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## A NOTE ON THE DEPTH OF A SOURCE ALGEBRA OVER ITS DEFECT GROUP

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*Dedicated to the memory of Professor John Clark*

**ABSTRACT.** By results of Boltje and Külshammer, if a source algebra  $A$  of a principal  $p$ -block of a finite group with a defect group  $P$  with inertial quotient  $E$  is a depth two extension of the group algebra of  $P$ , then  $A$  is isomorphic to a twisted group algebra of the group  $P \rtimes E$ . We show in this note that this is true for arbitrary blocks. We observe further that the results of Boltje and Külshammer imply that  $A$  is a depth two extension of its hyperfocal subalgebra, with a criterion for when this is a depth one extension. By a result of Watanabe, this criterion is satisfied if the defect groups are abelian.

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Let  $p$  be a prime and  $\mathcal{O}$  a complete local principal ideal domain with an algebraically closed residue field  $k$  of characteristic  $p$ , allowing the case  $\mathcal{O} = k$ . We will make without further comment use of the fact that by [9, II, Prop. 8], the canonical group homomorphism  $\mathcal{O}^\times \rightarrow k^\times$  splits canonically, and hence group cohomology with coefficients in  $k^\times$  can be viewed as cohomology with coefficients in  $\mathcal{O}^\times$ . Following terminology in [4], a ring extension  $B \rightarrow A$  is called *of depth one* if  $A$  is isomorphic, as a  $B$ - $B$ -bimodule, to a direct summand of  $B^n$  for some positive integer  $n$ , and a ring extension  $B \rightarrow A$  is called *of depth two* if  $A \otimes_B A$  is isomorphic, as an  $A$ - $B$ -bimodule, to a direct summand of  $A^n$ , for some positive integer  $n$ . Tensoring by  $A \otimes_B -$  shows that a ring extension of depth one is also an extension of depth two.

Let  $A$  be a source algebra of a block algebra over  $\mathcal{O}$  of a finite group, with a defect group  $P$ . Boltje and Külshammer showed in [2, 2.4] that if  $A$  is isomorphic to a twisted group algebra of the form  $\mathcal{O}_\alpha(P \rtimes E)$  for some  $p'$ -subgroup  $E$  of  $\text{Aut}(P)$  and some  $\alpha \in H^2(E; k^\times)$ , inflated trivially to  $P \rtimes E$ , then the canonical map  $\mathcal{O}P \rightarrow A$  is an extension of depth two. Moreover, they showed that the converse holds for principal blocks. The following result shows that this converse holds for arbitrary

blocks. See for instance [10, §11, §38] and [5, §6, §7] for background material on the Brauer homomorphism  $\text{Br}_P$  and fusion in source algebras.

**Theorem 1.** *Let  $G$  be a finite group,  $b$  a block of  $\mathcal{O}G$ ,  $P$  a defect group of  $b$  and  $A = i\mathcal{O}Gi$  a source algebra of  $b$ , where  $i$  is a primitive idempotent in the  $P$ -fixed point algebra  $(\mathcal{O}Gb)^P$  such that  $\text{Br}_P(i) \neq 0$ . The following are equivalent:*

- (i) *The ring extension  $\mathcal{O}P \rightarrow A$  induced by the canonical map  $P \rightarrow A^\times$  is of depth two.*
- (ii) *The ring extension  $kP \rightarrow k \otimes_{\mathcal{O}} A$  induced by the canonical map  $P \rightarrow A^\times$  is of depth two.*
- (iii) *There is an isomorphism of interior  $P$ -algebras  $A \cong \mathcal{O}_\alpha(P \rtimes E)$  for some  $p'$ -subgroup  $E$  of  $\text{Aut}(P)$  and some  $\alpha \in H^2(E; k^\times)$  inflated trivially to  $P \rtimes E$ .*
- (iv) *There is an isomorphism of interior  $P$ -algebras  $k \otimes_{\mathcal{O}} A \cong k_\alpha(P \rtimes E)$  for some  $p'$ -subgroup  $E$  of  $\text{Aut}(P)$  and some  $\alpha \in H^2(E; k^\times)$  inflated trivially to  $P \rtimes E$ .*

**Proof.** The equivalence of (iii) and (iv) is an immediate consequence of results of Puig (either apply [7, 14.6] over both  $\mathcal{O}$  and  $k$ , or use the lifting property [6, 7.8] for source algebras). Statement (iv) implies (i) and (ii) by Boltje and Külshammer [2, 2.4]. The implication (i)  $\Rightarrow$  (ii) is trivial. It suffices to show that (ii) implies (iv). We may therefore assume that  $\mathcal{O} = k$ . Suppose that (ii) holds but that (iv) does not hold. As an  $A$ - $kP$ -bimodule,  $A$  is indecomposable since  $1_A = i$  is primitive in  $A^P$ . Thus, if (ii) holds, then the Krull-Schmidt theorem implies that any indecomposable direct summand of  $A \otimes_{kP} A$  as an  $A$ - $kP$ -bimodule is isomorphic to  $A$  as an  $A$ - $kP$ -bimodule. Now if (iv) does not hold, then by [7, 14.6], there is a proper subgroup  $Q$  of  $P$  and an injective group homomorphism  $\varphi$  from  $Q$  to  $P$  such that the indecomposable  $kP$ - $kP$ -bimodule  $kP \otimes_{kQ} (\varphi kP)$  is isomorphic to a direct summand of  $A$  as a  $kP$ - $kP$ -bimodule. Thus  $A \otimes_{kQ} (\varphi kP)$  is isomorphic to a direct summand of  $A \otimes_{kP} A$  as an  $A$ - $kP$ -bimodule, and hence so is  $Aj \otimes_{kQ} (\varphi kP)$ , where  $j$  is a primitive idempotent in  $A^Q$ . Since  $Aj$  is indecomposable as an  $A$ - $kQ$ -bimodule, so is the  $k(G \times Q)$ -module  $kGj$ . Green's indecomposability theorem implies that the  $k(G \times P)$ -module  $kGj \otimes_{kQ} (\varphi kP)$  is indecomposable. Using that multiplication by  $i$  yields a Morita equivalence between  $kGb$  and  $A$  it follows that the  $A$ - $kP$ -bimodule  $Aj \otimes_{kQ} (\varphi kP)$  is also indecomposable, hence isomorphic to  $A$  as an  $A$ - $kP$ -bimodule, by the above. Since  $\text{Br}_P(i) \neq 0$  this is, however, only possible if  $Q = P$ , a contradiction.  $\square$

For the sake of completeness, we mention that the depth of an extension  $D \rightarrow A$ , where  $D$  is a hyperfocal subalgebra (cf. [8]) in a source algebra  $A$  of a block of a finite group, can be determined essentially as an application of the methods from [1] and [2]. The first statement of the following proposition is a special case of [1, 1.5].

**Proposition 2.** *Let  $A$  be a source algebra of a block of a finite group algebra over  $\mathcal{O}$  with defect group  $P$ , and let  $D$  be a hyperfocal subalgebra of  $A$ . The following hold.*

- (i) *The extension  $D \rightarrow A$  is of depth two.*
- (ii) *The extension  $D \rightarrow A$  is of depth one if and only if  $P$  acts by inner automorphisms on  $D$ .*

**Proof.** As mentioned above, statement (i) is a special case of [1, 1.5], as  $A$  is  $P/Q$ -graded, with  $D$  as 1-component. Since the argument is short and some parts of the notation will be useful in the proof of (ii), we sketch this briefly. We identify  $P$  with its canonical image in  $A^\times$ . The following definitions and facts on the hyperfocal subalgebra  $D$  of  $A$  are from [8]. The subalgebra  $D$  is  $P$ -stable, and the group  $Q = P \cap D^\times$  is the  $\mathcal{F}$ -hyperfocal subgroup of  $P$ , where  $\mathcal{F}$  is the fusion system of  $A$  on  $P$ . An immediate consequence of these properties is that  $D$  is indecomposable as an  $\mathcal{O}$ -algebra. Indeed, we have  $D^P \subseteq A^P$ , which is local, and hence  $P$  permutes the blocks of  $D$  transitively. But we also have  $\text{Br}_P(1_A) \neq 0$ , and hence  $D$  has a unique block. By [8, Theorem 1.8] we have  $A = \bigoplus_{u \in [P/Q]} Du$ , where  $[P/Q]$  is a set of representatives in  $P$  of  $P/Q$ . Since  $D$  is  $P$ -stable, this is a decomposition of  $A$  as a  $D$ - $D$ -bimodule. Thus  $A \otimes_D A = \bigoplus_{u \in [P/Q]} A \otimes_D Du$  is a decomposition of  $A \otimes_D A$  as an  $A$ - $D$ -bimodule. For  $u \in P$ , a trivial verification shows that the  $A$ - $D$ -bimodule  $A \otimes_D Du$  is isomorphic to  $A$  via the map sending  $a \otimes du$  to  $adu$ , where  $a \in A$  and  $d \in D$ . Thus any indecomposable direct summand of the  $A$ - $D$ -bimodule  $A \otimes_D A$  is isomorphic to a direct summand of  $A$  as an  $A$ - $D$ -bimodule. This proves (i). The summands  $Du$  in the  $D$ - $D$ -bimodule decomposition  $A = \bigoplus_{u \in [P/Q]} Du$  are all indecomposable as  $D$ - $D$ -bimodules. Indeed,  $D$  is indecomposable by the above, and  $Du$  is isomorphic to the image of  $D$  under the Morita equivalence on  $\text{mod}(D \otimes_{\mathcal{O}} D^{\text{op}})$  obtained from twisting the right  $D$ -module structure by the automorphism induced by conjugation with  $u$ . Thus the extension  $D \rightarrow A$  is of depth one if and only if  $Du \cong D$  as  $D$ - $D$ -bimodules, for all  $u \in [P/Q]$ , hence for all  $u \in P$ . By standard facts on automorphisms (cf. [3, §55A]) this is equivalent to the condition that  $u$  induces an inner automorphism of  $D$ , for all  $u \in P$ . This proves (ii).  $\square$

In conjunction with a result of Watanabe [11], this yields the following consequence.

**Corollary 3.** *With the notation of Proposition 2, if  $P$  is abelian, then the extension  $D \rightarrow A$  is of depth one.*

**Proof.** By [11, Theorem 2], if  $P$  is abelian, then  $P$  acts as inner automorphisms on  $D$ . Thus the result follows from Proposition 2 (ii).  $\square$

**Remark 4.** *What we have called depth two in this note is called right  $D2$  in [4, 3.1], with left  $D2$  being the obvious analogue, requiring  $A \otimes_B A$  to be a direct summand, as a  $B$ - $A$ -bimodule, of  $A^n$  for some positive integer  $n$ . It is easy to see directly that left and right  $D2$  are equivalent conditions for the extensions  $\mathcal{O}P \rightarrow A$  and  $D \rightarrow A$  considered in the results above; this follows also from a more general result in [4, 6.4]. See [2, §2.3] for a related discussion.*

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