$

(here it is assumed that the upper nonzero entry of the second column is entry $(3,2)$ and using similar arguments as above we take the final form of the matrices

$$A_1^{(5)} = A_2^{(5)} = \begin{bmatrix} 1 & 0 & 0 \\ c_{21} & 0 & 0 \\ c_{31} & 1 & 0 \\ c_{41} & d_{41} & 1 \\ c_{51} & d_{51} & f_{51} \end{bmatrix} = A_c \tag{5.48}$$

The above procedure may be readily generalised for A_1, A_2 of any dimension and the result follows. □

The relation defined by the restricted column transformations is clearly an equivalence relation. From the uniqueness of the matrix A_c of the above theorem we readily have the following.

Theorem 5.4.2 *The matrix A_c in (5.48) is in canonical form under the restricted column transformations. This canonical form will be referred to as C–canonical form.* □

Remark 5.4.1 *The C–canonical form is obtained by transformations which are a subset of the transformations leading to the Hermite form, since no permutation and addition of columns to columns with greater column index are allowed.* □

The above canonical form combined with the transformations corresponding to the stabilizer of $L(s)$ are the main tools for the development of the canonical forms in the following sections.

5.5 A semi canonical form of $T(s)$

In this section the construction of a semi canonical form for $\mathcal{S}(E, A, B)$ is developed. In this form the matrices E and B are in canonical form, but A is not, since it is not uniquely defined. The development of the above form is necessary because it provides the transformations leading to the canonical forms with outputs and the canonical forms without outputs in the case where the reachability indices of $\mathcal{S}(E, A, B)$ are equal. The semi canonical form is developed on the reachability pencil $T(s)$.

In section 5.2 it was shown that the reachability pencil $T(s)$ is equivalent to the pseudo canonical pencil (5.11). Thus, without loss of generality we may always assume that

$$T(s) = \left[\begin{array}{c|c} L(s) & 0 \\ \hline sK - \Lambda & -I \end{array}\right] \tag{5.49}$$

The above pencil is not unique since $sK - \Lambda$ is not unique. Thus the problem of finding the canonical form of $T(s)$ under coordinate transformations is equivalent to the problem of finding a canonical form for the pencil $sK - \Lambda$ in (5.49) under the transformation group preserving the block structure of (5.49) i.e. leaves the blocks (1,1), (1,2), (2,2) invariant. This transformation group is obviously a subgroup of the group defined by (5.9). In what follows we are going to investigate the form of the transformation matrices preserving the block structure (5.49).

Proposition 5.5.1 *Let $T(s)$ and $T'(s)$ be two pencils of the form (5.49) in the same orbit. Then they are related by strict equivalence transformations of the following form:*
If $PT\hat{Q} = T'$ then

$$P = \begin{bmatrix} P_1 & 0 \\ P_3 & I \end{bmatrix}, \ \hat{Q} = \begin{bmatrix} Q & 0 \\ 0 & I \end{bmatrix} \tag{5.50}$$

where $(P_1, Q) \in Stab(L(s))$.

Proof: The block diagonal form of $\hat{Q}$ with the identity matrix as the $(2,2)$ block readily follows from the definition of the coordinate transformations (see (5.9)). Let

$$\begin{bmatrix} P_1 & P_2 \\ P_3 & P_4 \end{bmatrix} \begin{bmatrix} L(s) & 0 \\ sK - \Lambda & -I \end{bmatrix} \begin{bmatrix} Q & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} L(s) & 0 \\ sK' - \Lambda' & -I \end{bmatrix}$$

The above may be expanded to the following equations:

$$P_1 L(s) Q + P_2 (sK - \Lambda) Q = L(s) \tag{5.51}$$

$$P_2 = 0 \tag{5.52}$$

$$P_3 L(s) Q + P_4 (sK - \Lambda) Q = sK' - \Lambda' \tag{5.53}$$

$$P_4 = I \tag{5.54}$$

Equations (5.51) and (5.52) yield that $(P_1, Q) \in \text{Stab}(L(s))$ and from (5.54) the result follows. □

In order to proceed with the transformations leading to the pseudo canonical form it is convenient to consider a partitioning of $sK - \Lambda$ conformable to the block-partitioning of $L(s)$. Let

$$sK - \Lambda = [sK_1 - \Lambda_1, sK_2 - \Lambda_2, \ldots, sK_\ell - \Lambda_\ell] \tag{5.55}$$

where

$$K_i \in \Re^{\ell \times r_i} \text{ and } \Lambda_i \in \Re^{\ell \times r_i}, \ i = 1, \ldots, \ell, \ r_i = \varepsilon_i + 1 \tag{5.56}$$

and denote by $\underline{k}^i_j$, k^i_{qj} and $\underline{\lambda}^i_j$, λ^i_{qj} the columns and the entries of K_i and Λ_i respectively. Then $T(s)$ may be written as follows

$$T(s) = \left[\begin{array}{c|c|c|c|c} L_{\varepsilon_1}(s) & & & & \\ \hline & L_{\varepsilon_2}(s) & & & \\ \hline & & \ddots & & \\ \hline & & & L_{\varepsilon_\ell}(s) & \\ \hline sK_1 - \Lambda_1 & sK_2 - \Lambda_2 & \cdots & sK_\ell - \Lambda_\ell & -I_\ell \end{array}\right] \tag{5.57}$$

where

$$K_i = [\underline{k}_1^i, \underline{k}_2^i, \ldots, \underline{k}_{r_i}^i] = \begin{bmatrix} k_{11}^i & k_{12}^i & \cdots & k_{1r_i}^i \\ k_{21}^i & k_{22}^i & \cdots & k_{2r_i}^i \\ \vdots & \vdots & & \vdots \\ k_{\ell 1}^i & k_{\ell 2}^i & \cdots & k_{\ell r_i}^i \end{bmatrix} \tag{5.58}$$

and

$$\Lambda_i = [\underline{\lambda}_1^i, \underline{\lambda}_2^i, \ldots, \underline{\lambda}_{r_i}^i] = \begin{bmatrix} \lambda_{11}^i & \lambda_{12}^i & \cdots & \lambda_{1r_i}^i \\ \lambda_{21}^i & \lambda_{22}^i & \cdots & \lambda_{2r_i}^i \\ \vdots & \vdots & & \vdots \\ \lambda_{\ell 1}^i & \lambda_{\ell 2}^i & \cdots & \lambda_{\ell r_i}^i \end{bmatrix} \tag{5.59}$$

Consider now the following strict–equivalence transformations on $T(s)$

$$PT(s)\hat{Q} = \begin{bmatrix} P_1 & 0 \\ P_3 & I \end{bmatrix} \begin{bmatrix} L(s) & 0 \\ sK - \Lambda & -I \end{bmatrix} \begin{bmatrix} Q & 0 \\ 0 & I \end{bmatrix} = T'(s) \tag{5.60}$$

where $(P_1, Q) \in \text{Stab}(L(s))$. We have the following.

Proposition 5.5.2 *The matrix pencil $T'(s)$ in (5.60) has the following form*

$$T'(s) = \left[\begin{array}{c|c|c|c|c} L_{\varepsilon_1}(s) & & & & \\ \hline & L_{\varepsilon_2}(s) & & & \\ \hline & & \ddots & & \\ \hline & & & L_{\varepsilon_\ell}(s) & \\ \hline sK_1' - \Lambda_1' & sK_2' - \Lambda_2' & & sK_\ell' - \Lambda_\ell' & -I_\ell \end{array}\right] = \left[\begin{array}{c|c} L(s) & 0 \\ \hline sK' - \Lambda' & -I \end{array}\right]$$

and the elementary column and row operations induced by the transformations in (5.60) *are the following:*

(i) $s(\alpha, i)$*: multiplication of the columns of the $sK_i - \Lambda_i$ block by the scalar α*

(ii) $c(\alpha, i, j, k)$*: addition of α times the columns r_i, $r_i - 1, \ldots, 1$ of $sK_i - \Lambda_i$ to the columns $r_j - k$, $r_j - k - 1, \ldots$, $r_j - k - r_i + 1$ of $sK_j - \Lambda_j$, $k \leq r_j - r_i$, $r_j \geq r_i$. This transformation is defined only in the case where $r_j \geq r_i$*

(iii) $p(i, j)$*: permutation of the blocks $sK_i - \Lambda_i$, $sK_j - \Lambda_j$ where $r_j = r_i$*

(iv) $r(\alpha, i, j, k)$*: addition of α times the j-th row of block $L_{\varepsilon_i}(s)$ to the k-th row of $sK - \Lambda$*

Proof: Since $(P_1, Q) \in \text{Stab}(L(s))$ it readily follows that the top row block of $T'(s)$ is $[L(s), 0]$. The form of elementary column operations $s(\alpha, j)$, $c(\alpha, i, j, k)$, $p(i, j)$ follows from proposition 5.3.1. The form of elementary operations $r(\alpha, i, j, k)$ follows directly from (5.60). □

We are now ready to construct the pseudo canonical form of $T(s)$ proceeding systematically using the transformations of the above proposition.

The first step is to eliminate the $1, 2, \ldots, r_i - 1$ columns of the matrices K_i using transformations of the type $r(\alpha, i, j, k)$. The resulting pencil has the form

$$T_2(s) = \left[\begin{array}{c|c|c|c|c} L_{\varepsilon_1}(s) & & & & \\ \hline & L_{\varepsilon_2}(s) & & & \\ \hline & & \ddots & & \\ \hline & & & L_{\varepsilon_\ell}(s) & \\ \hline sK_1 - \Lambda_1 & sK_2 - \Lambda_2 & & sK_\ell - \Lambda_\ell & -I_\ell \end{array}\right] \tag{5.61}$$

where

$$K_i = \left[\begin{array}{cc} 0_{\ell \times (r_i - 1)}, & \underline{k}^i_{r_i} \end{array}\right] \tag{5.62}$$

Note that K_i and $\underline{k}^i_{r_i}$ in (5.62), (5.61) are different than those appearing in (5.57) and (5.58) but the same notation is used for simplicity.

Next, by transformations of the form $s(\alpha, j)$ and $c(\alpha, i, j, k)$ we take an equivalent pencil $T(s)$ of the form (5.61) and (5.62) such that the matrix

$$K^* = [\underline{k}^1_{r_1}, \underline{k}^2_{r_2}, \ldots, \underline{k}^\ell_{r_\ell}] \tag{5.63}$$

is in $\mathcal{C}$–canonical form. Let now $r_i = r_{i+1} = r_{i+2} = \ldots = r_i + p_i$ for some $i \in \{1, \ldots, \ell\}$. By transformations of the form $c(\alpha, j)$, $r(\alpha, j, i, 0)$, $p(i, j)$ on $T(s)$ we take a pencil $T_4(s)$ of the form (5.61), (5.62) such that the submatrices of K^* consisting of the columns $i, i+1, \ldots, i+p_i$ $i \in \{1, \ldots, \ell\}$ are in the usual echelon form for constant matrices. Note that all other columns of K^* remain unaltered by the latter transformations. The matrix obtained after the above transformations of all sets of equal r_i, is canonical and will be referred to as $\mathcal{K}$*–canonical form* of $[\underline{k}^1_{r_1}, \underline{k}^2_{r_2}, \ldots, \underline{k}^\ell_{r_\ell}]$
We may now state the following.

Proposition 5.5.3 *The coefficient matrix of* s *of* $T_4(s)$ *is in canonical form*

Proof: We have that

$$T_4(s) = \left[\begin{array}{c|c} L(s) & 0 \\ \hline sK - \Lambda & -I \end{array}\right] \tag{5.64}$$

where

$$K = [0\ \underline{k}_{r_1}^1 | \cdots | 0\ \underline{k}_{r_\ell}^\ell] \tag{5.65}$$

and

$$K^* = [\underline{k}_{r_1}^1, \ldots, \underline{k}_{r_i}^i, \ldots, \underline{k}_{r_i+1}^{i+1}, \underline{k}_{r_i+p_i}^{i+p_i}, \ldots, \underline{k}_{r_\ell}] \tag{5.66}$$

The coefficient of s is

$$\left[\begin{array}{c|c|c|c|c} L_{\varepsilon_1}^1 & & & & 0 \\ \hline & L_{\varepsilon_2}^1 & & & 0 \\ \hline & & \ddots & & \vdots \\ \hline & & & L_{\varepsilon_\ell}^1 & 0 \\ \hline 0\ \underline{k}_{r_1}^1 & 0\ \underline{k}_{r_2}^2 & & 0\ \underline{k}_{r_\ell}^\ell & 0 \end{array}\right] \tag{5.67}$$

where $L_{\varepsilon_i}(s) = sL_{\varepsilon_i}^1 - L_{\varepsilon_i}^2$.

The matrix K was obtained by invertible column transformations and it is in canonical form since K^* is in $\mathcal{K}$–canonical form and the result follows. □

The above result has as a direct consequence the following:

Theorem 5.5.1 *The matrices E and B, where*

$$E = \left[\begin{array}{c|c|c|c} L_{\varepsilon_1}^1 & & & \\ \hline & L_{\varepsilon_2}^1 & & \\ \hline & & \ddots & \\ \hline & & & L_{\varepsilon_\ell}^1 \\ \hline 0\ \underline{k}_{r_1}^1 & 0\ \underline{k}_{r_2}^2 & & 0\ \underline{k}_{r_\ell}^\ell \end{array}\right], \quad B = \left[\begin{array}{c} 0 \\ \hline I_\ell \end{array}\right]$$

are in canonical form. □

We have thus provided a semi canonical form of the triple $\mathcal{S}(E, A, B)$ where the matrices E and B are canonical. The matrix A is not, in general, in canonical form since the operations leading to the matrices of the theorem 5.5.1 do not guarantee the uniqueness of A.

5.6 Canonical form for systems with outputs

In this section the construction of the canonical form of the quadruple (E, A, B, C), or equivalently of the pencil $P(s)$ is developed. It is shown that the canonical form with outputs is directly related to the echelon form of the composite matrix of a coprime and column reduced MFD of the transfer function $G(s) = C(sE - A)^{-1}B$. This relationship between the transfer function and the descriptor model with outputs is expected, since transfer function is an invariant of the strict equivalence transformations.

In order to obtain the canonical form of (E, A, B, C) we are going to use the system matrix $P(s)$. The following preliminary result is necessary for the description of the transformations leading to the canonical form.

Proposition 5.6.1 *Let*

$$T(s) = \left[\begin{array}{ccc|c} L_{\varepsilon_1}(s) & & & \\ & \ddots & & \\ & & L_{\varepsilon_\ell}(s) & \\ \hline sK_1 - \Lambda_1 & \cdots & sK_\ell - \Lambda_\ell & -I_\ell \end{array}\right] \tag{5.68}$$

There exist transformations of the type $s(\alpha, j), c(\alpha, i, j, k), p(i, j), r(\alpha, i, j, k)$ such that the pencil $sK - \Lambda$ has the form:

$K_i = [0, \underline{k}^i_{r_i}]$, K^* *(defined in (5.63))is in $\mathcal{K}$−canonical form and Λ_i is defined as follows: If ρ_i is the row index of the first nonzero entry of $\underline{k}^i_{r_i}, \underline{k}^i_{r_i} \neq 0$, then the entries $\lambda^i_{\rho_{i,\nu}} = 0, \nu > r_i$*

Proof: For the sake of simplicity and in order to avoid complicated notations we are going to show the transformations leading to the above form of $T(s)$, by means of a general example.

Let $T(s)$ be

$$T(s) = \left[\begin{array}{ccc|ccccc|cc} s & -1 & 0 & & & & & & & \\ 0 & s & -1 & & & & & & & \\ \hline & & & s & -1 & 0 & 0 & 0 & & \\ & & & 0 & s & -1 & 0 & 0 & & \\ & & & 0 & 0 & s & -1 & 0 & & \\ & & & 0 & 0 & 0 & s & -1 & & \\ \hline \lambda^1_{11} & \lambda^1_{12} & s - \lambda^1_{13} & \lambda^2_{11} & \lambda^2_{12} & \lambda^2_{13} & \lambda^2_{14} & \lambda^2_{15} & -1 & 0 \\ \lambda^1_{21} & \lambda^1_{22} & sk^1_{32} - \lambda^1_{32} & \lambda^2_{21} & \lambda^2_{22} & \lambda^2_{23} & \lambda^2_{24} & s - \lambda^2_{25} & 0 & -1 \end{array}\right]$$

By applying the transformations $c(-\lambda^2_{15}, 1, 2, 1)$ we get

$$\left[\begin{array}{ccc|ccccc|cc}
s & -1 & 0 & & & & & & & \\
0 & s & -1 & & & & & & & \\
\hline
 & & & s & -1 & 0 & 0 & 0 & & \\
 & & & 0 & s & -1 & 0 & 0 & & \\
 & & & 0 & 0 & s & -1 & 0 & & \\
 & & & 0 & 0 & 0 & s & -1 & & \\
\hline
\lambda^1_{11} & \lambda^1_{12} & s-\lambda^1_{13} & \lambda^2_{11} & -\lambda^2_{15}\lambda^1_{11}+\lambda^2_{12} & -\lambda^2_{15}\lambda^1_{12}+\lambda^2_{13} & -\lambda^2_{15}s-\lambda^2_{15}\lambda^1_{13}+\lambda^2_{14} & \lambda^2_{15} & -1 & 0 \\
\lambda^1_{21} & \lambda^1_{22} & sk^1_{32}-\lambda^1_{32} & \lambda^2_{21} & -\lambda^2_{15}\lambda^1_{21}-\lambda^2_{22} & -\lambda^2_{15}\lambda^1_{22}+\lambda^2_{23} & -s\lambda^2_{15}k^1_{32}-\lambda^2_{15}\lambda^1_{32}+\lambda^2_{24} & s-\lambda^2_{25} & 0 & -1
\end{array}\right]$$

Now applying the transformations $r(\lambda^2_{15},2,4,1)$, $r(\lambda^2_{15}k^1_{32},2,4,2)$ we have

$$\left[\begin{array}{ccc|ccccc|cc}
s & -1 & 0 & & & & & & & \\
0 & s & -1 & & & & & & & \\
 & & & s & -1 & 0 & 0 & 0 & & \\
 & & & 0 & s & -1 & 0 & 0 & & \\
 & & & 0 & 0 & s & -1 & 0 & & \\
 & & & 0 & 0 & 0 & s & -1 & & \\
\hline
\lambda^1_{11} & \lambda^1_{12} & s-\lambda^1_{13} & \lambda^2_{11} & -\lambda^2_{15}\lambda^1_{11}+\lambda^2_{12} & -\lambda^2_{15}\lambda^1_{12}+\lambda^2_{13} & \lambda^2_{15}\lambda^1_{13}+\lambda^2_{14} & 0 & -1 & 0 \\
\lambda^1_{21} & \lambda^1_{22} & sk^1_{32}-\lambda^1_{32} & \lambda^2_{21} & -\lambda^2_{15}\lambda^1_{21}+\lambda^2_{22} & -\lambda^2_{15}\lambda^1_{22}+\lambda^2_{23} & \lambda^2_{15}\lambda^1_{32}+\lambda^2_{24} & s-\lambda^2_{25}\ \lambda^2_{15}k^1_{32} & 0 & -1
\end{array}\right]$$

Applying the same procedure to the above pencil we finally take the equivalent pencil.

$$\left[\begin{array}{ccc|ccccc}
s & -1 & 0 & & & & & \\
0 & s & -1 & & & & & \\
\hline
 & & & s & -1 & 0 & 0 & 0 \\
 & & & 0 & s & -1 & 0 & 0 \\
 & & & 0 & 0 & s & -1 & 0 \\
 & & & 0 & 0 & 0 & s & -1 \\
\hline
\mathrm{x} & \mathrm{x} & s-\mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} & 0 & 0 \\
\mathrm{x} & \mathrm{x} & s\mathrm{x}-\mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} & s-\mathrm{x}
\end{array}\right]$$

The above derivation may be readily generalised. □

Proposition 5.6.2 *If the pencil $T(s)$ is as in the previous proposition, then a basis matrix of $Ker_{\Re(s)}\{T(s)\}$ which is in echelon form, may be derived by inspection from the pencil $L(s)$ and the coefficients of $sK - \Lambda$.*

Proof: A basis matrix of $\mathrm{Ker}_{\Re(s)}\{T(s)\}$ has the form

$$\left[\begin{array}{c} N(s) \\ \hline D(s) \end{array}\right] = \left[\begin{array}{c|c|c} 1 & & \\ s & & \\ \vdots & & \\ s^{r_1-1} & & \\ \hline & \ddots & \\ \hline & & 1 \\ & & s \\ & & \vdots \\ & & s^{r_\ell-1} \\ \hline \underline{k}^1_{r_1} s^{r_1} + \cdots & \cdots & \underline{k}^\ell_{r_\ell} s^{r_\ell} + \cdots \end{array}\right]$$

where $[\underline{k}^1_{r_1}, \ldots, \underline{k}^\ell_{r_\ell}]$ is in $\mathcal{K}$–canonical form and the columns corresponding to equal reachability indices are in Hermite form. Note that the other coefficients of the polynomial entries of $D(s)$ are obtained by inspection and they are equal to λ^k_{ij}. Then the high order coefficient matrix has full col–rank i.e. $[N^T(s), D^T(s)]^T$ is column reduced. The latter is also coprime since it contains a constant submatrix formed by the rows $1, r_1+1, \ldots, r_1+\ldots+r_{\ell-1}+1$ which is the unity matrix. Finally $[N^T(s), D^T(s)]^T$ is in echelon form because the pivot indices are

$$p_i = \begin{cases} r_1 + \ldots + r_i & \text{if } \underline{k}^i_{r_i} = 0 \\ n + q_i & \text{if } \underline{k}^i_{r_i} \neq 0 \end{cases}$$

where q_i are the pivot indices of the matrix K^* which is in $\mathcal{K}$–canonical form and the result follows. □

As it was mentioned in section 5.2 we are going to find a canonical form for the system $P(s)$ starting from the pseudo canonical form

$$P(s) = \left[\begin{array}{cc} L(s) & 0 \\ sK - \Lambda & -I \\ \hline C & 0 \end{array}\right] \tag{5.69}$$

Similarly to the case of $T(s)$ we are going to consider transformations of the type

$$\left[\begin{array}{cc|c} P_1 & 0 & \\ P_3 & -I & \\ \hline & & I \end{array}\right] \left[\begin{array}{c|c} L(s) & 0 \\ sK - \Lambda & -I \\ \hline C & 0 \end{array}\right] \left[\begin{array}{cc} Q & 0 \\ 0 & I \end{array}\right] \tag{5.70}$$

where $(P_1, Q) \in \mathrm{Stab}(L(s))$. In what follows we are going to use the following partitioning on C.

$$C = [C_1, \ldots, C_\ell] \tag{5.71}$$

where

$$C_i = \begin{bmatrix} c^i_{11} & c^i_{12} & \cdots & c^i_{1r_i} \\ c^i_{21} & c^i_{22} & \cdots & c^i_{2r_i} \\ \vdots & \vdots & & \vdots \\ c^i_{m1} & c^i_{m2} & \cdots & c^i_{mr_i} \end{bmatrix} = [\underline{c}^i_1, \underline{c}^i_2, \ldots, \underline{c}^i_{r_i}] \tag{5.72}$$

From (5.70) we see that Q affects the output matrix C in exactly the same way it affects $sK - \Lambda$. Thus, whatever was mentioned for the elementary column transformations induced on the blocks $sK_i - \Lambda_i$ holds also for the blocks C_i.

The steps of the construction of the canonical form will be clarified with the use of a general example. Throughout the description of the steps we are going to use the following notation. By x and y we shall denote constant numbers which are not fixed zeros or 1's. By $s\mathsf{x}-\mathsf{y}$ we denote a general binomial in the indeterminate s. The matrix pencil $sK - \Lambda$ will be written in the form

$$sK - \Lambda = \left[\begin{array}{ccccc|c|ccccc} \mathsf{x} & \mathsf{x} & \cdots & \mathsf{x} & s\mathsf{x}-\mathsf{y} & & \mathsf{x} & \mathsf{x} & \cdots & \mathsf{x} & s\mathsf{x}-\mathsf{y} \\ \vdots & \vdots & & & & \cdots & \mathsf{x} & \mathsf{x} & \cdots & \mathsf{x} & s\mathsf{x}-\mathsf{y} \\ \mathsf{x} & \mathsf{x} & \cdots & \mathsf{x} & s\mathsf{x}-\mathsf{y} & & \mathsf{x} & \mathsf{x} & \cdots & \mathsf{x} & s\mathsf{x}-\mathsf{y} \end{array}\right] \tag{5.73}$$

This notation will be used in all the intermediate forms of the pencil $P(s)$. The entries of the output matrix C will be denoted by c^k_{ij} and this notation will be used in all the stages of the derivation of the canonical form in order to avoid complicated notation. Thus, the element c^k_{ij} in two different stages is not, in general, the same number. We start with

$$P(s) = \left[\begin{array}{ccc|cccc|c|ccccc|ccc} s & -1 & 0 & & & & & & & & & & & & & \\ 0 & s & -1 & & & & & & & & & & & & & \\ \hline & & & s & -1 & 0 & 0 & & & & & & & & & \\ & & & 0 & s & -1 & 0 & & & & & & & & & \\ & & & 0 & 0 & s & -1 & & & & & & & & & \\ \hline & & & & & & & \ddots & & & & & & & & \\ \hline & & & & & & & & s & -1 & 0 & 0 & 0 & & & \\ & & & & & & & & 0 & s & -1 & 0 & 0 & & & \\ & & & & & & & & 0 & 0 & s & -1 & 0 & & & \\ & & & & & & & & 0 & 0 & 0 & s & -1 & & & \\ \hline \mathsf{x} & \mathsf{x} & s\mathsf{x}-\mathsf{y} & \mathsf{x} & \mathsf{x} & \mathsf{x} & s\mathsf{x}-\mathsf{y} & & \mathsf{x} & \mathsf{x} & \mathsf{x} & \mathsf{x} & s\mathsf{x}-\mathsf{y} & -1 & & \\ \mathsf{x} & \mathsf{x} & s\mathsf{x}-\mathsf{y} & \mathsf{x} & \mathsf{x} & \mathsf{x} & s\mathsf{x}-\mathsf{y} & & \mathsf{x} & \mathsf{x} & \mathsf{x} & \mathsf{x} & s\mathsf{x}-\mathsf{y} & & \ddots & \\ & \vdots & & & & \vdots & & & & & \vdots & & & & & -1 \\ \hline c^1_{11} & c^1_{12} & c^1_{13} & c^2_{11} & c^2_{12} & c^2_{13} & c^2_{14} & \cdots & c^\ell_{11} & c^\ell_{21} & c^\ell_{31} & c^\ell_{41} & c^\ell_{51} & & & \\ & \vdots & & & & \vdots & & & & & \vdots & & & & & \end{array}\right] \tag{5.74}$$

Following the procedure described in 5.6.1 we may transform $P(s) = \left[\begin{array}{c} T(s) \\ \hline [C,0] \end{array}\right]$ to a form such that the the echelon form of the basis matrix of $\mathrm{Ker}_{\Re(s)}\{T(s)\}$ is related to the entries of $T(s)$ as in proposition 5.6.2. As an example consider the system matrix

$$P(s) = \left[\begin{array}{cc|ccccc|ccccccc|ccc}
s & -1 & & & & & & & & & & & & & & & \\ \hline
 & & s & -1 & 0 & 0 & 0 & & & & & & & & & & \\
 & & 0 & s & -1 & 0 & 0 & & & & & & & & & & \\
 & & 0 & 0 & s & -1 & 0 & & & & & & & & & & \\
 & & 0 & 0 & 0 & s & -1 & & & & & & & & & & \\ \hline
 & & & & & & & s & -1 & 0 & 0 & 0 & 0 & 0 & & & \\
 & & & & & & & 0 & s & -1 & 0 & 0 & 0 & 0 & & & \\
 & & & & & & & 0 & 0 & s & -1 & 0 & 0 & 0 & & & \\
 & & & & & & & 0 & 0 & 0 & s & -1 & 0 & 0 & & & \\
 & & & & & & & 0 & 0 & 0 & 0 & s & -1 & 0 & & & \\
 & & & & & & & 0 & 0 & 0 & 0 & 0 & s & -1 & & & \\ \hline
\mathsf{x} & s\mathsf{x}-\mathsf{y} & \mathsf{x} & \mathsf{x} & \mathsf{x} & \mathsf{x} & s\mathsf{x}-\mathsf{y} & \mathsf{x} & \mathsf{x} & \mathsf{x} & \mathsf{x} & \mathsf{x} & \mathsf{x} & s\mathsf{x}-\mathsf{y} & -1 & 0 & 0 \\
\mathsf{x} & s\mathsf{x}-\mathsf{y} & \mathsf{x} & \mathsf{x} & \mathsf{x} & \mathsf{x} & s\mathsf{x}-\mathsf{y} & \mathsf{x} & \mathsf{x} & \mathsf{x} & \mathsf{x} & \mathsf{x} & \mathsf{x} & s\mathsf{x}-\mathsf{y} & 0 & -1 & 0 \\
\mathsf{x} & s\mathsf{x}-\mathsf{y} & \mathsf{x} & \mathsf{x} & \mathsf{x} & \mathsf{x} & s\mathsf{x}-\mathsf{y} & \mathsf{x} & \mathsf{x} & \mathsf{x} & \mathsf{x} & \mathsf{x} & \mathsf{x} & s\mathsf{x}-\mathsf{y} & 0 & 0 & -1 \\ \hline
c^1_{11} & c^1_{12} & c^2_{11} & c^2_{12} & c^2_{13} & c^2_{14} & c^2_{15} & c^3_{11} & c^3_{12} & c^3_{13} & c^3_{14} & c^3_{15} & c^3_{16} & c^3_{17} & & & \\
c^1_{11} & c^1_{12} & c^2_{11} & c^2_{12} & c^2_{13} & c^2_{14} & c^2_{15} & c^3_{21} & c^3_{22} & c^3_{23} & c^3_{24} & c^3_{25} & c^3_{26} & c^3_{27} & & &
\end{array}\right] \tag{5.75}$$

We assume that $\mathrm{rank}\{E\} = 12$ and that (without loss of generality) the columns with indices $r_1 + r_2 = 7$ and $r_1 + r_2 + r_3 = 14$ are linearly dependent on the column with index $r_1 = 2$. Using transformations of the form $c(\alpha, i, j, k)$, $r(\alpha, i, j, k)$, $s(\alpha, i)$ we may bring the pencil (5.75) to the form

$$P^{(1)}(s) = \left[\begin{array}{cc|ccccc|ccccccc|ccc}
s & -1 & & & & & & & & & & & & & & & \\ \hline
 & & s & -1 & 0 & 0 & 0 & & & & & & & & & & \\
 & & 0 & s & -1 & 0 & 0 & & & & & & & & & & \\
 & & 0 & 0 & s & -1 & 0 & & & & & & & & & & \\
 & & 0 & 0 & 0 & s & -1 & & & & & & & & & & \\ \hline
 & & & & & & & s & -1 & 0 & 0 & 0 & 0 & 0 & & & \\
 & & & & & & & 0 & s & -1 & 0 & 0 & 0 & 0 & & & \\
 & & & & & & & 0 & 0 & s & -1 & 0 & 0 & 0 & & & \\
 & & & & & & & 0 & 0 & 0 & s & -1 & 0 & 0 & & & \\
 & & & & & & & 0 & 0 & 0 & 0 & s & -1 & 0 & & & \\
 & & & & & & & 0 & 0 & 0 & 0 & 0 & s & -1 & & & \\ \hline
\mathsf{x} & s-\mathsf{y} & \mathsf{x} & \mathsf{x} & \mathsf{x} & \mathsf{x} & \mathsf{x} & \mathsf{x} & \mathsf{x} & \mathsf{x} & \mathsf{x} & \mathsf{x} & \mathsf{x} & \mathsf{x} & -1 & 0 & 0 \\
\mathsf{x} & s\mathsf{x}-\mathsf{y} & \mathsf{x} & \mathsf{x} & \mathsf{x} & \mathsf{x} & \mathsf{x} & \mathsf{x} & \mathsf{x} & \mathsf{x} & \mathsf{x} & \mathsf{x} & \mathsf{x} & \mathsf{x} & 0 & -1 & 0 \\
\mathsf{x} & s\mathsf{x}-\mathsf{y} & \mathsf{x} & \mathsf{x} & \mathsf{x} & \mathsf{x} & \mathsf{x} & \mathsf{x} & \mathsf{x} & \mathsf{x} & \mathsf{x} & \mathsf{x} & \mathsf{x} & \mathsf{x} & 0 & 0 & -1 \\ \hline
c^1_{11} & c^1_{12} & c^2_{11} & c^2_{12} & c^2_{13} & c^2_{14} & c^2_{15} & c^3_{11} & c^3_{12} & c^3_{13} & c^3_{14} & c^3_{15} & c^3_{16} & c^3_{17} & & & \\
c^1_{11} & c^1_{12} & c^2_{11} & c^2_{12} & c^2_{13} & c^2_{14} & c^2_{15} & c^3_{21} & c^3_{22} & c^3_{23} & c^3_{24} & c^3_{25} & c^3_{26} & c^3_{27} & & &
\end{array}\right] \tag{5.76}$$

Observe that the coefficients of s in the columns 7 and 14 of the matrix $sK - \Lambda$ are zero. In general, using transformations of the type $c(\cdot,\cdot,\cdot,\cdot)$, $r(\cdot,\cdot,\cdot,\cdot)$, $p(\cdot,\cdot)$we may bring the system matrix $P(s)$ to the form of $P^{(1)}(s)$ such that the coefficient matrix of s in the rows 12, 13, 14 of $P^{(1)}(s)$ is in the $\mathcal{K}$–canonical form. Now, following the method of proposition 5.6.1 we may transform the pencil (5.76) to the following form

$$P^{(2)}(s) = \left[\begin{array}{cc|ccccc|ccccccc|ccc}
s & -1 & & & & & & & & & & & & & & & \\
\hline
 & & s & -1 & 0 & 0 & 0 & & & & & & & & & & \\
 & & 0 & s & -1 & 0 & 0 & & & & & & & & & & \\
 & & 0 & 0 & s & -1 & 0 & & & & & & & & & & \\
 & & 0 & 0 & 0 & s & -1 & & & & & & & & & & \\
\hline
 & & & & & & & s & -1 & 0 & 0 & 0 & 0 & 0 & & & \\
 & & & & & & & 0 & s & -1 & 0 & 0 & 0 & 0 & & & \\
 & & & & & & & 0 & 0 & s & -1 & 0 & 0 & 0 & & & \\
 & & & & & & & 0 & 0 & 0 & s & -1 & 0 & 0 & & & \\
 & & & & & & & 0 & 0 & 0 & 0 & s & -1 & 0 & & & \\
 & & & & & & & 0 & 0 & 0 & 0 & 0 & s & -1 & & & \\
\hline
\times & s-y & \times & \times & 0 & 0 & 0 & \times & \times & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
\times & sx-y & \times & \times & \times & \times & \times & \times & \times & \times & \times & \times & \times & \times & 0 & -1 & 0 \\
\times & sx-y & \times & \times & \times & \times & \times & \times & \times & \times & \times & \times & \times & \times & 0 & 0 & -1 \\
\hline
c^1_{11} & c^1_{12} & c^2_{11} & c^2_{12} & c^2_{13} & c^2_{14} & c^2_{15} & c^3_{11} & c^3_{12} & c^3_{13} & c^3_{14} & c^3_{15} & c^3_{16} & c^3_{17} & & & \\
c^1_{21} & c^1_{22} & c^2_{21} & c^2_{22} & c^2_{23} & c^2_{24} & c^2_{25} & c^3_{21} & c^3_{22} & c^3_{23} & c^3_{24} & c^3_{25} & c^3_{26} & c^3_{27} & & &
\end{array}\right] \tag{5.77}$$

Observe that the entries λ^2_{13}, λ^2_{14}, λ^2_{15}, λ^3_{13}, λ^3_{14}, λ^3_{15}, λ^3_{16}, λ^3_{17} are zero (see proposition 5.6.1) and $k^1_{1,\sigma_1} = 1$. Thus, the corresponding basis matrix of $\mathrm{Ker}_{\Re(s)}\{T(s)\}$ formed as in proposition 5.6.2 is in echelon form.

Since the system $\mathcal{S}(E, A, B, C)$ is observable at infinity, the matrix $[E^T, C^T]^T$ has full column rank. Then, from (5.77) it is clear that the matrix formed by the rightmost columns of the blocks C_i where $\underline{k}_{r_i} = 0$, must have full column rank. In the present example this means that

$$\mathrm{rank}\{\begin{bmatrix} c^2_{15} & c^3_{17} \\ c^2_{25} & c^3_{27} \end{bmatrix}\} = 2 \tag{5.78}$$

The matrix in (5.78) may be transformed to the $\mathcal{C}$–canonical form of section 5.4 by appropriate column transformations. Without loss of generality, we assume that $c^2_{15} \neq 0$. Then the canonical form of the matrix in (5.78) is

$$\begin{bmatrix} 1 & 0 \\ c^2_{25} & 1 \end{bmatrix} \tag{5.79}$$

Note that c^2_{25} in (5.78) is not necessarily the same as c^2_{25} in (5.78) but we use the same notation for simplicity. By transformations of the type $s(\alpha, 2)$, $s(\alpha, 3)$, $c(\alpha, 2, 3, 0)$, we put $P^{(2)}(s)$ in the form

$$
P^{(3)}(s) = \left[\begin{array}{cc|ccccc|ccccccc|ccc}
s & -1 & & & & & & & & & & & & & & & \\
\hline
 & & s & -1 & 0 & 0 & 0 & & & & & & & & & & \\
 & & 0 & s & -1 & 0 & 0 & & & & & & & & & & \\
 & & 0 & 0 & s & -1 & 0 & & & & & & & & & & \\
 & & 0 & 0 & 0 & s & -1 & & & & & & & & & & \\
\hline
 & & & & & & & s & -1 & 0 & 0 & 0 & 0 & 0 & & & \\
 & & & & & & & 0 & s & -1 & 0 & 0 & 0 & 0 & & & \\
 & & & & & & & 0 & 0 & s & -1 & 0 & 0 & 0 & & & \\
 & & & & & & & 0 & 0 & 0 & s & -1 & 0 & 0 & & & \\
 & & & & & & & 0 & 0 & 0 & 0 & s & -1 & 0 & & & \\
 & & & & & & & 0 & 0 & 0 & 0 & 0 & s & -1 & & & \\
\hline
\mathrm{x} & s-\mathrm{y} & \mathrm{x} & \mathrm{x} & 0 & 0 & 0 & \mathrm{x} & \mathrm{x} & \boxed{\mathrm{x}} & \boxed{\mathrm{x}} & 0 & 0 & 0 & -1 & 0 & 0 \\
\mathrm{x} & s\mathrm{x}-\mathrm{y} & \mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} & 0 & -1 & 0 \\
\mathrm{x} & s\mathrm{x}-\mathrm{y} & \mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} & 0 & 0 & -1 \\
\hline
c^1_{11} & c^1_{12} & c^2_{11} & c^2_{12} & c^2_{13} & c^2_{14} & 1 & c^3_{11} & c^3_{12} & c^3_{13} & c^3_{14} & c^3_{15} & c^3_{16} & 0 & & & \\
c^1_{21} & c^1_{22} & c^2_{21} & c^2_{22} & c^2_{23} & c^2_{24} & c^2_{25} & c^3_{21} & c^3_{22} & c^3_{23} & c^3_{24} & c^3_{25} & c^3_{26} & 1 & & &
\end{array}\right] \tag{5.80}
$$

Observe that the columns $\underline{c}^2_{r_2}$ and $\underline{c}^3_{r_3}$ form matrix (5.79). The entries λ^3_{13} and λ^3_{14} (in the boxes) are not fixed to zero. We may eliminate this λ^3_{14} by using transformations of the type $c(\alpha,1,3,4)$, $c(\alpha,1,3,5)$, $r(\alpha,3,\cdot,\cdot)$. The resulting system matrix is

$$
P^{(4)}(s) = \left[\begin{array}{cc|ccccc|ccccccc|ccc}
s & -1 & & & & & & & & & & & & & & & \\
\hline
 & & s & -1 & 0 & 0 & 0 & & & & & & & & & & \\
 & & 0 & s & -1 & 0 & 0 & & & & & & & & & & \\
 & & 0 & 0 & s & -1 & 0 & & & & & & & & & & \\
 & & 0 & 0 & 0 & s & -1 & & & & & & & & & & \\
\hline
 & & & & & & & s & -1 & 0 & 0 & 0 & 0 & 0 & & & \\
 & & & & & & & 0 & s & -1 & 0 & 0 & 0 & 0 & & & \\
 & & & & & & & 0 & 0 & s & -1 & 0 & 0 & 0 & & & \\
 & & & & & & & 0 & 0 & 0 & s & -1 & 0 & 0 & & & \\
 & & & & & & & 0 & 0 & 0 & 0 & s & -1 & 0 & & & \\
 & & & & & & & 0 & 0 & 0 & 0 & 0 & s & -1 & & & \\
\hline
\mathrm{x} & s-\mathrm{y} & \mathrm{x} & \mathrm{x} & 0 & 0 & 0 & \mathrm{x} & \mathrm{x} & \boxed{\mathrm{x}} & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
\mathrm{x} & s\mathrm{x}-\mathrm{y} & \mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} & 0 & -1 & 0 \\
\mathrm{x} & s\mathrm{x}-\mathrm{y} & \mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} & 0 & 0 & -1 \\
\hline
c^1_{11} & c^1_{12} & c^2_{11} & c^2_{12} & c^2_{13} & c^2_{14} & 1 & c^{3}11 & c^3_{12} & c^3_{13} & c^3_{14} & c^3_{15} & c^3_{16} & 0 & & & \\
c^1_{21} & c^1_{22} & c^2_{21} & c^2_{22} & c^2_{23} & c^2_{24} & c^2_{25} & c^3_{21} & c^3_{22} & c^3_{23} & c^3_{24} & c^3_{25} & c^3_{26} & 1 & & &
\end{array}\right] \tag{5.81}
$$

Continuing along the lines of the above procedure we end up with a system pencil of the form

$$
P^{(5)}(s) = \left[\begin{array}{cc|ccccc|ccccccc|ccc}
s & -1 &&&&&&&&&&&&&&& \\
\hline
& & s & -1 & 0 & 0 & 0 &&&&&&&&&& \\
& & 0 & s & -1 & 0 & 0 &&&&&&&&&& \\
& & 0 & 0 & s & -1 & 0 &&&&&&&&&& \\
& & 0 & 0 & 0 & s & -1 &&&&&&&&&& \\
\hline
&&&&&&& s & -1 & 0 & 0 & 0 & 0 & 0 &&& \\
&&&&&&& 0 & s & -1 & 0 & 0 & 0 & 0 &&& \\
&&&&&&& 0 & 0 & s & -1 & 0 & 0 & 0 &&& \\
&&&&&&& 0 & 0 & 0 & s & -1 & 0 & 0 &&& \\
&&&&&&& 0 & 0 & 0 & 0 & s & -1 & 0 &&& \\
&&&&&&& 0 & 0 & 0 & 0 & 0 & s & -1 &&& \\
\hline
\mathsf{x} & s-\mathsf{y} & \mathsf{x} & \mathsf{x} & 0 & 0 & 0 & \mathsf{x} & \mathsf{x} & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
\mathsf{x} & s\mathsf{x}-\mathsf{y} & \mathsf{x} & \mathsf{x} & \mathsf{x} & \mathsf{x} & \mathsf{x} & \mathsf{x} & \mathsf{x} & \mathsf{x} & \mathsf{x} & \mathsf{x} & \mathsf{x} & \mathsf{x} & 0 & -1 & 0 \\
\mathsf{x} & s\mathsf{x}-\mathsf{y} & \mathsf{x} & \mathsf{x} & \mathsf{x} & \mathsf{x} & \mathsf{x} & \mathsf{x} & \mathsf{x} & \mathsf{x} & \mathsf{x} & \mathsf{x} & \mathsf{x} & \mathsf{x} & 0 & 0 & -1 \\
\hline
c^1_{11} & c^1_{12} & c^2_{11} & c^2_{12} & c^2_{13} & c^2_{14} & 1 & c^3_{11} & c^3_{12} & c^3_{13} & c^3_{14} & 0 & 0 & 0 &&& \\
c^1_{21} & c^1_{22} & c^2_{21} & c^2_{22} & c^2_{23} & c^2_{24} & c^2_{25} & c^3_{21} & c^3_{22} & c^3_{23} & c^3_{24} & c^3_{25} & c^3_{26} & 1 &&&
\end{array}\right] \tag{5.82}
$$

Remark 5.6.1 *The above system matrix has the following special characteristics*

(i) The matrix K^* is in the $\mathcal{K}$–canonical form

(ii) The matrix formed by the columns $\underline{c}^i_{r_i}$, where $\underline{k}^i_{r_i} = 0$ is in the $\mathcal{C}$–canonical form of section 5.4

(iii) If φ_i is the smallest row index of $\underline{c}^i_{r_i}$, where $\underline{k}^i_{r_i} = 0$ such that $c^i_{\varphi_i, r_i} \neq 0$ then $c^j_{\varphi_i, q} = 0,\ j > i,\ q \geq r_i$. □

For the pencil $P^{(5)}(s)$ in (5.82) we have: $c^3_{15} = c^3_{16} = c^3_{17} = 0$, $c^2_{15} = 1, c^3_{27} = 1$ and $k^1_{1,r_1} = 1$.

Let the pencil $P^{(5)}(s)$ in (5.82) be denoted by

$$
P^{(5)}(s) = \left[\begin{array}{c|c} L(s) & 0 \\ sK^{(5)} - \Lambda^{(5)} & -I_\ell \\ \hline C^{(5)} & 0 \end{array}\right] = \left[\begin{array}{c} T^{(5)}(s) \\ \hline [C^{(5)}, 0] \end{array}\right] \tag{5.83}
$$

and $[S^T(s), D^T(s)]^T$ be the basis matrix of the right null space of $T^{(5)}(s)$. Then the input–state transfer function is

$$
(sE^{(5)} - A^{(5)})^{-1} B^{(5)} = S(s)(D^{(5)})^{-1}(s) \tag{5.84}
$$

where

$$
sE^{(5)} - A^{(5)} = \left[\begin{array}{c} L(s) \\ sK^{(5)} - \Lambda^{(5)} \end{array}\right], \quad B^{(5)} = \left[\begin{array}{c} 0 \\ I_\ell \end{array}\right] \tag{5.85}
$$

and the input–output transfer function $G(s) = C^{(5)}(sE^{(5)} - A^{(5)})^{-1}B^{(5)}$ may be written in MFD form as follows

$$G(s) = C^{(5)}S(s)(D^{(5)}(s))^{-1} \tag{5.86}$$

Let $T_G(s)$ be the composite matrix of the MFD (5.86) i.e.

$$T_G(s) = \begin{bmatrix} C^{(5)}S(s) \\ D^{(5)}(s). \end{bmatrix} \tag{5.87}$$

From (5.82) it follows, by inspection, that

$$\begin{bmatrix} S(s) \\ D^{(5)}(s) \end{bmatrix} = \left[\begin{array}{c|c|c} 1 & & \\ s & & \\ \hline & 1 & \\ & s & \\ & s^2 & \\ & s^3 & \\ & s^4 & \\ \hline & & 1 \\ & & s \\ & & s^2 \\ & & s^3 \\ & & s^4 \\ & & s^5 \\ & & s^6 \\ \hline s^2 + \cdots & s\mathrm{x} + \cdots & s\mathrm{x} + \cdots \\ s^2\mathrm{x} + \cdots & s^4\mathrm{x} + \cdots & s^6\mathrm{x} + \cdots \\ s^2\mathrm{x} + \cdots & s^4\mathrm{x} + \cdots & s^6\mathrm{x} + \cdots \end{array}\right] \tag{5.88}$$

By obvious row operations on $P^{(5)}(s)$ of the type $r(\alpha, i, j, k)$ we may eliminate the entries of the vectors $\underline{\lambda}^i_{r_i}$ of the blocks with $\underline{k}^i_{r_i} = 0$. The resulting pencil has the form

$$P^{(6)}(s) = \left[\begin{array}{cc|ccccc|ccccccc|ccc} s & -1 & & & & & & & & & & & & & & & \\ \hline & & s & -1 & 0 & 0 & 0 & & & & & & & & & & \\ & & 0 & s & -1 & 0 & 0 & & & & & & & & & & \\ & & 0 & 0 & s & -1 & 0 & & & & & & & & & & \\ & & 0 & 0 & 0 & s & -1 & & & & & & & & & & \\ \hline & & & & & & & s & -1 & 0 & 0 & 0 & 0 & 0 & & & \\ & & & & & & & 0 & s & -1 & 0 & 0 & 0 & 0 & & & \\ & & & & & & & 0 & 0 & s & -1 & 0 & 0 & 0 & & & \\ & & & & & & & 0 & 0 & 0 & s & -1 & 0 & 0 & & & \\ & & & & & & & 0 & 0 & 0 & 0 & s & -1 & 0 & & & \\ & & & & & & & 0 & 0 & 0 & 0 & 0 & s & -1 & & & \\ \hline \mathrm{x} & s-\mathrm{y} & \mathrm{x} & \mathrm{x} & 0 & 0 & 0 & \mathrm{x} & \mathrm{x} & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ \mathrm{x} & s\mathrm{x}-\mathrm{y} & \mathrm{x} & \mathrm{x} & \mathrm{x} & s\mathrm{x}-\mathrm{y} & 0 & \mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} & s\mathrm{x}-\mathrm{y} & 0 & 0 & -1 & 0 \\ \mathrm{x} & s\mathrm{x}-\mathrm{y} & \mathrm{x} & \mathrm{x} & \mathrm{x} & s\mathrm{x}-\mathrm{y} & 0 & \mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} & s\mathrm{x} & 0 & 0 & 0 & -1 \\ \hline c^1_{11} & c^1_{12} & c^2_{11} & c^2_{12} & c^2_{13} & c^2_{14} & 1 & c^3_{11} & c^3_{12} & c^3_{13} & c^3_{14} & 0 & 0 & 0 & & & \\ c^1_{11} & c^1_{12} & c^2_{11} & c^2_{12} & c^2_{13} & c^2_{14} & c^2_{15} & c^3_{21} & c^3_{22} & c^3_{23} & c^3_{24} & c^3_{25} & c^3_{26} & 1 & & & \end{array}\right] =$$

$$= \left[\begin{array}{cc} L(s) & 0 \\ sK^{(6)} - \Lambda^{(6)} & -I \\ \hline C^{(6)} & 0 \end{array}\right] \tag{5.89}$$

The next step is the following: by elementary row and column operations (permutations) we put the system pencil $P^{(6)}(s)$ to the form

$$P^{(7)}(s) = \left[\begin{array}{c|c|c} \tilde{L}(s) & 0 & \\ s\tilde{K} - \tilde{\Lambda} & -I_p & \\ \hline s\hat{K} & & -I_\ell \\ \hline \tilde{C}_1 & \tilde{C}_2 & \end{array}\right] \tag{5.90}$$

where $p = n - rank\{E\}$. Note that the above operations do not correspond to operations induced by the stabilizer Stab$(L(s))$. However they are used, in order to relate the system pencil with the realisation theory of the previous Chapter and facilitate the proof of the canonicity of the form $P^{(6)}(s)$. The form $P^{(7)}(s)$ and all subsequent forms will be used temporarily and then we shall return to the equivalent form $P^{(6)}(s)$.

The pencil $\tilde{L}(s)$ is a block diagonal pencil with c.m.i. blocks $\tilde{L}_{\sigma_i}(s)$ on the diagonal. The dimensions of these blocks are as follows:

$$\sigma_i = \begin{cases} r_i & \text{if } \underline{k}^i_{r_i} \neq 0 \\ r_i - 1 & \text{if } \underline{k}^i_{r_i} = 0 \end{cases} \tag{5.91}$$

The pencil $s\tilde{K} - \tilde{\Lambda}$ has the form:

$$s\tilde{K} - \tilde{\Lambda} = [s\tilde{K}^1 - \tilde{\Lambda}^1, \ldots, s\tilde{K}^\ell - \tilde{K}^\ell] \tag{5.92}$$

where $s\tilde{K}^i - \tilde{\Lambda}^i = sK^i - \Lambda^i$ if $\underline{k}^i_{r_i} \neq 0$ and if $\underline{k}^i_{r_i} = 0$ $s\tilde{K}^\ell - \tilde{\Lambda}_i$ consists only from the first $r_i - 1$ columns of $sK^i - \Lambda^i$. $\tilde{C}_1 = [\tilde{C}^1_1, \ldots, \tilde{C}^\ell_1]$, $\tilde{C}^i_1 = C_i$ if $\underline{k}^i_{r_i} \neq 0$ and $\tilde{C}^i_1$ consists from the first $r_i - 1$ columns of C^i if $\underline{k}^i_{r_i} = 0$. The matrix C_2 is formed from the last columns of the blocks C_i for which $\underline{k}^i_{r_i} = 0$ and finally the matrix $\hat{K}$ has the form

$$\hat{K} = [\hat{K}_1, \ldots, \hat{K}_\ell]$$

where

$$\hat{K}_i = 0_{p\times r_i} \text{ if } \underline{k}^i_{r_i} \neq 0$$

$$\hat{K}_i = [0_{p\times(r_i-2)}, e_i] \text{ if } \underline{k}^i_{r_i} = 0, \; [e_1, \cdots, e_p] = I_p$$

For the pencil (5.89), (5.90) has the form

$$
P^{(7)}(s) = \left[\begin{array}{cc|cccc|cccccc|cc|ccc}
s & -1 & & & & & & & & & & & & & & & \\
\hline
 & & s & -1 & 0 & 0 & & & & & & & & & & & \\
 & & 0 & s & -1 & 0 & & & & & & & & & & & \\
 & & 0 & 0 & s & -1 & & & & & & & & & & & \\
\hline
 & & & & & & s & -1 & 0 & 0 & 0 & 0 & & & & & \\
 & & & & & & 0 & s & -1 & 0 & 0 & 0 & & & & & \\
 & & & & & & 0 & 0 & s & -1 & 0 & 0 & & & & & \\
 & & & & & & 0 & 0 & 0 & s & -1 & 0 & & & & & \\
 & & & & & & 0 & 0 & 0 & 0 & 0 & -1 & & & & & \\
\hline
 & & 0 & 0 & 0 & s & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & & & \\
 & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & s & 0 & -1 & & & \\
\hline
\mathsf{x} & s-\mathsf{y} & \mathsf{x} & \mathsf{x} & 0 & 0 & \mathsf{x} & \mathsf{x} & 0 & 0 & 0 & 0 & & & -1 & 0 & 0 \\
\mathsf{x} & s\mathsf{x}-\mathsf{y} & \mathsf{x} & \mathsf{x} & \mathsf{x} & s\mathsf{x}-\mathsf{y} & \mathsf{x} & \mathsf{x} & \mathsf{x} & \mathsf{x} & \mathsf{x} & s\mathsf{x}-\mathsf{y} & & & 0 & -1 & 0 \\
\mathsf{x} & s\mathsf{x}-\mathsf{y} & \mathsf{x} & \mathsf{x} & \mathsf{x} & s\mathsf{x}-\mathsf{y} & \mathsf{x} & \mathsf{x} & \mathsf{x} & \mathsf{x} & \mathsf{x} & s\mathsf{x}-\mathsf{y} & & & 0 & 0 & -1 \\
\hline
c^1_{11} & c^1_{12} & c^2_{11} & c^2_{12} & c^2_{13} & c^2_{14} & c^3_{11} & c^3_{12} & c^3_{13} & c^3_{14} & 0 & 0 & 1 & 0 & & & \\
c^1_{21} & c^1_{21} & c^2_{21} & c^2_{22} & c^2_{23} & c^2_{24} & c^3_{21} & c^3_{22} & c^3_{23} & c^3_{24} & c^3_{25} & c^3_{26} & c^2_{25} & 1 & & &
\end{array}\right]
$$

The above pencil may be further transformed by row operations (reordering of the rows) to the following form

$$
P^{(8)}(s) = \left[\begin{array}{cc|cccc|cccccc|cc|ccc}
s & -1 & & & & & & & & & & & & & 0 & 0 & 0 \\
\mathsf{x} & s-\mathsf{y} & \mathsf{x} & \mathsf{x} & 0 & 0 & \mathsf{x} & \mathsf{x} & 0 & 0 & 0 & 0 & & & -1 & 0 & 0 \\
\hline
 & & s & -1 & 0 & 0 & & & & & & & & & 0 & 0 & 0 \\
 & & 0 & s & -1 & 0 & & & & & & & & & 0 & 0 & 0 \\
 & & 0 & 0 & s & -1 & & & & & & & & & 0 & 0 & 0 \\
\mathsf{x} & s\mathsf{x}-\mathsf{y} & \mathsf{x} & \mathsf{x} & \mathsf{x} & s\mathsf{x}-\mathsf{y} & \mathsf{x} & \mathsf{x} & \mathsf{x} & \mathsf{x} & \mathsf{x} & s\mathsf{x}-\mathsf{y} & & & 0 & -1 & 0 \\
\hline
 & & & & & & s & -1 & 0 & 0 & 0 & 0 & & & 0 & 0 & 0 \\
 & & & & & & 0 & s & -1 & 0 & 0 & 0 & & & 0 & 0 & 0 \\
 & & & & & & 0 & 0 & s & -1 & 0 & 0 & & & 0 & 0 & 0 \\
 & & & & & & 0 & 0 & 0 & s & -1 & 0 & & & 0 & 0 & 0 \\
 & & & & & & 0 & 0 & 0 & 0 & 0 & -1 & & & 0 & 0 & 0 \\
\mathsf{x} & s\mathsf{x}-\mathsf{y} & \mathsf{x} & \mathsf{x} & \mathsf{x} & s\mathsf{x}-\mathsf{y} & \mathsf{x} & \mathsf{x} & \mathsf{x} & \mathsf{x} & \mathsf{x} & s\mathsf{x}-\mathsf{y} & & & 0 & 0 & -1 \\
\hline
 & & 0 & 0 & 0 & s & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & & & \\
 & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & s & 0 & -1 & & & \\
\hline
c^1_{11} & c^1_{12} & c^2_{11} & c^2_{12} & c^2_{13} & c^2_{14} & c^3_{11} & c^3_{12} & c^3_{13} & c^3_{14} & 0 & 0 & 1 & 0 & & & \\
c^1_{21} & c^1_{21} & c^2_{21} & c^2_{22} & c^2_{23} & c^2_{24} & c^3_{21} & c^3_{22} & c^3_{23} & c^3_{24} & c^3_{25} & c^3_{26} & c^2_{25} & 1 & & &
\end{array}\right]
$$

Clearly, $P^{(8)}(s)$ is a realisation of a transfer function (see Chapter 4) $G(s) = N(s)D^{-1}(s)$ with

$$
T_G(s) = \left[\begin{array}{c} N(s) \\ D(s) \end{array}\right] = \left[\begin{array}{c} C_cS(s) \\ D_c(s) \end{array}\right] = \left[\begin{array}{c} N(s) \\ D_c(s) \end{array}\right]
$$

Proposition 5.6.3 *The matrix $T_G(s)$ is a minimal basis of the vector space spanned by its columns. Furthermore, it is in echelon canonical form.*

Proof: First it is shown that $T_G(s)$ is column reduced and has no finite zeros. If

$$
T_G^{hc} = \left[\begin{array}{c} N_{hc} \\ D_{hc} \end{array}\right]
$$

is the high order coefficient matrix of $T_G(s)$ then

(i) If $\underline{k}^i_{r_i} \neq 0$ then $\underline{k}^i_{r_i} = [0,\ldots,0,1,\mathrm{x},\mathrm{x},\mathrm{x}]^T$. Then, the i–th column of D_{hc} is equal to $\underline{k}^i_{r_i}$ and the i–th column of N_{hc} is zero.

(ii) If $\underline{k}^i_{r_i} = 0$ then the i–th column of N_{hc} is equal to $\underline{c}^i_{r_i}$.

An immediate consequence of this is that the high order coefficient matrix of $T_G(s)$ has full column rank and thus, $T_G(s)$ is column reduced.

The coprimeness at finite s of $N(s)$ and $D_c(s)$ arises from the fact that the system with system matrix $P^{(8)}(s)$ is a realisation obtained from $T_G(s)$ and from the assumption that $\mathcal{S}(E,A,B,C)$ is minimal. Thus $T_G(s)$ is a minimal basis. From the construction of the matrix $P^{(6)}(s)$ we see that $\underline{\lambda}^i_{\rho_i,\nu} = 0$, $\nu > \sigma_i$, $j > \sigma_i$. Then all the entries of $T_G(s)$ laying on the same row to the pivot elements and have column index greater than the column index of the pivot elements have degree lower than the degree of the pivot element. Thus, $T_G(s)$ is in echelon form [For., 1975]. □

The matrix T_G^{hc} for the system under study is

$$T_G^{hc} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ \mathrm{x} & \mathrm{x} & \mathrm{x} \\ \mathrm{x} & \mathrm{x} & \mathrm{x} \end{bmatrix}$$

and the pivot indices of $T_G(s)$ are $p_1 = 3$, $p_2 = 1$, $p_3 = 2$. It is clear that the pivots are defined directly by the nonzero $\underline{k}^i_{r_i}$ and $\underline{c}^i_{r_i}$.

Observe that the matrix $T_G(s)$ is equal to the matrix $T_G^{(6)}(s)$ which corresponds to $P^{(6)}(s)$ and that the pencil $P^{(8)}(s)$ is obtained from $P^{(6)}(s)$ by appropriate permutations of the columns and rows. Thus, $P^{(8)}(s)$ and $P^{(6)}(s)$ are uniquely defined from the echelon form of the composite matrix of the transfer function of $\mathcal{S}(E,A,B,C)$. The main result of this section follows.

Theorem 5.6.1 *The pencil $P_c(s) = P^{(6)}(s)$ is in canonical form.*

Proof: In order to prove that $P_c(s) = P^{(6)}(s)$ is in canonical form we have to show that it is uniquely defined and that every system $\mathcal{S}(E',A',B',C')$ related by strict system equivalence transformations to the original system $\mathcal{S}(E,A,B,C)$ may be transformed to the form defined by $P_c(s)$.

Since $\mathcal{S}(E,A,B,C)$ and $\mathcal{S}(E',A',B',C')$ are related by strict equivalence transformations they have the same transfer function. Thus, the echelon form of the composite matrix $T_G(s) = [N^T(s), D^T(s)]^T$ of the transfer function is common to both systems. Since the echelon form is canonical [Pop., 1969], [For., 1975], and therefore unique, and $P_c(s)$ is uniquely defined from this echelon composite matrix it follows that $P_c(s)$ is in canonical form. □

Remark 5.6.2 *Clearly, $P^{(8)}(s)$ is also in canonical form. However, we prefer to define as canonical the quadruple (E, A, B, C) corresponding to $P^{(6)}(s)$ for the following reason: The invariants of the system are the controllability indices σ_i of (E, A, B), the reachability indices ε_i of (E, A, B) and the entries of $sK^{(6)} - \Lambda^{(6)}$ and $C^{(6)}$.*

The canonical form $P^{(6)}(s)$ has the advantage over $P^{(8)}(s)$ that it shows clearly the reachability indices of the system and allows the immediate classification of the controllability indices into proper and nonproper [Gl.-Luer., 1990], [Kar. & Hel., 1990] *as follows.*

The reachability indices are the column minimal indices of the pencil $L(s)$ and thus, they may be identified immediately since $L(s)$ is in canonical form.

The controllability indices may be directly identified as proper and nonproper from (5.91). *If $\sigma_i = r_i$, σ_i is a proper index and if $\sigma_i = r_i - 1$ then σ_i is nonproper. Thus, inspection of $\underline{k}^i_{r_i}$ yields this classification.*

The controllability indices σ_i coincide with the column degrees of $T_G(s)$. On the other hand, the continuous invariants i.e. the entries of $sK_c - \Lambda_c$, C_c are uniquely defined from the coefficients of the polynomial entries of $T_G(s)$. □

Remark 5.6.3 *In order to find the canonical form of $\mathcal{S}(E, A, B, C)$ we may find any MFD of $G(s) = C(sE - A)^{-1}B$, form the composite matrix $[N^T(s), D^T(s)]^T$ and find a minimal basis of the col–span of this matrix in echelon form. Then, the canonical form is obtained by inspection.* □

Remark 5.6.4 *The problem of finding the echelon form of a given MFD is in general complicated. In* [For., 1975] *a general procedure for finding the echelon form is given. Algorithms for finding echelon forms are given in* [Kun., Kai. & Mor., 1977,] *and* [Kail., 1980].

The procedure of the present paper for finding the canonical form of $\mathcal{S}(E, A, B, C)$ and the realisation method of the previous chapter may be used as an alternative method for finding the echelon form of a given MFD. The steps of this method are outlined below.

Given $G(s)$, find any coprime and column reduced MFD $G(s) = N(s)D^{-1}(s)$. Then find a minimal realisation following the method of Chapter 4. Next, find the canonical form following the steps described in the present Chapter. The echelon form of $[N^T(s), D^T(s)]^T$ is then obtained by inspection. □

5.7 Systems with equal reachability indices

In this section the problem of finding canonical forms for the triple (E, A, B) under restricted system equivalence transformations is considered. The problem is solved for the case of systems $\mathcal{S}(E, A, B)$ with all the reachability indices equal. As it was mentioned in section 5.2, this problem is essentially the problem of finding canonical forms of the pencil

$$T(s) = \left[\begin{array}{c|c} L(s) & \\ \hline sK - \Lambda & -I_\ell \end{array}\right] \tag{5.93}$$

under the transformations of the type (5.50). The procedure of the derivation of the canonical form is described below: The main steps of the reduction to the canonical form are

Procedure 1

(i) Following the procedure described in section 5.5 bring $T(s)$ to a form such that K^* is in echelon form.

(ii) Let $\varphi = \max_{\underline{\lambda}^i_{\varphi_i} \neq 0,\ \underline{k}^i_{r_i} = 0} \{\varphi_i\}$. Use transformations of the type $c(\alpha, i, j, k)$, $s(\alpha, i)$ to obtain a pencil with the matrix formed by $\underline{\lambda}^i_{\varphi_i} \neq 0$ with $\underline{k}^i_{r_i} = 0$, in the Hermite canonical form of constant matrices.

(iii) Using the pivot elements of the above Hermite form, eliminate the corresponding entries of all the blocks with $\underline{k}^i_{r_i} \neq 0$.

The above procedure is clarified below with the help of a general example. Consider the pencil

$$T(s) = \left[\begin{array}{c|c} L(s) & 0 \\ \hline sK^{(1)} - \Lambda^{(1)} & -I_\ell \end{array}\right] \tag{5.94}$$

where K^* is in echelon form and let $T(s)$ be as follows

$$T(s) = \left[\begin{array}{cccc|cccc|cccc|ccc} s & -1 & 0 & 0 & & & & & & & & & & & \\ 0 & s & -1 & 0 & & & & & & & & & & & \\ 0 & 0 & s & -1 & & & & & & & & & & & \\ \hline & & & & s & -1 & 0 & 0 & & & & & & & \\ & & & & 0 & s & -1 & 0 & & & & & & & \\ & & & & 0 & 0 & s & -1 & & & & & & & \\ \hline & & & & & & & & s & -1 & 0 & 0 & & & \\ & & & & & & & & 0 & s & -1 & 0 & & & \\ & & & & & & & & 0 & 0 & s & -1 & & & \\ \hline \mathrm{x} & \mathrm{x} & \mathrm{x} & s-\mathrm{y} & \mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} & -1 & 0 & 0 \\ \mathrm{x} & \mathrm{x} & \mathrm{x} & s\mathrm{x}-\mathrm{y} & \mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} & 0 & -1 & 0 \\ \mathrm{x} & \mathrm{x} & \mathrm{x} & s\mathrm{x}-\mathrm{y} & \mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} & \mathrm{x} & 0 & 0 & -1 \end{array}\right] \tag{5.95}$$

where, without loss of generality it is assumed that $\lambda^2_{14} \neq 0$. In order to put $[\underline{\lambda}^2_{r_2}, \underline{\lambda}^3_{r_3}]$ in Hermite canonical form we use transformations of the type $s(\alpha, i)$, $c(\alpha, i, j, k)$. The resulting pencil has the form

$$T^{(2)}(s) = \left[\begin{array}{cccc|cccc|cccc|ccc} s & -1 & 0 & 0 & & & & & & & & & & & \\ 0 & s & -1 & 0 & & & & & & & & & & & \\ 0 & 0 & s & -1 & & & & & & & & & & & \\ \hline & & & & s & -1 & 0 & 0 & & & & & & & \\ & & & & 0 & s & -1 & 0 & & & & & & & \\ & & & & 0 & 0 & s & -1 & & & & & & & \\ \hline & & & & & & & & s & -1 & 0 & 0 & & & \\ & & & & & & & & 0 & s & -1 & 0 & & & \\ & & & & & & & & 0 & 0 & s & -1 & & & \\ \hline \mathsf{x} & \mathsf{x} & \mathsf{x} & s-\mathsf{y} & \mathsf{x} & \mathsf{x} & \mathsf{x} & \boxed{1} & \mathsf{x} & \mathsf{x} & \mathsf{x} & 0 & -1 & 0 & 0 \\ \mathsf{x} & \mathsf{x} & \mathsf{x} & s\mathsf{x}-\mathsf{y} & \mathsf{x} & \mathsf{x} & \mathsf{x} & 0 & \mathsf{x} & \mathsf{x} & \mathsf{x} & \boxed{1} & 0 & -1 & 0 \\ \mathsf{x} & \mathsf{x} & \mathsf{x} & s\mathsf{x}-\mathsf{y} & \mathsf{x} & \mathsf{x} & \mathsf{x} & \mathsf{x} & \mathsf{x} & \mathsf{x} & \mathsf{x} & \mathsf{x} & 0 & 0 & -1 \end{array}\right] \tag{5.96}$$

Obviously, the matrix

$$[\underline{\lambda}^2_{r_2},\ \underline{\lambda}^3_{r_3}] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \mathsf{x} & \mathsf{x} \end{bmatrix} \tag{5.97}$$

is in Hermite canonical form. The pivot elements are the 1's in the boxes in (5.96). Now apply transformations of the type $c(\alpha, i, j, k)$ to take the pencil

$$T^{(3)}(s) = \left[\begin{array}{cccc|cccc|cccc|ccc} s & -1 & 0 & 0 & & & & & & & & & & & \\ 0 & s & -1 & 0 & & & & & & & & & & & \\ 0 & 0 & s & -1 & & & & & & & & & & & \\ \hline & & & & s & -1 & 0 & 0 & & & & & & & \\ & & & & 0 & s & -1 & 0 & & & & & & & \\ & & & & 0 & 0 & s & -1 & & & & & & & \\ \hline & & & & & & & & s & -1 & 0 & 0 & & & \\ & & & & & & & & 0 & s & -1 & 0 & & & \\ & & & & & & & & 0 & 0 & s & -1 & & & \\ \hline \mathsf{x} & \mathsf{x} & \mathsf{x} & s & \mathsf{x} & \mathsf{x} & \mathsf{x} & 1 & \mathsf{x} & \mathsf{x} & \mathsf{x} & 0 & -1 & 0 & 0 \\ \mathsf{x} & \mathsf{x} & \mathsf{x} & s\mathsf{x} & \mathsf{x} & \mathsf{x} & \mathsf{x} & 0 & \mathsf{x} & \mathsf{x} & \mathsf{x} & 1 & 0 & -1 & 0 \\ \mathsf{x} & \mathsf{x} & \mathsf{x} & s\mathsf{x}-\mathsf{y} & \mathsf{x} & \mathsf{x} & \mathsf{x} & \mathsf{x} & \mathsf{x} & \mathsf{x} & \mathsf{x} & \mathsf{x} & 0 & 0 & -1 \end{array}\right] \tag{5.98}$$

Observe that $\lambda^1_{14} = \lambda^1_{24} = 0$. The canonicity of the form obtained by the above procedure is proved below.

Theorem 5.7.1 *The pencil obtained by applying* **Procedure 1** *to the original pencil $P(s)$ is canonical.*

Proof: Consider two pencils $T_1(s)$ and $T_2(s)$ of the same equivalence class. Apply to both **Procedure 1** to take $T_1^c(s)$ and $T_2^c(s)$ respectively. From proposition 5.5.3 we have that the coefficient matrix of s is in canonical form. Furthermore, since the matrix formed by the columns $\underline{\lambda}^i_{r_i}$ with $\underline{k}^i_{r_i} = 0$ is in Hermite canonical form it follows that the pencils $T_1^c(s)$ and $T_2^c(s)$ have the same matrix $[\underline{\lambda}^i_{r_i}]$, where $\underline{k}^i_{r_i} = 0$. Thus the fixed zero elements of the columns nonzero $\underline{k}^i_{r_i}$ are the same. Since $T_1^c(s)$ and $T_2^c(s)$ are in the same orbit they are related by strict equivalence transformations. The uniqueness of the form obtained by **Procedure 1** follows from the observation that no transformation of the form $s(\cdot,\cdot)$, $c(\cdot,\cdot,\cdot,\cdot)$, $r(\cdot,\cdot,\cdot,\cdot)$,$p(\cdot,\cdot)$ may be applied to $T_1^c(s)$, $T_2^c(s)$ without destroying the structure (i), (ii), (iii) obtained by **Procedure 1**. Thus $T_1^c(s) = T_2^c(s)$ and they are in canonical form. □

Remark 5.7.1 *In the case of the canonical form with outputs this form is directly related to the echelon form of the composite matrix of an MFD of the transfer function. This connection is possible because the transfer function is an invariant of $\mathcal{S}(E, A, B, C)$ under strict equivalence transformations.*

For the case of the canonical form of $\mathcal{S}(E, A, B)$ the corresponding transfer function is the input–state transfer function which is not invariant under strict equivalence transformations. This is the main difficulty arising in the problem of finding canonical form in the general case where the controllability indices of the system do not have equal values. Note that no arguments related to the MFD of the state–input transfer function were used in the development of the canonical form of $\mathcal{S}(E, A, B)$. However it is straightforward to see that the canonical form may be directly related to an MFD of the input–state transfer function with composite matrix in echelon form. □

Remark 5.7.2 *The canonical form obtained allows a direct classification of the controllability indices of $\mathcal{S}(E, A, B)$ in the same way it was done to that of the canonical form with outputs (see remark* 5.6.2*).* □

5.8 Examples on the canonical forms

Because of the complexity of the transformations leading to the canonical form, we include this section with two fully worked examples in order to clarify the methods developed in this Chapter.

Example 5.8.1 In this example we find the canonical form for a system with outputs. Consider the system matrix $P(s)$ of the system obtained in the example of Chapter 4.

$$P(s) = \left[\begin{array}{cc|ccccc|c|cc}
s & -1 & 0 & 0 & 0 & 0 & 0 & & 0 & 0 \\
1 & s & 3 & 0 & 1 & 0 & 0 & & -1 & 0 \\
\hline
 & & s & -1 & 0 & 0 & 0 & & 0 & 0 \\
 & & 0 & s & -1 & 0 & 0 & & 0 & 0 \\
 & & 0 & 0 & s & -1 & 0 & & 0 & 0 \\
 & & 0 & 0 & 0 & s & -1 & & 0 & 0 \\
2 & 1 & 3 & 0 & 4 & 0 & 1 & & 0 & -1 \\
\hline
0 & s & 0 & 0 & 0 & 0 & s & -1 & & \\
\hline
0 & 0 & 3 & 15 & 1 & 8 & 0 & 1 & & \\
0 & 5 & 3 & 0 & 0 & 0 & 0 & 1 & &
\end{array}\right]$$

The above is strictly equivalent to

$$P(s) = \left[\begin{array}{ccc|ccccc|cc}
s & -1 & 0 & & & & & & & \\
0 & s & -1 & & & & & & & \\
\hline
 & & & s & -1 & 0 & 0 & 0 & & \\
 & & & 0 & s & -1 & 0 & 0 & & \\
 & & & 0 & 0 & s & -1 & 0 & & \\
 & & & 0 & 0 & 0 & s & -1 & & \\
\hline
1 & s & 0 & 3 & 0 & 1 & -1 & -s & -1 & 0 \\
2 & 1 & 0 & 3 & 0 & 4 & -2 & 0 & 0 & -1 \\
\hline
0 & 0 & 1 & 3 & 15 & 1 & 8 & 0 & & \\
0 & 5 & 1 & 3 & 0 & 0 & 0 & -5 & &
\end{array}\right] = \left[\begin{array}{c|c}
L(s) & \\
\hline
sK - \Lambda & -I_2 \\
\hline
C &
\end{array}\right]$$

which is in the form (5.14). We are going to describe in detail the transformations leading to the canonical form.

Apply the transformation $r(-1,1,2,1)$ to bring $P(s)$ to the form:

$$P^{(1)}(s) = \left[\begin{array}{ccc|ccccc|cc}
s & -1 & 0 & & & & & & & \\
0 & s & -1 & & & & & & & \\
\hline
 & & & s & -1 & 0 & 0 & 0 & & \\
 & & & 0 & s & -1 & 0 & 0 & & \\
 & & & 0 & 0 & s & -1 & 0 & & \\
 & & & 0 & 0 & 0 & s & -1 & & \\
\hline
1 & 0 & 1 & 3 & 0 & 1 & -1 & -s & -1 & 0 \\
2 & 1 & 0 & 3 & 0 & 4 & -2 & 0 & 0 & -1 \\
\hline
0 & 0 & 1 & 3 & 15 & 1 & 8 & 0 & & \\
0 & 5 & 1 & 3 & 0 & 0 & 0 & -5 & &
\end{array}\right]$$

The matrix $[\underline{k}^1_{r_1},\ \underline{k}^2_{r_2}] = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}$ is not in the $\mathcal{C}$–canonical form. Apply the transformation $s(-1,2)$ and take

$$P^{(2)}(s) = \left[\begin{array}{ccc|ccccc|cc}
s & -1 & 0 & & & & & & & \\
0 & s & -1 & & & & & & & \\
\hline
 & & & s & -1 & 0 & 0 & 0 & & \\
 & & & 0 & s & -1 & 0 & 0 & & \\
 & & & 0 & 0 & s & -1 & 0 & & \\
 & & & 0 & 0 & 0 & s & -1 & & \\
\hline
1 & 0 & 1 & -3 & 0 & -1 & 1 & s & -1 & 0 \\
2 & 1 & 0 & -3 & 0 & -4 & 2 & 0 & 0 & -1 \\
\hline
0 & 0 & 1 & -3 & -15 & -1 & -8 & 0 & & \\
0 & 5 & 1 & -3 & 0 & 0 & 0 & 5 & &
\end{array}\right]$$

Now $[\underline{k}^1_{r_1},\ \underline{k}^1_{r_2}] = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is in $\mathcal{C}$–canonical form and by applying $r(1,1,2,1)$ we take

$$P^{(3)}(s) = \left[\begin{array}{ccc|ccccc|cc}
s & -1 & 0 & & & & & & & \\
0 & s & -1 & & & & & & & \\
\hline
 & & & s & -1 & 0 & 0 & 0 & & \\
 & & & 0 & s & -1 & 0 & 0 & & \\
 & & & 0 & 0 & s & -1 & 0 & & \\
 & & & 0 & 0 & 0 & s & -1 & & \\
\hline
1 & s & 0 & -3 & 0 & -1 & 1 & s & -1 & 0 \\
2 & 1 & 0 & -3 & 0 & -4 & 2 & 0 & 0 & -1 \\
\hline
0 & 0 & 1 & -3 & -15 & -1 & -8 & 0 & & \\
0 & 5 & 1 & -3 & 0 & 0 & 0 & 5 & &
\end{array}\right]$$

In order to eliminate the entry $c^2_{14} = -8$, we apply $r(-1,1,2,1)$, $c(8,1,2,1)$. The resulting pencil is

$$P^{(4)}(s) = \left[\begin{array}{ccc|ccccc|cc}
s & -1 & 0 & & & & & & & \\
0 & s & -1 & & & & & & & \\
\hline
 & & & s & -1 & 0 & 0 & 0 & & \\
 & & & 0 & s & -1 & 0 & 0 & & \\
 & & & 0 & 0 & s & -1 & 0 & & \\
 & & & 0 & 0 & 0 & s & -1 & & \\
\hline
1 & 0 & 1 & -3 & 8 & -1 & 9 & s & -1 & 0 \\
2 & 1 & 0 & -3 & 16 & 4 & 2 & 0 & 0 & -1 \\
\hline
0 & 0 & 1 & -3 & -15 & -1 & 0 & 0 & & \\
0 & 5 & 1 & -3 & 0 & 40 & 8 & 5 & &
\end{array}\right]$$

Next, we apply $c(1,1,2,2)$ in order to eliminate $c^2_{13} = -1$ and we take

$$P^{(5)}(s) = \left[\begin{array}{ccc|ccccc|cc}
s & -1 & 0 & & & & & & & \\
0 & s & -1 & & & & & & & \\
\hline
 & & & s & -1 & 0 & 0 & 0 & & \\
 & & & 0 & s & -1 & 0 & 0 & & \\
 & & & 0 & 0 & s & -1 & 0 & & \\
 & & & 0 & 0 & 0 & s & -1 & & \\
\hline
1 & 0 & 1 & -2 & 8 & 0 & 9 & s & -1 & 0 \\
2 & 1 & 0 & -1 & 17 & 4 & 2 & 0 & 0 & -1 \\
\hline
0 & 0 & 1 & -3 & -15 & 0 & 0 & 0 & & \\
0 & 5 & 1 & -3 & 5 & 41 & 8 & 5 & &
\end{array}\right]$$

Now, by applying $r(1,1,2,1)$ we take

$$P^{(6)}(s) = \left[\begin{array}{ccc|ccccc|cc}
s & -1 & 0 & & & & & & & \\
0 & s & -1 & & & & & & & \\
\hline
 & & & s & -1 & 0 & 0 & 0 & & \\
 & & & 0 & s & -1 & 0 & 0 & & \\
 & & & 0 & 0 & s & -1 & 0 & & \\
 & & & 0 & 0 & 0 & s & -1 & & \\
\hline
1 & s & 0 & -2 & 8 & 0 & 9 & s & -1 & 0 \\
2 & 1 & 0 & -1 & 17 & 4 & 2 & 0 & 0 & -1 \\
\hline
0 & 0 & 1 & -3 & -15 & 0 & 0 & 0 & & \\
0 & 5 & 1 & -3 & 5 & 41 & 8 & 5 & &
\end{array}\right]$$

Then by column and row permutations we transform the above to

$$P^{(7)}(s) = \left[\begin{array}{cc|ccccc|c|cc}
s & -1 & & & & & & & 0 & 0 \\
1 & s & -2 & 8 & 0 & 9 & s & & -1 & 0 \\
\hline
 & & s & -1 & 0 & 0 & 0 & & 0 & 0 \\
 & & 0 & s & -1 & 0 & 0 & & 0 & 0 \\
 & & 0 & 0 & s & -1 & 0 & & 0 & 0 \\
 & & 0 & 0 & 0 & s & -1 & & 0 & 0 \\
2 & 1 & -1 & 17 & 4 & 2 & 0 & & 0 & -1 \\
\hline
0 & s & 0 & 0 & 0 & 0 & 0 & -1 & & \\
\hline
0 & 0 & -3 & -15 & 0 & 0 & 0 & 1 & & \\
0 & 5 & -3 & 5 & 41 & 8 & 5 & 1 & &
\end{array}\right]$$

Then,

$$T_G(s) = \left[\begin{array}{c|c} s^2 & -15s-3 \\ s^2+5s & 5s^4+8s^3+41s^2+5s-3 \\ \hline s^2+1 & s^5+9s^3+8s-2 \\ s+2 & 2s^3+4s^2+17s-1 \end{array}\right] = \left[\frac{CS(s)}{D(s)}\right]$$

It may be readily verified that the above matrix is in echelon canonical form. The canonical form of $\mathcal{S}(E, A, B, C)$ is the form corresponding to $P^{(6)}(s)$.

Example 5.8.2 In this example the canonical form without outputs is derived. First, a constructive method for transforming $T(s)$ in the form

$$T(s) = \begin{bmatrix} L(s) & 0 \\ sK - \Lambda & -I \end{bmatrix}$$

is applied and then the canonical form of $\mathcal{S}(E, A, B)$ is found. The same procedure is followed to derive the canonical form of a different triple (E', A', B') in the same orbit and it is verified that the method leads to the same canonical element.

Consider the singular system with matrices

$$E = \left[\begin{array}{ccc|ccccc} 0&1&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ \hline 0&0&0&1&0&0&0&0 \\ 0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&1 \end{array}\right], \quad A = \left[\begin{array}{ccc|ccccc} 1&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0 \\ \hline 0&0&0&2&1&0&0&0 \\ 0&0&0&0&2&1&0&0 \\ 0&0&0&0&0&2&0&0 \\ 0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&0 \end{array}\right], \quad B = \left[\begin{array}{cc} 1&0 \\ 2&0 \\ 0&1 \\ \hline 0&0 \\ 0&0 \\ 1&0 \\ 0&0 \\ 0&1 \end{array}\right]$$

let $K = (1 \cdot E - A)$. Then the canonical form of the pair $(K^{-1}E, K^{-1}B)$ is

$$E_2 = \left[\begin{array}{cccc|cccc} \frac{-31}{11} & \frac{-27}{11} & \frac{-5}{11} & \frac{2}{11} & \frac{-1}{11} & \frac{-3}{11} & \frac{-3}{11} & \frac{-1}{11} \\ 1&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0 \\ \hline \frac{4}{11} & \frac{-8}{11} & \frac{4}{11} & 0 & \frac{20}{11} & \frac{-7}{11} & \frac{-2}{11} & 0 \\ 0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&1&0 \end{array}\right], \quad B_2 = \left[\begin{array}{cc} 1&0 \\ 0&0 \\ 0&0 \\ 0&0 \\ \hline 0&1 \\ 0&0 \\ 0&0 \\ 0&0 \end{array}\right]$$

and $(I + (s-1)E_2, -B_2) = (sE_2 - A_2, -B_2)$

$$
\left[\begin{array}{cccc|cccc|cc}
\frac{-31}{11}s+\frac{42}{11} & \frac{-27}{11}s+\frac{27}{11} & \frac{-5}{11}s+\frac{5}{11} & \frac{2}{11}s-\frac{2}{11} & \frac{-1}{11}s+\frac{1}{11} & \frac{-3}{11}s+\frac{3}{11} & \frac{-1}{11}s+\frac{3}{11} & \frac{-1}{11}s+\frac{1}{11} & -1 & 0 \\
s-1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & s-1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & s-1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
\frac{4}{11}s-\frac{4}{11} & \frac{-8}{11}s+\frac{8}{11} & \frac{4}{11}s-\frac{4}{11} & 0 & \frac{20}{11}s-\frac{9}{11} & \frac{-7}{11}s+\frac{7}{11} & \frac{-2}{11}s+\frac{2}{11} & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & s-1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & s-1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & s-1 & 1 & 0 & 0
\end{array}\right]
$$

The above is strictly equivalent to

$$
\left[\begin{array}{cccc|cccc|cc}
s-1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & s-1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & s-1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & s-1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & s-1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & s-1 & 1 & 0 & 0 \\
\hline
\frac{-31}{11}s+\frac{42}{11} & \frac{-27}{11}s+\frac{27}{11} & \frac{-5}{11}s+\frac{5}{11} & \frac{2}{11}s-\frac{2}{11} & \frac{-1}{11}s+\frac{1}{11} & \frac{-3}{11}s+\frac{3}{11} & \frac{-3}{11}s+\frac{3}{11} & \frac{-1}{11}s+\frac{1}{11} & -1 & 0 \\
\frac{4}{11}s-\frac{4}{11} & \frac{-8}{11}s+\frac{8}{11} & \frac{4}{11}s-\frac{4}{11} & 0 & \frac{20}{11}s-\frac{9}{11} & \frac{-7}{11}s+\frac{7}{11} & \frac{-2}{11}s+\frac{2}{11} & 0 & 0 & -1
\end{array}\right]
$$

Now, multiply the above pencil from the left and right by

$$
\left[\begin{array}{cccccccc}
1 & 0 & 0 & & & & & \\
1 & -1 & 0 & & & & & \\
1 & -2 & 1 & & & & & \\
 & & & 1 & 0 & 0 & & \\
 & & & 1 & -1 & 0 & & \\
 & & & 1 & -2 & 1 & & \\
 & & & & & & 1 & 0 \\
 & & & & & & 0 & 1
\end{array}\right]
\quad \text{and} \quad
\left[\begin{array}{cccccccccc}
1 & 0 & 0 & 0 & & & & & & \\
1 & -1 & 0 & 0 & & & & & & \\
1 & -2 & 1 & 0 & & & & & & \\
1 & -3 & 3 & -1 & & & & & & \\
 & & & & 1 & 0 & 0 & 0 & & \\
 & & & & 1 & -1 & 0 & 0 & & \\
 & & & & 1 & -2 & 1 & 0 & & \\
 & & & & 1 & -3 & 3 & -1 & & \\
 & & & & & & & & 1 & 0 \\
 & & & & & & & & 0 & 1
\end{array}\right]
$$

respectively. The resulting pencil is the following:

$$
\left[\begin{array}{cccc|cccc|cc}
s & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & s & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & s & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & s & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & s & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & s & -1 & 0 & 0 \\
\hline
\frac{-61}{11}s+\frac{72}{11} & \frac{31}{11}s-\frac{31}{11} & \frac{1}{11}s-\frac{1}{11} & \frac{-2}{11}s+\frac{2}{11} & \frac{-8}{11}s+\frac{8}{11} & \frac{12}{11}s-\frac{12}{11} & \frac{-6}{11}s+\frac{6}{11} & \frac{1}{11}s-\frac{1}{11} & -1 & 0 \\
0 & 0 & \frac{4}{11}s-\frac{4}{11} & 0 & s & s-1 & \frac{-2}{11}s+\frac{2}{11} & 0 & 0 & -1
\end{array}\right]
$$

Consider now the following strictly equivalent pencil obtained by dividing the first and second block by 11

$$
\left[\begin{array}{cccc|cccc|cc}
s & -1 & 0 & 0 & & & & & & \\
0 & s & -1 & 0 & & & & & & \\
0 & 0 & s & -1 & & & & & & \\
\hline
 & & & & s & -1 & 0 & 0 & & \\
 & & & & 0 & s & -1 & 0 & & \\
 & & & & 0 & 0 & s & -1 & & \\
\hline
-61s+72 & 31s-31 & s-1 & -2s+2 & -8s+8 & 12s-12 & -6s+6 & s-1 & -1 & 0 \\
0 & 0 & 4s-4 & 0 & 11s & 11s-11 & -2s+2 & 0 & 0 & -1
\end{array}\right]
$$

Permuting the col-blocks we get

$$
\left[\begin{array}{cccc|cccc|cc}
s & -1 & 0 & 0 & & & & & & \\
0 & s & -1 & 0 & & & & & & \\
0 & 0 & s & -1 & & & & & & \\
\hline
 & & & & s & -1 & 0 & 0 & & \\
 & & & & 0 & s & -1 & 0 & & \\
 & & & & 0 & 0 & s & -1 & & \\
\hline
-8s+8 & 12s-12 & -6s+6 & s-1 & -61s+2 & 31s-31 & s-1 & -2s+2 & -1 & 0 \\
11s & 11s-11 & -2s+2 & 0 & 0 & 0 & 4s-4 & 0 & 0 & -1
\end{array}\right]
$$

Note that the above is obtained by permutation of the two first column blocks followed by a permutation of the first two row blocks.

The next equivalent pencil is

$$\left[\begin{array}{cccc|cccc|cc}
s & -1 & 0 & 0 & & & & & & \\
0 & s & -1 & 0 & & & & & & \\
0 & 0 & s & -1 & & & & & & \\
\hline
 & & & & s & -1 & 0 & 0 & & \\
 & & & & 0 & s & -1 & 0 & & \\
 & & & & 0 & 0 & s & -1 & & \\
\hline
-8s & 12s-12 & -6s+6 & s-1 & -77s+88 & 55s-55 & -11s+11 & 0 & -1 & 0 \\
11s & 11s-11 & -2s+2 & 0 & 22s & 22s-22 & 0 & 0 & 0 & -1
\end{array}\right]$$

The above is obtained by adding $2\times$ the first column block to the second column block and performing the appropriate row operations on the first two row blocks. Now by addition of the appropriate multiples of the rows of the first two blocks we eliminate the coefficients of s on the first three columns of each of the two lower column blocks:

$$\left[\begin{array}{cccc|cccc|cc}
s & -1 & 0 & 0 & & & & & & \\
0 & s & -1 & 0 & & & & & & \\
0 & 0 & s & -1 & & & & & & \\
\hline
 & & & & s & -1 & 0 & 0 & & \\
 & & & & 0 & s & -1 & 0 & & \\
 & & & & 0 & 0 & s & -1 & & \\
\hline
8 & -20 & 18 & s-7 & 88 & -132 & 66 & -11 & -1 & 0 \\
0 & 0 & 13 & -2 & 0 & 0 & 22 & 0 & 0 & -1
\end{array}\right]$$

Next, divide the second column block by -11 such that the rightmost entry is 1

$$\left[\begin{array}{cccc|cccc|cc}
s & -1 & 0 & 0 & & & & & & \\
0 & s & -1 & 0 & & & & & & \\
0 & 0 & s & -1 & & & & & & \\
\hline
 & & & & s & -1 & 0 & 0 & & \\
 & & & & 0 & s & -1 & 0 & & \\
 & & & & 0 & 0 & s & -1 & & \\
\hline
8 & -20 & 18 & s-7 & -8 & 12 & -6 & 1 & -1 & 0 \\
0 & 0 & 13 & -2 & 0 & 0 & -2 & 0 & 0 & -1
\end{array}\right]$$

Add $7\times$ second col-block to the first col-block in order to eliminate 7 in the entry $(7,4)$ of the above matrix

$$\left[\begin{array}{cccc|cccc|cc}
s & -1 & 0 & 0 & & & & & & \\
0 & s & -1 & 0 & & & & & & \\
0 & 0 & s & -1 & & & & & & \\
\hline
 & & & & s & -1 & 0 & 0 & & \\
 & & & & 0 & s & -1 & 0 & & \\
 & & & & 0 & 0 & s & -1 & & \\
\hline
-48 & 64 & -24 & s & -8 & 12 & -6 & 1 & -1 & 0 \\
0 & 0 & -1 & -2 & 0 & 0 & -2 & 0 & 0 & -1
\end{array}\right]$$

Consider now another pencil in the orbit of $(sE-A,B)$:

$$\left[\begin{array}{cccc|cccc|cc}
s & -1 & 0 & 0 & & & & & & \\
0 & s & -1 & 0 & & & & & & \\
0 & 0 & s & -1 & & & & & & \\
\hline
 & & & & s & -1 & 0 & 0 & & \\
 & & & & 0 & s & -1 & 0 & & \\
 & & & & 0 & 0 & s & -1 & & \\
\hline
\frac{17}{11}s-\frac{40}{11} & \frac{-9}{11}s+\frac{27}{11} & \frac{-1}{11}s+\frac{3}{11} & \frac{2}{11}s-\frac{6}{11} & \frac{-8}{99}s+\frac{8}{33} & \frac{4}{33}s-\frac{4}{11} & \frac{-2}{33}s+\frac{2}{11} & \frac{1}{99}s-\frac{1}{33} & 1 & 0 \\
0 & 0 & \frac{-4}{11}s+\frac{12}{11} & 0 & \frac{1}{3}s & \frac{1}{9}s-\frac{1}{3} & \frac{-2}{99}s+\frac{2}{33} & 0 & 0 & 1
\end{array}\right] = s\tilde{E}-\tilde{A}$$

The above pencil is obtained from the original by the following strict equivalent transformations

$$RP(sT^{-1}\tilde{K}^{-1}ET - T^{-1}\tilde{K}^{-1}AT)Q$$

where $\tilde{K} = 3E - A$, T is the similarity transformation such that $(T^{-1}\tilde{K}^{-1}ET, T^{-1}B)$ is in the Popov canonical form

$$P = \left[\begin{array}{cccccccc}
1\ 0 & & & & & & & \\
 & 1 & 0 & 0 & & & & \\
 & 3 & -1 & 0 & & & & \\
 & 9 & -6 & 1 & & & & \\
 & & & & 1 & & & \\
 & & & & & 1 & 0 & 0 \\
 & & & & & 3 & -1 & 0 \\
 & & & & & 9 & -6 & 1
\end{array}\right], \quad Q = \left[\begin{array}{cccccccccc}
1 & 0 & 0 & 0 & & & & & & \\
3 & -1 & 0 & 0 & & & & & & \\
9 & -6 & 1 & 0 & & & & & & \\
27 & -27 & 9 & -1 & & & & & & \\
 & & & & 1 & 0 & 0 & 0 & & \\
 & & & & 3 & -1 & 0 & 0 & & \\
 & & & & 9 & -6 & 1 & 0 & & \\
 & & & & 27 & -27 & 9 & -1 & & \\
 & & & & & & & & 1 & 0 \\
 & & & & & & & & 0 & 1
\end{array}\right]$$

The above pencil is equivalent to

$$\left[\begin{array}{cccc|cccc|cc}
s & -1 & 0 & 0 & & & & & & \\
0 & s & -1 & 0 & & & & & & \\
0 & 0 & s & -1 & & & & & & \\
\hline
 & & & & s & -1 & 0 & 0 & & \\
 & & & & 0 & s & -1 & 0 & & \\
 & & & & 0 & 0 & s & -1 & & \\
\hline
17s-40 & -9s+27 & -s+3 & 2s-6 & -8s+24 & 12s-36 & -6s+18 & s-3 & -1 & 0 \\
0 & 0 & -4s+12 & 0 & 33s & 11s-33 & -2s+6 & 0 & 0 & -1
\end{array}\right]$$

Permutations on the col–blocks give:

$$\left[\begin{array}{cccc|cccc|cc}
s & -1 & 0 & 0 & & & & & & \\
0 & s & -1 & 0 & & & & & & \\
0 & 0 & s & -1 & & & & & & \\
\hline
 & & & & s & -1 & 0 & 0 & & \\
 & & & & 0 & s & -1 & 0 & & \\
 & & & & 0 & 0 & s & -1 & & \\
\hline
-8s+24 & 12s-36 & -6s+18 & s-3 & 17s-40 & -9s+27 & -s+3 & 2s-6 & -1 & 0 \\
33s & 11s-33 & -2s+6 & 0 & 0 & 0 & -4s+12 & 0 & 0 & -1
\end{array}\right]$$

Now, adding $-2\times$ 1st col-block to the 2nd col-block we get

$$\left[\begin{array}{cccc|cccc|cc}
s & -1 & 0 & 0 & & & & & & \\
0 & s & -1 & 0 & & & & & & \\
0 & 0 & s & -1 & & & & & & \\
\hline
 & & & & s & -1 & 0 & 0 & & \\
 & & & & 0 & s & -1 & 0 & & \\
 & & & & 0 & 0 & s & -1 & & \\
\hline
-8s+24 & 12s-36 & -6s+18 & s-3 & 33s-88 & -33s+99 & 11s-33 & 0 & -1 & 0 \\
33s & 11s-33 & -2s+6 & 0 & -66s & -22s+66 & 0 & 0 & 0 & -1
\end{array}\right]$$

Now add the appropriate multiples of the rows of the top blocks to the bottom block such that the xs′ are eliminated on the first three columns of the two bottom col-blocks

$$\left[\begin{array}{cccc|cccc|cc}
s & -1 & 0 & 0 & & & & & & \\
0 & s & -1 & 0 & & & & & & \\
0 & 0 & s & -1 & & & & & & \\
\hline
 & & & & s & -1 & 0 & 0 & & \\
 & & & & 0 & s & -1 & 0 & & \\
 & & & & 0 & 0 & s & -1 & & \\
\hline
24 & -44 & 30 & s-9 & -88 & 132 & -66 & 11 & -1 & 0 \\
0 & 0 & 17 & -2 & 0 & 0 & -22 & 0 & 0 & -1
\end{array}\right]$$

Divide the second col–block by 11

$$\left[\begin{array}{cccc|cccc|cc}
s & -1 & 0 & 0 & & & & & & \\
0 & s & -1 & 0 & & & & & & \\
0 & 0 & s & -1 & & & & & & \\
\hline
 & & & & s & -1 & 0 & 0 & & \\
 & & & & 0 & s & -1 & 0 & & \\
 & & & & 0 & 0 & s & -1 & & \\
\hline
24 & -44 & 30 & s-9 & -8 & 12 & -6 & 1 & -1 & 0 \\
0 & 0 & 17 & -2 & 0 & 0 & -2 & 0 & 0 & -1
\end{array}\right]$$

Add $9\times$ second-col-block to first col-block

$$\left[\begin{array}{cccc|cccc|cc}
s & -1 & 0 & 0 & & & & & & \\
0 & s & -1 & 0 & & & & & & \\
0 & 0 & s & -1 & & & & & & \\
\hline
 & & & & s & -1 & 0 & 0 & & \\
 & & & & 0 & s & -1 & 0 & & \\
 & & & & 0 & 0 & s & -1 & & \\
\hline
-48 & 64 & -24 & s & -8 & 12 & -6 & 1 & -1 & 0 \\
0 & 0 & -1 & -2 & 0 & 0 & -2 & 0 & 0 & -1
\end{array}\right]$$

The above form is identical to the form obtained for the first system $\mathcal{S}(E_2, A_2, B_2)$.

5.9 Conclusions

In this chapter the problem of canonical forms for minimal singular systems has been considered. The transformation group considered is the strict equivalence group. Two types of systems were studied. First the systems $\mathcal{S}(E, A, B, C)$ (systems with outputs) were considered. For those systems a canonical form was derived and the relation between this form and the echelon form of the composite matrix of any MFD was established. This result is a generalisation of the work of Forney to the case of singular systems. The derivation of the canonical form was based on the fact that the input–output transfer function is invariant under strict equivalence transformations.

The second type of systems studied here is that of the systems $\mathcal{S}(E, A, B)$ i.e. systems without output equation. The problem of Popov type canonical forms for this type of systems was solved for the case where the reachability indices of the system are equal. For the general case where the reachability indices are not necessarily equal, a semi canonical form has been obtained .The main difficulty arising in the general case is that the input–state transfer function is not invariant under strict equivalence transformations. The general case is the subject of future research.

Chapter 6

FIRST ORDER REALISATIONS OF AUTOREGRESSIVE EQUATIONS

6.1 Introduction

In Chapter 4 the problem of obtaining a generalised state–space representation from a given transfer function $G(s)$ was considered. Our requirement there, was to find a system described by the equations $E\dot{x} = Ax + Bu$, $y = Cx$ having a transfer function equal to a given transfer function i.e. the system $\mathcal{S}(E, A, B, C)$ and the system described by $G(s)$ were transfer equivalent.

In the present Chapter we consider the problem of obtaining a first order representation from a given autoregressive equation $T(\sigma)w(t) = 0$, such that the first order representation is externally equivalent to the autoregressive equation. This problem is different from the realisation problem under transfer equivalence for two reasons. First, because transfer equivalence does not necessarily mean external equivalence and second, because systems described in autoregressive form may not admit transfer function descriptions and in general the external signals are not distinguished into inputs and outputs.

The first order realisations considered in this Chapter are of descriptor type $E\dot{\xi} = A\xi + Bu$, $y = C\xi + Du$ where E, A are not necessarily square and of pencil type $F\dot{\xi} = G\xi$, $w = H\xi$. The vector $w(t)$ is the vector of the external variables of the system. If some of these variables are labeled as inputs and the other as outputs then we may in general consider that $w(t) = [u(t), y(t)]$. This partitioning does not mean that the outputs may be explicitly expressed as functions of the inputs. When we consider the above partitioning we may obtain a descriptor realisation. When we consider the external variables vector without partitioning we can take a pencil realisation.

The problem of finding first order representations of general external form descriptions was first solved by Kuijper and Schumacher in [Kuij. & Sch., 1990]. Bonilla [Bon., 1991] also obtained descriptor realisations from a given set of differential equations.

It is the purpose of the present Chapter to provide an alternative method for first order realisations of a given set of linear differential equations. The results are similar to those of Kuijper and Schumacher. However, the algorithm proposed is much simpler and the realisation is directly obtained by inspection from the coefficients of the polynomial entries of $T(s)$. This brings our approach closer to the methodology of realisation of transfer functions. Furthermore, the proposed method leads to such representations that we may easily identify the structural invariants of the system. Such invariants are the observability indices which may be directly identified in the case where the realisation is in descriptor form.

The structure of the Chapter is as follows: First, an ARMA first order representation is obtained from the given autoregressive equations. This representation is used as an intermediate step to both descriptor and pencil realisations. Then, a realisation method leading to descriptor representation is proposed and the minimality of this realisation is examined. Next, a pencil type realisation is obtained and finally a descriptor realisation without feedthrough term is found. For all of the above, the issue of minimality is examined and it is shown that if we start from row reduced autoregressive equations matrix, our method leads to minimal first order model.

Another topic examined in this Chapter is the relation of the row degrees of the autoregressive equation matrix and the row minimal indices of the observability pencil $[sE^T - A^T, C^T]^T$ of the descriptor realisation and a generalisation of the observability indices to the case of implicit systems is proposed.

6.2 Statement of the problem and preliminary results

Consider the autoregressive equation

$$T(\sigma)w(t) = 0 \tag{6.1}$$

where $T(s) \in \Re^{p\times(m+\ell)}[s]$ and σ denotes the derivative operator $\frac{d}{dt}$. The above equation is a set of differential equations describing the behaviour of a dynamical system. The solutions (trajectories) of this equation are defined as the behaviour $\mathcal{B}$ of the system (6.1). The vector $w(t)$ is defined as the vector of external variables.

When we work in the framework of systems of type (6.1) the external variables are not necessarily partitioned into inputs and outputs. If we label ℓ of the external variables as inputs and m as outputs we obtain (possibly after reordering of the entries of $w(t)$) the following *oriented* system

$$\begin{bmatrix} N(\sigma) & D(\sigma) \end{bmatrix} \begin{bmatrix} u(t) \\ y(t) \end{bmatrix} = 0 \tag{6.2}$$

The partitioning of the variables $w(t)$ into inputs and outputs does not necessarily mean that the output $y(t)$ can be expressed explicitly in terms of the input $u(t)$.

The problem of realisation in descriptor form is, given the system (6.2), to find a system of differential equations of the form

$$E\dot{\xi} = A\xi + Bu \tag{6.3}$$

$$y = C\xi + Du \tag{6.4}$$

such that it induces the same behaviour $[u^T, y^T]^T$ to (6.2). The variables ξ are auxiliary and according to the terminology of [Wil., 1991] are called *latent variables.* The transformation of (6.2) to (6.3), (6.4) is obtained by appropriate choice of latent variables. Our aim is to obtain a realisation of the form (6.3), (6.4) which is minimal. In the case of descriptor representations, minimality is defined in terms of three numbers: the number of states (latent variables), the number of equations and the rank defficiency of the matrix E in (6.3). For descriptor systems we have the following criteria of minimality.

Proposition 6.2.1 [Kui. & Sch., 1991]. *The descriptor system* (6.3), (6.4) *is minimal under external equivalence if and only if the following conditions hold:*

(i) $[E, B]$ is surjective

(ii) $\begin{bmatrix} E \\ C \end{bmatrix}$ is injective

(iii) $A\mathrm{Ker}\{E\} \subseteq \mathrm{Im}\{E\}$

(iv) $\begin{bmatrix} sE - A \\ C \end{bmatrix}$ has no finite zeros □

The first and second conditions above correspond to the requirement of observability and reachability at infinity. Condition (ii) means that the dynamical part of the descriptor representation ($E\dot{x} = Ax + Bu$) does not contain any nondynamic state variables [Ver., Lev. & Kail., 1981] while (iv) corresponds to the classic condition for observability in finite s. Note that finite reachability is not a requirement for the minimality under external equivalence. For a discussion of this, see [Wil., 1991], [Kui. & Sch., 1991].

The second form of first order differential equations that (6.1) may be transformed, is the pencil form or pencil representation [Kuij. & Sch., 1990].

$$F\dot{z} = Gz \tag{6.5}$$

$$w = Hz \tag{6.6}$$

Realisation (6.5), (6.6) in pencil form is obtained from (6.1). Note that the pencil form does not require partitioning of the external variables vector into inputs and outputs. The minimality of (6.5), (6.6) under external equivalence may be inspected by using the following criteria [Kuij. & Sch., 1990].

Proposition 6.2.2 *A pencil representation of the form* (6.5), (6.6) *is minimal under external equivalence if and only if*

(i) F has full row rank

(ii) $\begin{bmatrix} F \\ H \end{bmatrix}$ has full column rank

(iii) The pencil $\begin{bmatrix} sF - G \\ H \end{bmatrix}$ does not have finite Smith zeros. □

In this Chapter we are going to obtain realisations of both of the above types. For the case of descriptor systems we may incorporate the nondynamic part of the system which is expressed by the matrix D in (6.4) and take descriptor equations without feedthrough term i.e. equation (6.4) is replaced by the following output equation

$$y = C\xi \tag{6.7}$$

Then the conditions for minimality are given by the following result [Kuij., 1992].

Proposition 6.2.3 *The descriptor representation* (6.3), (6.7) *is minimal under external equivalence if and only if*

(i) $[E, B]$ *has full row rank*

(ii) $\begin{bmatrix} E \\ C \end{bmatrix}$ *has full column rank*

(iii) The pencil $\begin{bmatrix} sE - A \\ C \end{bmatrix}$ *does not have finite Smith zeros.* □

Note that criteria for the minimality of descriptor representations with feedthrough term and without it differ only in (iii) of proposition 6.2.1. This criterion expresses the absence of nondynamic variables.

6.3 An ARMA realisation of $T(s)$

In this section a first order representation which is externally equivalent to (6.1), is derived. This realisation is used as an intermediate step towards the descriptor and pencil realisations. We start from $T(\sigma)w = 0$. Note that $T(s)$ is a polynomial matrix.

Consider the first order ARMA system.

$$\tilde{R}(\sigma)x(t) = T_c w(t) \tag{6.8}$$

and T_c is obtained from:

$$S(s) = \text{block–diag}\{\cdots, [1, s \cdots s^{\sigma_i}], \cdots\},\ T(s) = S(s)T_c(s) \tag{6.9}$$

where σ_i are the row degrees of $T(s)$ and

$$\tilde{R}(s) = \text{block-diag}\{L_{n_i}(s)\} \tag{6.10}$$

where $L_{\eta_i}(s)$ is the standard row minimal index block of dimensions $(\sigma_i + 1) \times \sigma_1$ i.e.

$$L_{\sigma_i}(s) = \begin{bmatrix} s & \cdots & 0 \\ -1 & \ddots & \vdots \\ \vdots & \ddots & s \\ 0 & & -1 \end{bmatrix}_{(\sigma_i+1)\times\sigma_i} \tag{6.11}$$

Our aim is to show that (6.8) is externally equivalent to autoregressive equation (6.1). In order to prove this we shall make use of the following important result [Kuij. & Sch., 1990].

Lemma 6.3.1 *Consider a behaviour given by the equations:*

$$P(\sigma)\xi = 0$$

$$w = Q(\sigma)\xi$$

It is always possible to find matrices $V(s)$ and $T(s)$ such that

(i) $V(s) \in \mathcal{R}^{(n+q-r)\times n}[s]$, $T(s) \in \mathcal{R}^{(n+q-r)\times q}[s]$

(ii) $V(s)$ *and* $T(s)$ *are left coprime*

(iii) $[V(s),\ T(s)][P^T(s),\ Q^T(s)]^T = 0$

If $V(s)$ *satisfy the above properties, then the equation*

$$T(\sigma)w(t) = 0$$

induces the same external behaviour, to

$$P(\sigma)\xi = 0$$

$$w = Q(\sigma)\xi$$

In addition

$$\text{Ker}\{T(s)\} = Q(s)\text{Ker}\{P(s)\}$$

where $r = \text{rank}_{\mathcal{R}(s)}\{[P^T(s),\ Q^T(s)]^T\}$, $P(s)$ *has* n *rows,* $Q(s)$ *has* q *rows* □

The representation (6.8) is of the form

$$P(\sigma)\xi = 0 \tag{6.12}$$

$$w = Q(\sigma)\xi \tag{6.13}$$

where

$$P(\sigma) = [\tilde{R}(\sigma), T_c],\ Q(\sigma) = [0\ I] \tag{6.14}$$

and

$$\xi = \begin{bmatrix} x \\ w \end{bmatrix} \tag{6.15}$$

We may now state the following result.

Proposition 6.3.1 *Equations* (6.1) *and* (6.8) *induce the same external behaviour* B.

Proof: We have to prove conditions *(i)-(iii)* of lemma 6.3.1 for

$$P(s) = [\tilde{R}(s),\ T_c],\ \xi = [x^T(s),\ w^T(s)]^T$$

$$Q(s) = [0\ I]$$

From (6.10),(6.11) it follows that

$$\begin{bmatrix} P(s) \\ Q(s) \end{bmatrix} = \begin{bmatrix} \tilde{R}(s) & T_c \\ 0 & I \end{bmatrix}$$

Consider now the following matrix:

$$M(s) = \left[\begin{array}{c|c} M_1(s) & M_1(s)T_c \\ 0 & I \\ \hline S(s) & T(s) \end{array}\right]$$

where

$$M_1(s) = \text{block}-\text{diag}\left\{\ldots, \left[\begin{array}{c|ccccc} 0 & -1 & -s & -s^2 & \ldots & -s^{\sigma_i-1} \\ & & \ddots & \ddots & & \vdots \\ \vdots & & & \ddots & \ddots & -s^2 \\ & & & & -1 & -s \\ 0 & & & & & -1 \end{array}\right], \ldots\right\}$$

$$S(s) = \text{block} - \text{diag}\{\ldots, [1\ s \ldots s^{\sigma_i}], \ldots\}$$

and it is easy to check that

$$M(s) = \left[\frac{P(s)}{Q(s)}\right] = \begin{bmatrix} I \\ 0 \end{bmatrix}$$

Now, if $V(s) = S(s)$, it follows that conditions *(i)-(iii)* of lemma 6.3.1 are satisfied and the result follows. □

Remark 6.3.1 *Note that* $[S(s), T(s)]$ *is a basis matrix of the left null space of* $\begin{bmatrix} P(s) \\ G(s) \end{bmatrix}$.

□

Remark 6.3.2 *The ARMA realisation obtained in this section is an intermediate step towards the realisations of this chapter. As it will be shown in the next chapter, it is convenient for use in problems related to interconnections of behavioural systems. Such a problem is considered in the next chapter.*

6.4 Realisation in descriptor form

In this section the realisation in descriptor form is obtained by appropriate reordering of the equation of the ARMA representation and the introduction of some new internal variables which is necessary in order to express the output $y(t)$ explicitly in terms of the internal variables and the input variables. We proceed as follows: We start from

$$\tilde{R}(\sigma)x(t) = T_c w(t) \tag{6.16}$$

The above may be multiplied from the left by a (unimodular) permutation matrix and give

$$R(\sigma)x(t) = \begin{bmatrix} T_{\ell c} \\ T_{hc} \end{bmatrix} w(t) \tag{6.17}$$

where T_{hc} is the high order coefficient matrix of $T(s)$ and $T_{\ell c}$ the lower order coefficient and $R(s)$ has the following structure:

$$R(s) = \left[\frac{R_1(s)}{R_2(s)}\right], \tag{6.18}$$

$$R_1(s) = \text{block} - \text{diag}\{sI_{\sigma_i} - \hat{A}_i\} \tag{6.19}$$

$$\hat{A}_i = \begin{array}{c} \overbrace{\phantom{\begin{matrix} 0\cdots & \cdots 0 \end{matrix}}}^{\sigma_i} \\ \left[\begin{matrix} 0\cdots & \cdots & 0 \\ & & \vdots \\ I_{\sigma_i-1} & & 0 \end{matrix}\right] \sigma_i \end{array} \tag{6.20}$$

$$R_2(s) = \text{block} - \text{diag}\left\{\left[e^t_{\sigma_i}\right]\right\} \tag{6.21}$$

Equations (6.17) and (6.16) have identical sets of solutions since permutation is just reordering of the equations and thus, does not affect the solutions. From the above it follows that (6.17) has the form

$$\left[\begin{array}{ccc} I_{\sigma_1} & & \\ & \ddots & \\ & & I_{\sigma_\rho} \\ \hline & 0 & \end{array}\right] \underline{\dot{x}}(t) = \left[\begin{array}{ccc} \hat{A}_1 & & \\ & \ddots & \\ & & \hat{A}_\rho \\ \hline 0\cdots 1 & & \\ & & \\ & & 0\cdots 1 \end{array}\right] \underline{x}(t) + \left[\begin{array}{c|c} T_{\ell 1} & T_{\ell 2} \\ \hline T_{h1} & T_{h2} \end{array}\right] \left[\begin{array}{c} \underline{u}(t) \\ \underline{y}(t) \end{array}\right] \tag{6.22}$$

where $T_{\ell c}$ and T_{hc} are partitioned conformably to the partitioning of $w(s)$ into inputs and outputs, as follows

$$T_{\ell c} = [T_{\ell 1},\ T_{\ell 2}],\ T_{hc} = [T_{h1},\ T_{h2}] \tag{6.23}$$

Then, (6.22) may be written in the form:

$$\left[\begin{array}{c} I_\sigma \\ \hline 0 \end{array}\right] \dot{x}(t) = \left[\begin{array}{c} \hat{A} \\ \hline F \end{array}\right] x(t) + \left[\begin{array}{c|c} T_{\ell 1} & T_{\ell 2} \\ \hline T_{h1} & T_{h2} \end{array}\right] \left[\begin{array}{c} u(t) \\ \hline y(t) \end{array}\right] \tag{6.24}$$

where $\sigma = \sum_{i=1}^{\rho} \sigma_i$ and

$$F = \text{block} - \text{diag}\{\ldots [0,\ldots,0,1]_{1\times\sigma_i},\ldots\},\ \hat{A} = \text{block–diag}\{\hat{A}_i\} \tag{6.25}$$

Let $q = \text{rank}\{T_{h2}\} \leq \rho$. Equation (6.24) is equivalent to

$$\left[\begin{array}{cc} I & 0 \\ 0 & Q \end{array}\right] \left[\begin{array}{c} I_\sigma \\ \hline 0 \end{array}\right] \dot{x} = \left[\begin{array}{cc} I & 0 \\ 0 & Q \end{array}\right] \left[\begin{array}{c} \hat{A} \\ \hline F \end{array}\right] x(t) + \left[\begin{array}{cc} I & 0 \\ 0 & Q \end{array}\right] \left[\begin{array}{c|c} T_{\ell 1} & T_{\ell 2} \\ \hline T_{h1} & T_{h2} \end{array}\right] \left[\begin{array}{c} u(t) \\ y(t) \end{array}\right] \tag{6.26}$$

where Q is an invertible constant matrix such that

$$QT_{h2} = \left[\begin{array}{c|c} I_q & \bar{T}_{h2} \\ \hline 0 & 0 \end{array}\right] \tag{6.27}$$

Remark 6.4.1 *Note that in order to transform $[T_{h1}, T_{h2}]$ to the form (6.27), reordering and relabeling of the output vector may be required. This does not affect the generality as it will be shown later.* □

Now, (6.26) may be written as

$$\left[\begin{array}{c} I_\sigma \\ \hline 0 \end{array}\right] \dot{x}(t) = \left[\begin{array}{c} \hat{A} \\ \hline QF \end{array}\right] x(t) + \left[\begin{array}{c|c} T_{\ell 1} & T_{\ell 2} \\ \hline QT_{h1} & \left[\begin{array}{c|c} I_q & \bar{T}_{h2} \\ \hline 0 & 0 \end{array}\right] \end{array}\right] \left[\begin{array}{c} u(t) \\ y(t) \end{array}\right]$$

or

$$\left[\begin{array}{c} I_\sigma \\ \hline 0 \\ \hline 0 \end{array}\right] \dot{x}(t) = \left[\begin{array}{c} \hat{A} \\ \hline F_1 \\ \hline F_2 \end{array}\right] x(t) + \left[\begin{array}{cc} T_{\ell 1} & T_{\ell 2} \\ T_1 & 0 \\ T_2 & [I_q, \ \bar{T}_{h2}] \end{array}\right] \left[\begin{array}{c} u(t) \\ \hline y_a(t) \\ \hline y_b(t) \end{array}\right] \tag{6.28}$$

where

$$QF = \left[\begin{array}{c} F_1 \\ F_2 \end{array}\right] \tag{6.29}$$

The above may be split into the following:

$$\left[\begin{array}{c} I_\sigma \\ 0 \end{array}\right] \dot{x}(t) = \left[\begin{array}{c} \hat{A} \\ F_1 \end{array}\right] x(t) + \left[\begin{array}{cc} T_{\ell 1} & T_{\ell 2} \\ T_1 & 0 \end{array}\right] \left[\begin{array}{c} u(t) \\ y_a(t) \\ y_b(t) \end{array}\right] \tag{6.30}$$

$$\underline{0} = F_2 x(t) + T_2 u(t) + I_q y_a(t) + \bar{T}_{h2} y_b(t) \tag{6.31}$$

Now, define the new state-variable $z(t) = y_b(t)$, then (6.31) gives:

$$\underline{0} = \left[\begin{array}{cc} F_2 & \bar{T}_{h2} \\ 0 & -I \end{array}\right] \left[\begin{array}{c} x(t) \\ z(t) \end{array}\right] + \left[\begin{array}{c} T_2 \\ 0 \end{array}\right] u(t) + \left[\begin{array}{cc} I & 0 \\ 0 & I \end{array}\right] \left[\begin{array}{c} y_a(t) \\ y_b(t) \end{array}\right] \tag{6.32}$$

or,

$$\left[\begin{array}{c} y_a(t) \\ y_b(t) \end{array}\right] = \left[\begin{array}{cc} -F_2 & -\bar{T}_{h2} \\ 0 & I \end{array}\right] \left[\begin{array}{c} x(t) \\ z(t) \end{array}\right] + \left[\begin{array}{c} -T_2 \\ 0 \end{array}\right] u(t) \tag{6.33}$$

or,

$$y(t) = C\xi(t) + Du(t) \tag{6.34}$$

where

$$\xi(t) = \begin{bmatrix} x(t) \\ \xi(t) \end{bmatrix} \tag{6.35}$$

and

$$C = \begin{bmatrix} -F_2 & -T_{h2} \\ 0 & I \end{bmatrix}, \ D = \begin{bmatrix} -T_2 \\ 0 \end{bmatrix} \tag{6.36}$$

Now, by substitution of (6.33) to (6.30) we get

$$\begin{bmatrix} I_\sigma \\ 0 \end{bmatrix} \dot{x} = \begin{bmatrix} \hat{A} \\ F_1 \end{bmatrix} x(s) + \begin{bmatrix} T_{\ell 1} \\ T_1 \end{bmatrix} u +$$

$$+ \begin{bmatrix} T_{\ell 2} \\ 0 \end{bmatrix} \begin{bmatrix} -F_2 & -\bar{T}_{h2} \\ 0 & I \end{bmatrix} \begin{bmatrix} x(s) \\ z(s) \end{bmatrix} + \begin{bmatrix} -T_2 \\ 0 \end{bmatrix} u(s)$$

or

$$\begin{bmatrix} I_\sigma & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} \hat{A} & 0 \\ F_1 & 0 \end{bmatrix} + \begin{bmatrix} T_{\ell 2} \\ 0 \end{bmatrix} \begin{bmatrix} -F_2 & -\bar{T}_{h2} \\ 0 & I \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} +$$

$$+ \begin{bmatrix} T_{\ell 2} \\ 0 \end{bmatrix} \begin{bmatrix} -T_2 \\ 0 \end{bmatrix} + \begin{bmatrix} T_{\ell 1} \\ T_1 \end{bmatrix} u(s) \tag{6.37}$$

Summarising the above we have that the matrices of the descriptor realisation of $T(\sigma)w(t) = 0$ are the following.

$$E = \begin{bmatrix} I_\sigma & 0 \\ 0 & 0 \end{bmatrix}, \ A = \begin{bmatrix} \hat{A} & 0 \\ F_1 & 0 \end{bmatrix} + \begin{bmatrix} T_{\ell 2} \\ 0 \end{bmatrix} \begin{bmatrix} -F_2 & -\overline{T}_{h2} \\ 0 & I \end{bmatrix} \tag{6.38}$$

$$B = \begin{bmatrix} T_{\ell 2} \\ 0 \end{bmatrix} \begin{bmatrix} -T_2 \\ 0 \end{bmatrix} + \begin{bmatrix} T_{\ell 1} \\ T_1 \end{bmatrix}, \ C = \begin{bmatrix} -F_2 & -T_{h2} \\ 0 & I \end{bmatrix} \tag{6.39}$$

$$D = \begin{bmatrix} -T_2 \\ 0 \end{bmatrix} \tag{6.40}$$

where $\sigma = \sum_{i=1}^{\rho} \sigma_i$.

Remark 6.4.2 *As it was mentioned in remark* 6.4.1 *we may need to perform some permutations on the output vector entries in order to obtain* (6.27) *and thus all subsequent equations and formulas. In that case, the output matrix is given by* $\hat{P}^{-1}\hat{C}$ *where* $\hat{P}$ *is the permutation matrix corresponding to the reordering of the entries of* $y(t)$. □

Remark 6.4.3 *The above realisation is externally equivalent to* (6.17) *since the transformations from* (6.17) *to* (6.38), (6.39) *include only renaming of variables and permutations of the equations.* □

Next we consider the issue of the minimality of the realisation (6.38)-(6.40). As in the case of the realisation of transfer functions we have that minimality of the realisation is dependent on the matrix $T(s)$ we start from. This is shown below.

Proposition 6.4.1 *Descriptor realisation* (6.38)-(6.40) *is minimal if and only if the matrix* $T(s)$ *is column reduced.*

Proof: We have to examine when the conditions of proposition 6.2.1 hold true.
We start from condition (ii):
Condition rank $\begin{bmatrix} E \\ C \end{bmatrix}$ = full column, is obvious since

$$\begin{bmatrix} E \\ C \end{bmatrix} = \left[\begin{array}{c|c} I_\sigma & 0 \\ 0 & 0 \\ \hline -F_2 & -\bar{T}_{h2} \\ 0 & I \end{array}\right]$$

Next we examine the zero structure of $\begin{bmatrix} sE - A \\ C \end{bmatrix}$.
We have that

$$\begin{bmatrix} sE - A \\ C \end{bmatrix} = \begin{bmatrix} sI - \hat{A} + T_{\ell 2}^1 F_2 & T_{\ell 2}\begin{bmatrix} \overline{T}_{h2} \\ I \end{bmatrix} \\ -F_1 & I \\ F_2 & -T_{h2} \\ 0 & I \end{bmatrix}$$

where $T_{\ell 2} = [T_{\ell 2}^1, T_{\ell 2}^2]$. The above pencil is equivalent to

$$\begin{bmatrix} sI - \hat{A} & 0 \\ F_1 & 0 \\ F_2 & 0 \\ 0 & I \end{bmatrix}$$

Now, $\begin{bmatrix} F_1 \\ \hline F_2 \end{bmatrix}$ has full rank and from the form of $sI - \hat{A}$ we may see that

$$\text{rank}\begin{bmatrix} sI - \hat{A} \\ F_1 \\ F_2 \end{bmatrix} = \text{rank}\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} + \#\hat{\eta}_i = \sum_{\sigma_i} \forall s \in C$$

where $\hat{\eta}_i = \{\sigma_i / \sigma_i > 1\}$.

Therefore $\begin{bmatrix} sE - A \\ C \end{bmatrix}$ has no finite zeros which proves condition (iv).

To prove (iii) observe that a basis matrix $\overline{E}$ for Ker$\{E\}$ is the following

$$\overline{E} = \begin{bmatrix} 0 \\ I \end{bmatrix}$$

The matrix A may be written (see (6.38))

$$\begin{bmatrix} \hat{A} & 0 \\ F_1 & 0 \end{bmatrix} + \begin{bmatrix} T_{\ell 2} \\ 0 \end{bmatrix} \begin{bmatrix} -F_2 & -\bar{T}_{h2} \\ 0 & I \end{bmatrix}$$

From the above we see that A has the form:$A = \begin{bmatrix} A_1' & A_2' \\ F_1 & 0 \end{bmatrix}$. Therefore, the matrix representation of the basis of A Ker $\{E\}$ is

$$A_{\overline{E}} = \begin{bmatrix} A_2' \\ 0 \end{bmatrix}$$

and

$$[E, A_{\overline{E}}] = \begin{bmatrix} I & 0 & A_i' \\ 0 & 0 & 0 \end{bmatrix}$$

From the above is obvious that A Ker $\{E\} \subseteq$ Im$\{E\}$ and the result follows.
From (6.38), (6.39) it follows that

$$E = \begin{bmatrix} I_\sigma & 0 \\ 0 & 0 \end{bmatrix}, \; B = \left[\begin{bmatrix} T_{\ell_2} \\ 0 \end{bmatrix} \; \begin{bmatrix} -T_2 \\ 0 \end{bmatrix} + \begin{bmatrix} T_{\ell_1} \\ T_1 \end{bmatrix} \right]$$

for $[E, B]$ to be surjective it suffices T_1 to have full row rank i.e. This is true from (6.28) and the row reducedness of $[N(s), D(s)]$. Thus (i) holds true. □

By simple inspection of the form of the matrices of the descriptor representations we have the dimensions of the minimal realisation as it is shown on the corollary below.

Corollary 6.4.1 *The dimensions of the minimal realisation are*

1. rank$\{E\} = \sigma$

2. #col$E = m -$rank$\{T_{h1}\} + \sigma$

3. $\#rowsE = \text{rank}\{T_{h1}\} + \sigma$

□

Example 6.4.1 Consider the autoregressive system $T(\sigma)w(t) = 0$ where

$$T(s) = \left[\begin{array}{cc|ccc} s^2 & 2 & s^2+1 & s^2 & s^2+2 \\ 1 & s-1 & 2 & 0 & 1 \end{array}\right]$$

Then (6.24) has the form

$$\left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \left[\begin{array}{ccc} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \left[\begin{array}{cc|ccc} 0 & 2 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 2 & 0 & 1 \\ \hline 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{array}\right] \left[\begin{array}{c} u_1 \\ u_2 \\ \hline y_1 \\ y_2 \\ y_3 \end{array}\right]$$

Note that $\text{rank}\{T_{h2}\} = \text{rank}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = 1$. By defining the new state variables $z_1(t) = y_2(t)$, $z_2(t) = y_3(t)$ we take

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

and thus,

$$C = \begin{bmatrix} 0 & -1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \; D = \begin{bmatrix} -1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

From (6.38) with

$$\hat{A} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \; F_1 = [0\ 0\ 1], \; T_2 = [1\ 0]$$

$$F_2 = [0\ 1\ 0], \; \overline{T}_{h2} = [1\ 1], \; T_{\ell 2} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 1 \end{bmatrix} \; T_1 = [0\ 1]$$

we take

$$A = \begin{bmatrix} 0 & -1 & 0 & -1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & -2 & -1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 2 \\ 0 & 0 \\ -1 & -1 \\ 0 & 1 \end{bmatrix}$$

The descriptor model is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \\ \dot{\xi}_3 \\ \dot{\xi}_4 \\ \dot{\xi}_5 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & -1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & -2 & -1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \\ \xi_5 \end{bmatrix} + \begin{bmatrix} -1 & 2 \\ 0 & 0 \\ -1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \\ \xi_5 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

It may be easily verified that this model is minimal.

6.5 Realisation in pencil form

In this section a realisation of $T(s)$ of the form

$$F\dot{\xi} = G\xi \tag{6.41}$$

$$w = H\xi \tag{6.42}$$

is obtained. It is shown that if $T(s)$ is row reduced, then the resulting representation is minimal. The procedure for the realisation is the following.

We start again from behavioural equations

$$\tilde{R}(\sigma)x(t) = T_c w(t) \tag{6.43}$$

and using the same arguments as in section 6.4 we obtain the form

$$R(\sigma)x(t) = \left[\frac{T_{\ell_c}}{T_{h_c}}\right] w(t) \tag{6.44}$$

The above may be written as

$$\left[\frac{\sigma I - \hat{A}}{\tilde{F}}\right] x(t) = \left[\frac{T_{\ell_c}}{T_{h_c}}\right] w(t) \tag{6.45}$$

Multiplying (6.45) from the left by the matrix $\begin{bmatrix} I & 0 \\ 0 & Q \end{bmatrix}$ where QT_{hc} is in row-echelon form we get

$$\begin{bmatrix} \sigma I - \hat{A} \\ \tilde{F}_1 \end{bmatrix} x(t) = \left[\frac{T_{\ell_c}}{T_{ehc}}\right] w(t) \tag{6.46}$$

where $T_{ech} = QT_{hc}$.

Without loss of generality we may assume that $T(s)$ is row reduced and T_{ech} has the form

$$T_{ech} = \left[\begin{array}{ccc|ccc} 1 & & & \times & \cdots & \times \\ & \ddots & & & & \\ & & 1 & \times & \cdots & \times \end{array}\right] = [I, T_{ech}^2] \tag{6.47}$$

If T_{ech} is not in form (6.47) we may always, by reordering the entries of the vector of external variables $w(t)$, take T_{ech} in form (6.47). Let $T_{\ell c} = [T_{\ell c}^1, T_{\ell c}^2]$. Then, by elementary row operations, (6.46) may be transformed to

$$\begin{bmatrix} \sigma I - \hat{A}' \\ \tilde{F}_1 \end{bmatrix} x(t) = \left[\begin{array}{c|c} 0 & \overline{T}_{\ell c}^2 \\ \hline I & T_{ech}^2 \end{array}\right] \left[\frac{w_1(t)}{w_2(t)}\right] \tag{6.48}$$

where

$$w(t) = [w_1(t), w_2(t)],\ \hat{A}' = \hat{A} - T_{\ell c}^1 \tilde{F}_1,\ \overline{T}_{\ell c}^2 = T_{\ell c}^2 - T_{\ell c}^1 T_{ech}^2 \tag{6.49}$$

and $w(t)$ is partitioned in an obvious way. Define now the new internal variables

$$z(t) = w_2(t) \tag{6.50}$$

Then (6.48) and (6.50) are equivalent to

$$\begin{bmatrix} I & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}(t) \\ \dot{z}(t) \end{bmatrix} = \begin{bmatrix} \hat{A}' & 0 \\ -\tilde{F}_1 & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} x(t) \\ z(t) \end{bmatrix} + \begin{bmatrix} 0 & \overline{T}_{\ell c}^2 \\ I & T_{ech}^2 \\ 0 & I \end{bmatrix} \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix} \tag{6.51}$$

Now multiplying the above from the left by the invertible matrix

$$\begin{bmatrix} I & 0 & -\overline{T}^2_{\ell c} \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \tag{6.52}$$

we take the equivalent equation

$$\left[\begin{array}{cc} I & 0 \\ \hline 0 & 0 \\ 0 & 0 \end{array}\right] \begin{bmatrix} \dot{x}(t) \\ \dot{z}(t) \end{bmatrix} = \left[\begin{array}{cc} \hat{A}' & -\overline{T}^2_{\ell c} \\ \hline -\tilde{F}_1 & 0 \\ 0 & -I \end{array}\right] \begin{bmatrix} x(t) \\ z(t) \end{bmatrix} + \left[\begin{array}{cc} 0 & 0 \\ \hline I & T^2_{ehc} \\ 0 & I \end{array}\right] \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix} \tag{6.53}$$

which may be separated into two equations

$$\begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} \dot{x}(t) \\ \dot{z}(t) \end{bmatrix} = \begin{bmatrix} \hat{A}' & -\overline{T}^2_{\ell c} \end{bmatrix} \begin{bmatrix} x(t) \\ z(t) \end{bmatrix} \tag{6.54}$$

$$\begin{bmatrix} \tilde{F}_1 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} x(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} I & T^2_{ech} \\ 0 & I \end{bmatrix} \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix} \tag{6.55}$$

The above is equivalent to

$$\begin{bmatrix} \tilde{F}_1 & -T^2_{ech} \\ 0 & I \end{bmatrix} \begin{bmatrix} x(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix} \tag{6.56}$$

Thus the pencil realisation has the following matrices

$$F = [I \ 0], \ G = [\hat{A}' \ -\overline{T}^2_{\ell c}] \tag{6.57}$$

$$H = \begin{bmatrix} \tilde{F}_1 & T^2_{ech} \\ 0 & I \end{bmatrix} \tag{6.58}$$

Remark 6.5.1 *In the case where reordering of the entries of* $w(t)$ *is needed in order to obtain* (6.47), *equation* (6.56) *is modified to*

$$\hat{P}^{-1} \begin{bmatrix} \tilde{F}_1 & -T^2_{ech} \\ 0 & I \end{bmatrix} \begin{bmatrix} x(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

where $\hat{P}$ *is the permutation matrix used to perform the reordering. Then* (6.57) *and* (6.58) *are modified analogously.* □

Note that the above realisation is externally equivalent to the original AR system since the transformations leading from (6.43) to the pencil form involve change of basis in the equation space and state–space and remaining of some variables. These operations do not affect the external behaviour.

For the minimality of the pencil realisation we have the following result.

Proposition 6.5.1 *The realisation* (6.57), (6.58) *is minimal.*

Proof: We have to prove (i), (ii) and (iii) of proposition 6.2.2. Condition (i) follows readily since

$$\begin{bmatrix} F \\ H \end{bmatrix} = \left[\begin{array}{cc} I & 0 \\ \hline \hat{F}_1 & -T^2_{ech} \\ 0 & I \end{array}\right]$$

has full column rank.

Condition (ii) is obviously satisfied since $F = [I\ 0]$.

In order to show that (iii) holds true we consider the pencil $[sF^T - G^T, H^T]^T$. From (6.57), (6.58) we have

$$\begin{bmatrix} sF - G \\ H \end{bmatrix} = \begin{bmatrix} sI - \hat{A}' & -T^2_{\ell c} \\ \tilde{F}_1 & -T^2_{ech} \\ 0 & I \end{bmatrix} \sim \begin{bmatrix} sI - \hat{A}' & 0 \\ \tilde{F}_1 & 0 \\ 0 & I \end{bmatrix} \tag{6.59}$$

where "$\sim$" denotes strict equivalence.

From the above it is clear that the Smith zeros of $[sF^T - G^T, H^T]^T$ are provided by the zeros of $[sI - \hat{A}'^T, \tilde{F}_1^T]^T$. Observe that using the second of (6.49) the pencil in (6.59) may be written as

$$\begin{bmatrix} sF - G \\ H \end{bmatrix} \sim \begin{bmatrix} sI - \hat{A} - T^1_{\ell c}\tilde{F}_1 & 0 \\ \tilde{F}_1 & 0 \\ 0 & I \end{bmatrix} \sim \begin{bmatrix} sI - \hat{A} & 0 \\ \tilde{F}_1 & 0 \\ 0 & I \end{bmatrix} \tag{6.60}$$

and since $\tilde{F}_1 = Q\tilde{F}$ (see (6.45), (6.46)) it follows that

$$\begin{bmatrix} sF - G \\ H \end{bmatrix} \sim \begin{bmatrix} R(s) & 0 \\ 0 & I \end{bmatrix} \tag{6.61}$$

and since $R(s)$ has only r.m.i. it follows that $[sF^T - G^T, H^T]^T$ has only r.m.i. and linear i.e.d. and the result follows. □

Corollary 6.5.1 *The row indices of* $[sF^T - G^T, H^T]^T$ *are equal to the row degrees of* $T(s)$. □

Example 6.5.1 Consider the system

$$T(s) = \begin{bmatrix} s^2 & s^2+1 & 0 \\ 1 & s+2 & 3 \end{bmatrix}$$

The equivalent ARMA system derived in section 6.3 is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 2 & 3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$$

Then we bring the above to the form (6.48)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & -1 \\ \hline 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$$

We have from the above

$$Q\tilde{F} = \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}, \; T_{ech} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \; T_{ech}^2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

By defining $z(t) = w_3(t)$ we take the pencil realisation

$$F = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \; G = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 3 \end{bmatrix}, \; H = \begin{bmatrix} 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

6.6 Descriptor realisations without feedthrough term

The topic of this section is the derivation of a descriptor form realisation of the type

$$E\dot{\xi} = A\xi + Bu, \; y = C\xi \tag{6.62}$$

This realisation does not have feedthrough term in the output equation. A consequence of this, is that the nondynamic state variables are incorporated in the dynamical equation (the first of (6.62)).

The realisation procedure follows along similar lines to the realisation with feedthrough term. The original equations are transformed to the ARMA equations as described in section 6.3 and then to equations (6.30) and (6.33) after the introduction of the new variables $z(t)$. We write again (6.30) and (6.33) to make the inspection easy

$$\begin{bmatrix} I_\sigma \\ 0 \end{bmatrix} \dot{x} = \begin{bmatrix} T_{\ell 1} & T_{\ell 2} \\ T_1 & 0 \end{bmatrix} x + \begin{bmatrix} u \\ \begin{bmatrix} y_a \\ y_b \end{bmatrix} \end{bmatrix} \tag{6.63}$$

$$\begin{bmatrix} y_a \\ y_b \end{bmatrix} = \begin{bmatrix} -F_2 & -\overline{T}_{h2} \\ 0 & I \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} -T_2 \\ 0 \end{bmatrix} u \tag{6.64}$$

By defining the new internal variables $\zeta(t)$ as

$$\zeta(t) = T_2 u(t) \tag{6.65}$$

the output equation (6.64) becomes

$$\begin{bmatrix} y_a(t) \\ y_b(t) \end{bmatrix} = \begin{bmatrix} -F_2 & -I & -\overline{T}_{h2} \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} x(t) \\ \zeta(t) \\ z(t) \end{bmatrix} \tag{6.66}$$

and taking into account all the additional variables $(z(t), \zeta(t))$ (6.64) may be written as follows

$$\begin{bmatrix} I_\sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{\zeta} \\ \dot{z} \end{bmatrix} = \left\{ \begin{bmatrix} \hat{A} & 0 & 0 \\ F_1 & 0 & 0 \\ 0 & I & 0 \end{bmatrix} + \begin{bmatrix} T_{\ell 2} \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} -F_2 & -T_2 & -T_{h2} \\ 0 & 0 & I \end{bmatrix} \right\} \begin{bmatrix} x \\ \zeta \\ z \end{bmatrix} + \begin{bmatrix} T_{\ell 1} \\ T_1 \\ T_2 \end{bmatrix} u \tag{6.67}$$

Clearly, (6.66) and (6.67) are in the descriptor form (6.62) with

$$E = \begin{bmatrix} I_\sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ A = \begin{bmatrix} \hat{A} & 0 & 0 \\ F_1 & 0 & 0 \\ 0 & I & 0 \end{bmatrix} + \begin{bmatrix} T_{\ell 2} \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} -F_2 & -T_2 & -T_{h2} \\ 0 & 0 & I \end{bmatrix} \tag{6.68}$$

$$B = \begin{bmatrix} T_{\ell 1} \\ T_1 \\ T_2 \end{bmatrix}, \ C = \begin{bmatrix} -F_2 & -I & -\overline{T}_{h2} \\ 0 & 0 & I \end{bmatrix} \tag{6.69}$$

Remark 6.6.1 *As in the case of the realisation with feedthrough term the external equivalence of the above is justified from the fact that the operation leading to this realisation are only renaming of variables which does not affect the external behaviour of the system.* □

Remark 6.6.2 *As in the case of descriptor realisation with feedthrough term we may need reordering of the outputs in order to obtain* (6.68), (6.69). *Then the output matrix is* $\hat{P}^{-1}C$ *where* $\hat{P}$ *is defined in remark* 6.4.2. □

Next we consider the minimality of the realisation (6.68), (6.69). Note that $T(s)$ is assumed to be row reduced.

Proposition 6.6.1 *The descriptor realisation* (6.68), (6.69) *is minimal.*

Proof:

(i) The matrix

$$\begin{bmatrix} E \\ C \end{bmatrix} = \begin{bmatrix} I_\sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ -F_2 & -I & 0 \\ 0 & 0 & I \end{bmatrix}$$

has obviously full column rank

(ii) The matrix

$$[E, B] = \left[\begin{array}{ccc|c} I_\sigma & 0 & 0 & T_{\ell 1} \\ 0 & 0 & 0 & T_1 \\ 0 & 0 & 0 & T_2 \end{array}\right]$$

has full row rank because $[T_1^T, T_2^T]^T$ is the high order coefficient matrix of $T(s)$ multiplied by an invertible matrix and $T(s)$ is row reduced.

(iii) The pencil

$$\begin{bmatrix} sE - A \\ C \end{bmatrix} = \left[\begin{array}{ccc} sI_\sigma - \hat{A} + T_{\ell 2}^1 F_2 & T_{\ell 2}^1 T_2 & T_{\ell 2}^1 T_{h2} + T_{\ell 2}^2 \\ F_1 & 0 & 0 \\ 0 & I & 0 \\ \hline -F_2 & -I & -T_{h2} \\ 0 & 0 & I \end{array}\right]$$

is strictly equivalent to

$$\begin{bmatrix} sI_\sigma - \hat{A} & 0 & 0 \\ F_1 & 0 & 0 \\ F_2 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$$

The result follows from the fact that $[sI_\sigma - \hat{A}^T, F_1^T, F_2^T]^T$ has only r.m.i.. □

Example 6.6.1 Consider the autoregressive system with

$$T(s) = \left[\begin{array}{ccc|ccc} 0 & 1 & 0 & s+2 & 0 & s+1 \\ 3 & s & s+4 & 0 & s-1 & 0 \\ s^2 & 0 & 0 & s & 0 & 0 \end{array}\right]$$

with $w(t) = [u_1(t), u_2(t), u_3(t), y_1(t), y_2(t), y_3(t)]^T$. The equation (6.63) for this system is

$$\left[\begin{array}{cccc} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\\ \hline 0&0&0&0\\0&0&0&0\\0&0&0&0 \end{array}\right]\begin{bmatrix}\dot{x}_1\\\dot{x}_2\\\dot{x}_3\\\dot{x}_4\end{bmatrix} = \left[\begin{array}{cccc} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&1&0\\ \hline 1&0&0&0\\0&1&0&0\\0&0&0&1 \end{array}\right]\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} + \left[\begin{array}{ccc|ccc} 0&1&0&2&0&1\\3&0&4&0&-1&0\\0&0&0&0&0&0\\0&0&0&1&0&0\\ \hline 0&0&0&1&0&1\\0&1&1&0&1&0\\1&0&0&0&0&0 \end{array}\right]\begin{bmatrix}u_1\\u_2\\u_3\\y_1\\y_2\\y_3\end{bmatrix}$$

We have

$$\hat{A} = \begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&1&0\end{bmatrix}, \; T_2 = \begin{bmatrix}1&0&0\\0&1&1\end{bmatrix}, \; T_1 = [1\ 0\ 0]$$

$$F_1 = [0\ 0\ 0\ 1], \; F_2 = \begin{bmatrix}1&0&0&0\\0&1&0&0\end{bmatrix}, \; \overline{T}_{h2} = \begin{bmatrix}1&0&1\\0&1&0\end{bmatrix}$$

$$T_{\ell 1} = \begin{bmatrix}0&1&0\\3&0&4\\0&0&0\\0&0&0\end{bmatrix}, \; T_{\ell 2} = \begin{bmatrix}2&0&1\\0&-1&0\\0&0&0\\1&0&0\end{bmatrix}$$

Then the realisation without feedthrough term is

$$\begin{bmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{bmatrix}\begin{bmatrix}\dot{\xi}_1\\\dot{\xi}_2\\\dot{\xi}_3\\\dot{\xi}_4\\\dot{\xi}_5\\\dot{\xi}_6\end{bmatrix} = \begin{bmatrix}-2&0&0&0&0&-1\\0&1&0&0&1&0\\0&0&0&0&0&0\\-1&0&1&0&0&-1\\0&0&0&1&0&0\\0&0&0&0&1&1\end{bmatrix}\begin{bmatrix}\xi_1\\\xi_2\\\xi_3\\\xi_4\\\xi_5\\\xi_6\end{bmatrix} + \begin{bmatrix}0&1&0\\3&0&4\\0&0&0\\0&0&0\\1&0&0\\0&-1&-1\end{bmatrix}\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \\ \xi_5 \\ \xi_6 \end{bmatrix}$$

6.7 The observability indices of the descriptor realisations

In the framework of the regular systems i.e. the systems where the pencil $sE - A$ is regular (state–space, singular systems) the observability indices may be related to the transfer function of the system as follows: If $G(s) = D^{-1}(s)N(s)$ is a coprime and column reduced MFD, then the observability indices are defined as the column minimal indices of the composite matrix $T_G(s) = [D(s), N(s)]$. The above result is well known for the case of state–space systems from the works of [For., 1975], [Pop., 1969] e.t.c.. For the case of regular singular systems this result was proven in Chapter 4.

In this section we are going to investigate the relation between the row indices (row–degrees) of the matrix $T(s)$, when it is row reduced, and the row minimal indices of the pencil $[sE^T - A^T, C^T]^T$. We have the following result:

Proposition 6.7.1 *Let the quadruple (E, A, B, C) be the matrices of a minimal realisation of the autoregressive equation $T(\sigma)w(t) = 0$. The matrix $T(s)$ is assumed to be row reduced. Then the row minimal indices of the pencil $[sE^T - A^T, C^T]^T$ are equal to the row degrees of $T(s)$.*

Proof: From (6.38), (6.39) we have

$$\begin{bmatrix} sE - A \\ C \end{bmatrix} = \begin{bmatrix} sI - \hat{A} + T_{\ell 2}^1 F_2 & T_{\ell 2} \begin{bmatrix} \overline{T}_{h2} \\ I \end{bmatrix} \\ -F_1 & I \\ F_2 & -T_{h2} \\ 0 & I \end{bmatrix} \tag{6.70}$$

where $T_{\ell 2}$ is partitioned as $[T_{\ell 2}^1, T_{\ell 2}^2]$ conformably to the row partitioning of A. It is straightforward to see that the above pencil is strictly equivalent to the pencil

$$\left[\begin{array}{cc} sI-\hat{A} & 0 \\ F_1 & 0 \\ F_2 & 0 \\ \hline 0 & I \end{array}\right] \tag{6.71}$$

Then, the row minimal indices of the above are clearly provided by the row minimal indices of the pencil

$$\begin{bmatrix} sI-\hat{A} \\ F_1 \\ F_2 \end{bmatrix} \sim \begin{bmatrix} sI-\hat{A} \\ F \end{bmatrix} \tag{6.72}$$

where "$\sim$" denotes strict equivalence [Gant., 1959]. For the definition of F see (6.25). Reordering the rows of (6.72) we have that it is strictly equivalent to the pencil $\tilde{R}(s)$ in (6.10) which obviously has row minimal indices equal to the row degrees of $T(s)$ and the result follows. □

A result related to the structure of the pencil $[sE-A,-B]$ is given below.

Proposition 6.7.2 *Let (E,A,B,C,D) be a minimal realisation of the autoregressive equation $T(\sigma)w(t)=0$. Then, the pencil $(sE-A,-B)$ does not have r.m.i..*

Proof: The proof will be given by contradiction arguments. Let $z^T(s)$ be a nonzero polynomial row–vector such that

$$z^T(s)[sE-A,-B]=0 \tag{6.73}$$

Since (E,A,B,C,D) is a realisation of an autoregressive system it follows that

$$E=\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \tag{6.74}$$

and thus, $[sE-A,-B]$ has the form

$$[sE-A,-B]=\left[\, s\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix},\ -\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \right] \tag{6.75}$$

Minimality of the above system implies that

$$A\ \mathrm{Ker}\ \{E\}\ \subseteq\ \mathrm{Im}\{E\} \tag{6.76}$$

From (6.74) we have that a basis matrix $\overline{E}$ for Ker $\overline{E}$ is

$$\overline{E} = \begin{bmatrix} 0 \\ I \end{bmatrix} \tag{6.77}$$

and thus, the basis matrix $A_{\overline{E}}$ of A Ker E has the form

$$A_{\overline{E}} = \begin{bmatrix} A_2 \\ A_4 \end{bmatrix} \tag{6.78}$$

For (6.76) to hold true we must have $A_4 = 0$. Then (6.73) may be written as

$$z^T(s)[sE - A, -B] = \begin{bmatrix} z_1^T(s), z_2^T(s) \end{bmatrix} \begin{bmatrix} sI - A_1 & -A_2 & -B_1 \\ A_3 & 0 & -B_2 \end{bmatrix} \tag{6.79}$$

From corollary 6.4.1 we have that the number of rows of the matrix $[A_3, B_2]$ is equal to rank T_{h1} (the rank defect of E). Now from (6.26) it is clear that rank $T_{h1} \leq \ell$ since T_{h1} has ℓ columns. Then, since B_2 has full row rank the pencil $(sE - A, -B)$ is strictly equivalent to the following pencil

$$\begin{bmatrix} sI - A_1' & -A_2 & -B_1 \\ 0 & 0 & -B_2 \end{bmatrix} \tag{6.80}$$

The above is obtained from $[sE - A, -B]$ by postmultiplication with an appropriate invertible matrix. From the above it is clear that $\text{rank}_{\Re(s)}[sE - A, -B]$ is full row and thus $z(s) = 0$ and the result follows. □

Next, the relation between the row indices of $T(s)$ and the r.m.i. of $[sF^T - G^T, H^T]^T$ is established.

Proposition 6.7.3 *When $T(s)$ is row reduced and $(sF - G, H)$ is a minimal realisation pair, the row minimal indices of the pencil $[sF^T - G^T, H^T]^T$ are equal to the row indices of $T(s)$.*

Proof: The result follows immediately from corollary 6.5.1. □

6.8 Conclusions

In this Chapter the problem of the realisation of autoregressive equations was considered. A new method for realisation in pencil and descriptor form was proposed. This method is simpler than the existing methods and allows the realisation by simple inspection of the autoregressive equations. It has been shown that it leads to minimal representations when the matrix $T(s)$ of the autoregressive equations is row reduced.

It was shown that the relation of the observability indices and the MFDs for systems with transfer function may be extended to the case of autoregressive systems if, instead of the composite matrix $[N(s), D(s)]$ of a left MFD, the matrix $T(s)$ is considered. The overall approach is simple and the construction of the realisation may be obtained by inspection from the given autoregressive equations. An auxiliary realisation in ARMA form has been obtained and it will be used as a design tool in the following chapter where the model matching problem is considered.

Chapter 7

MODEL MATCHING OF IMPLICIT SYSTEMS

7.1 Introduction

The model matching problem for systems described by transfer functions has been studied extensively in the literature. This problem consists in designing a compensator such that when it is interconnected in series to the plant, the overall composite system has a desired transfer function. In the framework of behavioural systems, instead of transfer equivalence, we have the notion of external equivalence discussed in Chapter 3. In this framework, the model matching problem consists in finding a behavioural system such that when it is interconnected to a given (behavioural) system, the external behaviour of the overall system matches a desired behaviour.

One convenient way of representing behaviours is by using autoregressive equations. In the case of the matching problem this is the appropriate representation since it appears to have similarities with the MFD description of systems described by transfer functions.

In the present Chapter we consider two types of the model matching problem for implicit systems. First, the problem of model matching of autoregressive systems is studied. We start from a given autoregressive system the "plant" and a desired behaviour the "model" which is described by autoregressive equations as well. We seek for a system (in AR form) to be interconnected with the plant and give the behaviour of the model. The structure of the controller (the row degrees of the AR model) is left free and is considered as a design parameter.

The second type of model matching problem considered, is the problem of finding a controller such that the overall interconnected system is $\mathcal{A}$-externally equivalent to a given model (see definitions 3.3.8 and 3.3.9).

The outline of the Chapter is the following. First, we consider the interconnected system obtained from the autoregressive equations of the controller and the plant. Then we transform this equation to a set of ARMA equations in a way similar to Chapter 6. The requirement for matching of the behaviour of the interconnected system and the model yields the necessary conditions for the solvability of the problem. Although in the general case sufficient conditions are not provided, in a special case the matching problem is solved completely and a constructive algorithm for the determination of solutions is given.

Next, the problem of model matching under $\mathcal{A}$-external equivalence is considered. As in the case of external equivalence, necessary conditions for the solvability are provided. These conditions turn out to be also sufficient in a special case where certain conditions are satisfied. It is also shown that in the case where we can define transfer function (i.e. the output may be expressed explicitly in terms of the input) the conditions for model matching under $\mathcal{A}$-external equivalence coincide with the conditions for model matching under transfer equivalence.

It must be mentioned that the model matching problem for autoregressive systems has been considered in a recent paper of Conte and Perdon in [Con. & Per., 1994]. The results of that paper are similar to ours for the general case. However the proof of the results is not constructive and thus, it is not convenient for design purposes. In the case where sufficient conditions can be produced, we provide a slight correction to the results of that paper.

The matching problem for behavioural systems has a significant difference from the model matching of classical systems. In the classical theory the inputs and outputs are fixed while in the theory of autoregressive systems no distinction is made between inputs and outputs. As a consequence the choice of inputs and outputs is a problem for the control engineer and gives freedom of choice of the type of interconnections such that the desired behaviour is achieved.

7.2 Statement of the AR–matching problem and preliminary results

Consider the autoregressive systems

$$\Sigma_T : \; T(\sigma)w(t) = [T_1(\sigma), T_2(\sigma)] \begin{bmatrix} u(t) \\ y(t) \end{bmatrix} = 0 \tag{7.1}$$

$$\Sigma_M : \; M(\sigma)w_M(t) = [M_1(\sigma), M_2(\sigma)]\,[u_M(t)y_M(t)] = 0 \tag{7.2}$$

where $T(s) \in \Re^{r_T \times \ell_C}[s]$ $M(s) \in \Re^{r_M \times \ell_M}[s]$, σ is the differentiation operator and the behaviour vectors $w(t)$ and $w_M(t)$ are partitioned into inputs $u(t)$,$u_M(t)$ and outputs $y(t)$, $y_M(t)$. The problem of model matching is to find a system

$$\Sigma_C : \; C(\sigma)w_C(t) = \begin{bmatrix} C_1(\sigma), C_2(\sigma) \end{bmatrix} \begin{bmatrix} u_C(t) \\ y_C(t) \end{bmatrix} = 0 \tag{7.3}$$

such that the interconnection of the output of Σ_C and the input of Σ_T results to a composite system with external behaviour identical to the external behaviour of Σ_M. Note that, without loss of generality, the matrices $M(s)$ and $T(s)$ are considered to have full row rank. When they do not have full row rank, we may consider only independent rows of them and take equivalent systems of equations. This problem may be viewed as a generalisation of the model matching problem for systems with transfer function descriptions.

In order to derive the solution of the model matching problem for autoregressive systems we proceed as follows. First, we transform the AR equations of Σ_T and Σ_M

to appropriate ARMA representations [Wil., 1991]. Then we obtain an ARMA representation of the composite system. The requirement for external equivalence of the composite ARMA system and the AR model imposes conditions on the matrices $M(s)$ and $T(s)$. These conditions are proved to be necessary for the solvability of the model matching problem. It is shown that in some cases the necessary conditions are also sufficient. Then, the system Σ_C is obtained in a straightforward way.

7.3 Model matching of AR systems

In this section the composite system arising from the series interconnection of two autoregressive systems is derived. Then, an ARMA realisation of the composite system is obtained. The requirement of external equivalence of the composite system and the model leads to necessary as well as sufficient conditions for the solvability of the model matching problem.

Let the plant and the controller described by

$$\Sigma_T = T(\sigma)w(t) = \begin{bmatrix} T_1(\sigma), T_2(\sigma) \end{bmatrix} \begin{bmatrix} u(t) \\ y(t) \end{bmatrix} = 0 \tag{7.4}$$

$$\Sigma_C = C(\sigma)w_C(t) = \begin{bmatrix} C_1(\sigma), C_2(\sigma) \end{bmatrix} \begin{bmatrix} u_C(t) \\ y_C(t) \end{bmatrix} = 0 \tag{7.5}$$

where the behaviour vectors are partitioned into inputs and outputs. We connect the above systems as follows: The output $y_C(t)$ of Σ_C is connected to the input $u(t)$ of Σ_T. Obviously this means that the partitioning of $w(t)$ and $w_C(t)$ is such that the number of inputs of Σ_T is equal to the number of outputs of Σ_C. This interconnection is expressed as the constraint

$$u(t) = y_C(t) \tag{7.6}$$

on the equations (7.4), (7.5).

From Chapter 6 (see section 6.3) we have that the systems Σ_T and Σ_C may be transformed to the ARMA systems $\Sigma(P,Q)$, $\Sigma(P_C,Q_C)$ respectively.

$$\Sigma(P,Q) : \begin{bmatrix} \tilde{R}(\sigma), T_{C1}, T_{C2} \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \\ y(t) \end{bmatrix} = 0 \tag{7.7}$$

$$\begin{bmatrix} u(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \\ y(t) \end{bmatrix} \tag{7.8}$$

$$\Sigma(P_C, Q_C): \left[\begin{array}{c} \tilde{R}(\sigma), C_{C1}, C_{C2} \end{array}\right] \left[\begin{array}{c} x_C(t) \\ u_C(t) \\ y_C(t) \end{array}\right] = 0 \tag{7.9}$$

$$\left[\begin{array}{c} u_C(t) \\ y_C(t) \end{array}\right] = \left[\begin{array}{ccc} 0 & I & 0 \\ 0 & 0 & I \end{array}\right] \left[\begin{array}{c} x_C(t) \\ u_C(t) \\ y_C(t) \end{array}\right] \tag{7.10}$$

where

$$\tilde{R}(s) = \text{block} - \text{diag}\{\ldots, L_{\sigma_i}(s), \ldots\} \tag{7.11}$$

$$\tilde{R}_C(s) = \text{block} - \text{diag}\{\ldots, L_{\sigma_i^C}(s), \ldots\} \tag{7.12}$$

$$L_i(s) = \left[\begin{array}{ccc} s & 0 & 0 \\ -1 & \cdots & \vdots \\ \vdots & -1 & s \\ 0 & 0 & -1 \end{array}\right]_{(i+1)\times i} \tag{7.13}$$

$$T(s) = S_C(s)[T_{C1}, T_{C2}], \; S(s) = \text{block} - \text{diag}\{\ldots, [1 \; s \ldots s^{\sigma_i}], \ldots\} \tag{7.14}$$

$$C(s) = S_C(s)[C_{C1}, C_{C2}], \; S_C(s) = \text{block} - \text{diag}\{\ldots, [1 \; s, \ldots s^{\sigma_i^C}], \ldots\} \tag{7.15}$$

whith σ_i and σ_i^C the row degrees of $T(s)$ and $C(s)$ respectively.

Then, taking into account the interconnection equation $u(t) = y_C(t)$ we obtain the following composite system

$$\left[\begin{array}{ccccc} \tilde{R}(\sigma) & 0 & T_{C1} & 0 & T_{C2} \\ 0 & \tilde{R}_C(\sigma) & C_{C2} & C_{C1} & 0 \end{array}\right] \left[\begin{array}{c} x(t) \\ x_C(t) \\ y_C(t) \\ \hline u_C(t) \\ y(t) \end{array}\right] = 0 \tag{7.16}$$

$$\left[\begin{array}{c} u_C(t) \\ y(t) \end{array}\right] = \left[\begin{array}{ccccc} 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \end{array}\right] \left[\begin{array}{c} x(t) \\ x_C(t) \\ y_C(t) \\ \hline u_C(t) \\ y(t) \end{array}\right] \tag{7.17}$$

Observe that the interconnected system has external variables the input of Σ_C and the output of Σ_T and as internal (latent) variables the direct sum of the internal variables of the two subsystems and the variables of interconnection.

If the system Σ_C is considered as the controller and Σ_T as the plant, then the model matching problem consists in finding $\tilde{R}(\sigma)$, C_{C1}, C_{C2} such that the system described by (7.16) and (7.17) has the same behaviour to a given system

$$M(\sigma)w_M(t) = 0 \tag{7.18}$$

Note that the row degrees of $C(s)$ are not prespecified.

Let

$$\tilde{P}(s) = \begin{bmatrix} \tilde{R}(s) & 0 & T_{C1} & 0 & T_{C2} \\ 0 & \tilde{R}_C(s) & C_{C2} & C_{C1} & 0 \end{bmatrix} \tag{7.19}$$

$$\tilde{Q}(s) = \begin{bmatrix} 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \tag{7.20}$$

$$S(s) = \text{block} - \text{diag}\{\ldots, [1\ s \ldots s^{\sigma_i}], \ldots\} \tag{7.21}$$

$$S_C(s) = \text{block} - \text{diag}\{\ldots, [1\ s \ldots s^{\sigma_i^C}], \ldots\} \tag{7.22}$$

The following proposition provides the conditions for the solvability of the model matching problem.

Proposition 7.3.1 *If the model matching problem is solvable, then the following conditions hold.*

(i) row–span $\{V_1(s)\} \subseteq \mathcal{N}_\ell\{\tilde{R}(s)\}$ =row-span$\{S(s)\}$

(ii) row–span $\{V_2(s)\} \subseteq \mathcal{N}_\ell\{\tilde{R}_C(s)\}$ =row-span$\{S_C(s)\}$

(iii) $V_1(s)T_{C1} + V_2(s)C_{C2} = 0$

(iv) $V_2(s)C_{C1} + M_1(s) = 0$

(v) $V_1(s)T_{C2} + M_2(s) = 0$

where $V_1(s)$, $V_2(s)$ are polynomial matrices.

Proof: Let the model matching problem be solvable and let $C(s)$ be the matrix of the autoregressive representation of the solution. According to lemma 6.3.1 if $[\hat{V}(s), \hat{C}(s)]$ is a polynomial basis matrix of the left null space of $[\tilde{P}^T(s), \tilde{Q}^T(s)]^T$ where $\hat{V}(s)$, $\hat{C}(s)$ are left coprime, then the system $\Sigma(\tilde{P}(\sigma), \tilde{Q}(\sigma))$ is externally equivalent to the autoregressive system

$$\hat{M}(\sigma)w(t) = 0 \tag{7.23}$$

Now, since $M(\sigma)$ describes a system equivalent to $\Sigma(\tilde{P}(\sigma), \tilde{Q}(\sigma))$, it follows that $M(s)$ and $\hat{M}(s)$ are unimodularly equivalent i.e. [Wil., 1991], [Wil., 1983] there exists unimodular matrix $\hat{U}(s)$ such that

$$M(s) = \hat{U}(s)\hat{M}(s) \tag{7.24}$$

Then, there exists a polynomial matrix

$$[V(s), M(s)] = \hat{U}(s)[\hat{V}(s), \hat{M}(s)] \tag{7.25}$$

such that

$$\left[\, V(s), M(s) \,\right] \left[\begin{array}{c} \tilde{P}(s) \\ \tilde{Q}(s) \end{array} \right] = 0 \tag{7.26}$$

Obviously, $[V(s), M(s)]$ is a basis matrix of the left null space of $[\tilde{P}^T(s), \tilde{Q}^T(s)]^T$ and the matrices $V(s)$, $C(s)$ are left coprime. Consider now the following partitioning of the matrix $[V(s), M(s)]$:

$$[V(s), M(s)] = [V_1(s), V_2(s), M_1(s), M_2(s)] \tag{7.27}$$

conformably to the row partitioning of the matrix $[\tilde{P}^T(s), \tilde{Q}^T(s)]^T$. Then (7.26) may be written as follows

$$\left[\, V_1(s), V_2(s), M_1(s), M_2(s) \,\right] \left[\begin{array}{ccccc} \tilde{R}(s) & 0 & T_{C1} & 0 & T_{C2} \\ 0 & \tilde{R}_C(s) & C_{C2} & C_{C1} & 0 \\ \hline 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \end{array} \right] = 0 \tag{7.28}$$

And from the above

$$V_1(s)\tilde{R}(s) = 0 \tag{7.29}$$

$$V_2(s)\tilde{R}_C(s) = 0 \tag{7.30}$$

$$V_1(s)T_{C1} + V_2(s)C_{C2} = 0 \tag{7.31}$$

$$V_2(s)C_{C1} + M_1(s) = 0 \tag{7.32}$$

$$V_1(s)T_{C2} + M_2(s) = 0 \tag{7.33}$$

The result follows from the observation that $S(s)$ and $S_C(s)$ are basis matrices of the left null space of $\tilde{R}(s)$ and $\tilde{R}_C(s)$ respectively. □

Proposition 7.3.2 *Necessary condition for the solvability of the model matching problem for autoregressive systems is*

$$\text{Ker }\{M_2(s)\} \supseteq \text{Ker }\{T_2(s)\} \tag{7.34}$$

Proof: Conditions (7.29)-(7.32) may always be satisfied by appropriate selection of C_{C1}, C_{C2} and $V(s)$. Indeed, if we choose

$$V_1 = S(s) \tag{7.35}$$

$$V_2(s) = S_C(s) = \text{block}-\text{diag}\{\ldots,[1\ s\ldots s^{\tau_i}],\ldots\} \tag{7.36}$$

where

$$\tau_i \geq \max\{\text{row}-\deg\{M_1(s), V_1(s)T_{C1}\}\} \tag{7.37}$$

there always exist C_{C1}, C_{C2} such that (7.29)-(7.32) are satisfied. In fact, this choice corresponds to a controller $C(s)$ with

$$C_1(s) = -M_1(s) \tag{7.38}$$

$$C_2(s) = -T_1(s) \tag{7.39}$$

Thus, only (7.33) constitutes a necessary condition for solvability. Condition (7.34) readily follows from (7.33). □

We now state a result which is an improvement of the above proposition.

Theorem 7.3.1 *Necessary condition for the solvability of the model matching problem is that the equation*

$$K(s)T_2(s) = -M_2(s) \tag{7.40}$$

has a polynomial solution $K(s)$.

Proof: From (7.29) it follows that there exists rational matrix $K(s)$ such that

$$V_1(s) = K(s)S(s) \tag{7.41}$$

Let

$$K(s) = [\underline{k}_1(s),\ldots,\underline{k}_r(s)] \tag{7.42}$$

where r is the number of rows of $S(s)$. Then we have

$$K(s)S(s) = \left[\ \underline{k}_1(s),\ldots,\underline{k}_r(s)\ \right]\begin{bmatrix} 1 & s & \cdots & s^{\sigma_1} & & & & \\ & & & & \ddots & & & \\ & & & & & 1 & s & \cdots & s^{\sigma_p} \end{bmatrix} =$$

$$= [\underline{k}_1(s)\ \underline{k}_1(s)s \ldots \underline{k}_1(s)s^{\sigma_1}| \ldots |\underline{k}_\rho(s)\ \underline{k}_\rho(s)s \ldots \underline{k}_\rho(s)s^{\sigma_\rho}] \tag{7.43}$$

From the above it is clear that if $K(s)$ is not polynomial, then $V_1(s)$ is not polynomial since the submatrix formed by its columns with indices $1, \sigma_1+2, \sigma_1+\sigma_2+3\ldots$ is $K(s)$. Thus, in order to have $V_1(s)$ polynomial, $K(s)$ must be polynomial.

Consider now the matrix $S(s)T_{C2}$. Clearly,

$$S(s)T_{C2} = T_2(s) \tag{7.44}$$

and the result follows. □

The following result provides criteria for an equation involving polynomial matrices to have a polynomial solution.

Lemma 7.3.1 [Vid., 1985] *Consider the equation*

$$A(s)X(s) = B(s) \tag{7.45}$$

where $A(s)$ and $B(s)$ are polynomial matrices. Then there exists $X(s)$ polynomial satisfying (7.45) *if and only if*

$$[A(s), B(s)] = [A(s), 0]U(s) \tag{7.46}$$

where $U(s)$ is a unimodular matrix.

Proof: Let

$$[A(s), B(s)] = [A(s), 0]U(s) = \left[\ A(s), B(s)\ \right] \begin{bmatrix} U_{11}(s) & U_{12}(s) \\ U_{21}(s) & U_{22}(s) \end{bmatrix} \tag{7.47}$$

then

$$B(s) = AU_{12}(s)$$

Conversely, if $B(s) = A(s)X(s)$ with $X(s)$ polynomial

$$[A(s), B(s)] = [A(s), AX(s)] = A(s)[I, X(s)] \sim \left[\ I, X(s)\ \right] \begin{bmatrix} I & -X(s) \\ 0 & I \end{bmatrix} = [A(s), 0]$$

where "$\sim$" denotes unimodular equivalence. □

From the above lemma and theorem 7.3.1 we have the following.

Proposition 7.3.3 *A necessary condition for the solvability of the model matching problem is*

$$\begin{bmatrix} T_2(s) \\ 0 \end{bmatrix} = U(s) \begin{bmatrix} T_2(s) \\ -M_2(s) \end{bmatrix} \tag{7.48}$$

where $U(s)$ is a unimodular matrix.

Proof: The result readily follows from the requirement for (7.33) to have a polynomial solution and from lemma 7.3.1 (in transposed form). □

Note that when $K(s)T_2(s) = -M_2(s)$ has a polynomial solution we may always find C_{C1}, C_{C2} such that (7.26) holds, by choosing $V_2(s) = S_C(s)$ as it is defined in (7.36).Thus, if the matching problem is solvable, the general form of a family of controllers is given by

$$C_2(s) = -K(s)T_1(s) \tag{7.49}$$

$$C_1(s) = -M_1(s) \tag{7.50}$$

Although the existence of polynomial solution of $K(s)T_2(s) = -M_2(s)$ is necessary for the existence of $C(s)$ such that (7.26) holds, it is not, in general, sufficient for the solution of the model matching problem. The reason for this is that the matrix $[V(s), M(s)]$ which annihilates $[\tilde{P}^T(s), \tilde{Q}^T(s)]^T$ is not necessarily a basis of the left null space of the latter matrix. The following results lead to a necessary and sufficient condition for the solvability of a special case of the model matching problem.

Proposition 7.3.4 *If $T_1(s)$ has full row rank then the matrix $[\tilde{R}(s), T_{C1}]$ has full row rank.*

Proof: Let $y^T(s)$ be a polynomial vector of the left null space of $[\tilde{R}(s), T_{C1}]$. Then, $y^T(s)$ must have the form

$$y^T(s) = \lambda^T(s)S(s) \tag{7.51}$$

where $\lambda^T(s)$ is polynomial nonzero vector. Then

$$y^T(s)[\tilde{R}(s), T_{C1}] = 0 \Rightarrow \lambda^T(s)S(s)[\tilde{R}(s), T_{C1}] = 0 \Rightarrow \lambda^T(s)[0, T_1(s)] = 0$$

or

$$\lambda^T(s)T_1(s) = 0 \tag{7.52}$$

which contradicts the assumption that $T_1(s)$ has full row rank and the result follows. □

Theorem 7.3.2 *Let the matrix $T_1(s)$ have full row rank. Then, necessary and sufficient condition for the solvability of the model matching problem is that the equation $K(s)T_2(s) = -M_2(s)$ has a polynomial solution.*

Proof: We consider only the sufficiency since necessity was proved in proposition 7.3.3.. We shall prove that when $T_1(s)$ has full rank and $K(s)T_2(s) = -M_2(s)$ has a polynomial solution $K(s)$, then we can choose $C(s)$ such that the row rank defect (over $\Re(s)$) of the matrix $[\tilde{P}^T(s), \tilde{Q}^T(s)]^T$ is equal to the number of rows of $[V(s), C(s)]$. Consider the matrix

$$\begin{bmatrix} \tilde{P}(s) \\ \tilde{Q}(s) \end{bmatrix} = \left[\begin{array}{ccccc} \tilde{R}(s) & 0 & T_{C1} & 0 & T_{C2} \\ 0 & \tilde{R}_C(s) & C_{C2} & C_{C1} & 0 \\ \hline 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \end{array}\right] \tag{7.53}$$

From proposition 7.3.4 it follows that the top row–block of the above matrix has full row rank. Thus, the rank defect of $[\tilde{P}^T(s), \tilde{Q}^T(s)]^T$ is less or equal to the number of rows of $C(s)$ i.e. it is less or equal to r_C. Now consider the matrix

$$[V(s), M(s)] = [K(s)S(s), S_C(s), M_1(s), M_2(s)] \tag{7.54}$$

and let r_M be the number of its rows. Note that $r_C = r_M$. Clearly, the above matrix is polynomial, has full row rank and does not have finite zeros i.e. $V(s)$, $M(s)$ are left coprime.

Now

$$\begin{bmatrix} V(s), M(s) \end{bmatrix} \begin{bmatrix} \tilde{P}(s) \\ \tilde{Q}(s) \end{bmatrix} = [0, 0, K(s)T_1(s) + C_2(s), 0, 0] \tag{7.55}$$

since $K(s)T_2(s) = -M_2(s)$. Now, by choosing $C_2(s) = -K(s)T_1(s)$ it follows that $[V(s), M(s)]$ annihilates $[\tilde{P}^T(s), \tilde{Q}^T(s)]^T$ and it is a basis matrix of the left null space of the latter matrix. □

The above provides a complete solution of the model matching problem in the case where $T_1(s)$ has full row rank.

Remark 7.3.1 *In* [Con. & Per., 1994] *the necessary and sufficient condition when $T_1(s)$ has full row rank is that $K(s)T_2(s) = -M_2(s)$ must have a rational solution. This is incorrect as we see from the above theorems. As it is shown in a following section this condition is necessary and sufficient if we consider $\mathcal{A}$-external equivalence instead of external equivalence.* □

7.4 Parametrisation of the solutions

In this section we study the parametrisation of the solutions of the model matching problem i.e. we derive the whole family of the controllers $C(s)$ solving the problem. This parametrisation is based on the parametrisation of the polynomial solutions of the equation

$$A(s)X(s) = B(s) \tag{7.56}$$

with $A(s)$ and $B(s)$ polynomial matrices. The following result gives the parametric form of the solutions of (7.56).

Proposition 7.4.1 *If the equation $A(s)X(s) = B(s)$ has polynomial solutions, then the general form of the solution is*

$$X(s) = X_0(s) + N_A(s)Y(s) \tag{7.57}$$

where $X_0(s)$ is a particular solution, $N_A(s)$ is a minimal basis matrix of the right null space of $A(s)$ and $Y(s)$ is any arbitrary polynomial matrix.

Proof: Let $X_0(s)$, $X(s)$ be two solutions of $A(s)X(s) = B(s)$. Then $B(s) = A(s)X(s) = A(s)X_0(s)$ and thus,

$$A(s)(X(s) - X_0(s)) = 0$$

The above means that

$$X(s) - X_0(s) = N_A(s)Y(s) \text{ or } X(s) = X_0(s) + N_A(s)Y(s)$$

with $Y(s)$ arbitrary polynomial matrix and the result follows. □

Corollary 7.4.1 *The general solution of $X(s)A(s) = B(s)$ ($X(s)$ polynomial) is*

$$X(s) = X_0(s) + Y(s)N_A(s)$$

where $N_A(s)$ is a minimal basis matrix of the left null space of $A(s)$. □

Applying the above parametrisation to the equation

$$K(s)T_2(s) = -M_2(s)$$

we have that the general solution $K(s)$ is given by

$$K(s) = K_0(s) + Y(s)N_{T2}(s) \tag{7.58}$$

where $N_{T2}(s)$ is a minimal basis matrix of $\text{Ker}\{T_2(s)\}$. Then we have the following parametrisation of the controllers.

Theorem 7.4.1 *If the model matching problem is solvable, then the solutions $C(s)$ are given in the following parametric form:*

$$C(s) = [C_1(s), C_2(s)] \tag{7.59}$$

where

$$C_1(s) = -M_1(s),\ C_2(s) = -(K_0(s) + Y(s)N_{T_2}(s)) \tag{7.60}$$

where $Y(s)$ is arbitrary polynomial matrix and $N_{T_2}(s)$ is a minimal basis matrix of the right null space of $T_2(s)$ and $K_0(s)$ is a particular solution of $K(s)T_2(s) = -M_2(s)$. □

Example 7.4.1 Let

$$T(s) = [s+1, s+3], \ M(s) = [s+5, s(s+3)]$$

be the matrices of the given "plant" and "model" autoregressive descriptions. We have that

$$T_1(s) = s+1, \ T_2(s) = s+3, \ M_1(s) = s+5, \ M_2(s) = s^2+3s$$

The matrix $T(s)$ has full row rank, and the equation $K(s)T_2(s) = -M_2(s)$ has the polynomial solution $K(s) = -s$. Thus, the matching problem is solvable. Since $K(s)T_1(s) = s^2 + s = V_1(s)T_{C1}$, from (7.36) and (7.37) we take

$$S_C(s) = [1, s, s^2]$$

From (7.49), (7.50) it follows that the controller has the following matrix

$$C(s) = [C_1(s), C_2(s)] = [-(s+5), s^2+s]$$

and

$$\begin{bmatrix} \tilde{P}(s) \\ \tilde{Q}(s) \end{bmatrix} = \left[\begin{array}{c|cc|c|c|c} s & & & 1 & 0 & 3 \\ -1 & & & 1 & 0 & 1 \\ \hline & s & 0 & 0 & -5 & 0 \\ & -1 & s & 1 & -1 & 0 \\ & 0 & -1 & 1 & 0 & 0 \\ \hline & & & & 1 & 0 \\ & & & & 0 & 1 \end{array}\right]$$

In order to verify that $\Sigma(\tilde{P}(\sigma), \tilde{Q}(\sigma))$ has the same external behaviour to Σ_M we consider the following transformation on $[\tilde{P}^T(\sigma), \tilde{Q}^T(\sigma)]^T$

$$U(s)\begin{bmatrix} \tilde{P}(s) \\ \tilde{Q}(s) \end{bmatrix} = \begin{bmatrix} U_{11}(s) & U_{12}(s) \\ U_{21}(s) & U_{22}(s) \end{bmatrix}\begin{bmatrix} \tilde{P}(s) \\ \tilde{Q}(s) \end{bmatrix} =$$

$$= \left[\begin{array}{ccccc|cc} 0 & -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -s & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 1 & s & 0 & 0 & 0 & 0 & -s-3 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline -s & -s^2 & 1 & s & s^2 & s+5 & s^2+3s \end{array}\right] \begin{bmatrix} \tilde{P}(s) \\ \tilde{Q}(s) \end{bmatrix} = \left[\begin{array}{cccccc} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & -s-1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & s+1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] = \left[\begin{array}{c} A(s) \\ \hline 0 \end{array}\right]$$

The matrix $A(s)$ has full row rank and $U(s)$ is unimodular. Then, according to lemma 6.3.1 the behaviour of $\Sigma(\tilde{P}(s), \tilde{Q}(s))$ is identical to the behaviour of the system $U_{22}(\sigma)w(t) = 0$ where $U_{22}(s) = [s+5, s(s+3)] = M(s)$.

In the case of transfer function equivalence the necessary and sufficient condition is that the equation $K(s)T_2(s) = -M_2(s)$ has a rational solution. The following example illustrates the differences between the model matching of behavioural systems and transfer function systems.

Example 7.4.2 Consider the system described by the differential equation

$$T: \; (\sigma+1)u(t) + (\sigma+3)y(t) = 0$$

This system may be described by the autoregressive equation

$$T(\sigma)w(t) = \left[\; \sigma+1, \sigma+3 \; \right] \begin{bmatrix} u(t) \\ y(t) \end{bmatrix} = 0$$

In the transfer function framework the system T has transfer function

$$G_T(s) = -\frac{s+1}{s+3}$$

If we consider the model matching problem for transfer function systems where the model is

$$M: \; (\sigma+5)u(t) + (\sigma+2)y(t)$$

which corresponds to the transfer function

$$G_M(s) = -\frac{s+5}{s+2}$$

then, the obvious solution to this problem is a controller with transfer function

$$G_C(s) = \frac{C_1(s)}{C_2(s)} = \frac{(s+3)(s+5)}{(s+1)(s+2)}$$

and the overall system has transfer function

$$G(s) = G_C(s)G_M(s) = -\left[\frac{s+3}{s+1} \cdot \frac{s+5}{s+2}\right]\frac{s+1}{s+3} = -\frac{s+5}{s+2}$$

The controller was designed such that its poles cancel the zeros of the plant and its zeros cancel the poles of the plant. Note that the solution of the equation

$$K(s)T_2(s) = -M_2(s)$$

is

$$K(s) = \frac{s+3}{s+1} \cdot (s+5)$$

which is not polynomial.

Consider now the situation where the above controller $C(s)$ is used for model matching of external behaviours. With model $M(s) = [M_1(s), M_2(s)] = [s+5, s+2]$ the composite system of the plant $[T_1(s), T_2(s)] = [s+1, s+3]$ and the controller $[C_1(s), C_2(s)] = [(s+3)(s+5), (s+1)(s+2)]$ yields an overall system with external behaviour

$$[(\sigma+5)(\sigma+3), (\sigma+2)(\sigma+3)] \begin{bmatrix} u(t) \\ y(t) \end{bmatrix} =$$

$$(\sigma+3)\,[\sigma+5, \sigma+3] \begin{bmatrix} u(t) \\ y(t) \end{bmatrix} = (s+3)T(\sigma) \begin{bmatrix} u(t) \\ y(t) \end{bmatrix}$$

The above behavioural system is not externally equivalent to the model since they are not unimodularly equivalent. As we see the term $(s+3)$ which is canceled in the case of transfer equivalence, is present in the external equivalence framework.

7.5 Model matching under $\mathcal{A}$–external equivalence

In this section we consider a modified version of the model matching problem. This problem consists in finding a controller $C(s)$ such that the overall interconnected system is $\mathcal{A}$-externally equivalent to a given model. This is a different problem than the problem studied in the previous sections of this Chapter since $\mathcal{A}$–external equivalence allows the extraction of common Smith zeros of the matrices $N(s)$, $D(s)$ where $T(s) = [N(s), D(s)]$. This definition of external equivalence coincides with the definition of transfer equivalence when transfer function may be defined. We proceed with the following preliminaries.

Consider the system described by the equations

$$P(\sigma)\xi(t) = 0, \; Q(\sigma)\xi(t) = w(t) \tag{7.61}$$

where $\xi(t)$ is the vector of internal variables and $w(t)$ the vector of external variables. Then we have

$$\begin{bmatrix} P(\sigma) \\ Q(\sigma) \end{bmatrix} \xi(t) = \begin{bmatrix} 0 \\ w(t) \end{bmatrix} \tag{7.62}$$

Definition 7.5.1 *Consider the systems $\Sigma(P_1(\sigma), Q_1(\sigma))$, $\Sigma(P_2(\sigma), Q_2(\sigma))$. Then they are $\mathcal{A}$-externally equivalent iff*

$$Q_1(s)\mathrm{Ker}\{P_1(s)\} = Q(s)\mathrm{Ker}\{P_2(s)\} \tag{7.63}$$

where $Q_i(s)$, $P_i(s)$ are considered as matrices over the field of rational functions and thus $Ker\{\cdot\}$ is a rational vector space. □

Remark 7.5.1 *This definition is essentially equivalent to the definition given in Chapter 3 for the $\mathcal{A}$-external equivalence.* □

Remark 7.5.2 *It is important to emphasise the difference between the definition of external equivalence in the sense of* [Wil., 1991], [Sch., 1988] *and definition* 7.5.1. *According to Willems two systems are considered as externally equivalent if*

$$Q_1\text{Ker}\{P_1\} = Q_2\text{Ker}\{P_2\} \tag{7.64}$$

where P_i, Q_i are considered as differential operators and $Ker\{P_i\}$ stands for the set of functions $f(t)$ satisfying $P_i(\sigma)f(t) = 0$. □

In order to proceed with the model matching under $\mathcal{A}$-external equivalence ($\mathcal{A}$-matching problem) we shall consider the interconnected system description derived in (7.16), (7.17).

This system was derived under the requirement of external equivalence and it can be used in the same case of $\mathcal{A}$-external equivalence, since external equivalence yields $\mathcal{A}$-external equivalence. We continue with the following result.

Lemma 7.5.1 *Let $A(s) = [0, I_r]$ and $B^T(s) = [E^T(s), D^T(s)]^T$ where $B(s)$ is polynomial and $D(s)$ has r rows. Then*

$$(A^T)^{-1}\ \text{Im}\ \{B^T(s)\} = D(s)\ \text{Ker}\ \{E(s)\} \tag{7.65}$$

Proof: Consider the space $A(s)$ Ker $\{B(s)\}$. Then, if $\overline{B}(s)$ is a basis of Ker$\{B(s)\}$,

$$(A(s)\ \text{Ker}\ \{B(s)\})^{\perp} = \{x(s)\ /\ x^T(s)A(s)\overline{B}(s) = 0\} = \{x(s)/\ A^T(s)x(s) \in \text{Im}\{B^T(s)\}\}$$

Since $A(s) = [0, I]$ it follows that if $x(s) \in (A^T)^{-1}\text{Im}\{B^T(s)\}$, then

$$\begin{bmatrix} 0 \\ I_r \end{bmatrix} x(s) = \begin{bmatrix} E(s) \\ D(s) \end{bmatrix} k(s) \tag{7.66}$$

Let $\overline{E}(s)$ be a basis matrix for Ker $\{E(s)\}$. Clearly, for (7.66) to hold, we must have

$$k(s) = \overline{E}(s)\pi(s) \tag{7.67}$$

where $\pi(s)$ is any rational vector. Equations (7.66), (7.67) yield

$$\begin{bmatrix} 0 \\ I_r \end{bmatrix} x(s) = \begin{bmatrix} E(s) \\ D(s) \end{bmatrix} k(s) = \begin{bmatrix} 0 \\ D(s) \end{bmatrix} k(s) \Leftrightarrow x(s) = D(s)k(s)$$

or equivalently

$$x(s) \in D(s)\mathrm{Ker}\{E(s)\} \tag{7.68}$$

and the result follows. □

Since the interconnected system has

$$\tilde{P}(s) = \begin{bmatrix} \tilde{R}(s) & 0 & T_{C1} & 0 & T_{C2} \\ 0 & \tilde{R}_C(s) & C_{C2} & C_{C1} & 0 \end{bmatrix} \tag{7.69}$$

$$\tilde{Q}(s) = \begin{bmatrix} 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \tag{7.70}$$

the above lemma yields:

$$(\tilde{Q}(s) \text{ Ker } \{\tilde{P}(s)\})^{\perp} = \begin{bmatrix} 0 & C_{C1}^T \\ T_{C2}^T & 0 \end{bmatrix} \text{ Ker } \{ \begin{bmatrix} R^T(s) & 0 \\ 0 & R_C^T(s) \\ T_{C1} & C_{C2}^T \end{bmatrix} \} \tag{7.71}$$

The $\mathcal{A}$–external behaviour of this model $M(s)$ is the rational vector space

$$\mathrm{Ker}\{[M_1(s), M_2(s)]\} \tag{7.72}$$

Then (7.71) and (7.63) yield.

$$(Q_M(s) \text{ Ker } \{P_M(s)\})^{\perp} = \text{Im } \{ \begin{bmatrix} M_1^T(s) \\ M_2^T(s) \end{bmatrix} \} \tag{7.73}$$

The above expresses the equality of the $\mathcal{A}$–external behaviours of the composite system plant–controller and the model. Let

$$R^* = \begin{bmatrix} R^T(s) & 0 \\ 0 & R_C^T(s) \\ T_{C1} & C_{C2}^T \end{bmatrix} \tag{7.74}$$

Then the basis matrix of Ker $\{R^*(s)\}$ has the following form

$$\Pi(s) = \begin{bmatrix} S^T(s)\Pi_1(s) \\ S_C^T(s)\Pi_2(s) \end{bmatrix} \tag{7.75}$$

The matrix version of (7.73) is (recall that $M(s)$ has full row rank)

$$\begin{bmatrix} 0 & C_{C1}^T \\ T_{C2}^T & 0 \end{bmatrix} \begin{bmatrix} S^T(s)\Pi_1(s) \\ S_C^T(s)\Pi_2(s) \end{bmatrix} \Lambda(s) = \begin{bmatrix} M_1^T(s) \\ M_2^T(s) \end{bmatrix} \tag{7.76}$$

where $\Lambda(s)$ is square and invertible rational matrix. The above and the fact that $R^*\Pi(s) = 0$ yield

$$\begin{bmatrix} T_1^T(s) & C_2^T(s) \\ 0 & C_1^T(s) \\ T_2^T(s) & 0 \end{bmatrix} \begin{bmatrix} \Lambda_1(s) \\ \Lambda_2(s) \end{bmatrix} = \begin{bmatrix} 0 \\ M_1^T(s) \\ M_2^T(s) \end{bmatrix} \tag{7.77}$$

or

$$T_1^T(s)\Lambda_1(s) + C_2^T(s)\Lambda_2(s) = 0 \tag{7.78}$$

$$C_1^T(s)\Lambda_2(s) = M_1^T(s) \tag{7.79}$$

$$T_2^T(s)\Lambda_1(s) = M_2^T(s) \tag{7.80}$$

where $\Lambda_1(s) = \Pi_1(s)\Lambda(s)$ and $\Lambda_2(s) = \Pi_2(s)\Lambda(s)$. We may now proceed to the main result:

Theorem 7.5.1 *Let $T_1(s)$ have full row rank. Then the model matching problem is solvable if and only if*

$$\text{Ker }\{T_2(s)\} \subseteq \text{Ker }\{M_2(s)\} \tag{7.81}$$

Proof: The necessity is obvious from (7.80). For the sufficiency, let (7.81) hold true or equivalently (7.79) have a solution $\Lambda_1(s)$.

Consider now the matrix

$$\begin{bmatrix} -T_1^T(s)\Lambda_1(s) \\ M_1^T(s) \end{bmatrix} \tag{7.82}$$

This is a rational matrix since $\Lambda_1(s)$ is rational. We may always express (7.82) in a MFD form as follows

$$\begin{bmatrix} -T_1^T(s)\Lambda_1(s) \\ M_1^T(s) \end{bmatrix} = A(s)B^{-1}(s) \tag{7.83}$$

If we write

$$A(s) = \begin{bmatrix} C_2^T(s) \\ C_1^T(s) \end{bmatrix}, \ B^{-1}(s) = \Lambda_2(s) \tag{7.84}$$

we have

$$-T_1^T(s)\Lambda_1(s) = C_2^T(s)\Lambda_2(s) \tag{7.85}$$

$$M_1^T(s) = C_1^T(s)\Lambda_2(s) \tag{7.86}$$

i.e. we obtain equations (7.78) and (7.79). This means that if (7.80) has a solution $\Lambda_1(s)$ we may always find matrices $C_1(s)$, $C_2(s)$, $\Lambda_2(s)$ such that (7.78), (7.79) and therefore (7.77), are satisfied. Then (7.77) may be written as

$$\begin{bmatrix} T_1^T(s) & C_2^T(s) \\ 0 & C_1^T(s) \\ T_2^T(s) & 0 \end{bmatrix} \begin{bmatrix} \Pi_1(s) \\ I \end{bmatrix} \Lambda_2(s) = \begin{bmatrix} 0 \\ M_1^T(s) \\ M_2^T(s) \end{bmatrix} \tag{7.87}$$

where $\Lambda_1(s) = \Pi_1(s)\Lambda_2(s)$. Note that such $\Pi_1(S)$ always exists since $\Lambda_2(s)$ is square and invertible. The matrix $[\Pi_1^T(s), I]$ is the basis matrix of $\mathrm{Ker}\{[T_1^T(s), C_2^T(s)]\}$. This may be shown as follows: Let

$$\mathrm{Im}\{\begin{bmatrix} \Pi_1(s) \\ I \end{bmatrix}\} \subset \mathrm{Ker}\{[T_1^T(s), C_2^T(s)]\} \tag{7.88}$$

where the inclusion is strict. Then the basis matrix of $\mathrm{Ker}\{[T_1^T(s), C_2^T(s)]\}$ has the form

$$\begin{bmatrix} \Pi_1(s) & \Pi_1'(s) \\ I & \Pi_2'(s) \end{bmatrix} \tag{7.89}$$

Then we have

$$[T_1^T(s), C_2^T(s)] \begin{bmatrix} \Pi_1(s) & \Pi_1'(s) \\ I & \Pi_2'(s) \end{bmatrix} = 0$$

or equivalently

$$[T_1^T(s), C_2^T(s)] \begin{bmatrix} \Pi_1(s) & \Pi_1'(s) - \Pi_1(s)\Pi_2'(s) \\ I & 0 \end{bmatrix} = 0 \tag{7.90}$$

The above yields

$$T_1^T(s)[\Pi_1'(s) - \Pi_1(s)\Pi_2'(s)] = 0 \tag{7.91}$$

This contradicts our assumption that $T_1(s)$ has full row rank. Consider now the matrix $[\Pi_1^T(s), I]^T$. Clearly,

$$\begin{bmatrix} R^T(s) & 0 \\ 0 & R_C^T(s) \\ T_{C1}^T & C_{C2}^T \end{bmatrix} \begin{bmatrix} S^T(s)\Pi_1(s) \\ S_C^T(s) \end{bmatrix} = 0 \tag{7.92}$$

and the matrix

$$\begin{bmatrix} S^T(s)\Pi_1(s) \\ S_C^T(s) \end{bmatrix} \tag{7.93}$$

is a basis matrix of the kernel of $R^*(s)$, since $[\Pi_1^T(s), I]^T$ is a basis matrix of the kernel of $[T_1^T(s), C_2^T(s)]$. Thus, we may go backwards from (7.76) to (7.73) and the result follows. $\square$

Remark 7.5.3 *Note that if $T_2(s)$ does not have full row rank, then it is not guaranteed that $[\Pi_1^T(s), I]^T$ is a basis matrix of the kernel of $[T_1^T(s), C_2^T(s)]$. This means that if we go backwards from* (7.76) *we end up with the following:*

$$(Q_M(s)\ Ker\ \{P_M(s)\})^{\perp} \supseteq Im\ \{\begin{bmatrix} M_1^T(s) \\ M_2^T(s) \end{bmatrix}\} \tag{7.94}$$

and thus the equality of the $\mathcal{A}$–external behaviour of the compensated system and the model is not guaranteed. In the case where $T_1(s)$ does not have full column rank but the column rank defect of the matrix $[T_1^T(s), C_2^T(s)]$ is equal to r_M (the number of rows of $M(s)$) where $C_2^T(s)$ is obtained from (7.85), *we have that $[\Pi_1^T(s), I]^T$ is a basis matrix of the kernel of $[T_1^T(s), C_2^T(s)]$. In this case we may construct the controller $C(s)$ such that model matching is obtained. This may be seen from the fact that we may go from* (7.77) *to* (7.73) *since $[S(s)\Pi_1(s), I]$ is a basis matrix for the kernel of $R^*(s)$.* □

Remark 7.5.4 *The model matching problem under $\mathcal{A}$–external equivalence is a direct extension of the model matching under transfer equivalence: To see this, let $G_M(s) = -M_2^{-1}(s)M_1(s)$, $G_C(s) = -C_2^{-1}(s)C_1(s)$, $G(s) = -T_2^{-1}(s)C_1(s)$ be the transfer functions of the model, controller and plant respectively. The requirement that the compensated system has transfer function $G_M(s)$ may be written as*

$$-M_2^{-1}(s)M_1(s) = T_2^{-1}(s)T_1(s)C_2^{-1}(s)C_1(s) \tag{7.95}$$

Clearly, the number of rows of $M_1(s)$ is equal to the number of rows of $T_1(s)$ and thus, the matrix $[T_1^T(s), C_2^T(s)]$ has column rank defect equal to r_M since $C_2(s)$ is square and invertible. Thus, if we see the transfer function systems as systems described by the equations

$$[M_1(s), M_2(s)]\begin{bmatrix} u_M(s) \\ y_M(s) \end{bmatrix} = 0,\ [C_1(s), C_2(s)]\begin{bmatrix} u_C(s) \\ y_C(s) \end{bmatrix} = 0,\ [T_1(s), T_2(s)]\begin{bmatrix} u(s) \\ y(s) \end{bmatrix} = 0 \tag{7.96}$$

it follows that the necessary and sufficient condition for the solvability of the model matching problem is $Ker\{T_2(s)\} \subseteq Ker\{M_2(s)\}$ [Em. & Haut., 1980]. □

Example 7.5.1 Let

$$T(s) = [T_1(s), T_2(s)] = \left[\begin{array}{cc|ccc} s^2 & s & s+1 & s^2+s+1 & s^2+3s+2 \\ s & 1 & s & s^2 & s^2+3s \end{array}\right]$$

$$M(s) = \left[\begin{array}{cccc|ccc} s+3 & 1 & 2s & 1 & 2 & 2s+1 & 2s+5 \\ 2 & s & s+1 & s & s^2+2s+2 & s^3+2s^2+4s+2 & s^3+5s^2+6s+4 \end{array}\right]$$

The equation $T_2^T(s)\Lambda_1(s) = M_2^T(s)$ has the solution

$$\Lambda_1(s) = \begin{bmatrix} \frac{1}{s+1} & \frac{1}{s} \\ 2 & s \end{bmatrix}$$

Then

$$\begin{bmatrix} -T_1^T(s)\Lambda_1(s) \\ M_1^T(s) \end{bmatrix} = \begin{bmatrix} \frac{-s^2+s+1}{s+1} & -3s^2 \\ \frac{-s^2+s+1}{s(s+1)} & -3s \\ \hline s+3 & 2 \\ 1 & s \\ 2s & s+1 \\ 1 & s \end{bmatrix}$$

or, in MFD form

$$\begin{bmatrix} -T_1^T(s)\Lambda_1(s) \\ M_1^T(s) \end{bmatrix} = \begin{bmatrix} s^3-s^2-s & -3s^2 \\ -s^2-s-1 & -3s \\ \hline s^3+4s^2+3s & 2 \\ s^2+s & s \\ 2s^3+2s^2 & s+1 \\ s^2+3 & s \end{bmatrix} \begin{bmatrix} s^2+s & 0 \\ 0 & 1 \end{bmatrix}^{-1} = A(s)B^{-1}(s)$$

Note that the column rank defect of $[T_1^T(s), C_2^T(s)]$ is equal to $r_M = 2$.Thus, (see (7.85), (7.86)) we have

$$C_2^T(s) = \begin{bmatrix} -s^3-s^2-3 & -3s^2 \\ -s^2-1 & -3s \end{bmatrix}, \; C_1^T(s) = \begin{bmatrix} s^3+4s^2+3s & 2s^2+2s \\ s^2+3 & s^3+s^2 \\ 2s^3+2s^2 & s^3+2s^2+3 \\ s^2+2 & s^3+s^2 \end{bmatrix}$$

and the controller has the following form

$$C(s) = \left[\begin{array}{cccc|cc} s^3+4s^2+3s & s^2+s & 2s^3+2s^2 & s^2+s & -s^3-s^2-s & -s^2-s-4 \\ 4 & s & s+1 & -3s^2 & -3s & \end{array}\right]$$

Example 7.5.2 Consider the matching problem for the plant and model as in example 7.4.2 i.e.

$$T(s) = [s+1, s+3], \; M(s) = [s+5, s+2]$$

The equation $T_1^T(s)\Lambda_1(s) = M_2^T(s)$ has the solution

$$\Lambda(s) = \frac{s+2}{s+3}$$

Thus

$$\begin{bmatrix} -T_1^T(s)\Lambda_1(s) \\ M_1^T(s) \end{bmatrix} = \begin{bmatrix} \frac{(s+1)(s+2)}{s+3} \\ s+5 \end{bmatrix} = \begin{bmatrix} (s+1)(s+2) \\ (s+5)(s+3) \end{bmatrix} \begin{bmatrix} s+3 & 0 \\ 0 & s+3 \end{bmatrix}^{-1}$$

and the resulting controller is

$$C(s) = [(s+5)(s+3), (s+1)(s+2)]$$

If we consider the model matching problem with $G_T = -\frac{s+1}{s+3} = -\frac{T_1(s)}{T_2(s)}$, $G_M = -\frac{s+5}{s+2} = -\frac{M_1(s)}{M_2(s)}$ being the transfer functions of the plant and the model then the controller is $G_C = -\frac{C_1(s)}{C_2(s)} = \frac{(s+5)(s+3)}{(s+1)(s+2)}$. The above transfer functions have the following composite matrices

$$T_T = [s+1, s+3],\ T_M = [s+5, s+2],\ T_C = [(s+5)(s+3), -(s+1)(s+2)]$$

From the above it is clear that the solutions of the $\mathcal{A}$-external equivalence matching problem coincide with the solutions of the transfer function model matching problem when the transfer function is defined (i.e. when the output may be explicitly defined from the input).

7.6 Conclusions

In this chapter the model matching problem for behavioural systems has been considered. This problem was defined as the problem of finding an appropriate behavioural system such that when it is interconnected to a given system, the overall system has a desired external behaviour. Necessary conditions for the solvability of the problem have been produced and in a special case where certain rank conditions are satisfied, it was shown that these conditions are also sufficient. Due to the constructive algebraic approach followed, it was possible to find a detailed algorithm for the determination of the solutions of the problem when these solutions exist. Finally, a parametrisation of the solutions of the problem has been given.

Next, a modified version of the model matching problem was considered. In this version, the model and the composite system are required to be equivalent in the sense of $\mathcal{A}$-external equivalence. For the general case, necessary and conditions for the solvability of the problem have been derived. For a special case of systems satisfying certain conditions the problem was solved entirely and it was shown that it is a straightforward generalisation of the model matching problem under transfer equivalence.

The problem of finding the necessary and sufficient conditions for the solvability of the problem in the general case of both types of model matching problem of this chapter is the topic of future research.

Chapter 8

THE DYNAMIC COVER PROBLEMS IN SYSTEM THEORY

8.1 Introduction

A number of important problems in control theory turn out to be equivalent to the geometric problem of covering a given subspace with another subspace with special properties. This problem is called the dynamic cover problem. In most of the cases, the covering space is required to be an (A, B)–invariant subspace. This is the case of the *standard cover problem.* The formal definition of this problem is the following: Given the linear maps $F : \Re^n \to \Re^n$, $G : \Re^\ell \to \Re^n$, $H : \Re^n \to \Re^m u$, find all subspaces $\mathcal{V} \subset \Re^n$ which satisfy the following:

$$F\mathcal{V} \subset \mathcal{V} + \mathrm{Im}\{G\} \tag{8.1}$$

$$\mathcal{J} \subset \mathcal{V} \subset \mathcal{W} \tag{8.2}$$

where $\mathcal{J}$ and $\mathcal{W}$ are given subspaces of $\Re^n$. Clearly, if the matrices F, G are considered to be the matrices of the state–space system

$$\dot{x}(t) = Fx(t) + Gu(t) \tag{8.3}$$

the cover problem is directly related to system theory.

It is the purpose of this Chapter to provide a brief review of the most important problems in control theory that may be formulated as the problem described by (8.1), (8.2). Such problems are the *linear functional observer problem* [Wonh. & Mor., 1972], the *the model matching problem* [Mor., 1973], [Mor., 1976], [Em. & Haut., 1980], the *deterministic identification problem* [Em., Sil. & Gl., 1977] and the *disturbance decoupling problem* [Wonh., 1979]. The above problems were shown to be equivalent to appropriately defined dynamic cover problems of the type (8.1), (8.2), or to modified versions of this problem. In this chapter it is also shown that the dynamic cover problems may be considered as a special type of the general class of *model projection problems* [Kar., 1994]. The structure of this Chapter is the following: First the problem of disturbance decoupling [Wonh., 1979] is considered and the formulation of this problem as a cover problem is described [Wonh., 1979]. There, the necessary and sufficient conditions for the solvability of the problem were given. These conditions are general and apply to all the other types of cover problems.

Next, the equivalence of disturbance decoupling and model matching problem is considered. In [Em. & Haut., 1980] it was shown that for every disturbance decoupling there exist a corresponding model matching problem which is solvable if the first problem is solvable and conversely.

Another problem that may be formulated as a dynamic cover problem is the deterministic identification of a discrete time system [Em., Sil. & Gl., 1977]. This problem consists in finding a minimal state–space system from a finite number of finite length input–output measurements. This problem was formulated as a dynamic cover problem. In this Chapter we give a brief outline of the cover approach as it was presented in [Em., Sil. & Gl., 1977].

The problem of the observer of a linear functional of the states was formulated as a geometric problem by Wonham and Morse in [Wonh. & Mor., 1972]. This is the next problem we consider in this Chapter. The formulation of this as a cover problem as it was given in [Wonh. & Mor., 1972] is shown, and an extension to the implicit systems framework is given. It is shown, that this problem is formulated as an extended cover problem where (A, B)–invariance of $\mathcal{V}$ in (8.1) is replaced by (A, E, B)–invariance.

Finally, the model projection problems are described briefly and it is shown that dynamic cover is a special type of this family of problems.

8.2 The disturbance decoupling problem

In this section the geometric formulation of the disturbance decoupling problem is given following [Wonh., 1979]. It is shown that this formulation is equivalent to a cover problem and that the solvability condition given in the above work is the solvability of the cover problem.

Consider the state–space system

$$\dot{x}(s) = Ax(t) + Bu(t) + Dd(t) \tag{8.4}$$

$$y(t) = Cx(t) \tag{8.5}$$

In the above system the signal $d(t)$ corresponds to unknown disturbance at the input and the constant matrix D corresponds to the mechanism the disturbance is fed into the system. The problem of disturbance decoupling consists in finding an appropriate constant state feedback such that the disturbance does not have any influence on the output of the system. The equation corresponding to feedback is $u(t) = Fx(t)$. Intuitively speaking, the feedback must be such that the trajectories of the state $x(t)$ corresponding to the input $d(t)$ are restricted in the subspace $\mathcal{K} \subset \text{Ker}\{C\}$. The following lemma provides the guidelines to the solution of the problem.

Lemma 8.2.1 [Wonh., 1979] *The system* (8.4), (8.5) *is disturbance decoupled if*

$$\langle A + BF | \mathcal{D} \rangle \subset \mathcal{K} \tag{8.6}$$

where $\langle A + BF | \mathcal{D} \rangle$ *denotes the controllable subspace of the pair* $(A + BF, D)$. □

By the introduction of the notion of the (A, B)–invariant subspace, Wonham gave the following solvability condition of the disturbance decoupling problem.

Theorem 8.2.1 [Wonh., 1979] *The disturbance decoupling problem is solvable if and only if*

$$\mathcal{V}^{max} \supset \mathcal{D} \tag{8.7}$$

*where $\mathcal{V}^{max}$ is the maximal (A, B)-invariant subspace contained in $\mathcal{K}$ =*Ker$\{C\}$. □

An equivalent transformation of the above result is the following:

Theorem 8.2.2 [Em. & Haut., 1980] *The disturbance decoupling problem is solvable if and only if there exists a subspace $\mathcal{V} \subset \mathcal{X}$ such that*

$$A\mathcal{V} \subset \mathcal{V} + \mathcal{B} \tag{8.8}$$

$$\mathcal{D} \subseteq \mathcal{V} \subseteq \mathcal{K} \tag{8.9}$$

□

The above, clearly shows that the disturbance decoupling problem may be formulated as a cover problem of the type (8.1), (8.2) with $F = A$, $G = B$, $\mathcal{J} = \mathcal{D}$.

In [Em. & Haut., 1980] a modified version of the disturbance decoupling problem was defined as follows: Given the system (8.4), (8.5) find matrices F, G such that if

$$u(t) = Fx(t) + Gd(t) \tag{8.10}$$

the disturbance $d(t)$ does not have any influence on the output of the system. The solvability conditions of this problem lead to a modified cover problem. These conditions are the following.

Theorem 8.2.3 [Em. & Haut., 1980] *The modified disturbance decoupling problem has a solution if and only if there exists a subspace $\mathcal{V}$ such that*

$$A\mathcal{V} \subset \mathcal{V} + \mathcal{B} \tag{8.11}$$

$$\mathcal{D} \subset \mathcal{V} + \mathcal{B} \tag{8.12}$$

$$\mathcal{V} \subset \mathcal{K} \tag{8.13}$$

□

This version of the cover problem differs from the problem (8.1), (8.2) in the requirement that the covering space of $\mathcal{D}$ is $\mathcal{V} + \mathcal{B}$ instead of $\mathcal{V}$. It will be shown later in this Chapter that (8.11), (8.12) also correspond to the problem of observers of linear functionals of the state.

8.3 The model matching problem

The model matching problem is the problem of finding an appropriate precompensator such that when it is connected in cascade to the plant, the overall transfer function of the interconnected system matches the transfer function of a given model. This is clearly an open loop design problem. In [Mor., 1976] it was shown that this problem may be formulated as an appropriately defined cover problem. In this section we follow along the lines of [Em. & Haut., 1980] in order to show that the model matching problem may be defined as a cover problem through the association of disturbance decoupling and model matching problems. We begin with the statement of the problem: Let $G(s)$ be the transfer function of a given plant. If $G_M(s)$ is another transfer function (the transfer function of the model) find a compensator with strictly proper transfer function $G_C(s)$ such that

$$G(s)G_C(s) = G_M(s) \tag{8.14}$$

The above problem is referred to as *exact model matching problem.* If the requirement of strict properness of $G_C(s)$ is replaced by the requirement of properness we have the *modified exact model matching problem.*

Before we proceed we need the following important result which provides an alternative characterisation of the (A, B)-invariant subspaces.

Theorem 8.3.1 [Em. & Haut., 1980] *Let (A, B, C) be the triple of the matrices corresponding to an observable realisation of the transfer function $G(s) = D^{-1}(s)N(s)$. Then $\mathcal{V} \subset \mathcal{X}$ is an (A, B)-invariant subspace contained in* Ker$\{C\}$ *if there exist constant matrices F_1, A_1 satisfying*

$$N(s)F_1 = \Psi(s)(sI - A_1) \tag{8.15}$$

where $\Psi(s)$ is a basis matrix of $S(s)\mathcal{V}$, where $S(s)$ =block-diag$\{\ldots, [1\ s \ldots s^{\sigma_i - 1}], \ldots\}$ and σ_i are the row degrees of $[D(s)\ N(s)]$. □

Consider now the rational matrix

$$\hat{G}(s) = [G(s), G_M(s)] \tag{8.16}$$

and find an observable realisation of this matrix. This realisation has the form

$$\hat{\mathcal{S}} : \dot{x}(t) = Ax + [B, D]u \tag{8.17}$$

$$y = Cx(t) \tag{8.18}$$

Clearly (8.17), (8.18) describe a state-space with disturbances at the input (let D be the matrix of disturbance in (8.4)). Thus, for any model matching problem we may find a corresponding disturbance decoupling problem.

Conversely, consider the disturbance decoupling problem defined by (8.4), (8.5). Then if

$$G(s) = C(sI - A)^{-1}B,\ G_M(s) = C(sI - A)^{-1}D \tag{8.19}$$

it readily follows that any disturbance decoupling problem has a corresponding exact matching problem. The following result provides the equivalence of the solvability of the disturbance decoupling and the model matching problem.

Theorem 8.3.2 [Em. & Haut., 1980] *Let* $\{\Psi(s)\}$ *be such that*

$$A\{\Psi(s)\} \subseteq \{\Psi(s)\} + \mathcal{B},\ \{\Psi(s)\} \subset \mathrm{Ker}C \tag{8.20}$$

where $\{\Psi(s)\}$ *denotes the vector space spanned by the columns of* $\Psi(s)$. *The above means that* $\{\Psi(s)\}$ *is an* (A, B)*-invariant subspace in* $\mathrm{Ker}\{C\}$. *Assume that there exist matrices* F_1, A_1 *such that*

$$N(s)F_1 = \Psi(s)(sI - A_1) \tag{8.21}$$

If $R(s) = S(s)D$ *and there exist matrices* B_1 *and* D_1 *such that*

$$R(s) = \Psi(s)B_1 + N(s)D_1 \tag{8.22}$$

then the proper matrix $G_C(s) = F_1(sI - A_1)^{-1}B_1 + D_1$ *is a solution to the modified exact model matching problem. Conversely if* $G_C(s)$ *is a solution to the modified exact model matching problem and*

$$\mathcal{S}_1 : \dot{x} = A_1x + B_1u \tag{8.23}$$

$$y = F_1x + D_1u \tag{8.24}$$

is a realisation of $G_C(s)$ *then there exists a matrix* $\Psi(s)$ *satisfying* (8.21). □

The above theorem means that the modified model matching problem is solvable if and only if the corresponding disturbance decoupling problem is solvable. Thus, according to the previous section the model matching problem may be formulated as a cover problem.

8.4 The deterministic identification problem

This section is based on [Em., Sil. & Gl., 1977]. In this paper it was shown that the problem of deterministic identification of a minimal discrete time state–space system from a finite number of finite length input–output measurements may be formulated as a dynamic cover problem. The cover problem considered is defined as follows.

Definition 8.4.1 [Em., Sil. & Gl., 1977] *Let $\mathcal{U}$, $\mathcal{V}$, $\mathcal{Y}$, $\mathcal{D}$ be given linear vector spaces in $\Re^n$. Let P^T be a linear transformation in $\Re^n$. Then a linear subspace Ψ is said to be a* **generalised dynamic cover** *if and only if it satisfies*

$$P^T\Psi \subset \Psi + \mathcal{U} + \mathcal{V} \tag{8.25}$$

$$\mathcal{Y} \subset \Psi + \mathcal{U} \tag{8.26}$$

$$\Psi \subset \mathcal{D} \tag{8.27}$$

□

Note that if $\mathcal{V} = \{0\}$ in (8.25) then we have the definition of the standard cover problem (8.1), (8.2).

The procedure of relating the identification to the cover problem is the following: The data for the identification are the single input sequence $\{u_i\}_{i=1}^N$ and output sequence $\{y_i\}_{i=1}^N$ ($u_i \in \Re^\ell$, $y_i \in \Re^m$). The problem is to find a discrete time system

$$x_{k+1} = Ax_k + Bu_k \tag{8.28}$$

$$y_k = Cx_k + Du_k \tag{8.29}$$

which gives rise to $\{u_i\}_{i=1}^N$, $\{y_i\}_{i=1}^N$ for a state sequence $\{x_i\}_{i=1}^{N+1}$.

The above state–space equations may be written as follows

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} x_1 & x_2 & \cdots & x_N \\ u_1 & u_2 & \cdots & u_N \end{bmatrix} = \begin{bmatrix} x_2 & x_3 & \cdots & x_{N+1} \\ y_1 & y_2 & \cdots & y_N \end{bmatrix} \tag{8.30}$$

By defining

$$X = [x_1, x_2, \ldots, x_N] \tag{8.31}$$

$$Y = [y_1, y_2, \ldots, y_N] \tag{8.32}$$

$$U = [u_1, u_2, \ldots, u_N] \tag{8.33}$$

$$V = [0 \ldots 0\ 1] \tag{8.34}$$

$$P^T = \begin{bmatrix} 0 & 0 & \cdots & \cdots & 0 \\ 1 & 0 & & & \\ 0 & 1 & 0 & & \vdots \\ \vdots & \vdots & \ddots & & \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \tag{8.35}$$

we readily take from (8.30) (after transposition)

$$P^T \mathrm{Im}\{X^T\} \subset \mathrm{Im}\{X^T\} + \mathrm{Im}\{U^T\} + \mathrm{Im}\{V^T\} \tag{8.36}$$

$$\mathrm{Im}\{Y^T\} \subset \mathrm{Im}\{X^T\} + \mathrm{Im}\{U^T\} \tag{8.37}$$

The above is clearly the cover problem defined by (8.25) and (8.26) with $\Psi = \mathrm{Im}\{X^T\}$, $\mathcal{U} = \mathrm{Im}\{U^T\}$, $\mathcal{V} = \mathrm{Im}\{V^T\}$, $\mathcal{Y} = \mathrm{Im}\{Y^T\}$ and $\mathcal{D} = \Re^n$.

The matrix form of (8.36), (8.37) is

$$P^T X^T = X^T A^T + U^T B^T + V^T Z^T \tag{8.38}$$

$$Y^T = X^T C^T + U^T D^T \tag{8.39}$$

Clearly, if there exists a subspace $\Psi = \mathrm{Im}\{X^T\}$ such that (8.36) and (8.37) hold, there exist matrices A, B, C, D, Z satisfying (8.38) and (8.39) and thus the system (8.28), (8.29) is a system giving rise to the sequences $\{u_i\}_{i=1}^N$, $\{y_i\}_{i=1}^N$. The matrix Z corresponds to the term x_{N+1} in (8.29).

The identification problem may be generalised to the case where we consider more than one input–output sequences. Consider the case where we have p different input–output sequences of length N_i. Then if

$$X = [X_1, X_2, \ldots, X_p] \tag{8.40}$$

$$U = [U_1, U_2, \ldots, U_p] \tag{8.41}$$

$$Y = [Y_1, Y_2, \ldots, Y_p] \tag{8.42}$$

$$V = \begin{bmatrix} V_1 & & & \\ & V_2 & & \\ & & \ddots & \\ & & & V_p \end{bmatrix} \tag{8.43}$$

$$P = \begin{bmatrix} P_1 & & & \\ & P_2 & & \\ & & \ddots & \\ & & & P_N \end{bmatrix}, \quad N = \sum_{i=1}^{p} N_i \tag{8.44}$$

We may end up with a cover problem defined by (8.36), (8.37). We have the following theorem.

Theorem 8.4.1 [Em., Sil. & Gl., 1977] *The problem of finding a system $S(A, B, C, D)$ which realises a given set of input-output sequences from some set of initial states is equivalent to finding a subspace Ψ satisfying* (8.25), (8.26),(8.27) *with $\mathcal{D} = \Re^n$.* □

The cover problem may be related to the problem of partial realisations [Kal., 1969], [Dick., Morf & Kail., 1974] as follows.

Theorem 8.4.2 [Em., Sil. & Gl., 1977] *The problem of finding a linear system $\mathcal{S}(A, B, C)$ with $B = [b_1, \ldots, b_\ell]$, such that the Markov parameters $CA^{i-1}b_j$ have values h_{ij}, is equivalent to finding a subspace Ψ satisfying*

$$P^T\Psi \subset \Psi + \mathrm{Im}\{V^T\} \tag{8.45}$$

$$\mathrm{Im}\{H^T\} \subset \Psi \tag{8.46}$$

where P and V are as in (8.43), (8.44).

$$H = [h_{11}, h_{12}, \ldots, h_{1N_1}, \ldots, h_{\ell 1}, h_{\ell 2}, \ldots, h_{\ell N_\ell}] \tag{8.47}$$

and $N = \sum_{i=1}^{\ell} N_i$. □

Note that the cover problem formulation (8.45), (8.46) is identical to the formulation of the problem (8.1), (8.2) with $\mathcal{W} = \Re^n$.

8.5 Observers of linear functionals

In this section the formulation of the observer problem as a dynamic cover problem is considered. This problem is the problem of designing an asymptotic observer of a linear functional of the states. There are several approaches towards the solution of this problem [Luen., 1966], [Fort. & Wil., 1972]. The geometric formulation of this problem was given by Wonham and Morse in [Wonh. & Mor., 1972]. We start with a brief description of the problem of observer of linear functionals. Consider the observable state–space system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) && (8.48)\\ y(t) &= Cx(t) && (8.49) \end{aligned}$$

The observer of a linear functional of the states is a dynamical system having as inputs the input and the output of the system (8.48), (8.49) and as output an estimate of a linear of the states. The dynamical equations describing the observer are the following:

$$\begin{aligned} \dot{z}(t) &= Fz(t) + Gy(t) + Hu(t) && (8.50)\\ w(t) &= Mz(t) + Ny(t) && (8.51) \end{aligned}$$

From the above we readily take

$$\dot{z}(t) - T\dot{x}(t) = Fz(t) + (GC - TA)x(t) + (H - TB)u(t) \tag{8.52}$$

If we choose the matrices G, F, H such that

$$GC - TA = FT, \; H = TB \tag{8.53}$$

equation (8.52) becomes

$$\dot{z}(t) - T\dot{x}(t) = F(z(t) - Tx(t)) \tag{8.54}$$

If the matrix F is stable, then $z(t)$ converges asymptotically to the vector $Tx(t)$.

Definition 8.5.1 [Fort. & Wil., 1972] *The output $w(t)$ of the observer is said to estimate $Kx(t)$ and* (8.50), (8.51) *is said to be an observer of the linear functional $Kx(t)$ if*

$$\lim_{t\to\infty} \frac{d^j}{dt^j}[w(t) - Kx(t)] = 0, \; j = 0, 1, 2, \ldots$$

independently of $u(t)$, $x(0)$, $z(0)$. □

Lemma 8.5.1 [Fort. & Wil., 1972] *Let the system* (8.48), (8.49) *be observable. Then $w(t)$ estimates $Kx(t)$ if and only if there exists T such that $z(t)$ estimates $Tx(t)$ and*

$$K = [M \; N] \begin{bmatrix} T \\ C \end{bmatrix} \tag{8.55}$$

□

From the above it follows that the observer problem consists in finding matrices F, H, M, N such that (8.53) and (8.55) are satisfied and, in addition, F has stable eigenvalues. Equation (8.55) is satisfied if and only if

$$\mathcal{R}\{\begin{bmatrix} T \\ C \end{bmatrix}\} \supseteq \mathcal{R}\{K\} \tag{8.56}$$

where $\mathcal{R}\{\cdot\}$ denotes the row–range. Now, (8.56) is equivalent to

$$\mathrm{Ker}\{\begin{bmatrix} T \\ C \end{bmatrix}\} \subseteq \mathrm{Ker}\{K\} \tag{8.57}$$

Note that the order of the observer is equal to the number of rows of T. Consider now the transposed version of the first of (8.53)

$$A^T T^T = T^T F^T + C^T G^T \tag{8.58}$$

If there exist matrices F, G satisfying the above, then the columns of T^T span an (A, B)–invariant subspace. From (8.57) it follows

$$\mathrm{Ker}\{T\} \cap \mathrm{Ker}\{C\} \subseteq \mathrm{Ker}\{K\} \tag{8.59}$$

or

$$\mathcal{T}' + \mathcal{C}' \subset \mathcal{K}' \tag{8.60}$$

where $\mathcal{T}' = \mathrm{Im}\{T^T\}$, $\mathcal{C}' = \mathrm{Im}\{C^T\}$ and $\mathcal{K}' = \mathrm{Im}\{K^T\}$. Now, equations (8.58) and (8.60) define the problem of finding a subspace $\mathcal{V}$ such that

$$\hat{A}\mathcal{V} \subseteq \mathcal{V} + \hat{\mathcal{B}} \tag{8.61}$$

$$\mathcal{V} + \hat{\mathcal{B}} \supseteq \hat{\mathcal{K}} \tag{8.62}$$

where $\mathcal{V} = \mathcal{T}'$, $\hat{A} = A^T$, $\hat{\mathcal{B}} = \mathcal{C}'$, $\hat{\mathcal{K}} = \mathcal{K}'$. From the above it follows that the observer problem may be formulated as the cover problem defined by (8.61), (8.62). This problem is the problem of finding an (A, B)–invariant subspace $\mathcal{V}$ such that (8.62) is satisfied. Next, we consider the generalisation of the observer of linear functionals to the case of implicit descriptor systems. The given system is

$$E\dot{x}(t) = Ax(t) + Bu(t) \tag{8.63}$$

$$y(t) = Cx(t) \tag{8.64}$$

The proposed observer is of the type (8.50), (8.51). Note that although the system is implicit, the observer is a standard state–space system. Using similar arguments to the state–space system case we may show that the observer of linear functionals problem for descriptor systems consists in finding matrices G, F, H, M, N such that

$$GC - TA = FTE \tag{8.65}$$

$$H = TB \tag{8.66}$$

$$K = [M, N]\begin{bmatrix} TE \\ C \end{bmatrix} \tag{8.67}$$

Following along the lines of the analysis of the state–space systems case it may be readily seen that the observer problem for descriptor systems is equivalent to the following geometric problem: Find a subspace $\mathcal{V}$ such that

$$\hat{A}\mathcal{V} \subseteq \hat{E}\mathcal{V} + \hat{\mathcal{B}} \tag{8.68}$$

$$\hat{E}\mathcal{V} + \hat{\mathcal{B}} \supseteq \hat{\mathcal{K}} \tag{8.69}$$

Note that the order of the observer is equal to the dimension of the subspace $\mathcal{V}$. The problem defined by (8.68), (8.69) may be defined as follows: Find an $(\hat{A}, \hat{E}, \hat{B})$-invariant subspace such that (8.69) is satisfied. This problem belongs to the family of the *extended cover problems.* These problems are considered in the following Chapter.

For the case of descriptor systems, observers of linear functionals of the state may be used for the observation of the whole state as it is shown below.

We assume that the quintuple (E, A, B, C, D) was obtained as a minimal realisation of a given autoregressive equation $T\left(\frac{d}{dt}\right) w(t) = 0$. Thus, the matrices E, A, B, C, D have the following form (see Chapter 6, [Kuij. & Sch., 1990]).

$$E = \begin{bmatrix} I_\sigma & 0 \\ 0 & 0 \end{bmatrix} \tag{8.70}$$

$$A = \begin{bmatrix} \hat{A} & 0 \\ F_1 & 0 \end{bmatrix} + \begin{bmatrix} T_{t2} \\ 0 \end{bmatrix} \begin{bmatrix} -F_2 & -\bar{T}_{h2} \\ 0 & I \end{bmatrix} \tag{8.71}$$

$$B = \begin{bmatrix} T_{t2} \\ 0 \end{bmatrix} \begin{bmatrix} -T_2 \\ 0 \end{bmatrix} + \begin{bmatrix} T_{t1} \\ T_1 \end{bmatrix} \tag{8.72}$$

$$C = \begin{bmatrix} -F_2 & -T_{h2} \\ 0 & I \end{bmatrix} \tag{8.73}$$

$$D = \begin{bmatrix} -T_2 \\ 0 \end{bmatrix} \tag{8.74}$$

The state vector of the realisation is

$$\xi(t) = \begin{bmatrix} x(t) \\ z(t) \end{bmatrix} \tag{8.75}$$

Below, it is shown that some of the states of the descriptor system may be obtained directly (without the use of a dynamical system), since they are linear combinations of the external signals $u(t)$ and $y(t)$. In Chapter 6 it was shown that the above matrices were obtained in a reversible way from (6.22). Observing now the output equation $y(t) = C\xi(t) + Du(t)$ it readily follows that the vector $z(t)$ is defined as the component $y_b(t)$ of the outputs and thus, it is directly available. Furthermore if we partition the vector $x(t)$ in (8.75) as follows

$$x(t) = \begin{bmatrix} x_1^1 \\ \vdots \\ x_{\sigma_1}^1 \\ \hline \vdots \\ \hline x_1^\rho \\ \vdots \\ x_{\sigma_\rho}^\rho \end{bmatrix} \tag{8.76}$$

we see that the states $x_{\sigma_1}^1, \cdots, x_{\sigma_\rho}^\rho$ may be expressed as linear combinations of the inputs and outputs and may be observed directly. The components of $x(t)$ with indices $x_{\sigma_i - j}^i$ are not directly available from the inputs and outputs. Thus, the observer must be designed such that it reconstructs the above states. Therefore it is sufficient to estimate the vector $K\xi(t)$ where

$$K = [\text{block-diag}\{[I_{\sigma_i - 1}, 0]\}, 0] \tag{8.77}$$

Note that we considered a system with feedthrough term D. This does not change the overall approach to the observer problem.

8.6 Model Projection Problems

In this section the general class of model projection problems (MPP) [Kar., 1994] is discussed and it is shown that the dynamic cover problem belongs to this type of problems. We are going to focus on the *constant external model projection problem* (CEMPP). This problem is defined on the system Σ, referred to as *progenitor model*, with transfer function $H(s)$, or $S(A,B,C,D)$ model may be stated as follows: Find $K \in R^{m \times q}, L \in R^{p \times \ell}, m \leq q, \ell \leq p$, $rank(K) = m$, $rank(L) = \ell$ such that

$$\Theta(s) = KH(s)L \in R_{pr}^{m \times \ell}(s) \tag{8.78}$$

where $\Theta(s)$ is some "desirable" model to be specified, or equivalently in state space terms

$$\tilde{S}_\Theta(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}) : \tilde{A} = A, \tilde{B} = BL, \tilde{C} = KC, \tilde{D} = KDL \tag{8.79}$$

where $\tilde{S}_\Theta$ is a realisation of $\Theta(s)$, or a desirable model. The system Σ_Θ obtained from Σ under the (K, L) projecting pair will be referred to as an *input-output projected system* and the whole family of such systems that corresponds to all (K, L) possible pairs will be denoted by $\{\Sigma\}$. Of special interest here is the *Matrix Pencil Transformation Problem* (MPTP) i.e. the problem transforming the CEMPP to an equivalent problem of the matrix pencil setup.

For the case of strictly proper systems, the full C-EMPP (and thus also the partial) may be studied as an equivalent matrix pencil theory problem. In fact, let us assume that $S(A,B,C)$ is the progenitor model, $rank(B) = p$, $rank(c) = q$ and let $(B^{\dagger}, N)$, $(C^{\dagger}, M)$ be pairs of left inverse, left annihilator for B, right inverse, right annihilator for C respectively ($B^{\dagger}B = I_p$, $NB = 0$, $CC^{\dagger} = I_q$, $CM = 0$). We first note:

Lemma 8.6.1 *Let $K \in R^{m\times q}$, $L \in R^{p\times \ell}$, $rank(K) = m < q$, $rank(L) = \ell < p$ and let Q, R be such that*

$$KR = K[K^{\dagger}, K^{\perp}] = [I_m, 0],\ R \in R^{q\times q},\ |R| \neq 0 \tag{8.80}$$

$$QL = \begin{bmatrix} L^{\dagger} \\ L^{\perp} \end{bmatrix} L = \begin{bmatrix} I_\ell \\ 0 \end{bmatrix},\ Q \in R^{p\times p},\ |Q| \neq 0 \tag{8.81}$$

For any $\tilde{C} = KC$, $\tilde{B} = BL$ pair, there exist matrices $\tilde{Q}$, $\tilde{R} \in R^{n\times n}$, $|\tilde{Q}| \neq 0, |\tilde{R}| \neq 0$, such that

$$\tilde{C}\tilde{R} = \tilde{C}[C^{\dagger}K^{\dagger}, C^{\dagger}K^{\perp}, M] = [I_m, 0_{q-m}, 0_{n-q}],\ R \in R^{q\times q},\ |R| \neq 0 \tag{8.82}$$

$$\tilde{Q}\tilde{B} = \begin{bmatrix} L^{\dagger}B^{\dagger} \\ L^{\perp}B^{\dagger} \\ N \end{bmatrix} \tilde{B} = \begin{bmatrix} I_\ell \\ 0_{p-\ell} \\ 0_{n-p} \end{bmatrix} \tag{8.83}$$

and for any pair $\tilde{C}, \tilde{B}$ defined as above, the $(\tilde{C}^{\dagger}, \tilde{M})$, $(\tilde{N}, \tilde{B}^{\dagger})$ pairs are:

$$\tilde{B}^{\dagger} = L^{\dagger}B^{\dagger},\ \tilde{C}^{\dagger} = C^{\dagger}K^{\dagger},\ \tilde{M} = [M, C^{\dagger}K^{\perp}],\ \tilde{N} = \begin{bmatrix} N \\ L^{\perp}B^{\dagger} \end{bmatrix} \tag{8.84}$$

□

Using the above we may describe the essential pencils of the input-output projected system $S_\theta(\tilde{A}, \tilde{B}, \tilde{C})$ as shown below:

Proposition 8.6.1 *Let $S(A,B,C)$ be a progenitor model, (K,L), a projecting pair and let $\tilde{S}(\tilde{A}, \tilde{B}, \tilde{C})$ be the resulting input-output projecting system. For the $\tilde{S}(\tilde{A}, \tilde{B}, \tilde{C})$ the following properties hold true:*

(i) *The pencils $R(s) = sN - NA$, $\tilde{R}(s) = s\tilde{N} - \tilde{N}\tilde{A}$ of $S(A,B)$, $\tilde{S}(\tilde{A}, \tilde{B})$ are related as:*

$$\tilde{R}(s) = \begin{bmatrix} R(s) \\ sL^{\perp}B^{\dagger} - L^{\perp}B^{\dagger}A \end{bmatrix} \tag{8.85}$$

(ii) *The pencils $T(s) = sM - AM$, $\tilde{T}(s) = s\tilde{M} - \tilde{A}\tilde{M}$ of $S(A,C)$, $\tilde{S}(\tilde{A}, \tilde{C})$ are related as:*

$$\tilde{T}(s) = [T(s); sC^{\dagger}K^{\perp} - AC^{\dagger}K^{\perp}] \tag{8.86}$$

(iii) *The pencils* $Z(s) = sNM - NAM$, $\tilde{Z}(s) = s\tilde{N}\tilde{M} - \tilde{N}\tilde{A}\tilde{M}$ *of* $S(A,B,C)$, $\tilde{S}(\tilde{A},\tilde{B},\tilde{C})$ *are related as:*

$$\tilde{Z}(s) = \begin{bmatrix} Z(s) & sNC^{\dagger}K^{\perp} - NAC^{\dagger}K^{\perp} \\ sL^{\perp}B^{\dagger}M - L^{\perp}B^{\dagger}AM & sL^{\perp}B^{\dagger}C^{\dagger}K^{\perp} \end{bmatrix} \tag{8.87}$$

□

Given that $\tilde{R}(s)$, $\tilde{T}(s)$, $\tilde{Z}(s)$ define the controllability, observability, zero properties of $\tilde{S}(\tilde{A},\tilde{B},\tilde{C})$; Proposition 8.6.1 implies that that the C-EMPP is equivalent to augmentation of existing pencils and thus it is a problem of transformation of Kronecker invariants defined below:

Definition 8.6.1 : *Let* $sF - G \in R^{m\times n}[s]$. *Determining the Kronecker structure of the pencils*

$$[sF - G;\ A(s)],\ \begin{bmatrix} sF - G \\ B(s) \end{bmatrix},\ \begin{bmatrix} sF - G & A(s) \\ B(s) & C(s) \end{bmatrix} \tag{8.88}$$

where $A(s)$, $B(s)$ *and* $C(s)$ *are given dimension but otherwise free pencils, as function of the Kronecker structure of* $sF - G$, *will be called a Kronecker Structure Transformation Problem (KSTP), by column, row augmentation. If the pencils* $A(s)$, $B(s)$, $C(s)$ *are not free, but come from certain families, then the corresponding KSTP are called restricted–KSTP (R-KSTP).* □

The R–KSTP problem is related to the generalised dynamic cover problem as follows: If $[sF - G, A(s)] = (sN - NA)V$, where V is a given matrix representing the basis of a subspace $\mathcal{V}$, then the Kronecker invariants of $[sF - G, A(s)]$ determine the nature of $\mathcal{V}$. When is (A, B)-invariant the pencil $sF - G$ has i.e.d., c.m.i. and possibly n.z.r.m.i. [Kar., 1979]. If $V = [J, T]$, where J is the basis matrix of a given subspace $\mathcal{J}$ then it is clear that $\mathcal{V}$ is a dynamic cover of $\mathcal{J}$ as it is defined by (8.1), (8.2), with $\mathcal{W} = \mathcal{X}$. The pencil $[sF - G, A(s)]$ may be written as

$$[sF - G, A(s)] = [sNJ - NAJ, sNT - NAT] \tag{8.89}$$

From the above it is clear that the cover problem may be seen as a special case of the R-KSTP where $A(S) = sNT - NAT$ with N, A given and T the matrix to be found such that the overall pencil has a Kronecker structure corresponding to (A, B)-invariant subspace. This is an outline of the matrix pencil formulated problem which is discussed extensively in the following chapter.

8.7 Conclusions

Some important problems in system theory that may be formulated as appropriately defined cover problems were briefly described in this Chapter. These problems are the disturbance decoupling problem, the model matching, the deterministic identification and the problem of designing an observer of a linear function of the state. The latter problem was considered also for the case of implicit descriptor systems and it has been shown (for the descriptor case) that it may be formulated as an extended cover problem. The family of the model projection problems has also been considered in this chapter and shown that the dynamic cover problem is a special case of this family.

Chapter 9

A MATRIX PENCIL APPROACH TO THE GENERALISED COVER PROBLEMS

9.1 Introduction

In the previous Chapter we have seen that a number of important problems in control theory may be formulated as appropriate dynamic cover problems.

In the literature there is a small number of publications dealing with the solution of the cover problems. Antoulas in [Ant., 1983] gives a solution by means of partial realisations. In this paper it is shown that the covering spaces (the solutions of the cover problem) may be derived as the reachable subspaces of state–space realisations of appropriately defined Hankel matrices. The approach of the paper is rather complicated and deals only with the standard cover problem, i.e. the problem of covering a given subspace by an (A, B)–invariant subspace.

Another paper considering the cover problem is [Em., Sil. & Gl., 1977]. In this paper it is shown that the deterministic identification problem may be viewed as a generalised cover problem.

The standard cover problem that has been considered so far, belongs to a more general class of problems that arise within the general area of selection of input, output schemes for a given system [Kar., 1994]. Although the formulation of these problems is geometric in nature (find a certain type of invariant subspace that covers a given subspace and is contained in another one), their solvability and parametrisation of solution is closer in nature to problems of invariant structure assignment. The matrix pencil framework [Kar. & Kouv., 1979], [Jaf. & Kar., 1981] for the characterisation of invariant subspaces of the geometric theory [Wil., 1981], [Wonh., 1979] seems to be more suitable for the study of such problems, since it brings together the geometric and Kronecker invariant structure aspects of the problem; furthermore, the constructive nature of the matrix pencil tools allows the computation and parametrisation of solutions in a simple manner. Extending the matrix pencil framework to this new family of geometric problems is essential in the effort to provide unifying matrix pencil tools for the geometric synthesis methods. An integral part of this approach is the splitting of the overall problem into a Kronecker invariant transformation problem by matrix pencil augmentation and a matrix pencil realisation problem. The first deals with the study of the effect of adding matrix pencil columns to a given pencil on the resulting Kronecker structure; the second is equivalent to a problem of generating a given space restricted pencil [Kar. & Kouv., 1979] for a given system.

In this Chapter it is shown that the matrix pencil augmentation–realisation problem may be reduced to the solution of linear systems of equations. The set of the solutions of these equations provides a parametric representation of the basis matrices of the families of subspaces solving the cover problem.

The contribution of this Chapter is the following: First it provides a unification of the standard cover problem and the extended cover problem, i.e. the problem where the covering spaces are almost (A, B)–invariant subspaces, almost reachability subspaces, coasting, sliding spaces e.t.c. (see [Wil., 1981]). Second, the cover problem is solved for the case of general descriptor systems (singular or implicit).

The structure of the Chapter is the following: First, a restricted version of the cover problem is considered. This is the case where the restriction pencil of the covering space is not characterised by row minimal indices. Next, the overall cover problem is considered and an algorithm for the solution is given. The algorithm is essentially a search for solutions of the problem among the subspaces of all possible dimensions.

The extension of the cover problem to the case of the almost invariant spaces is considered next. This extension is obtained by using the concept of duality.

Finally, an alternative solution of the matrix pencil formulated problem is proposed. The problem is formulated as the problem of the solution of a set of multilinear equations. This method may be used alternatively to the first method provided in this Chapter, which is based on matrix pencil theory.

9.2 Preliminary definitions and statement of the problem

Let $\mathcal{S}(E, A, B, C)$ be the system described by the following descriptor equations

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t) && (9.1)\\ y(t) &= Cx(t) && (9.2) \end{aligned}$$

where $E \in \Re^{p\times n}$, $A \in \Re^{p\times n}$, $B \in \Re^{p\times \ell}$ and $C \in \Re^{m\times n}$. It is assumed that both matrices B and C have full rank. If N is a left annihilator of B (i.e. a basis matrix for the $\mathcal{N}_\ell\{B\}$ and $B^\dagger$ is a left inverse of $B, (B^\dagger B = I_\ell)$, then (9.1), (9.2) are equivalent to

$$\begin{aligned} NE\dot{x} &= NAx && (9.3)\\ u &= B^\dagger E\dot{x} - B^\dagger Ax && (9.4) \end{aligned}$$

where (9.3) is a "feedback free" system description and the associated pencil $R(s) = sNE - NA$ is known as the *input–state restriction pencil* [Kar. & Kouv., 1979] of the system.

Throughout the chapter we shall assume that the system (9.1), (9.2) is minimal in the sense of [Kui. & Sch., 1991]. This assumption does not impose any limitation since minimality is a property of the representation and not a fundamental characteristic of the system. Note that when a descriptor representation is minimal in this sense, it is not necessarily reachable in the sense of Kalman. Thus, when the system is a regular state–space system, minimality does not imply nonexistence of input decoupling zeros. The following proposition is of technical importance for the development of the matrix pencil method for the solution of the cover problem.

Proposition 9.2.1 *When the system $\mathcal{S}(E, A, B)$ is minimal, the input–state restriction pencil $R(s)$ does not have r.m.i.*

Proof: In Chapter 6 it was shown that the pencil $[sE - A, -B]$ does not have r.m.i. (see proposition 6.7.2). Then, the pencil

$$\begin{bmatrix} N \\ B^{\dagger} \end{bmatrix} [sE - A, -B] = \begin{bmatrix} sNE - NA & 0 \\ sB^{\dagger}E - B^{\dagger}A & I \end{bmatrix}$$

does not have r.m.i. either, since $[N^T, (B^{\dagger})^T]^T$ is square, invertible matrix. The block form of the above pencil shows that all the r.m.i. of $[sE - A, -B]$ are provided by the r.m.i. of $sNE - NA$ and since $[sE - A, -B]$ is left regular, the result follows. □

Before we proceed with the formal definition of the cover problem we summarise some of the basic theory of the fundamental subspaces of state–space systems i.e. systems of the form (9.1), (9.2) where $E = I$.

(i) *(A, B)–invariant subspaces*: The family of the (A, B)–invariant subspaces is characterised by the property:

$$A\mathcal{V} \subseteq \mathcal{V} + \mathcal{B} \tag{9.5}$$

where $\mathcal{B} = \mathrm{Im}\{B\}$. The (A, B)–invariant subspaces may be defined in dynamical terms as follows: A subspace $\mathcal{V} \subseteq \mathcal{X}$ is (A, B)–invariant if we can find appropriate input to the system $\mathcal{S}(A, B)$ such that when we start from an initial state $x_0 \in \mathcal{V}$, the state trajectory $x(t)$ remains in $\mathcal{V}$ for every t. An equivalent characterisation is that there exists a matrix F such that the subspace $\mathcal{V}$ is $(A + BF)$–invariant, i.e.

$$(A + BF)\mathcal{V} \subseteq \mathcal{V} \tag{9.6}$$

(ii) *Reachability subspaces*: A subspace $\mathcal{V} \subseteq \mathcal{X}$ is a reachability subspace if there exists matrix F such that [Wonh., 1979]

$$\mathcal{V} = \langle A + BF | \mathcal{B} \cap \mathcal{V} \rangle \tag{9.7}$$

The dynamical characterisation of the reachability subspaces is the following: $\mathcal{V}$ is a reachability subspace if for every state x_1, there exists input such that the state is driven from the initial state x_0 to the final x_1 in finite time and with the trajectory $x(t)$ remaining in $\mathcal{V}$ $\forall\, t$.

(iii) *Almost (A,B)–invariant subspaces*: A subspace $\mathcal{V} \subseteq \mathcal{X}$ is almost (A,B)–invariant if and only if $\forall\, x_0 \in \mathcal{V}$ and $\forall\, \varepsilon > 0$ there exists an input such that the resulting trajectory $x(t)$ satisfies $\inf_{v\in\mathcal{V}} \|x(t) - v\| < \varepsilon$. The above means that $\mathcal{V}$ is almost (A,B)–invariant, if starting from an initial state $x_0 \in \mathcal{V}$, the trajectory can remain arbitrarily close to the subspace $\mathcal{V}$.

(iv) *Almost reachability subspaces*: A subspace $\mathcal{V} \subseteq \mathcal{X}$ is almost reachability subspace if and only if $\forall\, x_0,\ x_1 \in \mathcal{V}$, some $T > 0$ and $\forall\, \varepsilon > 0$ $\exists u(t)$ such that $x(0) = x_0$, $x(T) = x_1$ and $\inf_{v\in\mathcal{V}} \|x(t) - v\| < \varepsilon \ \ \forall\, t$.

When we have a subspace $\mathcal{K} \subset \mathcal{X}$ we may define the maximal (A,B)–invariant subspace $\mathcal{V}^{max}$ contained in $\mathcal{K}$. This subspace has maximal dimension among all the (A,B)–invariant subspaces contained in $\mathcal{K}$ and was first defined by [Wonh., 1979], [Bas. & Mar., 1969]. Note that this space is unique. The computation of $\mathcal{V}^{max}$ may be obtained through a recursive algorithm with initial space the zero space [Wonh., 1979]. A similar definition may be given for the maximal reachability subspace $\Re^{max}$ contained in $\mathcal{K}$ and there exists an algorithm for its computation [Wonh., 1979].

As in the case of (A,B)–invariant subspace we may define the maximal almost (A,B)–invariant subspace as well as the maximal almost reachability subspace contained in a given subspace $\mathcal{K} \subset \mathcal{X}$.

In the case where E in (9.1) is singular or rectangular matrix, we have the descriptor representations. In this case we may extend the definition of the (A,B)–invariant subspaces to the (A,E,B)–invariant subspaces. The geometric definition of these spaces is given by the following relation. A subspace $\mathcal{V}$ is (A,E,B)–invariant if

$$A\mathcal{V} \subseteq E\mathcal{V} + \mathcal{B} \tag{9.8}$$

It is important to note that the dynamical characterisation of the (A,B)–invariant subspaces cannot be extended to the (A,E,B)–invariant subspaces in a straightforward way. This is because in the case of implicit systems there may be no solution $x(t)$ of the descriptor equations for a given initial condition $x(0)$ or, if there exists a solution it may not be unique. For this reason, the definition of the fundamental subspaces for descriptor implicit systems will be given in terms of the invariants of the restriction pencil $R_{\mathcal{V}}(s)$. The characterisation of the subspaces via matrix pencils allows the extension of the definitions of the state–space descriptions to the implicit descriptions.

The key tool for this characterisation is the *$\mathcal{V}$-restriction pencil* $R_{\mathcal{V}} = sNEV - NAV$ where V is a basis matrix of $\mathcal{V}$ [Kar., 1979]. The type of the subspace $\mathcal{V}$ is related to the Kronecker invariant of $R_{\mathcal{V}}(s)$ according to the following table [Jaf. & Kar., 1981]:

$\mathcal{V}$	Type of invariants of $R_{\mathcal{V}}(s)$			
	n.z.r.m.i.	c.m.i.	f.e.d.	i.e.d.
Arbitrary	yes	yes	yes	yes
Almost (A, E, B)-invariant	no	yes	yes	yes
(A, E, B)-invariant	no	yes	yes	no
Almost reachability	no	yes	no	yes
Reachability	no	yes	no	no
Coasting	no	no	yes	no
Jordan struct. (A, E, B)-invariant	no	yes	$(s - s_o)^{\kappa}$	no
Sliding	no	no	no	yes

Table 1. Matrix pencil characterisation of fundamental subspaces.

Definition 9.2.1 *Let $R_{\mathcal{V}} = sNEV - NAV$ be the restriction pencil of the subspace $\mathcal{V}$. Then, the spectrum of $\mathcal{V}$ is defined as the set of the finite and infinite elementary divisors of $R_{\mathcal{V}}(s)$, including the multiplicities.* □

A family of cover problems of the geometric theory are defined below.

Definition 9.2.2 *Let $\mathcal{X}$ be the state-space of the $\mathcal{S}(E, A, B)$ system and let $\mathcal{J} \subseteq \mathcal{W} \subseteq \mathcal{X}$. Finding all subspaces $\mathcal{V}$ of $\mathcal{X}$ such that*

(i) $\mathcal{V}$ is (A, E, B)-invariant, i.e. $A\mathcal{V} \subseteq E\mathcal{V} + \mathcal{B}$ and

$$\mathcal{J} \subseteq \mathcal{V} \subseteq \mathcal{W} \tag{9.9}$$

is known as the **standard cover problem** [Ant., 1983], [Em., Sil. & Gl., 1977].

(ii) $\mathcal{V}$ is subspace with infinite spectrum and (9.9) *is also satisfied, will be referred to as* **extended cover problem.**

(iii) $\mathcal{V}$ is any of the invariant types of subspaces in (i), (ii) and $\mathcal{W} = \mathcal{X}$, then the problem will be called **partial cover problem.** □

The extended cover problems form an integral part of the investigation of Model Projection Problems (MPP) [Kar., 1994], which arise in the study of selection of control structures. Our approach is based on the matrix pencil characterisation of the (A, E, B)-invariant subspaces [Jaf. & Kar., 1981], [Kar., 1979].

The main idea underlying the matrix pencil approach to the study of the cover problems is the following: Let J be the basis matrix of the subspace to be covered. Since $\mathcal{V}$ is the covering subspace, then $\mathcal{V} = \mathcal{J} \bigoplus \mathcal{T}$ where $\mathcal{T}$ is some appropriate subspace, or in matrix form

$$V = [J, T] \tag{9.10}$$

The restriction pencil of the covering subspace is then

$$R_{\mathcal{V}}(s) = sNEV - NAV = (sNE - NA)[J, T] \tag{9.11}$$

From the above expression, it is clear that the general family of cover problems are equivalent to problems of Kronecker structure assignment defined below.

Kronecker Structure Assignment Problem (KSAP): Given the $\mathcal{J}$-restriction pencil $R_{\mathcal{J}}(s) = sNEJ - NAJ$, find an appropriate $\mathcal{T}$-restriction pencil $R_{\mathcal{T}}(s) = sNET - NAT$ such that the column augmented pencil $R_{\mathcal{V}}(s)$ in (9.11) has a certain type invariant structure.

The general Kronecker structure assignment problem may be naturally divided to the following two subproblems:

Matrix Pencil Augmentation Problem (MPAP): Given the pencil $sF - G \in \Re^{m\times\kappa}[s]$, find the conditions for the existence of a pencil $s\overline{F} - \overline{G} \in \Re^{m\times p}[s]$ such that the pencil

$$P(s) = [sF - G, s\overline{F} - \overline{G}] \tag{9.12}$$

has a given set of invariants.

Matrix Pencil Realisation Problem (MPRP): Given the pencil $sNE - NA \in \Re^{(n-\ell)\times n}[s]$, find the conditions under which there exists $T \in \Re^{n\times p}$ such that

$$sNET - NAT = sF - G \tag{9.13}$$

The above two problems are integral parts of the KSAP and will be examined here.

The above family of structure assignment problems deal with assignment of certain types of invariants, rather than the assignment of exact values of pencil invariants; in this sense they are extensions of the zero assignment problems considered so far [Kar. & Gian., 1989].

9.3 Kronecker invariant transformation by matrix pencil augmentation

In this section, we examine a number of results related to the transformation of the types of strict equivalence (SE)–invariants of a matrix pencil by addition of columns (rows). We consider first an important property established for a general polynomial matrix by [Thom., 1979] and presented here for the case of matrix pencils.

Theorem 9.3.1 *Let $P(s) = sF - G$ be a matrix pencil and let $sf - g$ be a column pencil and let $P'(s) = [sF - G, sf - g]$. If $\theta_i(s)$, $i = 1, \dots, \kappa$, $\zeta_j(s)$, $j = 1, \dots, \kappa$ or $\kappa + 1$ are the invariant polynomials of $P(s)$, $P'(s)$ respectively, then*

(a) If $\text{rank}_{\Re(s)}\{P(s)\} < \text{rank}_{\Re(s)}\{P'(s)\}$ *then the following interlacing property holds*

$$\zeta_1(s)/\theta_1(s)/\zeta_2(s)/\theta_2(s)/\dots/\theta_\kappa(s)/\zeta_{\kappa+1}(s) \tag{9.14}$$

(b) If $\text{rank}_{\Re(s)}\{P(s)\} = \text{rank}_{\Re(s)}\{P'(s)\}$ *then the interlacing property holds*

$$\zeta_1(s)/\theta_1(s)/\zeta_2(s)/\theta_2(s)/\dots/\theta_\kappa(s) \tag{9.15}$$

□

Note that in the above a/b denotes that a divides b. Some further result is stated below.

Proposition 9.3.1 *Consider the pencil $sF - G$ and augment it by a single column $sf - g$ such that its rank is increased. Then the sets of the i.e.d and f.e.d. of the original pencil are subsets of the i.e.d. and f.e.d. of the augmented pencil.*

Proof: From theorem 9.3.1 it follows that the invariant polynomials of the original and the augmented pencils are related by the interlacing inequalities (9.14). The invariant factors ζ_i, $i = 1, \dots, \ell + 1$ and ε_i, $i = 1, \dots, \ell$ can be factorised as follows:

$$\zeta_i(s) = (s - \alpha_i^1)^{\lambda_{i,1}}(s - \alpha_i^2)^{\lambda_{i,2}} \cdots (s - \alpha_i^{\rho_i})^{\lambda_{i,\rho_i}} \tag{9.16}$$

$$\varepsilon_i(s) = (s - \beta_i^1)^{\mu_{i,1}}(s - \beta_i^2)^{\mu_{i,2}} \cdots (s - \beta_i^{\phi_i})^{\mu_{i,\phi_i}} \tag{9.17}$$

The factors $(s - \alpha_i^j)^{\lambda_{i,j}}$ and $(s - \beta_\kappa^\ell)^{\mu_{\kappa,\ell}}$ are the f.e.d. of the augmented and the original pencil respectively.

From the interlacing inequalities (9.14) it is clear that

$$\varepsilon_j(s)/\zeta_{j+1}(s) \tag{9.18}$$

i.e. $\zeta_{j+1}(s)$ can be expressed as

$$\zeta_{j+1}(s) = x_j(s)\varepsilon_j(s) \tag{9.19}$$

or

$$\zeta_{j+1}(s) = x_j[(s-\beta_j^1)^{\mu_{j,1}}(s-\beta_j^2)^{\mu_{j,2}}\cdots(s-\beta_j^{\phi_j})^{\mu_{j,\phi_j}}] \tag{9.20}$$

The above yields that all the f.e.d. of $sF-G$ are f.e.d. of the augmented pencil $[sF-G, sf-g]$ and the result follows. □

The case of the i.e.d. may be proved similarly, taking the "dual" pencil $F-\hat{s}G$. It should be mentioned that the multiplicities of the common elementary divisors of the two pencils may be different, since the polynomial $x_i(s)$ may have some of its roots equal to the roots of $\varepsilon_i(s)$.

An obvious consequence of the above is the following result.

Proposition 9.3.2 *Consider the pencil* $[sF-G, sf-g]$.

(i) If the additional column is linearly dependent, on the columns of $sF-G$*, the number of the c.m.i. is increased by one and the number of the r.m.i. remains unchanged.*

(ii) If the additional column is linearly independent, then the number of the c.m.i. remains unchanged and the number of the r.m.i. is reduced by one.

Proof: The number of c.m.i. and r.m.i. of $sF-G$ is equal to the dimension of the right and left null space of $sF-G$ respectively.

(i) If the additional column of $sf-g$ is linearly dependent on the columns of $sF-G$ then rank$\{sF-G\}$=rank$\{[sG-G, sf-g]\}$ and therefore the dimension of the right null space of $sF-G$ is increased by one while the dimension of the left null space remains the same. From the above it follows that the number of the c.m.i. is increased by one and the number of the r.m.i. remains unchanged.

(ii) In the case where the additional column is linearly independent from the columns of $sF-G$ we have that rank$\{[sF-G, sf-g]\}$=rank$\{sF-G\}+1$ and therefore, the dimension of the right null space remains unchanged. The dimension of the null space is reduced by one since it is equal to the number of rows of the augmented pencil minus the rank of that pencil. □

From the above proposition and theorem 9.3.1 it follows that when the rank of the pencil $sF - G$ is increased by 1 with the addition of a single column, the result is the elimination of one r.m.i. and the possible change of the structure of the f.e.d/i.e.d.. Thus, when we want to eliminate the r.m.i. of a pencil, it is necessary to augment it by a number of linearly independent columns equal to the number of the r.m.i..

Consider now the general pencil $sF - G$ and without loss of generality, we may assume to be in the Kronecker canonical form.

$$[sF - G, s\overline{F} - \overline{G}] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & L_\eta(s) & 0 & 0 & 0 \\ 0 & 0 & L_\epsilon(s) & 0 & 0 \\ 0 & 0 & 0 & D_\infty(s) & 0 \\ 0 & 0 & 0 & 0 & D_f(s) \end{bmatrix} \tag{9.21}$$

where the blocks L_ϵ, L_η, D_∞, D_f correspond to all the nonzero c.m.i., i.e.d., f.e.d. respectively.

Proposition 9.3.3 *The number of the zero r.m.i. of the augmented pencil* $[sF - G, \overline{F} - \overline{G}]$ *cannot exceed the number of the zero r.m.i. of the pencil* $sF - G$.

Proof: The number of the z.r.m.i. of $sF - G$ is equal to the dimension of the left null space of the matrix $[F, G]$ and the number of z.r.m.i. of the augmented pencil is the dimension of the left null space of the matrix $[F, G, \overline{F}, \overline{G}]$. But

$$\mathcal{N}_\ell\{[F, G, \overline{F}, \overline{G}]\} = \mathcal{N}_\ell\{[F, G]\} \cap \mathcal{N}_\ell\{[\overline{F}, \overline{G}]\} \subseteq \mathcal{N}_\ell\{[F, G]\} \tag{9.22}$$

and therefore

$$\dim\mathcal{N}_\ell\{[F, G, \overline{F}, \overline{G}]\} \leq \dim\mathcal{N}_\ell\{[F, G]\}$$

and the result follows. □

Lemma 9.3.1 *Let* $sF - G$ *be the restriction pencil of the system* (9.1), (9.2) *on a subspace* $\mathcal{V}$. $\mathcal{V}$ *is an* (A, B)*–invariant subspace if and only if*

$$\mathcal{N}_\ell\{F\} \subseteq \mathcal{N}_\ell\{G\} \tag{9.23}$$

Proof: We may assume $sF - G$ in Kronecker canonical form without loss of generality. Necessity: If $\mathcal{V}$ is (A, E, B)–invariant subspace, then $sF - G$ is characterised only by f.e.d., c.m.i. and possibly z.r.m.i. We may consider a typical case, without loss of generality i.e.

$$F = \left[\begin{array}{c|c|c|c} 0 & 0 & 0 & \\ \hline I_\alpha & 0 & 0 & \\ \hline 0 & I_p & 0 & \\ \hline 0 & 0 & I_\varepsilon & 0 \end{array}\right], \quad G = \left[\begin{array}{c|c|c|c} 0 & 0 & 0 & \\ \hline J_\alpha & 0 & 0 & \\ \hline 0 & J_p & 0 & \\ \hline 0 & 0 & 0 & I_\varepsilon \end{array}\right] \tag{9.24}$$

where we have h z.r.m.i., $(s-\tau_1)^\alpha$, $\alpha \neq 0$, $(s-\tau_2)^p$, $p \neq 0$, f.e.d. and ε c.m.i. Clearly

$$\mathcal{N}_\ell\{F\} = \mathcal{N}_\ell\{[F, G]\} \subseteq \mathcal{N}_\ell\{G\} \tag{9.25}$$

and strict equality holds only when we have a zero e.d.; otherwise, i.e. if $\mathcal{V}$ has no zero f.e.d. then

$$\mathcal{N}_\ell\{F\} \subseteq \mathcal{N}_\ell\{G\} \tag{9.26}$$

Sufficiency: To prove the sufficiency we use contradiction arguments. Thus, let us assume that (9.25) holds true. If $sF - G$ has a nonzero r.m.i. and possibly i.e.d., then we have a Kronecker form of a typical type for $\hat{s}^q$, $\eta > 0$ as

$$F = \left[\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline \begin{array}{c} I_\eta \\ \hline 0 \end{array} & 0 & 0 \\ \hline 0 & J_q & 0 \\ \hline 0 & 0 & F' \end{array}\right], \quad G = \left[\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline \begin{array}{c} 0 \\ \hline I_\eta \end{array} & 0 & 0 \\ \hline 0 & I_q & 0 \\ \hline 0 & 0 & G' \end{array}\right] \tag{9.27}$$

where J_q is the standard Jordan block of $q \times q$ dimensions corresponding to the zero eigenvalue.

Note:

(i) If we have $\hat{s}_q$ i.e.d. then there exists a vector $v_q^t = [0, \cdots, 0, 1]^T$ such that $v_q^T J_q = 0$ and thus a vector $y^T = [0, \cdots, 0, v_q^T, 0, \cdots, 0]^T$ such that $y^T F = 0$ but $y^t G \neq 0$. This clearly contradicts the assumption that $\mathcal{N}_\ell\{F\} \subseteq \mathcal{N}_\ell\{G\}$.

(ii) If we have a nonzero r.m.i. of value ε, then there exist vectors $v_1^T = [1, \cdots, 0, 0]^T$, $v_\eta^T = [0, 0, \cdots, 0, 1]^T$ for which

$$v_1^T \left[\begin{array}{c} 0 \\ \hline I_\eta \end{array}\right] = 0, \quad v_\eta^T \left[\begin{array}{c} I_\eta \\ \hline 0 \end{array}\right] = 0 \tag{9.28}$$

and thus, vectors $y_1^T = [0, \cdots, 0, v_1^T, 0, \cdots, 0]^T$, $y^T = [0, \cdots, 0, v_\eta^T, 0, \cdots, 0]^T$ for which

$$y_1^T F \neq 0 \;\text{ and }\; y_1^T G = 0 \tag{9.29}$$

$$y_\eta^T F \neq 0 \;\text{ and }\; y_\eta^T G = 0 \tag{9.30}$$

Thus, there exist vectors y_1, y_η such that $y_1 \in \mathcal{N}_\ell\{G\}$ and $y_1 \notin \mathcal{N}_\ell\{G\}$ and $y_\eta \in \mathcal{N}_\ell\{G\}$ and $y_\eta \notin \mathcal{N}_\ell\{G\}$; these conditions clearly imply that the presence of i.e.d. and n.z.r.m.i. contradicts the $\mathcal{N}_\ell\{F\} \subseteq \mathcal{N}_\ell\{G\}$ condition. □

The above lemma yields the following.

Lemma 9.3.2 *Let $sF - G$ be the restriction pencil of $\mathcal{V}$. The subspace $\mathcal{V}$ is (A, E, B)-invariant if and only if*

$$\mathrm{Im}\{F\} \supseteq \mathrm{Im}\{G\} \tag{9.31}$$

Proof: From lemma 9.3.2 we have that $sF - G$ does not have i.e.d. and n.z.r.m.i. if and only if $\mathcal{N}_\ell\{F\} \subseteq \mathcal{N}_\ell\{G\}$. Let $g \in \mathrm{Im}\{G\}$. Then

$$y^T g = 0, \quad \forall\, y^T \in \mathcal{N}_\ell\{G\} \tag{9.32}$$

Since $\mathcal{N}_\ell\{F\} \subseteq \mathcal{N}_\ell\{G\}$ it follows that g annihilates all $y^T \in \mathcal{N}_\ell\{F\}$. Therefore $g \in \mathrm{Im}\{F\}$ and the result follows. □

Proposition 9.3.4 *Necessary condition for the augmented pencil $[sF - G, s\overline{F} - \overline{G}]$ to have no i.e.d. and no n.z.r.m.i. is that the number of columns of $s\overline{F} - \overline{G}$ is greater or equal to the total number of the n.z.r.m.i. and i.e.d. of $sF - G$.*

Proof: From proposition 9.3.2 it follows that in order to eliminate the n.z.r.m.i., we need at least equal number of linearly independent columns. Obviously, the minimal number of the additional columns is obtained when the composite pencil $[sF - G, s\overline{F} - \overline{G}]$ has equal number of z.r.m.i., to the number of the z.r.m.i. of the original pencil $sF - G$. From proposition 9.3.1 it follows that as long as we augment the pencil by linearly independent columns, the resulting pencil is characterised by i.e.d.. Since we keep the number of the z.r.m.i. unchanged, we can assume that the composite pencil has the form

$$[sF - G, s\overline{F} - \overline{G}] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & L_\eta(s) & 0 & 0 & 0 & sK_2 - M_2 \\ 0 & 0 & L_\epsilon(s) & 0 & 0 & sK_3 - M_3 \\ 0 & 0 & 0 & D_\infty(s) & 0 & sK_4 - M_4 \\ 0 & 0 & 0 & 0 & D_f(s) & sK_5 - M_5 \end{bmatrix} \tag{9.33}$$

where L_η, L_ϵ, D_∞, D_f are the nonzero r.m.i., nonzero c.m.i., i.e.d. and f.e.d blocks respectively.

The structure of that pencil as far as the n.z.r.m.i. and the i.e.d. are concerned, is identical to the structure of the pencil

$$[s\overline{F}-\overline{G}] = \begin{bmatrix} 0 & L_\eta(s) & 0 & 0 & 0 & sK_2-M_2 \\ 0 & 0 & L_\epsilon(s) & 0 & 0 & sK_3-M_3 \\ 0 & 0 & 0 & D_\infty(s) & 0 & sK_4-M_4 \\ 0 & 0 & 0 & 0 & D_f(s) & sK_5-M_5 \end{bmatrix} \tag{9.34}$$

This matrix pencil cannot be characterised by zero r.m.i. since $\mathcal{N}_l\{[\overline{F},\overline{G}]\}=\{0\}$. Therefore pencil (9.34) is not characterised by i.e.d. and n.z.r.m.i. only if the matrix $\overline{F}$ is left regular. From the form of the pencil (9.34) we can see that the matrix $\overline{F}$ can have full rank only if the matrix that consists of the rows of the pencil $s\overline{F}-\overline{G}$ that correspond to the bottom rows of the blocks of the n.z.r.m.i. and i.e.d. has full rank. Since the number of the rows of that matrix is equal to the total number of i.e.d. and n.z.r.m.i. of the pencil $sF-G$, the result follows. □

One of the major issues in characterising the solvability of the extended cover problems is the investigation of the conditions under which the resulting pencil after augmentation has no n.z.r.m.i.. By assuming the pencil in the canonical form we have:

$$[sF-G,\overline{F}-\overline{G}] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & sK_1-M_1 \\ 0 & L_\eta(s) & 0 & 0 & 0 & sK_2-M_2 \\ 0 & 0 & L_\epsilon(s) & 0 & 0 & sK_3-M_3 \\ 0 & 0 & 0 & D_\infty(s) & 0 & sK_4-M_4 \\ 0 & 0 & 0 & 0 & D_f(s) & sK_5-M_5 \end{bmatrix} \tag{9.35}$$

Now it is obvious that necessary and sufficient condition for $P'(s)$ to have any type of r.m.i., is that the subpencil

$$P''(s) = \begin{bmatrix} 0 & sK_1-M_1 \\ L_\eta(s) & sK_2-M_2 \end{bmatrix} \tag{9.36}$$

to provide this type of r.m.i. since the rest of the blocks are left regular. We may summarise as follows:

Proposition 9.3.5 *Necessary and sufficient conditions for $P''(s)$ to have all its r.m.i. with values strictly less than those in the $L_\eta(s)$ block, or $P''(s)$ has no r.m.i. are:*

(i) If $\mathcal{R}(K_1,M_1)$, $\mathcal{R}(K_2,M_2)$ are the $\Re(s)$-row spaces of the pencils sK_1-M_1, sK_2-M_2 respectively, then

$$\mathcal{R}(K_1,M_1)\cap\mathcal{R}(K_2,M_2)=\{0\} \tag{9.37}$$

(ii) The pencil $[sK_2 - M_2, L_\eta]$ is left regular.

(iii) All r.m.i. of $sK_1 - M_1$ are strictly less than those of L_η, or the pencil $sK_1 - M_1$ is left regular if $P''(s)$ has no r.m.i..

Proof: Let $y^T(s) = [y_1^T(s), y_2^T(s)]$ be an $\Re[s]$ vector in $\mathcal{N}_\ell(P''(s))$. Then we have

$$[y_1^T(s), y_2^T(s)][sK_1 - M_1, sK_2 - M_2] = 0,\ y_2^T(s)L_\eta(s) = 0$$

or equivalently

$$y_2^T(s)L_\eta(s) = 0 \tag{9.38}$$

$$y_1^T(s)(sK_1 - M_1) = -y_2^T(s)(sK_2 - M_2) \tag{9.39}$$

From condition (9.39) we see that either $y_2^T(s) \neq 0$, or $y_2^T(s) = 0$. We distinguish the following cases:

(i) $y_2^T(s) \neq 0$. In this case, if $\overline{n}$ is the minimal of the degrees in $L_n(s)$ block, then $\partial\{y_2^T(s)\} \geq \overline{n}$. It is thus a necessary condition that $y_2^T(s) = 0$ for the degree of $y(s)$ to be less than $\overline{n}$.

(ii) If $y_2^T(s) = 0$, then (9.39) is reduced to

$$y_1^T(s)(sK_1 - M_1) = 0 \tag{9.40}$$

and it is necessary that $\mathcal{N}_\ell$ $(sK_1 - M_1)$ is either $\{0\}$, or if it is nonzero, then its r.m.i. are strictly less than $\overline{n}$. Thus, necessary conditions are

$$y_2^T(s) = 0 \text{ and } \mathcal{N}_\ell\{sK_1 - M_1\} = \{0\}$$

or the r.m.i. of $sK_1 - M_1$ are strictly less than $\overline{n}$.

For $y_2^T(s) = 0$ we must determine the necessary conditions for this to happen. From equation (9.39) we have that:

(a) If $y_2^T(s) \neq 0$ and $y_1^T(s) \neq 0$ then

$$\mathcal{R}(K_1, M_1) \cap \mathcal{R}(K_2, M_2) \neq \{0\} \tag{9.41}$$

(b) If $y_2^T(s) \neq 0$ and $y_1^T(s) = 0$, then by (9.38) and $y_1^T(s) = 0$ in (9.39) we have

$$y_2^T(s)[sK_2 - M_2, L_\eta(s)] = 0 \tag{9.42}$$

It is clear that from (a) and (b) above that for $y_2^T(s) = 0$ it is necessary that both (9.41) and (9.42) conditions to be true, which proves the necessity.

To prove the sufficiency we argue as follows:

$$\mathcal{R}(K_1, M_1) \cap \mathcal{R}(K_2, M_2) = 0 \tag{9.43}$$

implies that condition (9.39) yields

$$y_1^T(s)(sK_1 - M_1) \;=\; 0 \tag{9.44}$$

$$y_2^T(s)(sK_2 - M_2) \;=\; 0 \tag{9.45}$$

and from (9.45) and (9.38) we have

$$y_2^T(s)[sK_2 - M_2, L_\eta(s)] = 0 \tag{9.46}$$

which since $[sK_2 - M_2, L_\eta(s)]$ is left regular implies $y_2^T(s) = 0$. Since $sK_1 - M_1$ is either left regular, or has r.m.i. with values strictly less than $\overline{n}$ the sufficiency is established. □

9.4 The matrix pencil realisation problem

The analysis of the previous section has assumed that the pencil used in the augmentation process, $s\overline{F} - \overline{G}$, is arbitrary; however, this pencil is generated from the input-state pencil of the system as

$$(sNE - NA)T = s\overline{F} - \overline{G} \tag{9.47}$$

or equivalently as a solution of the system

$$\begin{bmatrix} \overline{F} \\ \overline{G} \end{bmatrix} = \begin{bmatrix} NE \\ NA \end{bmatrix} T \tag{9.48}$$

The problem of matrix pencil realisation is equivalent to finding a T, when (N, E, A), (F, G) are given such that (9.48) is satisfied. Our present version of the problem is equivalent to generating an appropriate $\mathcal{T}$–restriction pencil for the given system. Clearly, this problem, does not always have a solution i.e. not any pair $(\overline{F}, \overline{G})$ may be created as a $\mathcal{T}$–restriction of a pair (N, NA); this problem is a generalisation of the zero assignment problem [Kar., 1990]. Clearly, the family of pairs $(\overline{F}, \overline{G})$ provide the necessary input to the Matrix Pencil Augmentation Problem.

In the case of the cover problem the matrices F, G, N, E, A are given and the problem is to find T such that (9.47) is satisfied. An obvious result for the solvability of this problem is:

Remark 9.4.1 *The matrix pencil realisation problem is solvable if and only if*

$$\text{col}-\text{span}\{\begin{bmatrix} \overline{F} \\ \overline{G} \end{bmatrix}\} \subseteq \text{col}-\text{span}\{\begin{bmatrix} NE \\ NA \end{bmatrix}\}T \tag{9.49}$$

□

Proposition 9.4.1 *If $2p-2\ell \leq n$ and $\mathcal{S}(E, A, B)$ is reachable, the matrix pencil realisation problem is always solvable.*

Proof: Since the system $\mathcal{S}(E, A, B)$ is reachable, the pencil $sNE - NA$ is characterised only by c.m.i. and has the following canonical form:

$$sNE - NA = \text{block}-\text{diag}\{\cdots \begin{bmatrix} s & -1 & 0 & \cdots & 0 \\ 0 & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & s & -1 \end{bmatrix} \cdots\} \tag{9.50}$$

where the dimensions of the blocks are $(\varepsilon_i - 1) \times \varepsilon_i$ and ε_i are the reachability indices of the triple (E, A, B). From the form of the above pencil we can easily see that the matrix $[E^T N^T, A^T, N^T]^T$ has always full rank. The dimensions of $[E^T N^T, A^T, N^T]^T$ are $(2p - 2\ell) \times n$. Then if $2p - 2\ell \leq n$ the equation

$$\begin{bmatrix} \overline{F} \\ \overline{G} \end{bmatrix} = \begin{bmatrix} NE \\ NA \end{bmatrix} T \tag{9.51}$$

is always solvable with respect to T and the result follows. □

Remark 9.4.2 *For reachable systems with $2p-2\ell \leq n$, any particular cover problem is equivalent to a matrix pencil augmentation problem as discussed in the previous section; otherwise, the Matrix Pencil Realisation Problem becomes an essential part of the overall cover problem.* □

9.5 Left regular solutions and the overall cover problem

In this section some special cases of the cover problem are investigated and some sufficient conditions for the solvability of the general case of the cover problem are given. The left regular cover problem is defined as that where the resulting augmented pencil has no left null space. For such cases a parametrisation of the solution spaces is also given. Note that a special case of the left regular case is when the resulting pencil is square and regular. This is defined as the *regular case*.

First we tackle the cover problem corresponding to the case where the subspaces are (A, E, B)-invariant and the restriction pencil has no r.m.i. at all. Some preliminary results are given below:

Proposition 9.5.1 *If the restriction pencil $sNEJ - NAJ$ of the given subspace $\mathcal{J}$ has no zero r.m.i., then the restriction pencil of any solution of the cover problem is not characterised by r.m.i. at all.*

Proof: From proposition 9.3.3 it follows that, since the number of the z.r.m.i. if $sNEJ - NAJ$ is zero, then any augmentation of that pencil is not characterised by z.r.m.i.. □

Proposition 9.5.2 *Let $\mathcal{L} \subset \Re^n$, $dim\{\mathcal{L}\} = p - \ell$, L be a basis matrix of $\mathcal{L}$. If the restriction pencil $R_{\mathcal{V}}(s)$ has full rank (over $\Re(s)$) and has no i.e.d., then:*

(i) $\mathcal{L} + \mathcal{J}$ is a solution of the partial cover problem

(ii) Any subspace defined as

$$\mathcal{L}' = \mathcal{L} + \hat{\mathcal{L}} + \mathcal{J} \tag{9.52}$$

where $\hat{\mathcal{L}}$ is arbitrary is also a solution of the partial cover problem.

Proof: Let $L \in \Re^{n \times (p-\ell)}$, such that $sNEL - NAL$ is regular and has no i.e.d.; clearly the restriction pencil $[sNEL - NAL, sNEJ - NAJ]$ has no r.m.i. and thus $\mathcal{L}$ is a solution of the partial cover problem which proves (i).

For any $\hat{\mathcal{L}} \in \Re^{n \times \kappa}$ matrix the augmented pencil

$$(sNE - NA)[L, \hat{L}, J] = [sNEL - NAL, sNE\hat{L} - NA\hat{L}, sNEJ - NAJ] \tag{9.53}$$

has an $(p - \ell) \times (p - \ell)$ subpencil, which is regular and thus, the pencil $(sNE - NA)[L, \hat{L}, J]$ has no r.m.i.. Given that $(sNE - NA)L$ is regular and has no i.e.d., we have that NEL has full rank and thus also $NE[L, \hat{L}, J]$; the latter shows that $(sNE - NA)[L, \hat{L}, J]$ has also no i.e.d.. The space $\mathcal{L}' = \mathcal{L} + \hat{\mathcal{L}} + \mathcal{J}$ is thus a solution to the partial cover problem. □

The specific solution defined by the space $\mathcal{L}$ for which the pencil $sNEL - NAL$ is regular and has no i.e.d. will be referred to as *squaring* solution and conditions for its existence will be examined next.

Remark 9.5.1 *The family $\mathcal{L}' = \mathcal{L} + \hat{\mathcal{L}} + J$ where $\mathcal{L}$ is a squaring solution does not necessarily cover the whole set of solutions of the partial cover problem; even for the squaring partial cover problem, different $\mathcal{L}$ squaring solutions, in general lead to different families. The squaring partial cover problem mentioned above may be formally stated as follows: Given the pencil $sNE - NA$, find L such that*

$$\det\{(sNE - NA)L\} \neq 0, \; \det(NL) \neq 0 \tag{9.54}$$

The above conditions combined yield that the squaring problem is solvable if and only if L is such that

$$\deg \det\{(sNE - NA)L\} = n - \ell \tag{9.55}$$

or equivalently

$$\det(NEL) \neq 0 \tag{9.56}$$

□

When the matrix E is singular or nonsquare the solvability of the (A, E, B)–invariant subspace partial cover problem is not always guaranteed. This is shown next.

Proposition 9.5.3 *Let the restriction pencil $sNE - NA$ be in the Kronecker canonical form and J partitioned according to the block structure of the restriction pencil i.e.*

$$sNE - NA = \left[\begin{array}{c|c|c|c} 0 & 0 & 0 & 0 \\ \hline 0 & L_\epsilon(s) & 0 & 0 \\ \hline 0 & 0 & D_\infty(s) & 0 \\ \hline 0 & 0 & 0 & D_f(s) \end{array}\right], \quad J = \left[\begin{array}{c} J_1 \\ \hline J_\epsilon \\ \hline J_\infty \\ \hline J_f \end{array}\right] \tag{9.57}$$

then the (A, E, B)–invariant subspace cover problem may have a solution only if $J_\infty = 0$.

Proof: Let $T = [T_1^T, T_\epsilon^T, T_\infty^T, T_f^T]^T$ be a solution of the cover problem. The restriction pencil of the covering space $\mathcal{V} = \mathcal{J} + \mathcal{T}$ has the form

$$R_{\mathcal{V}}(s) = \left[\begin{array}{c|c} 0 & 0 \\ \hline L_\epsilon(s)J_\epsilon & L_\epsilon(s)T_\epsilon \\ \hline D_\infty(s)J_\infty & D_\infty(s)T_\infty \\ \hline D_f(s)J_f & D_f(s)T_f \end{array}\right] \tag{9.58}$$

Consider now the subpencil of the above

$$D_\infty(s)[J_\infty, T_\infty] = s\Lambda[J_\infty, T_\infty] - [J_\infty, T_\infty] \tag{9.59}$$

where Λ is a block diagonal matrix with Jordan canonical blocks corresponding to zero eigenvalues, in the diagonal. If $J_\infty \neq 0$ then the pencil $s\Lambda[J_\infty, T_\infty] - [J_\infty, T_\infty]$ will always have i.e.d. whatever the T_∞ and the result follows. □

In the case where $J_\infty = 0$ the (A, E, B)–invariant subspace problem is equivalent to an appropriately defined cover problem for a state–space system, as it is shown by the following result.

Proposition 9.5.4 *Consider the system described by the triple (E, A, B). If the partial cover (A, E, B)–invariant subspace problem is solvable, then the problem may be reduced to an (A, B)–invariant subspace partial cover problem.*

Proof: From the previous proposition we have that if the restriction pencil $sNE - NA$ has i.e.d. then J_∞ must be zero matrix. On the other hand, solvability of the cover problem implies that $sNE - NA$ has no nonzero r.m.i. because otherwise the restriction pencil has nonzero r.m.i. Thus, the state–input restriction pencil may have only f.e.d., i.e.d., c.m.i. and zero r.m.i. Thus,

$$sNE - NA = \left[\begin{array}{c|c|c|c} 0 & 0 & 0 & 0 \\ \hline 0 & L_\epsilon(s) & 0 & 0 \\ \hline 0 & 0 & D_\infty(s) & 0 \\ \hline 0 & 0 & 0 & D_f(s) \end{array}\right] \left[\begin{array}{c|c} J_1 & T_1 \\ \hline J_\epsilon & T_\epsilon \\ \hline 0 & 0 \\ \hline J_f & T_f \end{array}\right] =$$

$$\left[\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline 0 & L_\epsilon(s) & 0 \\ \hline 0 & 0 & D_f(s) \end{array}\right] \left[\begin{array}{c|c} 0 & 0 \\ \hline J_\epsilon & T_\epsilon \\ \hline J_f & T_f \end{array}\right] \tag{9.60}$$

From the above it is clear that the original cover problem has a solution if and only if the cover problem defined by the system with restriction pencil

$$R'(s) = \left[\begin{array}{c|c} L_\epsilon(s) & 0 \\ \hline 0 & D_f(s) \end{array}\right] \tag{9.61}$$

The subspace to be covered has the following basis matrix

$$\left[\begin{array}{c} J_\epsilon \\ \hline J_f \end{array}\right] \tag{9.62}$$

The restriction pencil (9.62) corresponds to a regular state–space system with controllability indices equal to the column minimal indices of $R'(s)$ plus 1 and input decoupling zero structure identical to the zero structure of $D_f(s)$ and the result follows. □

The result provided by the above proposition allows us to consider state–space systems instead of descriptor systems for the discussion of the (A, E, B)–invariant subspace cover. Thus, for the rest of this section we are going to consider the (A, B)–invariant subspace cover problem i.e. the case where $E = I$. Note that in this case $p = n$.

Lemma 9.5.1 *The matrix NL has full rank if and only if*

$$\mathcal{L} \cap \mathcal{B} = \{0\} \tag{9.63}$$

□

Proposition 9.5.5 *Necessary condition for* (9.56) *to be true is that*

$$\dim\{\mathcal{L}\} \leq n - \ell \tag{9.64}$$

□

Theorem 9.5.1 *The squaring partial cover problem for state–space systems is always solvable.*

Proof: We can always find L such that (9.63) with $E = I$ is satisfied. □

The solution of the squaring cover problem is considered next. Condition (9.63) is equivalent to

$$\det[B, L] \neq 0 \tag{9.65}$$

where L is the basis–matrix of $\mathcal{L}$. The above is equivalent to

$$\det\{Q[B, L]\} \neq 0 \tag{9.66}$$

where Q is any invertible matrix. Since $\operatorname{rank}(B) = \ell$ we can always choose Q such that

$$QB = \begin{bmatrix} B_1^* \\ 0 \end{bmatrix} = B^* \tag{9.67}$$

where B^* is an $\ell \times \ell$ invertible matrix. Then (9.65) is equivalent to

$$\det(\begin{bmatrix} B_1^* & L_1^* \\ 0 & L_2^* \end{bmatrix}) \neq 0 \tag{9.68}$$

where

$$\begin{bmatrix} L_1^* \\ L_2^* \end{bmatrix} = QL = L^* \tag{9.69}$$

Relation (9.68) is equivalent to

$$\det(B_1^*) \cdot \det(L_2^*) \tag{9.70}$$

$$\det(L_2^*) \neq 0 \tag{9.71}$$

since B^* is invertible. Note that L^* is an arbitrary $\ell \times (n - \ell)$, L_2^* is an $(n - \ell) \times (n - \ell)$ matrix. Note that $n = p$, since $E = I$. Let now, W be the basis matrix of $\mathcal{W}$ and $w = \dim(\mathcal{W})$. Then, since $\mathcal{L} \subset \mathcal{W}$

$$\text{rank}[W, L] = \text{rank}[W] \tag{9.72}$$
$$\text{rank}[QW, QL] = \text{rank}[QW] \tag{9.73}$$
$$\text{rank}[W^*, L^*] = \text{rank}[W^*] \tag{9.74}$$

where

$$W^* = QW, \ L^* = QL \tag{9.75}$$

From (9.75)

$$\text{rank}(\begin{bmatrix} W_1^* & L_1^* \\ W_2^* & L_2^* \end{bmatrix}) = \text{rank}(W^*) \tag{9.76}$$

The above is equivalent to the existence of a matrix K of dimensions $w \times (n - \ell)$ such that

$$W^*K = L^* \tag{9.77}$$

or

$$\begin{bmatrix} W_1^* \\ W_2^* \end{bmatrix} K = \begin{bmatrix} L_1^* \\ L_2^* \end{bmatrix} \tag{9.78}$$

or

$$W_1^*K = L_1^*, \ W_2^*K = L_2^* \tag{9.79}$$

where L_2^* must be invertible. This analysis leads to:

Proposition 9.5.6 *Necessary and sufficient condition for the invertibility of L_2^* is that*

$$\text{rank}\{W_2^*\} = n - \ell \tag{9.80}$$

Proof: The necessity is obvious. For the sufficiency, if we assume that (9.80) holds true, we can choose

$$K = (W_2^*)^T \tag{9.81}$$

and the result follows. □

The matrices K that satisfy the requirement of the invertibility of L_2^* can be found as follows. From (9.79) we have that K must be such that the intersection of its columns space with the null space of W_2^* must be the zero space or, in matrix form

$$\det[\hat{W}, K] \neq 0 \tag{9.82}$$

where $\hat{W}$ is the basis matrix of the null space of W_2^* and has dimensions $w \times (w - n + \ell)$. From (9.80) we have that $\text{rank}(\hat{W}) = w - n + \ell$. Then there exists a nonsingular matrix P such that

$$P\hat{W} = \begin{bmatrix} \hat{W}_1^* \\ 0 \end{bmatrix} = \hat{W}^* \tag{9.83}$$

where $\hat{W}^*$ is an $(w-n+\ell)\times(w-n+\ell)$ invertible matrix. Now, (9.82) is equivalent to

$$\det(\begin{bmatrix} \hat{W}_1^* & K_1^* \\ 0 & K_2^* \end{bmatrix}) \neq 0 \tag{9.84}$$

or

$$\det(\hat{W}_1^*)\det(K_2^*) \neq 0 \tag{9.85}$$

where

$$\begin{bmatrix} K_1^* \\ K_2^* \end{bmatrix} = PK \tag{9.86}$$

Provided that (9.80) holds true, we can always find K such that L_2^* is invertible, by choosing K_2^* to be invertible. The expression for the matrix L that satisfies (9.56) and (9.72) simultaneously is

$$L = WP^{-1}\begin{bmatrix} K_1^* \\ K_2^* \end{bmatrix} \tag{9.87}$$

Next we are going to investigate (9.80) further and obtain an equivalent condition in terms of the matrices B and W. Consider the matrix

$$[B, W] \tag{9.88}$$

Then

$$Q[B,W] = \begin{bmatrix} B_1^* & W_1^* \\ 0 & W_2^* \end{bmatrix} \in R^{n\times(\ell+w)} \tag{9.89}$$

and B_1^* is invertible. Obviously, rank $[B_1^*, W_1^*] = \ell$ and all the nonzero rows of W_2^* are linearly independent of the rows of $[B_1^*, W_1^*]$. Thus,

$$\text{rank}\begin{bmatrix} B_1^* & W_1^* \\ 0 & W_2^* \end{bmatrix} = \text{rank}[B_1^*, W_1^*] + \text{rank}[0, W_2^*] \tag{9.90}$$

and since B_1^* is invertible

$$\text{rank}\begin{bmatrix} B_1^* & W_1^* \\ 0 & W_2^* \end{bmatrix} = \text{rank}[B_1^*] + \text{rank}[W_2^*] \tag{9.91}$$

We may now state the following theorem.

Theorem 9.5.2 *Necessary and sufficient condition for the solvability of the squaring cover problem is the following*

$$\dim\{\mathcal{B}\} \cap \{\mathcal{W}\} = \ell + w - n \tag{9.92}$$

and the general solution is (9.87) *where K_1^* is completely arbitrary and K_2^* is an arbitrary nonsingular matrix.*

Proof: From (9.91) we get that (9.80) holds true if and only if $\text{rank}\{[B, W]\} = n$ or equivalently if and only if (9.92) holds true. □

Theorem 9.5.3 *The left regular cover problem is solvable if and only if the subspace $\mathcal{W}$ is an (A, B)–invariant subspace and the $\mathcal{W}$–restricted pencil is not characterised by z.r.m.i.. If the problem is solvable, then the solutions have the following form*

$$\mathcal{T} = \mathcal{L} + \mathcal{J} + \hat{\mathcal{L}} \tag{9.93}$$

where $\mathcal{L}$ has a basis matrix given in (9.87) *and $\hat{\mathcal{L}}$ is an arbitrary subspace of $\mathcal{W}$.*

Proof: Let the left regular cover problem be solvable. Then from proposition 9.5.2 we have that the squaring problem is solvable. Let $\mathcal{L}$ be a solution of the squaring problem. Then there exists a subspace $\hat{\mathcal{L}} \subseteq \mathcal{W}$ such that $\mathcal{W} = \mathcal{L} \oplus \hat{\mathcal{L}}$. Since the $\mathcal{L}$-restricted pencil is characterised by i.e.d. and r.m.i., it follows that the $\mathcal{W}$-restricted pencil does not have i.e.d. and r.m.i. and therefore $\mathcal{W}$ is an (A, B)-invariant subspace not characterised by r.m.i..

Conversely let $\mathcal{W}$ be a subspace such that the $\mathcal{W}$-restricted pencil has neither i.e.d. nor r.m.i.. Then $\mathcal{W}$ is a solution to the problem and the result follows. □

9.6 The general cover problem

In this section we consider the general case i.e. the case where the restriction pencil of the covering space $\mathcal{V} = \mathcal{J} \bigoplus \mathcal{T}$ may have zero r.m.i. In this case we may find, in general, solutions of lower dimensions than that of the left regular case.

From lemma 9.3.1 we have that the general pencil $[sF - G, s\bar{F} - \bar{G}]$ has no i.e.d. and n.z.r.m.i. if and only if

$$\mathcal{N}_\ell\{[F, \bar{F}]\} \subseteq \mathcal{N}_\ell\{[G, \bar{G}]\} \tag{9.94}$$

The above applied to the augmented restriction pencil $[sNE - NA][J, T]$ gives

$$\mathcal{N}_\ell\{[NEJ, NET]\} \subseteq \mathcal{N}_\ell\{[NAJ, NAT]\} \tag{9.95}$$

Proposition 9.6.1 *Let* (9.95) *hold true. Then*

$$\mathcal{N}_\ell\{[NEJ, NET]\} \subseteq \mathcal{N}_\ell\{[NAJ, NEJ]\} \tag{9.96}$$

Proof: Let $y^T \in \Re^{p-\ell}$ be such that $y^T[NEJ, NET] = 0$. From (9.95) it follows that $y^T[NAT] = 0$ and thus $y^T \in \mathcal{N}_\ell\{[NEJ, NAJ]\}$ and the result follows. □

Let $\text{rank}\{[NEJ, NET]\} = \rho$. Then (9.96) yields

$$K\Psi[NEJ, NET] = 0 \qquad (9.97)$$

where $K \in \Re^{k\times\psi}$, is a full row rank matrix and $\psi = \text{rank}\{\Psi\}$ where Ψ is a basis matrix of $\mathcal{N}_\ell\{[NEJ, NAJ]\}$. Then, if $K\Psi$ is a basis matrix for $\mathcal{N}_\ell\{[NEJ, NET]\}$ it is clear that $k = p - \ell - \rho$. Let T be a solution to the partial cover problem. Then, from proposition 9.6.1 and (9.96),(9.97) it follows that T must satisfy the following equations

$$K\Psi NET = 0, \; K\Psi NAT = 0 \qquad (9.98)$$

Since $K\Psi$ is a basis matrix of $\mathcal{N}_\ell\{[NEJ, NET]\}$ the matrix $[NEJ, NET]$ must have full column rank (as long as $p - \ell \leq \rho$) or equivalently

$$C_\rho[NEJ, NET] \neq 0 \qquad (9.99)$$

where $C_\rho(\cdot)$ denotes the ρ–th compound matrix [Mark. & Mink, 1964]. In order to find T, we have to solve (9.98) under the constraint (9.99). Note that the solution (if it exists) is generally parametric since the number of the unknowns in the equation (9.98) is greater to the number of equations. These parametric solutions yield the parametrisation of the solutions of the cover problem. The following definitions are necessary for the parametrisation of the solutions and they indicate the families of solutions into which we are going to partition the set of all the solutions.

Definition 9.6.1 *The set of solutions $\mathcal{T}$ with $\dim\{\mathcal{T}\} = \tau$ and $\text{rank}\{[NEJ, NET]\} = \rho$ will be referred to as $\mathcal{S}(\tau, \rho)$.* □

Definition 9.6.2 *The set of solutions $\mathcal{T}$ of the cover problem with $\dim\{\mathcal{T}\} = \tau$ will be referred to as $\mathcal{S}(\tau)$.* □

Definition 9.6.3 *The set of solutions $\mathcal{T}$ of the cover problem for given spaces $\mathcal{J}$ and $\mathcal{W}$ will be referred to as $\mathcal{S}(\mathcal{J}, \mathcal{W})$. In the case of the partial cover problem, the family of the solutions will be referred to as $\mathcal{S}(\mathcal{J})$.* □

In order to find all $\mathcal{S} \in \mathcal{S}(\tau, \rho)$ we have to solve (9.98) and (9.99) for all possible K. To this end, we parametrize K as follows: Since K has full row rank, then $C_k(K) \neq 0$. Thus, $\mathcal{S}(\tau, \rho)$ is obtained by solving (9.98)–(9.99) for each one of the following cases: $C_k(i) \neq 0, i = 1, \ldots, \begin{pmatrix} p-\ell \\ k \end{pmatrix}$ where $C_k(i)$ is the ith entry of the vector $C_k(K)$. For each i we take a family $\mathcal{S}(\tau, \rho)$ of solutions of the cover problem. Clearly $\mathcal{S}(\tau) = \bigcup_{i=1}^{\begin{pmatrix} p-\ell \\ k \end{pmatrix}} \mathcal{S}(\tau, \rho)$. The above procedure yields all the solutions $\mathcal{V} = \mathcal{S} \oplus \mathcal{T}$ where rank$\{NE[J,T]\} = \rho$. The next step is to find the solutions $\mathcal{S}(\tau, \rho+1)$. In order to find all the subspaces $\mathcal{T}$ of dimension τ, which constitute solutions to the cover problem, we repeat the above by increasing ρ up to ρ_{max} =min$\{p-\ell, j+\tau\}$. The next step is to increase τ and repeat the above until $\tau = n-j$ since then, $\mathcal{S} \oplus \mathcal{T} = \mathcal{X}$.

Summarizing we give the algorithm of searching for solutions of the cover problem in a pseudocode format

for $\tau := \varphi$ to $n-j$ do
  begin
  for $\rho := 1$ to min $\{p-\ell, j+\rho\}$ do
    begin
    for $i := 1$ to $\begin{pmatrix} p-\ell \\ k \end{pmatrix}$
      begin
      solve $K\Psi NET = 0$, $K\Psi NAT = 0$ with K such that $C_k(i) \neq 0$
      i:=i+1
      end
      $\rho = \rho + 1$
    end
  $\tau := \tau + 1$
  end

The initial value of τ above is φ, the total number of the n.z.r.m.i. and i.e.d. of $R_{\mathcal{J}}(s)$, since this is the lower bound for dim$\{\mathcal{T}\}$ (see proposition 9.3.4). If we consider the restriction pencil $R_{\mathcal{T}}(s) = sNET - NAT$ in Kronecker canonical form, we may obtain the solution of the cover problem in a systematic way. The issue is discussed below.

From proposition 9.5.4 we have that the cover problem may be always reduced to an appropriately defined cover problem where the restriction pencil $sNE - NA$ does not have i.e.d. and thus $sNE - NA$ has the form

$$sNE - NA = \left[\begin{array}{c|c|c} & & \\ \hline & L_\epsilon(s) & \\ \hline & & D_f(s) \end{array}\right] \tag{9.100}$$

Let J and T be partioned according to the partitioning of $sNE - NA$ i.e.

$$J = \left[\begin{array}{c} J_0 \\ \hline J_\epsilon \\ \hline J_f \end{array}\right], \; T = \left[\begin{array}{c} T_0 \\ \hline T_\epsilon \\ \hline T_f \end{array}\right] \tag{9.101}$$

where

$$J_\epsilon = \begin{bmatrix} J_\epsilon^1 \\ \vdots \\ J_\epsilon^\ell \end{bmatrix}, \; T_\epsilon = \begin{bmatrix} T_\epsilon^1 \\ \vdots \\ T_\epsilon^\ell \end{bmatrix}, \; J_f = \begin{bmatrix} J_f^1 \\ \vdots \\ J_f^q \end{bmatrix}, \; T_f = \begin{bmatrix} T_f^1 \\ \vdots \\ T_f^q \end{bmatrix} \tag{9.102}$$

then

$$NE[J,T] = \left[\begin{array}{c|c} \hat{J}_\epsilon^1 & \hat{T}_\epsilon^1 \\ \vdots & \vdots \\ \hat{J}_\epsilon^\ell & \hat{T}_\epsilon^\ell \\ \hline J_f & T_f \end{array}\right] \left[\begin{array}{c|c} \hat{J}_\epsilon & \hat{T}_\epsilon \\ \hline J_f & T_f \end{array}\right], \; NA[J,T] = \left[\begin{array}{c|c} \tilde{J}_\epsilon^1 & \tilde{T}_\epsilon^1 \\ \vdots & \vdots \\ \tilde{J}_\epsilon^\ell & \tilde{T}_\epsilon^\ell \\ \hline \tilde{J}_f & \tilde{T}_f \end{array}\right] = \left[\begin{array}{c|c} \tilde{J}_\epsilon & \tilde{T}_\epsilon \\ \hline \tilde{J}_f & \tilde{T}_f \end{array}\right] \tag{9.103}$$

where

$$[\hat{J}_\epsilon^i, \hat{T}_\epsilon^i] = \begin{bmatrix} j_1^i & t_1^i \\ \vdots & \vdots \\ j_{\sigma_i-1}^i & t_{\sigma_i-1}^i \end{bmatrix}, \; [\tilde{J}_\epsilon^i, \tilde{T}_\epsilon^i] = \begin{bmatrix} j_2^i & t_2^i \\ j_3^i & t_3^i \\ \vdots & \vdots \\ j_{\sigma_i}^i & t_{\sigma_i}^i \end{bmatrix}, \; [\tilde{J}_f, \tilde{T}_f] = D_f^2[J_f, T_f] \tag{9.104}$$

$$D_f(s) = sI - D_f^2 \tag{9.105}$$

We distinguish two cases

(i) $\mathcal{N}_\ell\{NEJ\} \cap \mathcal{N}_\ell\{NAJ\} = \{0\}$. This is the case where the restriction pencil $sNEJ - NAJ$ does not have z.r.m.i. This case was considered in section 9.5.

(ii) $\mathcal{N}_\ell\{NEJ\} \cap \mathcal{N}_\ell\{NAJ\} = \{\Psi\} \neq \{0\}$. This is the case where the restriction pencil of the covering space $\mathcal{J} \oplus \mathcal{T}$ may have zero r.m.i.. This case will be discussed in this section.

From (9.103) and (9.104) it is clear that the equation $K\Psi NET = 0$ involves $(p-\ell)\tau$ unknowns and τk equations. Since $p - \ell > k$ it follows that the above homogeneous equation is always solvable. The solution T has $\tau[(p-\ell) - k]$ free parameters (the free unknowns). Without loss of generality we may choose as free parameters the first $\tau(p - \ell - k)$ entries of the matrix $[\hat{T}_\epsilon^T, T_f^T]^T$ i.e. the entries of the first $(p - \ell - k)$ rows of $[\hat{T}_\epsilon^T, T_f^T]^T$. Next we consider the constraint (9.99) on the entries of T. We have

$$C_\rho\{[NEJ, NET]\}J \neq 0 \text{ or } C_\rho\{\left[\begin{array}{c|c} \hat{J}_\epsilon^1 & \hat{T}_\epsilon^1 \\ \vdots & \vdots \\ \hat{J}_\epsilon^\ell & \hat{T}_\epsilon^\ell \\ J_f & T_f \end{array}\right]\} \neq 0 \tag{9.106}$$

Consider the first $p - \ell$ rows of $NE(J,T)$ (see (9.103)). Since this matrix has τ arbitrary columns we may always choose $[\hat{T}_\epsilon^T, T_f^T]^T$ such that (9.106) is satisfied. The next step is to solve the second of (9.98) such that the first of (9.98) is satisfied. If the solution does not contradict (9.106) we say that this solution gives the general form of the basis matrices of the family of subspaces $\mathcal{S}_i(\tau, \rho)$. The following examples illustrate the method described above.

Example 9.6.1 Consider the cover problem with c.m.i. of $sNE - NA$ equal to $\varepsilon_1 = 4$, $\varepsilon_2 = 2$ and $J = [1, -1, -1, 1, 1, 2]^T$. The pencil $sNJ - NAJ$ has two z.r.m.i. and one nonzero r.m.i. $\zeta = 1$. The pencil $sNE - NA$ is assumed to be in the Kronecker canonical form.

First we find the matrix Ψ.

$$\Psi = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 3 & 0 & 2 \end{bmatrix}$$

Since $R_{\mathcal{J}}(s)$ has one n.z.r.m.i. and no i.e.d., a lower bound of $\tau = \dim\{\mathcal{T}\}$ is 1. Thus, we start from $\tau = 1$ and $\rho = \text{rank}\{[NJ, NT]\} = 2$. We solve the equation

$$\Psi[NJ, NT] = 0$$

with respect to T. Since $\Psi NJ = 0$, the above is reduced to $\Psi NT = 0$ or

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 3 & 0 & 2 \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_5 \end{bmatrix} = 0$$

This is a system of two equations and four unknowns. Choosing as free unknowns the entries t_1 and t_2 we take the solution

$$t_3 = -t_1, \quad t_5 = \frac{-t_1 - 3t_2}{2}$$

The matrix $NE[J, T]$ is the following

$$NE[J, T] = \begin{bmatrix} 1 & t_1 \\ -1 & t_2 \\ -1 & t_3 \\ 1 & t_5 \end{bmatrix}$$

It is clear that the above matrix has rank 2 if $t1 \neq -t_2$. Continuing, we consider the equation $NA[J, T] = 0$ where the values of t_3 and t_5 are determined from the solution of $NE[J, T] = 0$, i.e. we have the equation

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 3 & 0 & 2 \end{bmatrix} \begin{bmatrix} t_2 \\ -t_1 \\ t_4 \\ t_6 \end{bmatrix} = 0$$

which has the solution

$$t_4 = -t_2, \quad t_6 = \frac{3t_1 - t_2}{2}$$

The solution of the above does not contradict the condition $t_1 \neq -t_2$ and thus, the basis matrix for the covering subspace $\mathcal{V}$ is

$$V = \begin{bmatrix} 1 & t_1 \\ -1 & t_2 \\ -1 & -t_1 \\ 1 & -t_2 \\ 1 & \frac{-t_1-3t_2}{2} \\ 2 & \frac{3t_1-t_2}{2} \end{bmatrix}$$

The above is a minimal solution. The uniqueness of this solution is examined next.

Let V' be the basis matrix of another minimal solution then

$$V' = [J, T'] = \begin{bmatrix} 1 & t_1' \\ -1 & t_2' \\ -1 & -t_1' \\ 1 & -t_2' \\ 1 & \frac{-t_1'-3t_2'}{2} \\ 2 & \frac{3t_1'-t_2'}{2} \end{bmatrix}$$

Consider the equation $VR = T'$. This equation has the solution

$$\begin{bmatrix} 1 & t_1 \\ -1 & t_2 \end{bmatrix}^{-1} \begin{bmatrix} t_1' \\ t_2' \end{bmatrix}$$

thus, the solution is unique.

We proceed now to the solution of dimension 3. The matrix T has the form

$$\begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \\ t_{31} & t_{32} \\ t_{41} & t_{42} \\ t_{51} & t_{52} \\ t_{61} & t_{62} \end{bmatrix}$$

First, we seek for solutions with $\rho = 2$. Applying the algorithm, we get the following solution

$$V' = [J, T'] = [J, T_1, T_2] = \begin{bmatrix} 1 & t_{11} & t_{12} \\ -1 & t_{21} & t_{22} \\ -1 & -t_{11} & -t_{12} \\ 1 & -t_{21} & -t_{22} \\ 1 & \frac{-t_{11}-3t_{21}}{2} & \frac{-t_{12}-3t_{22}}{2} \\ 2 & \frac{3t_{11}-t_{21}}{2} & \frac{3t_{12}-t_{22}}{2} \end{bmatrix}$$

As we have seen before, $\mathcal{V} = \mathcal{J} \bigoplus \mathcal{T} = \mathcal{V} = \mathcal{J} \bigoplus \mathcal{T}_1$ and thus this is the case of $\tau = 1$.

The next step is to take $\rho = 3$. Then the basis matrix of $\mathcal{N}_\ell\{[NEJ, NET]\}$ has the form

$$K\Psi = \begin{bmatrix} k_1, k_2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 3 & 0 & 2 \end{bmatrix}$$

Then $K\Psi NET = 0$ or

$$\begin{bmatrix} k_1 + k + 2, 3k_2, k_1, 2k_2 \end{bmatrix} \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \\ t_{31} & t_{32} \\ t_{51} & t_{52} \end{bmatrix} = 0$$

We assume that $C_1(k) = [k_1, k_2] \neq 0$ thus we have to examine the cases $k_1 \neq 0$ and $k_2 \neq 0$. When $k_2 \neq 0$ we have the equations

$$\begin{aligned} (k_1 + k_2)t_{11} + 3k_2t_{21} + k_1t_{31} + 2k_2t_{41} &= 0 \\ (k_1 + k_2)t_{12} + 3k_2t_{22} + k_1t_{32} + 2k_2t_{42} &= 0 \end{aligned}$$

since $k_2 \neq 0$, the solution is

$$\begin{aligned} t_{51} &= \frac{-(k_1 + k_2)t_{11} - 3k_2t_{21} - k_1t_{31}}{2k_2} \\ t_{52} &= \frac{-(k_1 + k_2)t_{12} - 3k_2t_{22} - k_1t_{32}}{2k_2} \end{aligned}$$

and $t_{11}, t_{12}, t_{21}, t_{22}, t_{31}, t_{32}$ are free parameters. Next we solve the equation $K\Psi NAT = 0$ and we get

$$\begin{aligned} t_{61} &= \frac{-(k_1 + k_2)t_{21} - 3k_2t_{31} - k_1t_{41}}{2k_2} \\ t_{62} &= \frac{-(k_1 + k_2)t_{22} - 3k_2t_{32} - k_1t_{42}}{2k_2} \end{aligned}$$

and the general form for the basis matrix of the covering space is

$$\begin{bmatrix} 1 & t_{11} & t_{12} \\ -1 & t_{21} & t_{22} \\ -1 & t_{31} & t_{22} \\ 1 & t_{41} & t_{42} \\ 1 & \frac{-(k_1+k_2)t_{11}-3k_2t_{21}-k_1t_{31}}{2k_2} & \frac{-(k_1+k_2)t_{12}-3k_2t_{22}-k_1t_{32}}{2k_2} \\ 2 & \frac{-(k_1+k_2)t_{21}-3k_2t_{31}-k_1t_{41}}{2k_2} & \frac{-(k_1+k_2)t_{22}-3k_2t_{32}-k_1t_{42}}{2k_2} \end{bmatrix}$$

The condition for $\rho = 3$ is

$$C_3\left\{\begin{bmatrix} 1 & t_{11} & t_{12} \\ -1 & t_{21} & t_{22} \\ -1 & t_{31} & t_{22} \\ 1 & \frac{-(k_1+k_2)t_{11}-3k_2t_{21}-k_1t_{31}}{2k_2} & \frac{-(k_1+k_2)t_{12}-3k_2t_{22}-k_1t_{32}}{2k_2} \end{bmatrix}\right\} \neq 0$$

may be always satisfied by appropriate selection of the free parameters. In order to find all the solutions of all dimensions we may proceed according to the algorithm presented above.

Example 9.6.2 In this example we consider the case where the given system is not reachable. The canonical form of the state–input restriction pencil is

$$sNE - NA = \left[\begin{array}{ccc|cc} s & -1 & 0 & & \\ 0 & s & -1 & & \\ \hline & & & s-2 & -1 \\ & & & 0 & s-2 \end{array}\right]$$

and the basis matrix of $\mathcal{J}$ is $J = [1, 1, -1, 1, 2]^T$ (expressed in the same coordinate system as the canonical form of the input–state restriction pencil). The restriction pencil $R_{\mathcal{J}}(s)$ has two z.r.m.i. and one n.z.r.m.i. We are going to solve the cover problem only for the case $\tau = 1$, $\rho = 2$ and therefore, $T = [t_1, t_2, t_3, t_4, t_5]^T$. We have

$$\Psi = \begin{bmatrix} -6 & 4 & 0 & 1 \\ -5 & 4 & 1 & 0 \end{bmatrix}$$

Then the equation $\Psi NET = 0$ yields

$$T = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ 5t_1 - 4t_2 \\ 6t_1 - 4t_2 \end{bmatrix}$$

The matrix $NE[J,T]$ has the form

$$NE[J,T] = \begin{bmatrix} 1 & t_1 \\ 1 & t_2 \\ 1 & 5t_1 - 4t_2 \\ 2 & 6t_1 - 4t_2 \end{bmatrix}$$

If we choose t_1, t_2 appropriately we may have $\rho = 2$. The equation $\Psi NAT = 0$ together with $\Psi NET = 0$ gives

$$T = \begin{bmatrix} t_1 \\ \frac{4t_1}{3} \\ \frac{6t_1}{3} \\ \frac{-t_1}{3} \\ \frac{2t_1}{3} \end{bmatrix}$$

The covering space is unique and has a basis matrix of the form

$$V = [J,T] = \begin{bmatrix} 1 & t_1 \\ 1 & \frac{4t_1}{3} \\ -1 & \frac{6t_1}{3} \\ 1 & \frac{-t_1}{3} \\ 2 & \frac{2t_1}{3} \end{bmatrix}$$

9.7 The overall cover problem

The overall cover problem arises in the case where $\mathcal{W} \subset \mathcal{X}$. In this section it is shown that the solvability condition is the following [Wonh., 1979].

Proposition 9.7.1 *The overall cover problem is solvable if and only if $\mathcal{J} \subseteq \mathcal{V}^*$, where $\mathcal{V}^*$ is the maximal (A, E, B)-invariant subspace contained in $\mathcal{W}$.* □

Note that $\mathcal{V}^*$ is uniquely defined [Wonh., 1979].

The solution procedure is similar to the solution of the partial cover problem. Since $\mathcal{J} \subseteq \mathcal{V}^*$ it follows that

$$J = V^* \bar{J} \tag{9.107}$$

where V^* is a basis matrix of $\mathcal{V}^*$ and $\bar{J}$ is a $v^* \times j$ ($v^* = \dim\{\mathcal{V}^*\}$), expressing the linear dependence of the basis vectors of $\mathcal{J}$ with respect to the basis vectors of $\mathcal{V}^*$. Then, the restriction pencil of $\mathcal{J}$ may be written in the form

$$sNEJ - NAJ = (sNV^* - NAV^*)\bar{J} \tag{9.108}$$

The (A, E, B)–invariance condition according to lemma 9.3.2 is

$$\mathrm{Im}\{[NEV^*\bar{J}, NEV^*\bar{T}]\} \supseteq \mathrm{Im}\{[NAV^*\bar{J}, NAV^*\bar{T}]\} \tag{9.109}$$

where $T = V^*\bar{T}$.

The overall cover problem may be reduced to an appropriate partial cover problem as it is shown below.

Proposition 9.7.2 *Let V be a solution of the overall cover problem defined by the system $\mathcal{S}(E, A, B)$ and the subspaces $\mathcal{J}$ and $\mathcal{W}$. Then, there exists a state–space system $\mathcal{S}(A', B')$ and a subspace $\bar{J}$ such that $\mathcal{V}$ is a solution of the partial cover problem defined by $\mathcal{S}(A', B')$ and $\bar{J}$.*

Proof: The restriction pencil $R_{\mathcal{V}^*}(s) = sNEV^* - NAV^*$ may have only f.e.d., c.m.i. and possibly z.r.m.i. since $\mathcal{V}^*$ is (A, E, B)–invariant subspace. Thus, if we consider $R_{\mathcal{V}^*}(s)$ in Kronecker form we may find a regular state–space system $\mathcal{S}(A', B')$ with $R_{\mathcal{V}^*}$ the input–state restriction pencil as follows: If

$$R_{\mathcal{V}^*}(s) = \left[\begin{array}{c|c} L_\epsilon(s) & \\ \hline & D_f(s) \end{array}\right] \tag{9.110}$$

then the state–space system $\mathcal{S}(A', B')$ where,

$$A' = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix}, \quad B' = \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \tag{9.111}$$

with (A_{11}, B_1) controllable in Luenberger canonical form, with controllability indices equal to the c.m.i. of $R_{\mathcal{V}^*}$ plus one and $(sI - A_{22}) = D_f(s)$. The above system is uncontrollable with input decoupling zero structure identical to the f.e.d. structure of $D_f(s)$. If N' is a left annihilator of B_1 then

$$R_{\mathcal{V}^*}(s) = sN' - N'A'$$

and the result follows. □

We continue with some properties of the solutions of the cover problem.

Proposition 9.7.3 *If $\mathcal{T}$ is a solution to the cover problem (partial or overall) then*

$$\mathrm{rank}\{sNE[J,T] - NA[J,T]\} = \mathrm{rank}\{NE[J,T]\} \tag{9.112}$$

Proof: Since T is a solution to the cover problem, $sNE[J,T] - NA[J,T]$ has no i.e.d. and the result follows. □

Proposition 9.7.4 *Let $\mathcal{T}$ be a solution of the cover problem with $\tau = dim\{\mathcal{T}\}$. Then if*

$$\text{rank}\{sNE[J,T] - NA[J,T]\} - \text{rank}\{sNEJ - NAJ\} = \tau \tag{9.113}$$

the sets of the f.e.d. of the original pencil divides the set of the f.e.d. of the augmented pencil.

Proof: We have that

$$\text{rank}\{sNEJ - NAJ\} = n - \ell - \#\text{r.m.i. of } sNEJ - NAJ = \rho_1 \tag{9.114}$$

Then if (9.114) holds it follows that every column of $sNET - NAT$ is linearly independent from the previous columns. The result follows from proposition 9.3.1 □

The class of the (A, E, B)–invariant subspaces covering $\mathcal{J}$ may include reachability subspaces, i.e. (A, B)–invariant subspaces with restriction pencil characterised only by c.m.i. [Jaf. & Kar., 1981].

Proposition 9.7.5 *Let $\mathcal{T}$ be a solution to the standard cover problem. Then*

(i) if $\rho < j + \tau$, the subspace $\mathcal{V} = \mathcal{J} + \mathcal{T}$ is an (A, E, B)-invariant subspace which contains reachability subspaces.

(ii) if $\rho = j + \tau$, the subspace $\mathcal{V} = \mathcal{J} + \mathcal{T}$ is a coasting subspace where $\rho = rank\{[NEJ, NET]\}$.

Proof: Let $\rho = \text{rank}\{[NEJ, NET]\}$. Then, since $\mathcal{V}$ is (A, E, B)–invariant subspace, it follows that $\rho = \text{rank}\{sNE[J,T] - NA[J,T]\}$. If $\rho < j + \tau$, then the pencil $sNE[J,T] - NA[J,T]$ has nonzero right null space and the result (i) follows.

If $\rho = j + \tau$ then the right null space is the zero space and the restriction pencil may have only f.e.d. and possibly z.r.m.i. which proves (ii). □

Since the controllability subspaces are (A, B)–invariant subspaces intersecting $\text{Im}\{B\}$ we have the following.

Corollary 9.7.1 *If $\rho < j + \tau$ then $\mathcal{V} \cap \text{Im}\{B\} \neq 0$.* □

9.8 The extended cover problems

In this section we consider the problem of covering a given subspace $\mathcal{J} \subset \mathcal{X}$ such that the covering space has finite as well as infinite spectrum. This type of spaces will be referred to as *infinite spectrum spaces.* The almost (A, B)–invariant subspace

[Wil., 1981] is a typical example of infinite spectrum space [Jaf. & Kar., 1981]. The treatment of the extended cover problem is similar to the standard problem. The approach is based on the matrix pencil characterisation of subspaces displayed in Table 1. As it is expected, the extended cover problem may be tackled by using arguments of duality (see [Gant., 1959]) between the finite and infinite elementary divisors of the restriction pencil. A important preliminary result which is important for the solution of the extended cover problems is the following:

Proposition 9.8.1 *The pencils $sF-G$ and $(s-c)F-G$ have the same sets of r.m.i., c.m.i., and i.e.d.. If $sF-G$ has a f.e.d. of the form $(s-\alpha)^q$ then the corresponding f.e.d. of $(s-c)F-G$ is $(s-(\alpha+c))^q$.*

Proof: The proof may be readily obtained from the Kronecker form of $sF-G$. □

We proceed now to the extended cover problem. Our aim is to find the general form of the basis matrix T of $\mathcal{T}$ such that the pencil $sNE[J,T]-NA[J,T]$ has f.e.d., i.e.d., c.m.i. and possibly n.z.r.m.i.. We consider first the partial case i.e. $\mathcal{W}=\mathcal{X}$. In order to reduce the extended problem to the standard problem we present the following proposition:

Proposition 9.8.2 *The matrix pencil $sF-G$ has no nonzero r.m.i. if and only if the pencil $F-\hat{s}(cF+G)$, $-c\notin\Phi(F,G)$ has only f.e.d., c.m.i. and possibly zero r.m.i., where $\Phi(F,G)$ denotes the spectrum of $sF-G$.*

Proof: From proposition 9.8.1 we have that the pencils $sF-G$ and $(s-c)F-G$ have the same sets of i.e.d., c.m.i., r.m.i. and if α is a f.e.d. of $sF-G$ then $\alpha+c$ is a f.e.d. of $(s-c)F-G$. Consider now the Kronecker form of $sF-G$. For the sake of simplicity we assume that it has one f.e.d. $(s-\alpha)^2$, one i.e.d. $\hat{s}^2$ and one c.m.i. $\varepsilon=2$ i.e.

$$sF-G=\left[\begin{array}{ccc|cc|cc} s & -1 & 0 & & & & \\ 0 & s & -1 & & & & \\ \hline & & & -1 & s & & \\ & & & 0 & -1 & & \\ \hline & & & & & s-\alpha & -1 \\ & & & & & 0 & s-\alpha \end{array}\right] \qquad (9.115)$$

then

$$(s-c)F-G=\left[\begin{array}{ccc|cc|cc} s-c & -1 & 0 & & & & \\ 0 & s-c & -1 & & & & \\ \hline & & & -1 & s-c & & \\ & & & 0 & -1 & & \\ \hline & & & & & s-(c+\alpha) & -1 \\ & & & & & 0 & s-(c+\alpha) \end{array}\right] \qquad (9.116)$$

The "dual" of the above is

$$F - \hat{s}(cF+G) = \left[\begin{array}{ccc|cc|cc} 1-\hat{s}c & -\hat{s} & 0 & & & & \\ 0 & 1-\hat{s}c & -\hat{s} & & & & \\ \hline & & & -\hat{s} & 1-\hat{s}c & & \\ & & & 0 & -\hat{s} & & \\ \hline & & & & & 1-\hat{s}(c+\alpha) & -\hat{s} \\ & & & & & 0 & 1-\hat{s}(c+\alpha) \end{array}\right] \tag{9.117}$$

Clearly, the above pencil has one c.m.i. $\varepsilon = 2$, one 0–e.d., $\hat{s}^2$, corresponding to the i.e.d. of $sF - G$ and one f.e.d. $(\hat{s} - (c+\alpha)^{-1})^2$ corresponding to the f.e.d. $(s-\alpha)^2$. Notice that the above pencil may not have i.e.d. since $-c \notin \Phi(F,G)$. Thus there is an 1–1 mapping between the invariants of the pencils $sF - G$ and $F - \hat{s}(cF+G)$. The result may be readily generalised to the case of a general pencil. □

Applying the above results to the matrix pencil formulated extended cover problem we readily take the following:

Proposition 9.8.3 *The restriction pencil $R_{\mathcal{V}}(s) = sNEV - NAV$ does not have nonzero r.m.i. and 0–e.d. if and only if the dual restriction pencil $\hat{R}_{\mathcal{V}}(\hat{s}) = NEV - \hat{s}NAV$ has no nonzero r.m.i. and i.e.d..* □

We may readily apply the above proposition to obtain a subset of the solutions of the extended cover problem as follows: Consider the "dual" state–input restriction pencil $\hat{R}(\hat{s}) = NE - \hat{s}NA$ and find the subspaces $\mathcal{V} = \mathcal{J} \bigoplus \mathcal{T}$ such that

$$\hat{R}_{\mathcal{V}}(\hat{s}) = NEV - \hat{s}NAV = -(\hat{s}NAV - NEV) \tag{9.118}$$

has no i.e.d. and n.z.r.m.i.. The problem is essentially the standard cover problem for the system $\mathcal{S}(A,E,B)$ and may be solved by following the method of section 9.6. The original restriction pencil $R_{\mathcal{V}}(s) = sNEV - NAV$ may have f.e.d., c.m.i., z.r.m.i. but no 0–e.d.. Thus, the solution of the cover problem defined by (9.118) yields all the solutions of the extended cover problem except those corresponding to subspaces $\mathcal{V}$ with spectrum including 0–e.d..

In order to find the solutions with spectrum including 0–e.d. we consider the following restriction pencil

$$\tilde{R}(\hat{s}) = NEV - \hat{s}(cNEV - NAV) = NEV - \hat{s}N(cE+A)V \tag{9.119}$$

where $-c \notin \Phi(R(s))$ and solve the standard cover problem for the system $\mathcal{S}(cE + A, E, B)$. The solutions of the cover problem defined by (9.119) provide all the solutions except those with spectrum including f.e.d. of the form $(s-c)^q$. Thus, in order to ensure that we have taken all the solutions it is necessary to solve two standard cover problems:The problem defined by the system $\mathcal{S}(-A,E,B)$ and the subspace $\mathcal{J}$ and the problem defined by $\mathcal{S}(cE-A,E,B)$ and $\mathcal{J}$.

As far as the overall extended cover problem is concerned, we have the analogous to the standard cover problem condition of solvability.

Proposition 9.8.4 *The overall extended cover problem is solvable if and only if $\mathcal{J} \subseteq \mathcal{V}^*_\infty$ where $\mathcal{V}^*_\infty$is the maximal finite–infinite spectrum subspace contained in $\mathcal{W}$.* □

An alternative approach to the cover problem using algebraic tools is considered next. The approach is based on the solvability of multilinear systems of equations using the theory of Groebner basis.

9.9 Polynomials and Groebner basis

In this section we give a brief background material about the multidimensional polynomials and the use of the Groebner Basis in the solution of polynomial equations in order to provide the appropriate mathematical tools for the next section, where the cover problem is formulated as a problem of solution of a system of multilinear equations. A detailed analysis of the Groebner Basis technique may be found in textbooks of computational algebraic geometry [Cox, Lit. & O'S., 1992].

Definition 9.9.1 *Let $f_1,\dots,f_p$ be polynomials in $\mathcal{R}[s_1,s_2,\dots,s_q]$. The the set of q-tuples defined by*

$$\mathbf{V}(f_1,\dots,f_p) = \\ \{(a_1,\dots,a_q)\in\mathcal{R}^q : f_i(a_1,\dots,a_q)=0,\ i=1,\dots,p\} \tag{9.120}$$

is called the affine variety defined by $f_1,\dots,f_p$. □

Now, it is clear that the affine variety $\mathbf{V}(f_1,\dots,f_p)$ is the set of all the solutions of the system of equations $f_1=f_2=\dots=f_p=0$.

Proposition 9.9.1 [Cox, Lit. & O'S., 1992] *Let the set of polynomials $f_1,\dots,f_p$ be a basis of an ideal in $\mathcal{R}[s_1,s_2,\dots,s_q]$. If $g_1,\dots,g_r$ is another basis of the same ideal we have that*

$$\mathbf{V}(f_1,\dots,f_p)=\mathbf{V}(g_1,\dots,g_r) \tag{9.121}$$

□

From the above we see that given a system of polynomial equations, we are free to use another system of equations, generating the same ideal, in order to find the solution (the affine variety corresponding to the system of equations).

A polynomial of one variable is a sum of monomials. The leading term is the term corresponding to the monomial with the higher degree and the ordering of the terms is obvious. In the case of polynomials in several variables, the ordering of the terms is not that obvious. A polynomial in several variables is the sum of monomials of the form $s_1^{a_1} s_2^{a_2} \ldots s_p^{a_p}$. The ordering of the monomials is determined by the p-tuple $(a_1, \ldots, a_p)$. The formal definition of the monomial ordering is the following:

Definition 9.9.2 [Cox, Lit. & O'S., 1992] *A monomial ordering on $\mathcal{R}[s_1, \ldots, s_p]$ is any relation $>$ on the set of polynomials of the form $s_1^{a_1} s_2^{a_2} \ldots s_p^{a_p}$, $a_i \geq 0$, satisfying*

(i) $>$ is a total ordering on $\mathcal{Z}^p_{\geq 0}$

(ii) if $(a_1, \ldots, a_p) > (b_1, \ldots, b_p)$ then $(a_1, \ldots, a_p) + (c_1, \ldots, c_p) > (b_1, \ldots, b_p) + (c_1, \ldots, c_p)$

(iii) Every subset of $\mathcal{Z}^p_{\geq 0}$ has a smallest element. □

A special type of ordering is the lexicographic ordering defined as follows:

Definition 9.9.3 *Let $a = (a_1, \ldots, a_p)$, $(b_1, \ldots, b_p) \in \mathcal{Z}^p_{\geq 0}$. We say $a >_{lex} b$ if the vector $a - b$ has its leftmost entry positive. Consider two monomials $s_1^{a_1} s_2^{a_2} \ldots s_p^{a_p}$ and $s_1^{b_1} s_2^{b_2} \ldots s_p^{b_p}$. We will say that $s_1^{a_1} s_2^{a_2} \ldots s_p^{a_p} >_{lex}$, $s_1^{b_1} s_2^{b_2} \ldots s_p^{b_p}$ if $a = (a_1, \ldots, a_p) >_{lex} (b_1, \ldots, b_p)$.* □

The lexicographic ordering plays an important role on the solution of systems of polynomial equations. Given a monomial ordering we may define the leading term of a polynomial as the greatest term corresponding to the ordering. Once a monomial ordering is chosen, every polynomial f has a unique leading term denoted by $\mathrm{LT}(f)$. Consider now an ideal $\mathcal{I}$ and a given monomial ordering. Let $\mathrm{LT}(\mathcal{I})$ denote the set of leading terms of elements of $\mathcal{I}$. This set is a set of monomials. The ideal generated by the elements of $\mathrm{LT}(\mathcal{I})$ is denoted by $\langle \mathrm{LT}(\mathcal{I}) \rangle$. We may now give the definition of the Groebner basis.

Definition 9.9.4 [Cox, Lit. & O'S., 1992] *Consider an ideal $\mathcal{I}$, a finite subset $\mathcal{G} = g_1, \ldots, g_t$ of $\mathcal{I}$ and fix a monomial ordering. We say that $\mathcal{G}$ is a Groebner Basis of the ideal $\mathcal{I}$ if*

$$\langle \mathrm{LT}(g_1), \ldots, \mathrm{LT}(g_t) \rangle = \langle \mathrm{LT}(\mathcal{I}) \rangle$$

□

Note that every nonzero ideal has a Groebner Basis.

The Groebner basis of an ideal is not unique. There is a special form of Groebner Basis, the Reduced Groebner Basis which is unique. An algorithm for finding the Groebner Basis of an ideal is the Buchberger's algorithm. [Buch., 1985], [Cox, Lit. & O'S., 1992].

The use of the Groebner Basis to the solution of a systems of polynomial equations is discussed below. Consider the system defined by the equations

$$f_1 = f_2 = \ldots = f_p = 0$$

The polynomials $f_1, f_2, \ldots, f_p$ generate an ideal $\mathcal{I}$. Now, let $\mathcal{G}$ be the Groebner Basis of $\mathcal{I}$. From proposition 9.9.1 it follows that the solutions of the given system of equations and the solutions of the system of equations defined by the polynomials of the Groebner basis are the same. When we use lexicographic monomial ordering, the use of the Groebner Basis, simplifies the solution considerably, because the equations we get have a nice form where some of the variables are eliminated from the equations in such a way that we may solve the system using the technique of "back-substitution" in a way similar to the well known Gauss elimination procedure for linear systems. An example of the Groebner basis technique is the following:

Example 9.9.1 [Cox, Lit. & O'S., 1992] Consider the system of equations

$$\begin{aligned} s_1^2 + s_2 + s_3 - 1 &= 0 \\ s_1 + s_2^2 + s_3 - 1 &= 0 \\ s_1 + s_2 + s_3^2 - 1 &= 0 \end{aligned}$$

A Groebner basis for the ideal generated by the left hand side polynomials is the following

$$\begin{aligned} g_1 &= s_1 + s_2 + s_3^2 - 1 \\ g_2 &= s_2^2 - s_2 - s_3^2 + s_3 \\ g_3 &= 2s_2 s_3^2 + s_3^4 - s_3^2 \\ g_4 &= s_3^6 - 4s_3^4 + 4s_3^3 - s_3^2 \end{aligned}$$

The system of equations that gives the same set of solutions to the original system, is

$$\begin{aligned} g_1 &= 0 \\ g_2 &= 0 \\ g_3 &= 0 \\ g_4 &= 0 \end{aligned}$$

The polynomial g_4 has one variable. Thus solving g_4 with respect to s_3 and substituting the roots to $g_3 = 0$ we get an equation with respect to s_2. Continuing this procedure of "back-substitution" we obtain all the solutions of the original system of equations.

The Groebner Basis technique is proven to be the appropriate tool for the parametric solution of the cover problem as it is shown in the next section.

9.10 The solution of the cover problem via Groebner basis

In this section an alternative method for the solution of the cover problem is proposed. This method is based on the matrix pencil formulation of the problem but differs from the method of section 9.6 in the final stage of the solution. The matrix pencil formulated problem may be further formulated as a problem of solution of multilinear equations. We are going to apply the method only to the standard cover problem for state–space systems, since the method is similar for the other cases.

From lemma 9.3.2 we have that $[sF - G, s\bar{F} - \bar{G}]$ has no i.e.d. and n.z.r.m.i. if and only if

$$\mathrm{Im}\{[F, \bar{F}]\} \supseteq \mathrm{Im}\{[G, \bar{G}]\} \tag{9.122}$$

Where $F = NJ$, $\bar{F} = NT$, $G = NAJ$, $\bar{G} = NAT$. Thus, (9.122) may be written

$$\mathrm{Im}\{[NJ, NT]\} \supseteq \mathrm{Im}\{[NAJ, NAT]\} \tag{9.123}$$

or equivalently

$$\mathrm{rank}\{[NJ, NT, NAJ, NAT]\} = \mathrm{rank}\{[NJ, NT]\} \tag{9.124}$$

Let $\mathrm{rank}\{[NJ, NT]\} = \rho$. Then (9.124) nay be written in terms of compound matrices as follows

$$C_{\rho+1}\{[NJ, NT, NAJ, NAT]\} = 0 \tag{9.125}$$

Note that $\rho \leq n - \ell$, since N has $n - \ell$ rows. The procedure of the solution of the cover problem is described below.

Since matrix T represents a basis of the subspace $\mathcal{T}$, it must have full column rank. Let τ be the number of the columns of T. Then

$$C_\tau\{T\} \neq 0 \tag{9.126}$$

Now, in order to ensure that $\mathcal{J} \cap \mathcal{T} = 0$ we must have

$$C_{j+\tau}\{[J, T]\} \neq 0 \tag{9.127}$$

where $j = \dim\{\mathcal{J}\}$. From proposition 9.3.4 we have that if φ is the total number of i.e.d. and n.z.r.m.i. of the restriction pencil $sNJ - NAJ$, then the dimension τ of a subspace $\mathcal{T}$ solving the cover problem must satisfy the condition

$$\tau \geq \varphi \tag{9.128}$$

Now, starting from $\tau = \varphi$, we consider the case where $\rho = j + 1$. Then, the condition $\text{rank}\{[NJ, NT]\} = \rho$ is equivalent to

$$C_\rho\{[NJ, NT]\} \neq 0 \text{ and } C_{\rho+1}\{[NJ, NT]\} = 0 \tag{9.129}$$

Next, we solve (9.124) with respect to T. We say that there exists a solution of dimension τ to the cover problem if the solution of (9.124) does not contradict (9.125),(9.126) and (9.129). In order to find all the subspaces of dimension τ which solve the cover problem, we repeat the above, increasing ρ up to $p = \min\{n - \ell, j + \tau\}$. The next step is to increase τ and repeat the procedure until $\tau = n - j$, since then $\mathcal{J} \oplus \mathcal{T} = \mathcal{X}$.

Summarising, the procedure for the derivation of all the solutions of the partial cover problem consists in solving the following equations

$$C_\rho\{[NJ, NT]\} \neq 0 \tag{9.130}$$

$$C_{\rho+1}\{[NJ, NT]\} = 0 \tag{9.131}$$

$$C_{\rho+1}\{[NJ, NT, NAJ, NAT]\} = 0 \tag{9.132}$$

$$C_\tau\{T\} \neq 0 \tag{9.133}$$

$$C_{j+\tau}\{[J, T]\} \neq 0 \tag{9.134}$$

for $\rho = 1, \ldots, \min\{n-\ell, j+\tau\}$, $\tau = \varphi+1, \ldots, n-j$, with respect to T. The above may be considered as a homogeneous system of polynomial equations in several variables. The indeterminates of the polynomials are the entries of the matrix T. The solution of the above systems is obtained via the Groebner Basis technique described in the previous section.

For the overall cover problem $\mathcal{J} \subset \mathcal{W} \subset \mathcal{X}$ we have

$$sNJ - NAJ = (sNV^* - NAV^*)\bar{J} \tag{9.135}$$

The (A, B)-invariance condition according to lemma 9.3.2 is

$$\text{Im}\{[NV^*\bar{J}, NV^*\bar{T}]\} \supseteq \text{Im}\{[NAV^*\bar{J}, NAV^*\bar{T}]\} \tag{9.136}$$

where $T = V^*\bar{T}$. The algorithm of the solution procedure is the same as in the case of the partial cover problem, i.e. solve the system of equations

$$C_\rho\{[NV^*\bar{J}, NV^*\bar{T}]\} \neq 0 \tag{9.137}$$

$$C_{\rho+1}\{[NV^*\bar{J}, NV^*\bar{T}]\} = 0 \tag{9.138}$$

$$C_{\rho+1}\{[NV^*\bar{J}, NV^*\bar{T}, NAV^*\bar{J}, NAV^*\bar{T}]\} = 0 \tag{9.139}$$

$$C_\tau\{\bar{T}\} \neq 0 \tag{9.140}$$

$$C_{j+\tau}\{[\bar{J}, \bar{T}]\} \neq 0 \tag{9.141}$$

for $\rho = 1, \ldots, \min\{n-\ell, j+\tau\}$, $\tau = \varphi+1, \ldots, v^* - j$, with respect to $\bar{T}$.

Remark 9.10.1 *The method described above gives a complete parametrisation of the (A,B)-invariant subspaces $\mathcal{V} = \mathcal{J} \oplus \mathcal{T}$. The equations* (9.130)–(9.134) *and* (9.137)–(9.141) *may have more than one set of solutions. Every set gives the parametric expressions of the basis matrices of $\mathcal{T}$.* □

Next we give an example to illustrate our method.

Example 9.10.1 [Ant., 1983] Consider the system $\mathcal{S}(A,B)$ with controllability indices $\sigma_1 = 4$ and $\sigma_2 = 2$. The subspace $\mathcal{J}$ to be covered has a basis matrix $J = [1,-1,-1,1,1,2]^T$. The Kronecker canonical form of the restriction pencil of $\mathcal{J}$ is $[0,0,s,-1]^T$ i.e. $\mathcal{J}$ is a r.m.i. subspace, with two z.r.m.i. and one n.z.r.m.i. $\eta = 1$. According to the algorithm presented we start with $\tau = 1$ and $\rho = 2$. Let $T = [t_{11}, t_{21}, t_{31}, t_{41}, t_{51}, t_{61}]^T$. Consider the equation

$$C_2\{[NJ, NT]\} = 0 \tag{9.142}$$

The above is equivalent to

$$\begin{aligned}
t_{11} + t_{21} &= 0\\
t_{11} + t_{31} &= 0\\
-t_{11} + t_{51} &= 0\\
t_{21} - t_{31} &= 0\\
-t_{21} - t_{51} &= 0\\
-t_{31} - t_{51} &= 0
\end{aligned}$$

The Groebner Basis of the ideal generated from the left-hand side polynomials of the above equations is the following

$$\begin{aligned}
g_1 &= -t_{31} - t_{51}\\
g_2 &= -t_{21} - t_{51}\\
g_3 &= -t_{11} + t_{51}
\end{aligned}$$

Thus, the solutions of (9.142) are

$$\begin{aligned} t_{11} &= t_{51} \\ t_{21} &= -t_{51} \\ t_{31} &= -t_{51} \end{aligned}$$

Therefore (9.130) is satisfied if any of the following holds true

$$\begin{aligned} t_{11} &\neq t_{51} \\ t_{21} &\neq -t_{51} \\ t_{31} &\neq -t_{51} \end{aligned}$$

The next step is to solve (9.132) i.e.

$$\mathcal{C}_3\{[NJ, NT, NAJ, NAT]\} = 0 \tag{9.143}$$

or equivalently to solve the system

$$\begin{aligned}
& t_{11} + 2t_{31} = 0 \\
& t_{21}^2 - t_{11}t_{31} - t_{21}t_{31} - t_{31}^2 + t_{11}t_{41} + t_{21}t_{41} = 0 \\
& -2t_{21}t_{21} - 2t_{41} = 0 \\
& t_{21}^2 - t_{11}t_{31} + t_{21}t_{31} - t_{31}^2 - t_{11}t_{41} + t_{21}t_{41} = 0 \\
& t_{11} + 3t_{21} + 2t_{51} = 0 \\
& -t_{21}^2 + t_{11}t_{31} - t_{21}t_{51} - t_{31}t_{51} + t_{11}t_{61} + t_{21}t_{61} = 0 \\
& -t_{21} - 3t_{31} - 2t_{61} = 0 \\
& 2t_{21}^2 - 2t_{11}t_{31} + t_{21}t_{51} - t_{31}t_{51} - t_{11}t_{61} + t_{21}t_{61} = 0 \\
& 3t_{11} + 3t_{31} = 0 \\
& -t_{21}t_{31} + t_{11}t_{41} - t_{21}t_{51} - t_{41}t_{51} + t_{11}t_{61} + t_{31}t_{61} = 0 \\
& -3t_{21} - 3t_{41} = 0 \\
& 2t_{21}t_{31} - 2t_{11}t_{41} - t_{21}t_{51} - t_{41}t_{51} + t_{11}t_{61} + t_{31}t_{61} = 0 \\
& 3t_{21} - t_{31} + 2t_{51} = 0 \\
& -t_{31}^2 + t_{21}t_{41} - t_{31}t_{51} + t_{41}t_{51} + t_{21}t_{61} - t_{31}t_{61} = 0 \\
& -3t_{31} + t_{41} - 2t_{61} = 0 \\
& t_{31}^2 - 2t_{21}t_{41} - t_{31}t_{51} - t_{41}t_{51} + t_{21}t_{61} + t_{31}t_{61} = 0
\end{aligned}$$

The corresponding Groebner basis is

$$\begin{aligned} f_1 &= -5t_{41} + 3t_{51} + t_{61} \\ f_2 &= 5t_{21} + 3t_{51} + t_{61} \\ f_3 &= -5t_{31} + t_{51} - 3t_{61} \\ f_4 &= -5t_{11} - t_{51} + 3t_{61} \end{aligned}$$

which leads to the following parametric set of solutions

$$t_{61} = \frac{3t_{11} - t_{21}}{2},\ t_{51} = \frac{-t_{11} - 3t_{21}}{2},\ t_{41} = -t_{21},\ t_{31} = -t_{11}$$

The parameters t_{11} and t_{21} are free and they may be chosen arbitrarily. The basis matrix of the subspace $\mathcal{T}$ has the following form

$$T = \begin{bmatrix} t_{11} \\ t_{21} \\ -t_{11} \\ -t_{21} \\ \frac{-t_{11}-3t_{21}}{2} \\ \frac{3t_{11}-t_{21}}{2} \end{bmatrix}$$

Now, choosing $t_{11} = t_{21} = -1$ we get the basis matrix of a covering space

$$V = \begin{bmatrix} 1 & -1 \\ -1 & -1 \\ -1 & 1 \\ 1 & 1 \\ 1 & 2 \\ 2 & -1 \end{bmatrix}$$

The corresponding restriction pencil is

$$sNV - NAV = s \begin{bmatrix} 1 & -1 \\ -1 & -1 \\ -1 & 1 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} -1 & -1 \\ -1 & 1 \\ 1 & 1 \\ 2 & -1 \end{bmatrix}$$

The above pencil has two z.r.m.i., two f.e.d at s=j and s=-j and has no i.e.d., i.e. $\mathcal{V}$ is a fixed spectrum (A, B)-invariant subspace.

9.11 Conclusions

In this chapter, the generalised dynamic cover problem has been considered. A matrix pencil method has been developed for the determination of all the solutions of the cover problem. It has been shown shown that the matrix pencil approach is the natural tool for the solution of the problem, since the nature of the subspaces of the state–space $\mathcal{X}$ is closely related to the type of the Kronecker invariants of the restriction pencil $R_{\mathcal{V}}(s)$.

The unification of the cover problems for state–space and general descriptor systems has been established and it has been shown that the restriction pencil–based methods allow the same treatment of both state–space and descriptor systems.

It has also been shown that the extended cover problems (the problems concerning subspaces with finite and infinite spectrum) may be reduced to standard cover problems in a straightforward manner.

Finally, an alternative method based on the solution of systems of multilinear equations has been developed where the use of Groebner basis theory was made.

An issue, which is a topic of future research, is the determination of the possible spectra of the solutions of the cover problem. This is the problem of choosing appropriately the parameters of the general forms of the basis matrices of the solutions such that the covering spaces have a desired spectrum.

Chapter 10

CONCLUSIONS
FURTHER RESEARCH

Conclusions–Further Research

In this thesis several problems from the framework of implicit systems have been studied and solved. The orientation of the work was towards the generalisation of some problems and results from the classical state–space and transfer function systems. Although the mathematical representations of implicit systems may be considered as direct extensions of transfer function of state–space systems we must be careful in proceeding to generalisations. The reason for this, is that in the case of implicit systems we have several definitions of the term "system" and "system equivalence". In this thesis the types of equivalence considered are transfer, external and $\mathcal{A}$–external equivalence. The problems considered in this thesis are: The problem of realisation of nonproper transfer functions in singular system form, and the closely related problem of canonical forms for singular systems with outputs under restricted system equivalence. The problem of realisation of autoregressive systems has been considered in the framework of external equivalence. A problem considered under both external and $\mathcal{A}$–external equivalence, is the model matching for implicit systems. Finally the cover and extended cover problems for implicit systems have been formulated and solved under the matrix pencil framework. The contribution of the thesis and the related topics for further research are discussed below.

In Chapter 4 the problem of realisation of a nonproper transfer function has been considered. First, the realisation was obtained by using the old technique of splitting the system into fast and slow subsystems and our attention was focused on the realisation of the polynomial part of the transfer function. Two new methods for obtaining this realisation (column and row) have been described. Next, a new proof has been given for the relation of the extended MacMillan degree of the composite matrix of a coprime and column reduced MFD of the transfer function and the dimensions of a minimal realisation. The main contribution of this thesis here, is the new realisation method from a given MFD. The proposed method treats finite and infinite frequencies in a unified way and does not require splitting of the system into fast and slow parts, or use of any transformation mapping infinity to a finite point. It has been shown, that if the MFD is coprime and column reduced i.e., if the corresponding composite matrix is a minimal basis of its column span, then the realisation is minimal. This result provides a complete generalisation of the results for strictly proper systems. The form of the obtained realisation allows us to distinguish directly the proper and nonproper controllability indices of the system. The relation between the Forney order of the MFD and the dimensions of the minimal system was also verified.

In Chapter 5, new canonical forms for singular systems have been obtained. These are canonical forms under restricted system equivalence for minimal singular systems with and without outputs. For systems without outputs the transformations leading to the canonical form have been described in detail and it has been shown that the canonical form is directly related to the echelon canonical form of a coprime and column reduced composite matrix of an MFD of the transfer function of the system. The canonical form has a block–companion form which is an extension of the Popov canonical form for state–space systems. The results of Popov and Forney relating minimal bases and realisations of state–space systems are completely extended to the case of singular systems. For systems without outputs, Popov type canonical forms have been obtained for systems with all the reachability indices equal. For the general case a semi canonical form has been found. The Popov type canonical form for singular systems without outputs in the general case is the subject of current research. Another useful result obtained is the derivation of the general form of the stabilizer of the canonical form of a pencil having only c.m.i..

Chapter 6 provides an alternative realisation of autoregressive systems under external equivalence. The realisations obtained are of descriptor (with and without feedthrough term) and pencil form and such realisations may be obtained by inspection from the matrix of the autoregressive system. Another result that has been produced in this Chapter is the generalisation of the observability indices to the case of descriptor behavioural systems. It has been shown that the row degrees of the matrix of the autoregressive system are equal to the r.m.i. of the observability pencil of the descriptor system. Furthermore, it was shown that these row degrees are also equal to the r.m.i. of the pencil realisation. The canonical realisation and canonical forms for nonsquare descriptor and pencil realisations are topics of further research.

In Chapter 7 a design problem in the implicit systems framework has been considered which is the problem of model matching. This problem is an extension of the transfer function case to the framework of external and $\mathcal{A}$–external behavioural systems. The contribution of this Chapter is that it has provided necessary as well as sufficient conditions for the existence of behavioural systems such that when they are interconnected to a given system the resulting system has a desired behaviour. The sufficiency of the conditions has been established for special types of systems satisfying certain conditions. It has been shown that the model matching problem under $\mathcal{A}$–external equivalence coincides with the problem under transfer equivalence. The derivation of necessary and sufficient conditions for the general case of behavioural systems is the subject of current research. Another subject of future research is the derivation of solutions of minimal MacMillan degree, such that the problem is considered in accordance to the problem for strictly proper transfer functions. Note that this Chapter is a first attempt to generalise the input–output control problems to the framework of implicit systems without transfer function.

In Chapters 8 and 9 the generalised dynamic cover problem has been considered. First (Chapter 8), a brief description of the control problems motivating the study of the cover problems has been given. Apart form the existing formulations of state–space problems as cover problems it has been shown that the problem of observer of linear functionals for implicit systems may be formulated as an appropriately defined extended dynamic cover problem. Furthermore it has been shown that the family of Model Projection Problems give rise to problems of the cover type. In Chapter 9 the cover problem has been formulated as a Matrix Pencil Augmentation–Realisation problem using the characterisation of the (A, B)–invariant subspaces via matrix pencils. The contribution of this Chapter may be summarised as follows: First the generalised cover problem has been extended to the case where the covering spaces are subspaces with infinite spectrum. The solution of this extended version of the problem has been obtained by a slight modification of the solution of the standard problem. Second, the matrix pencil approach allowed the unification of the cover problem for state–space and nonsquare descriptor systems. Note that in the case of descriptor system (A, B)–invariance is replaced by (A, E, B)–invariance. The matrix pencil approach to the cover problem has the advantage over other approaches that the solution is reduced to the solution of linear systems of equations and the parametric solutions of these yield a complete parametrisation of the solutions of the cover problem. In the last section of Chapter 9 the matrix pencil formulated cover problem was reduced to the problem of solving a system of multilinear equations. The appropriate tool for this approach is the Groebner basis.

Although the cover problem has been solved entirely, in the sense that a method for finding all the (A, B) or (A, E, B)–invariant spaces containing a given subspace and contained in another, there is an additional requirement encountered in many control problems. This is the requirement of stability of the spectrum of the solutions of the cover problem. Although sometimes we may choose (from the family of the solutions) some solutions with stable spectrum, in general we do not have a criterion for the characterisation of the families with stable spectra spaces. This characterisation is the topic for further research. Another topic for research is the exploration of the nature of the affine varieties arising from the multilinear formulation of the cover problem, which is linked to the Groebner basis approach of the problem.

NOTATION–ABBREVIATIONS

– $\Re, C$	: fields of real, complex numbers
– $\Re(s)$	: field of rational functions is s with real coefficients
– $\Re[s]$	: ring of polynomials in s with real coefficients
– $\Re_{pr}(s)$	: ring of proper rational functions
– $\Re[s_1, \cdots, s_n]$	: the set of polynomials in the variables $s_1, \cdots, s_n$
– $\Re^{m\times n}[s]$	: the set of $n \times n$ polynomial matrices
– $\Re^{m\times n}(s)$	: the set of $n \times n$ rational matrices
– $\mathcal{Z}^n_{\geq 0}$	: the set of ordered n–tuples of natural numbers
– $\mathcal{V}$	: denotes a vector space over the field of real numbers
– $\mathbf{V}(f_1, \cdots, f_p)$	: affine variety defined by the polynomials $f_1, \cdots, f_p$
– $>_{lex}$	: lexicographical ordering
– $\mathrm{LT}(\mathcal{I})$	: The set of leading terms of a polynomial ideal $\mathcal{I}$
– $\langle \mathrm{LT}(\mathcal{I}) \rangle$	: the ideal generated by $\mathrm{LT}(\mathcal{I})$
– $\dim\{\cdot\}$	: dimension of a vector space
– $\mathrm{Ker}\{\cdot\}$	: kernel, right null space
– $\mathrm{Im}\{\cdot\}$	: image
– $\mathcal{N}_\ell\{\cdot\}$	: left null space
– $\det\{\cdot\}$	: determinant
– $\deg\{\cdot\}$, $\partial\{\cdot\}$	: degree
– $\delta_M\{\cdot\}$	: MacMillan degree of a rational matrix
– $\delta_M^\infty\{\cdot\}$	: MacMillan degree of a rational matrix at infinity
– $\delta_M^*\{\cdot\}$	: extended MacMillan degree of a rational matrix
– $\delta_\infty\{\cdot\}$	: valuation at infinity of a rational matrix
– $\delta_\mathcal{F}$	: Forney dynamical order
– $\mathcal{P}_\infty$	: index at infinity
– $J_{r_i}(\lambda_i)$	: Jordan block with eigenvalue with algebraic multiplicity r_i
– A^{-1}	: inverse image of the operator A
– a/b	: a divides b
– $\mathcal{S}(A, B, C, D)$	: equations of a state–space system
– $\mathcal{S}(E, A, B, C, D)$	: equations of a descriptor system
– $\langle A/B \rangle$	: the controllable subspace of the pair (A, B)
– N	: left annihilator of B
– $B^\dagger$	: left inverse of B
– $R_\mathcal{V}(s)$	: $\mathcal{V}$–restricted pencil $sNEV - NAV$
– $\mathcal{C}_p\{\cdot\}$	: the p–th compound matrix
– $\mathcal{S}(\cdot,\cdot)$, $\mathcal{S}(\cdot)$	: subfamilies of the solution of the cover problem

– $Stab(\cdot)$	: the stabiliser
– c.m.i.	: column minimal index
– f.e.d.	: finite elementary divisor
– i.e.d.	: infinite elementary divisor
– n.z.r.m.i	: non–zero row minimal index
– r.m.i.	: row minimal index
– z.r.m.i.	: zero row minimal index
– AR	: autoregressive (representation)
– ARMA	: autoregressive moving average (representation)
– CEMPP	: constant external model projection problem
– KSAP	: Kronecker structure assignment problem
– KSTP	: Kronecker structure transformation problem
– MFD	: matrix fraction description
– MPAP	: matrix pencil augmentation problem
– MPP	: model projection problems
– MPRP	: matrix pencil realisation problem
– MPTP	: matrix pencil transformation problem
– PMD	: polynomial matrix description
– RKSTP	: restricted KSTP

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