Estimation of Multivariate Asset Models with Jumps

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Abstract

We propose a consistent and computationally efficient 2-step methodology for the estimation of multidimensional non-Gaussian asset models built using Lévy processes. The proposed framework allows for dependence between assets and different tail-behaviors and jump structures for each asset. Our procedure can be applied to portfolios with a large number of assets as it is immune to estimation dimensionality problems. Simulations show good finite sample properties and significant efficiency gains. This method is especially relevant for risk management purposes such as, for example, the computation of portfolio Value at Risk and intra-horizon Value at Risk, as we show in detail in an empirical illustration.

Keywords: Multivariate Lévy models, Jump models, Factor models, Principal Component, Risk Management.

JEL Classification: C15, C58, C61, C63, G11, G12.

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I. Introduction

The importance of modeling financial assets under realistic distributional assumptions away from normality has been highlighted in particular after the subprime financial crisis. Realistic modeling is especially relevant for risk management, given the extreme price movements, event risk, and sudden and large trading losses observed in financial data. Non-normality also directly affects the returns' tail distribution, which is crucial in the computation of regulatory capital requirements of financial institutions. Lévy processes offer a natural and robust approach for incorporating different distributional assumptions by means of discontinuous movements (commonly described as jumps), which can accommodate the levels of skewness and excess kurtosis observed in financial data, in particular over short horizons (see, e.g., Aït-Sahalia (2004)).

Although Lévy processes offer agile distribution modeling for asset prices, they also present significant estimation challenges especially in a multivariate setup; therefore, univariate models are generally used for portfolio analysis. For example, drawing on the flexible properties of the class of Lévy processes, Bakshi and Panayotov (2010) analyze portfolio tail risk measures on the basis of a pure jump Lévy model for portfolio returns. Bakshi and Panayotov (2010) setup is univariate in the sense that it models portfolio returns, rather than each individual asset in the given portfolio, and it captures the main portfolio characteristics while retaining tractability and computational efficiency. However, their univariate setting allows for neither the analysis of the impact of dependence between the components of the portfolio nor the measurement of each asset's individual risk contribution to the portfolio, which is particularly relevant for scenario analysis. A multivariate model, which caters to different tail behaviors for each asset in the portfolio, is ideal not only for risk management, but also for portfolio optimization and the analysis of multivariate structures such as basket options on equities and collateralized debt obligations.

In light of the above, in this paper we adopt a general multivariate setting for Lévy processes, which accounts for the impact of dependence between the components of the portfolio, and propose a consistent and computationally efficient model estimation procedure, suitable for portfolio risk measurement and management. After showing the theoretical validity of our method and testing its finite sample properties by simulations, we showcase its applicability to the computation of the risk measures Value at Risk (VaR) and intra-horizon Value at Risk (VaR-I).

Our estimation methodology is developed by combining the latest advances in the modeling of multivariate Lévy processes with the most recent developments in the estimation of latent factor models. Specifically, we adopt the multivariate construction of Ballotta and Bonfiglioli (2016) that models a portfolio of asset returns as a linear combination of independent Lévy processes representing systematic and idiosyncratic components. Since neither the common nor the idiosyncratic components driving the margin processes are directly observable, maximum likelihood estimation of the parameters is computationally burdensome and often unfeasible. Indeed, estimation of the parameters via a single maximization of the likelihood function presents significant issues in terms of implementation, in particular the curse of dimensionality due to the large number of parameters that are necessary to accommodate a multivariate model.

Thus, the first contribution of this paper is to propose a consistent and computationally efficient 2-step estimation procedure for the multivariate model of Ballotta and Bonfiglioli (2016), which overcomes the above mentioned dimensionality problems by means of the principal components method of Bai and Ng (2002) and Bai (2003). Specifically, principal component estimation is employed in step one to consistently estimate the common risk process; then in step two, we focus on the estimation of the parameters of the idiosyncratic components. In both steps, the estimation of the Lévy process parameters is based on the maximization of univariate sample likelihood functions. Hence, our procedure not only simplifies estimation by improving computational efficiency, but also solves the dimensionality problem while providing consistent estimation of the parameters. Our simulation study confirms the reliability and computational efficiency of our method in comparison with the (likely unfeasible) 1-step maximum likelihood approach in which all parameters of the multivariate Lévy process are estimated in a single step.

As a second contribution of this paper, we show how our methodology can be applied to the computation of portfolio risk measures. As in Bakshi and Panayotov (2010), specific attention is paid to VaR-I, which captures the exposure to losses throughout the investment horizon, contrasting with VaR, which is the industry standard for the estimation of regulatory capital requirements and measures the risk of possible losses at the end of a predetermined time horizon. As VaR-I incorporates information about the dynamic path of possible losses, it offers an ideal tool for dealing with intra-horizon risk over a multi-period investment horizon (see, e.g., Stulz (1996), Kritzman and Rich (2002), and Boudoukh, Richardson, Stanton, and Whitelaw (2004)). This is of paramount importance for monitored asset managers, leveraged investors, borrowers required to maintain a particular level of reserves as a condition of a loan agreement, or securities lenders required to deposit collateral. Our work, however, moves beyond Bakshi and Panayotov (2010) to a multivariate setting, which incorporates the impact of dependence between the components of the portfolio. The computation of the relevant risk measures in the multivariate setting is facilitated by the fact that the chosen factor construction gives immediate access to the portfolio's characteristic function; hence, we can use state-of-the-art numerical procedures required for the computation of VaR-I, such as the Fourier space time-stepping (FST) algorithm introduced by Jackson, Jaimungal, and Surkov (2008) for pricing path-dependent financial option contracts. Our methods avoid the implementation of numerical methods for partial integro-differential equations such as the ones used in Bakshi and Panayotov (2010), which might require approximations, especially for infinite activity jumps, that may lead to accuracy and stability problems (see Jackson et al. (2008) for further details). As an illustration, we provide a clear estimation procedure for portfolio VaR-I assuming a multivariate model following the normal inverse Gaussian (NIG) process of Barndorff-Nielsen (1997) and the Merton jump diffusion processes (MJD) of Merton (1976).

The third contribution of this paper is to show that the proposed framework also allows for the explicit computation of the risk contribution from each asset in the portfolio without the need for re-estimating the multivariate model. This is of relevance, for example, for active risk management of a portfolio as, by means of the proposed multivariate model, it is possible to assess changes in the portfolio risk profile resulting from one additional unit position of exposure to a given asset. This breakdown of the contribution to risk represents an invaluable "drill-down" exercise that enables managers to better control their risk profiles. We illustrate the application using the portfolio VaR-I previously mentioned.

The outline of the paper is as follows: In Section II, we review the most relevant features of the multivariate Lévy model under consideration and we discuss the estimation of the model, introducing the 2-step estimation procedure. In Section III, we assess the 2-step estimation procedure via simulations. Section IV illustrates how to compute the intra-horizon Value at Risk for a portfolio of assets following the proposed model, with an application in real data. Section V concludes. Detailed simulation results are presented in Appendix A, whilst Appendix B offers a brief review of the literature related to the present paper.

II. Multivariate Lévy Processes by Linear Combination: Model and Estimation

A. Model Specification

A Lévy process, L_t, on a filtered probability space is a continuous time process with independent and stationary increments, whose distribution is infinitely divisible. Lévy processes have attracted attention in the financial literature due to the fact that they accommodate distributions with nonzero higher moments (skewness and excess kurtosis), therefore allowing a more realistic representation of the stylized features of market quantities such as asset returns. Further, they represent a class of processes with known characteristic functions in virtue of the Lévy–Khintchine representation, so that $\mathbb{E}(\exp(iuL_t)) = \exp(t\varphi(u)), u \in \mathbb{R}$, with $\varphi(\cdot)$ denoting the so-called characteristic exponent. This feature in particular allows for the development of efficient numerical schemes for the approximation of potentially unknown distribution functions and derivative prices using Fourier inversion techniques.

In this setting, let us denote by P_t the price of a financial instrument represented as

$$P_t = P_0 \exp\left(L_t\right);$$

assuming that we observe the price process on an equally-spaced time grid $t = 1, 2, \ldots, T$, the log-return defined as

$$X_t = \log\left(\frac{P_t}{P_{t-1}}\right) = L_t - L_{t-1}$$

is a process with infinitely divisible distribution such that, for all t, X_t is distributed as L_1, and $X_1, X_2, \ldots, X_T$ are mutually independent. Hence, with a slight abuse of notation, we say that X_t is an independent and identically distributed (IID) process.

A convenient construction for an N-dimensional version of the Lévy process L_t is proposed in Ballotta and Bonfiglioli (2016) via a linear transformation of a vector of independent Lévy processes with components $\tilde{L}_t^{(n)}$, $n = 1, 2, \ldots, N$, each representing the idiosyncratic risk, and another independent Lévy process, $\tilde{L}_t^{(N+1)}$, modeling the common risk component, so that for $n = 1, 2, \ldots, N$, $L_t^{(n)} = \tilde{L}_t^{(n)} + a_n \tilde{L}_t^{(N+1)}$, $a_n \in \mathbb{R}$. Due to the property of independent and stationary increments of Lévy processes, the increments also respect the same linear transformation. In particular, let us denote by $Y_t^{(n)}, n = 1, \ldots, N$, Z_t the increments of $\tilde{L}_t^{(n)}$, $n = 1, 2, \ldots, N$, $\tilde{L}_t^{(N+1)}$, respectively. Then the following holds.

Proposition 1 Let $Z, Y^{(n)}, n = 1, \ldots, N$ be IID processes, with characteristic functions $\phi_Z(u;t)$ and $\phi_{Y^{(n)}}(u;t)$, for $n = 1, \ldots, N$, respectively. Then, for $a_n \in \mathbb{R}, n = 1, \ldots, N$

$$\boldsymbol{X}_t = (X_t^{(1)}, \ldots, X_t^{(N)})' = (Y_t^{(1)} + a_1 Z_t, \ldots, Y_t^{(N)} + a_N Z_t)' \tag{1}$$

is an IID process on $\mathbb{R}^N$ with characteristic function

$$\phi_{\boldsymbol{X}}(\boldsymbol{u};t) = \phi_Z\left(\sum_{n=1}^{N} a_n u_n; t\right) \prod_{n=1}^{N} \phi_{Y^{(n)}}(u_n;t), \quad \boldsymbol{u} \in \mathbb{R}^N.$$

It follows by conditioning on the systematic process, Z, that the joint probability density function of the multivariate IID process $\boldsymbol{X}_t$ is

$$f_{\boldsymbol{X}}(x_t^{(1)}, \ldots, x_t^{(N)}) = \int_{-\infty}^{\infty} f_{Y^{(1)}}(x_t^{(1)} - a_1 z) \cdot \ldots \cdot f_{Y^{(N)}}(x_t^{(N)} - a_N z) f_Z(z) dz. \tag{2}$$

We note that as the given multivariate model admits computable characteristic function, the joint distribution is always available (up to a Fourier inversion), even when the components' distributions, $f_{Y^{(1)}}, \ldots, f_{Y^{(N)}}, f_Z$, are not known analytically.

Proposition 1 implies that for each $X^{(n)}$, $n = 1, \ldots, N$, the process Z captures the systematic part of the risk originated by sudden changes affecting the whole market, while the process $Y^{(n)}$ represents the idiosyncratic shocks generated by company specific issues. Consequently, the components of $\boldsymbol{X}_t$ are dependent and may jump together. In particular, for each $t \geq 0$, the components of $\boldsymbol{X}_t$ are positively associated if the loading factors a_n for $n = 1, \ldots, N$ are all either positive or negative; otherwise, the components of $\boldsymbol{X}_t$ are negative quadrant dependent. The resulting pairwise linear correlation coefficient is

$$\rho_{n,m}^{\boldsymbol{X}} = \text{corr}(X_t^{(n)}, X_t^{(m)}) = \frac{a_n a_m \text{var}(Z_1)}{\sqrt{\text{var}(X_1^{(n)})}\sqrt{\text{var}(X_1^{(m)})}}, \tag{3}$$

which is well defined if all processes have finite moments of all order (specifically the variance). We note, in fact, from expression (3) that for fixed $a_n, a_m \neq 0$, $\rho_{n,m}^{\boldsymbol{X}} = 0$ if and only if Z is degenerate and the components are independent, whilst $\left|\rho_{n,m}^{\boldsymbol{X}}\right| = 1$ if and only if $Y^{(n)}$ and $Y^{(m)}$ are degenerate (i.e., there is no idiosyncratic factor in the components $X^{(n)}$ and $X^{(m)}$). Further, $\text{sign}(\rho_{n,m}^{\boldsymbol{X}}) = \text{sign}(a_n a_m)$; therefore, both positive and negative correlations can be accommodated. Finally, the resulting multivariate model shows nonzero indices of upper and lower tail dependence, which are controlled by the tail probabilities of the systematic risk process. For fuller details, we refer to Ballotta and Bonfiglioli (2016) and Ballotta, Deelstra, and Rayée (2017).

Several features of the construction in Proposition 1 are worth noticing. In the first place, this construction is relatively parsimonious in terms of the number of parameters involved, as this number grows linearly with the number of assets.

Further, the adopted modeling approach is quite flexible, as it can be applied to any Lévy process; indeed Proposition 1 allows for specifying any univariate Lévy process for the idiosyncratic and systematic risks. In this respect, we note that, differently from Ballotta and Bonfiglioli (2016), in this work we do not impose any convolution condition on the components aimed at recovering a known distribution for the margin processes, hence allowing for a more realistic portrayal of the asset log-return features and the dependence structure in place. Since factor models generally do not originate known marginal distributions (except in the Gaussian case), a large portion of the literature on multivariate Lévy processes focuses on finding suitable conditions for the model parameters under which this feature holds. However, as argued by several authors such as Eberlein, Frey, and von Hammerstein (2008), recovering known distributions for the margin processes, although intuitive, leads to a biased view of the dependence structure in place. This is because it reduces the flexibility of the factor model and fails to recognize the different tail-behaviors of the assets in the portfolio, which is an essential aspect in risk management, especially when it comes to the assessment of the marginal risk contribution of each individual asset in the portfolio. As observed by Ballotta et al. (2017) in a different context, the factor construction of Ballotta and Bonfiglioli (2016) retains its mathematical tractability and parsimonious parameter space regardless of the presence of these conditions.

Finally, the model is particularly tractable as the full description of the multivariate vector $\mathbf{X}_t$ only requires information on the univariate processes $Y_t^{(1)}, \ldots, Y_t^{(N)}$ and Z_t. However, from a practical point of view these sources of risk are not directly observable.

Thus, for the purpose of the estimation of the given multivariate Lévy model, discussed in the following section, we distinguish between 1-step and 2-step approaches, which depend on the estimation methods used for the common factor. The 1-step approach involves joint estimation of the parameters of the common factor and the idiosyncratic components; however, the maximization of the resulting likelihood function is feasible only if we consider a limited number of assets. The 2-step approach that we propose instead involves firstly the estimation of latent factors and loadings by principal component methods; conditioned on this information, the likelihood function admits a simple expression as a product of univariate densities. These facts simplify the estimation procedure to improve efficiency and solve the dimensionality problem, while providing consistent estimation of the parameters. We observe that this approach is facilitated by the lack of convolution conditions imposed on the model parameters as discussed above. An alternative possibility would be to consider the unobservable common factor as a latent factor whose dynamic is assigned so that the estimation procedure can be reduced to a (in general) non-Gaussian Kalman filtering problem. However, the application of these techniques is in general not straightforward and, in any case, does not solve the dimensionality problem.

We conclude by observing that in order to simplify the description of the model but without loss of generality, we assume that Z includes only one factor; however, all results in this paper can be generalized to a multifactor model as we explain in the following sections.

B. Model Estimation: A 2-Step Approach

From the joint density of the stock log-returns given by equation (2), it follows that the likelihood function of the sample $\boldsymbol{x} = \left\{ (x_t^{(1)}, \ldots, x_t^{(N)}) \right\}_{t=1,\ldots,T}$ is

$$L(\boldsymbol{x}, \boldsymbol{\theta}) = \prod_{t=1}^{T} \int_{-\infty}^{\infty} f_{Y^{(1)}}(x_t^{(1)} - a_1 z_t; \boldsymbol{\theta}_{Y^{(1)}}) \times \ldots \times f_{Y^{(N)}}(x_t^{(N)} - a_N z_t; \boldsymbol{\theta}_{Y^{(N)}}) f_Z(z_t; \boldsymbol{\theta}_Z) dz_t, \quad (4)$$

where $\boldsymbol{\theta} = [\boldsymbol{\theta}_{Y^{(1)}}, \ldots, \boldsymbol{\theta}_{Y^{(N)}}, \boldsymbol{\theta}_Z, \boldsymbol{a}]$ is the parameter set to be estimated.

Thus, all parameters of the chosen multivariate Lévy model can be estimated via a single maximization of the likelihood function (4). However, we note that this procedure presents significant issues in terms of implementation, in particular, the curse of dimensionality. This is caused by several elements: the dimension of the parameter space, due to a richer model parametrization; the number of assets N, which increases the complexity of the integrand function; the sample size T, which increases the number of integrals to be evaluated; and, in case of extensions to multifactor models, the number of common factors, which increases the dimension of the integral in equation (4). In addition, in the case of non-Gaussian dynamics, the density functions might not be known in closed forms and therefore have to be computed numerically. All these issues only exacerbate the numerical optimization, leading to imprecise parameter estimates and cases of false convergence. Finally, we note that in the case in which more systematic factors are assumed, there would be an infinite set of possible orthogonal factors and the maximum likelihood equations would have an infinite number of solutions returning the same value of the likelihood. This fundamental indeterminacy, called a problem of rotation, is discussed in Anderson (1957).

A valid alternative to an estimation procedure based on equation (4), which improves on the implementation issues mentioned above, exploits the independence of the common factor and the idiosyncratic processes. Indeed, the factor model (1), in virtue of standard results on the joint probability distribution of functions of random variables, gives access to a joint density for the pair $(\mathbf{X} = \mathbf{Y} + \mathbf{a}Z, Z)$ of the form

$$f_Z(z_t) \prod_{n=1}^{N} f_{Y^{(n)}} \left(x_t^{(n)} - a_n z_t \right).$$

This result, under the assumption of Z being observable (in a sense to be defined later), allows us to conveniently write the log-likelihood function of the sample $\boldsymbol{x}$ and $\boldsymbol{z} = \{z_t\}_{t=1\ldots T}$ as

$$\ln L(\boldsymbol{x}, \boldsymbol{z}; \boldsymbol{\theta}_Z, \boldsymbol{\theta}_{\mathbf{Y}}, \boldsymbol{a}) = \sum_{t=1}^{T} \ln f_Z(z_t; \boldsymbol{\theta}_Z) + \sum_{n=1}^{N} \sum_{t=1}^{T} \ln f_{Y^{(n)}} \left(x_t^{(n)} - a_n z_t; \boldsymbol{\theta}_{Y^{(n)}} \right), \quad (5)$$

the convenient feature of expression (5) being the separation of the log-likelihood of the systematic risk process from the log-likelihoods of the idiosyncratic components.

This leads to the following remark: if the systematic risk factor is observable, the additive structure of the log-likelihood function highlighted by expression (5) suggests that the optimization procedure for the model estimation can be performed in two steps, one for the systematic risk process and one for the idiosyncratic components. More specifically, the first step is represented by the following optimization with respect to the parameters of the observable systematic process Z, that is,

$$\max_{\boldsymbol{\theta}_Z} \ln L\left(\boldsymbol{z};\boldsymbol{\theta}_Z\right) = \max_{\boldsymbol{\theta}_Z} \sum_{t=1}^{T} \ln f_Z\left(z_t;\boldsymbol{\theta}_Z\right). \tag{6}$$

Given $\boldsymbol{\theta}_Z$, the second step consists of N independent maximizations, one for each instrument, of the likelihood of the idiosyncratic components with respect to the loading coefficients and the parameters of the idiosyncratic processes, which can be formalized as

$$\begin{aligned} \max_{\boldsymbol{\theta}_{Y^{(n)}},a_n} \ln L\left(x^{(n)} - a_n\boldsymbol{z};\boldsymbol{\theta}_{Y^{(n)}}\right) &= \max_{\boldsymbol{\theta}_{Y^{(n)}},a_n} \sum_{t=1}^{T} \ln f_{Y^{(n)}}\left(x_t^{(n)} - a_n z_t;\boldsymbol{\theta}_{Y^{(n)}}\right), \\ & \quad n = 1,\ldots,N. \end{aligned} \tag{7}$$

We notice that this estimation strategy, 'observe, divide, and conquer', allows us to solve the curse of dimensionality, because each maximization procedure involves only a subsection of the overall parameter space. In addition, generalizing the model for multiple factors has the minimal additional cost of solving more independent maximization problems. We also emphasize that once the common factor can be considered observable, our 2-step procedure is still based on the maximization of the likelihood function, and, therefore, the estimator retains all theoretical limiting properties, such as consistency, asymptotic normality, and efficiency.

In practice, though, the common factor is not directly observable. One way to proceed is to use observable proxy variables such as a well diversified index; however, these proxy variables are latent variables contaminated with an error that does not vanish, causing the estimation to lose all its theoretical limiting properties. To solve this problem, we propose an alternative approach based on the latest theoretical advances on factor models as in Bai and Ng (2002) and Bai (2003).

Allowing for a multifactor structure, our model (1) can be written as

$$X_t^{(n)} = a_n' Z_t + Y_t^{(n)}.$$

Hence, in terms of standard factor model notation, $X_t^{(n)}$ is the response variable, a_n is the $1 \times r$ vector of factor loadings specific to the cross-sectional unit n, Z_t is the $r \times 1$ vector of common factors, and $Y_t^{(n)}$ is the idiosyncratic error. In matrix notation, $\boldsymbol{X} = \boldsymbol{Z}\boldsymbol{a'} + \boldsymbol{Y}$, where the matrices $\boldsymbol{X}$ and $\boldsymbol{Y}$ are $T \times N$, $\boldsymbol{Z}$ is $T \times r$, and a is $N \times r$. Equation (1) can be recovered from this more general setting in the case of $r = 1$.

Bai and Ng (2002), (2008) and Bai (2003) propose a principal components method to consistently estimate factors, loadings, and the number of factors by solving the following optimization problem

$$\min_{a,Z}(NT)^{-1}\sum_{t=1}^{T}\sum_{n=1}^{N}\left(X_t^{(n)}-a_n'Z_t\right)^2,$$

subject to the normalization $a'a/N=\mathbb{I}_N$, where $\mathbb{I}_N$ is the N-dimensional identity matrix. The resulting (optimal) estimated loadings matrix, $\widetilde{a}$, is $\sqrt{N}$ times the eigenvectors associated with the r largest eigenvalues of the $N\times N$ matrix $\boldsymbol{X'X}$. Given $\widetilde{a}$, the factors can be estimated by $\widetilde{\boldsymbol{Z}}=\boldsymbol{X}\widetilde{\boldsymbol{a}}/N$.

Five main assumptions are required for consistent estimation of the factors (see Bai and Ng (2002), (2008), and Bai (2003) for further details).

Assumption 1. $\mathrm{E}\|Z_t\|^8\leq M\leq\infty$ and $(\sum_{t=1}^{T}Z_tZ_t')/T\xrightarrow{p}\Sigma_Z>0$, an $r\times r$ nonrandom matrix.

Assumption 2. The loading a_n is either deterministic or stochastic with $\mathrm{E}\|a_n\|^8\leq M\leq\infty$. In either case, $(\sum_{n=1}^{N}a_na_n')/N\xrightarrow{p}\Sigma_a>0$, an $r\times r$ nonrandom matrix as $N\to\infty$.

Assumption 3. $Y_t^{(n)}$ is weakly correlated, both over time and cross-sectionally.

Assumption 4. $\{a_n\}$, $\{Z_t\}$, and $\{Y_t^{(n)}\}$ are mutually independent. Dependence within each group is allowed.

Assumption 5. $\mathrm{E}\|N^{-\frac{1}{2}}\sum_{n=1}^{N}a_nY_t^{(n)}\|^8\leq M\leq\infty$ for all t.

In particular we note the following. According to Assumption 1, the factors Z_t are allowed to be non-IID, that is, some level of autocorrelation is permitted; by definition, X_t is an IID process, and thus Z_t is also an IID process and the assumptions holds. Assumption 2 rules out nonsignificant factors, or factors with trivial contribution to the variance of the response variables; factor loadings are assumed to be nonrandom variables, which is the case in our model. Assumption 3 allows for limited time series and cross-sectional dependence in the idiosyncratic components $Y_t^{(n)}$ and also heteroskedasticity in both the time and cross sectional dimensions (see Bai and Ng (2002) for formal details). This assumption is irrelevant in our model as $Y_t^{(n)}$ are also IID processes. However, it is important to note that in general some level of cross-sectional correlation is allowed, as under standard approximate factor models, common factors may not fully capture the total systematic variation on the response variables. Assuming that $Y_t^{(n)}$ are IID processes implies that all systematic variations are captured by the common factors (exact factor model); however, our factor estimation allows for some level of correlation left in the residuals, which may be relevant when dealing with real data. Assumption 4 is a standard assumption in factor analysis models and holds in our case by definition in Proposition 1. Assumption 5 allows some weak correlation between idiosyncratic errors and loadings and guarantees that the estimated factors uniformly converge to the space spanned by Z_t.

Under these assumptions, Bai and Ng (2002) show that estimated factors and loadings are consistent when $N,T\to\infty$. By means of Monte Carlo simulations, Bai (2003) shows also that the estimator has good finite sample properties for values of N as small[1] as 25.

Moreover, Bai and Ng (2008) show that the estimated factors $\widetilde{\boldsymbol{Z}}$ can be treated as observed variables in extremum estimation as MLE under suitable assumptions[2]. Thus, equations (6) and (7) can be rewritten as functions of estimated factors $\widetilde{\boldsymbol{Z}}$ and estimated loadings $\widetilde{\boldsymbol{a}}$, and the parameters $\boldsymbol{\theta}_Z$ and $\boldsymbol{\theta}_{Y(n)}$ can be consistently estimated by MLE.

Two additional comments are worth mentioning for the factor estimation. Firstly, in the case of multifactor models, the estimation of factors and loadings also requires estimation of the number of factors. Bai and Ng (2002) propose an estimation method based on classical model selection methods. More recently, Ahn and Horenstein (2013) proposed an estimation method based on the ratio of the eigenvalues of the matrix $\boldsymbol{X'X}/NT$, according to which, for eigv_k denoting the kth largest eigenvalue of the matrix $X'X/NT$, the number of factors is the value of k that maximizes the ratio criterion function

$$ER\left(k\right) = \frac{\mathrm{eigv}_k}{\mathrm{eigv}_{k+1}}.$$

The method is very simple and has very good small sample properties; hence, we adopt it for our empirical application in Section IV.

Secondly, as in any factor model with latent factors and loadings, the true factors and loadings can only be identified up to a scale. Specifically, the estimated factors $\widetilde{\boldsymbol{Z}}$ are consistent estimators of $\boldsymbol{ZH}$ where $\boldsymbol{Z}$ is the true factors and $\boldsymbol{H}$ is an invertible rotation matrix. This is known as the factor rotation problem, and it affects the interpretation of factors and loadings. However, our model (1) does not require factors and loadings to be separately identified, but only requires the identification of the components $\boldsymbol{Za'}$, which are in fact identifiable.

As mentioned above, the second step consists of N separate likelihood maximization problems as in equation (7). Since the loadings are estimated jointly with the factors, the objective function will only require estimation of the parameters $\boldsymbol{\theta}_{Y(n)}$.

Finally, a practitioners note: a proxy factor could be used instead of the estimated principal component factor. Although, as explained above, this method does not guarantee estimation consistency, it could nevertheless represent a practical alternative. Indeed, the optimization problem (7) can in this case be solved iteratively by maximizing first with respect to the idiosyncratic parameters and then with respect to the loading parameters until no further significant improvement in the objective functions is achieved. In particular, the loadings can be constrained to fit the covariance matrix to correctly recover the dependence structure as described below, and then the maximization of the likelihood in (7) is performed only with respect to the idiosyncratic parameters.

Specifically, the vector $\boldsymbol{a}$ can be initialized by fitting the non-diagonal entries of the sample covariance matrix to their theoretical counterparts predicted by the multivariate

[1]In the simulations performed by Bai (2003) the average correlation between the true factor and the estimated factor for $N = 25, T = 50$ over 2,000 repetitions is 0.98, improving as T increases.

[2]Additional technical assumptions required to prove consistency of estimated parameters are in most part also required for MLE estimation with observed variables; for technical details refer to Bai and Ng (2008).

model (1). This is achieved by solving

$$\min_{\boldsymbol{a}} \|\mathrm{cov}(\boldsymbol{X}) - \boldsymbol{\sigma}^2\|_F, \tag{8}$$

where $\|\cdot\|_F$ denotes the Frobenius norm,

$$\mathrm{cov}(\boldsymbol{X}) = \boldsymbol{a}\boldsymbol{a}'\mathrm{var}(Z_1) + \mathrm{diag}([\mathrm{var}(Y^{(1)}), \ldots, \mathrm{var}(Y^{(N)})]) \tag{9}$$

is the model covariance matrix (see eq. (3) as well), and $\boldsymbol{\sigma}^2$ denotes the sample covariance matrix (we set the diagonal entries equal to 0 in both). In expression (9), we can use either the sample variance of the increments of the proxy variable for Z or the parametric expression for the variance computed with the parameters estimated in Step 1; in the former case, this step turns out to be independent of the specification of the Lévy processes involved in the multivariate model construction.

III. Estimation Assessment

In this section we evaluate the performance of the 2-step estimation approach presented in Section II.B by simulations. Our objectives are to assess the efficiency gains of the 2-step approach in comparison with the 1-step approach and also to analyze the finite sample properties of the 2-step estimator. We focus on two particular specifications of the multivariate model (1): the case in which all the involved processes come from the normal inverse Gaussian (NIG) process of Barndorff-Nielsen (1997) with drift ("all-NIG") and the case in which all the involved processes are generated by Merton jump diffusion processes (MJD) of Merton (1976) ("all-MJD"). These jump structures were also considered by Ornthanalai (2014). The features of these processes are reviewed in the following of this section. At this stage we note that all required densities are generated via Fourier numerical inversion of the corresponding characteristic functions. The numerical inversion has been performed adopting the COS method introduced by Fang and Oosterlee (2008) in virtue of its high numerical accuracy.

The NIG process, introduced by Barndorff-Nielsen (1997), is a normal tempered stable process obtained by subordinating a (arithmetic) Brownian motion by an (unbiased) independent inverse Gaussian process. Its characteristic function reads

$$\phi(u;t) = \exp\left(i\mu t + \frac{t}{k}(1 - \sqrt{1 - 2iu\theta k + u^2\sigma^2 k})\right), \quad u \in \mathbb{R}, \tag{10}$$

for μ, $\theta \in \mathbb{R}$ and $\sigma, k > 0$.

It follows by differentiation of the (log of the) characteristic function that the first four cumulants of the NIG process are

$$\begin{aligned} c_1 &= (\mu + \theta)t, & c_2 &= \left(\sigma^2 + \theta^2 k\right)t, \\ c_3 &= 3\theta k\left(\sigma^2 + \theta^2 k\right)t, & c_4 &= 3k\left(\sigma^4 + 6\sigma^2\theta^2 k + 5\theta^4 k^2\right)t. \end{aligned}$$

From the above, we observe that θ primarily controls the sign of the skewness of the process distribution, σ affects the overall variability, and k primarily controls the kurtosis of the distribution. The drift parameter μ affects the mean of the distribution, which otherwise would be concordant with the skewness, allowing us to model return distributions with positive means and negative skewness as well (and vice versa). Finally, the tails of the distribution are characterized by a power-modified exponential decay, or semi-heavy tail (see, e.g., Cont and Tankov (2004)).

As the density function is known in (semi-)closed form (as it is expressed in terms of the modified Bessel function of the second kind, see, e.g., Cont and Tankov (2004)), the parameters of the NIG model can be estimated directly using maximum likelihood (ML) estimation, initialized via the method of moments based on the first four theoretical cumulants derived above.

A Lévy jump diffusion process has the form

$$\mu t + \sigma W_t + \sum_{i=1}^{N_t} J_i, \tag{11}$$

where W_t is a standard Brownian motion, N_t is a Poisson process with rate $\lambda > 0$ counting the jumps of the overall process, and $\{J_i\}_{i \in \mathbb{N}}$ are IID random variables describing the jump sizes. All the random quantities involved, W_t, N_t, and J_i (for all i), are assumed to be mutually independent. In the MJD model (Merton 1976) jump sizes are all normally distributed (i.e., $J_i \sim \mathrm{N}(\nu, \tau^2)$, $\nu \in \mathbb{R}$, $\tau > 0$, for all i). It follows that the characteristic function is

$$\phi(u;t) = \exp\left(iu\mu t - \frac{u^2\sigma^2}{2}t + \lambda t\left(e^{iu\nu - \frac{\tau^2 u^2}{2}} - 1\right)\right), \quad u \in \mathbb{R}. \tag{12}$$

The first four cumulants are

$$\begin{aligned} c_1 &= (\mu + \lambda\nu)\, t, & c_2 &= (\sigma^2 + \lambda(\nu^2 + \tau^2))t, \\ c_3 &= \lambda\nu(3\tau^2 + \nu^2)t, & c_4 &= \lambda(3\tau^4 + 6\tau^2\nu^2 + \nu^4)t. \end{aligned}$$

We can observe that the parameters λ, ν, and τ control the non-Gaussian part of the process; in particular, ν primarily controls the sign of skewness (the density function is symmetric when $\nu = 0$), whilst λ governs the jumps frequency and, therefore, the level of excess kurtosis. Further, the MJD process has an infinite Gaussian mixture distribution with mixing coefficients given by a Poisson distribution with parameter λ; hence, the probability density function can be expressed as a fast converging series. Finally, the tails are heavier than in the pure Gaussian case (see, e.g., Cont and Tankov (2004)).

We note that the estimation of the MJD model is far from trivial as the ML method requires a careful numerical optimization, as discussed in Honoré (1998). Consequently, in the numerical study we implement the expectation maximization (EM) algorithm in the formulation proposed by Duncan, Randal, and Thomson (2009), which has simple closed form solutions for the M-step.

We conclude this review by highlighting the main difference between the NIG and the MJD processes. Although they both cater to small movements occurring with a high frequency (i.e., they are infinite variation processes), in the MJD process these movements are generated by the Brownian motion and, therefore, are Gaussian (skewness and kurtosis are generated by the 'big' jumps controlled by the compound Poisson process part). In the NIG process, instead, these small movements are purely discontinuous and, therefore, their distribution is already skewed and leptokurtic.

Using the "all-NIG" and "all-MJD" processes described above we generate data for our simulations. The first objective of our simulation is to evaluate the computation efficiency gains of the 2-step approach versus the 1-step approach. Our second objective is to assess the error, bias, and efficiency of the parameter estimates obtained by adopting the proposed 2-step approach. We perform simulation experiments for different data generating processes, sample sizes and number of assets in the portfolio. The simulations show that the 2-step approach has very large computation efficiency gains relative to the 1-step approach. Our most conservative test shows that after controlling for errors, parameter estimation using the 2-step approach is more than 3,000 times faster than using the 1-step method. Moreover, the 2-step estimation has good finite sample properties with low bias and root mean squared errors that decrease with increased sample size. The number of assets included in the portfolio has a minimal impact on the estimation errors and the 2-step approach provides reliable estimates even for samples with more than 15 assets in the portfolio, case for which the 1-step approach becomes imprecise and often non-computable. Detailed simulation procedures and results including tables are presented in Appendix A.

IV. Application: Portfolio Risk Measures, VaR, and Intra-Horizon VaR

In this section, we illustrate our estimation method for the computation of portfolio risk measures like Value at Risk (VaR) and intra-horizon Value at Risk (VaR-I).

Trading portfolios of financial institutions can be adversely exposed to a multitude of risk factors that may lead to extreme losses. Therefore, asset managers are required to maintain a particular level of reserves as protection against these trading losses. VaR, defined as the lower tail percentile for the distribution of returns, is the market risk measure recommended by U.S. and international banking regulators to estimate the minimum capital requirements. For example, the Basel Capital Accord amended in 1996 established that the minimum capital requirement on a given day is equal to the sum of a charge to cover credit risk and a charge to cover general market risk, where the market-risk charge is equal to a multiple of the average reported 2-week VaRs in the last 60 trading days. A drawback of VaR estimates is that they do not take into consideration the magnitude of possible losses incurred before the end of the specified trading horizon. An improved alternative risk measure is the Var-I, as it takes into account the exposure to losses throughout the investment's life of the portfolio.

Estimation of tail risk measures as VaR and VaR-I crucially depends on modeling financial assets under realistic distributional assumptions. Market participants and regulators are likely to be concerned about the effects of event risk and sudden large trading losses or jumps. Therefore, the use of the traditional Brownian motion framework based on normality will likely understate such market risk. Lévy processes offer a natural and robust approach to incorporate jumps that can accommodate the levels of skewness and excess kurtosis observed in financial data, in particular over short horizons.

Moreover, asset managers will be concerned not only about the risk measures of the whole portfolio, but also about the risk contribution of each asset in the portfolio. For example, an active portfolio manager will be interested in evaluating the effect on the portfolio risk profile of a change of the portfolio weight of a given asset. This breakdown of the contribution to risk represents an invaluable 'drill-down' exercise that enables managers to better control their risk profiles. However, it requires a multivariate model on the one hand capable of incorporating the impact of dependence between the assets in the portfolio, and on the other hand sufficiently flexible to cater different returns' distributions for each asset in the portfolio, a feat that lies out of the range of a univariate setting.

Therefore, the methods developed in this paper are particularly suitable for the estimation of portfolio VaR and VaR-I, as they accommodate realistic distributional assumptions, including jumps, in a multivariate setting that allows for the evaluation of the risk contribution of each individual asset in the portfolio.

In this section we first provide a step-by-step general procedure to estimate VaR and VaR-I for portfolios following a multivariate Lévy model (1). Then we apply the proposed estimation method to a portfolio of the 20 most capitalized stocks in the S&P 500, using the two Lévy models' specifications introduced in Section III (i.e., the "all-NIG" model and the "all-MJD" model). For comparison, we also consider the case in which all assets follow a normal distribution ('all-Gaussian' model). After computing the model parameters and relevant risk measures, we assess the quality of our estimation and conclude the section with the identification of the risk contribution of each asset in the portfolio.

The results presented in the following of this section show that the multivariate Lévy models correctly capture the observed distribution of portfolio returns, with important improvements over the Gaussian model. Estimation under multivariate Lévy models in general provide most conservative risk estimates, with a VaR and VaR-I between 3.5% and 10% larger than using a Gaussian model. Finally, we identify that in our sample the assets with larger risk contribution to the portfolio risk are stocks of financial institutions.

A. Estimation of VaR and Var-I under Multivariate Lévy Models

The intra-horizon risk Value at Risk, VaR-I, is defined on the distribution of the minimum return. Thus, let X_t, for $t \in [0,T]$, be the real-valued random process describing possible paths of an instrument or portfolio log-return over the interval $[0,t]$; without loss of generality, we set $X_0 = 0$. For practical implementation, let us assume that the process is observed on an equally spaced time grid $0, \Delta, \dots, K\Delta = T$. In standard financial applications Δ is set at 1 day and $K\Delta = T$ is 10 days. Define the process of the minimum, M_k, up to the k-th monitoring date as $M_k := \min_{i=0,\dots,k} X_{i\Delta}$. The VaR-I at confidence level $(1-\alpha)$ is defined as the absolute value of the α-quantile of the distribution of the random variable

M_k, that is,

$$P(M_k \leq -\text{VaR-I}|X_0 = x) = \alpha. \tag{13}$$

The idea is that during the investment life the path of returns can reach high negative values, which investors may care about. In such cases, the left tail of the minimum return distribution better represents risk than the left tail of the return distribution itself.

While under the assumption of an arithmetic Brownian motion the distribution of the minimum return is analytically known (see Kritzman and Rich (2002) for the case with continuous monitoring, and Fusai, Abrahams, and Sgarra (2006) for the discrete monitoring one), under more general assumptions for the driving process it must be recovered numerically (see Fusai, Germano, and Marazzina (2016) for a review and comparison of different approaches in computing the distribution of the minimum under Lévy processes). To this purpose, we resort to the Fourier space time-stepping (FST) algorithm introduced by Jackson et al. (2008) for option pricing purposes. Our problem is indeed equivalent to finding the value of a down-and-out binary option, that is an option paying 1 if the underlying does not hit a certain lower barrier within a given time period, and 0 otherwise. However, due to the nature of the application under consideration, our computations are performed under the physical probability measure.

With a fixed arbitrary threshold y, the FST algorithm allows us to recover the value function

$$v(0, x) := \mathrm{E}[1_{\{M_K > y\}}|X_0 = x] = P(M_K > y|X_0 = x)$$

via backward recursion so that

$$\begin{aligned} v^K(x) &:= v(T, x) = 1_{\{x>y\}}, \\ v^{k-1}(x) &= \mathrm{FFT}^{-1}[\mathrm{FFT}[v^k(x)]e^{\varphi\Delta}]1_{\{x>y\}}, \quad k = 1, \ldots, K, \end{aligned} \tag{14}$$

where φ is the characteristic exponent of X_t, $\mathrm{FFT}(X)$ computes the discrete Fourier transform of the vector X using the fast Fourier transform (FFT) algorithm and $\mathrm{FFT}^{-1}(\cdot)$ denotes the inverse discrete Fourier transform. For further details on the FST algorithm, we refer to Jackson et al. (2008). Further, in virtue of the translation invariance property of Lévy processes, it follows that $v(0, x) = P(M_K > y - x|X_0 = 0)$. Hence, the computation of the $(1 - \alpha)$-VaR-I can be summarized in the following steps.

Step 1. Choose an arbitrary threshold y.

Step 2. Compute the function $v(0, x) = v^0(x)$ by means of the FST algorithm.

Step 3. Find the value x such that $v(0, x) = 1 - \alpha$.

Step 4. Compute the VaR-I as VaR-I$= -(y - x)$.

The implementation of the FST iteration (14) requires the expression of the characteristic function of the process X_t of the log-return of a portfolio of assets with weights w_n. In the given multivariate setting (1), for short time horizons, this expression can be easily derived in virtue of the approximation of the portfolio returns as linear combinations of the asset log-returns. Exploiting the independence of the idiosyncratic process, $Y^{(n)}$, $n = 1, \ldots, N$ and the systematic process, Z, we can in fact obtain

$$\begin{aligned} \mathrm{E}\left[\exp\left(iuX\right)\right] &= \mathrm{E}\left[\exp\left(iu\left(\sum_{n=1}^{N} w_n Y^{(n)} + Z\sum_{n=1}^{N} w_n a_n\right)\right)\right] \\ &= \left(\prod_{n=1}^{N} \phi_{Y^{(n)}}\left(uw_n\right)\right)\phi_Z\left(u\sum_{n=1}^{N} w_n a_n\right) \quad u \in \mathbb{R}, \end{aligned} \tag{15}$$

where we omit time subscripts to simplify the notation[3]. The characteristic functions in equation (15) are then chosen according to the specified model.

B. Estimation Results

We estimate the 10-days 99% VaR and VaR-I for an equally-weighted portfolio under the "all-NIG", "all-MJD" and 'all-Gaussian' (henceforth Gaussian) models. We include in the portfolio 20 of the most capitalized stocks[4] in the S&P 500 index. Our sample includes daily log-returns, from May 24, 2011 to May 20, 2013.

We start by estimating the parameters of the characteristic functions of X_t required for implementation of the FST iteration as described above. We apply the 2-step procedure presented in Section II.B. We estimate the number of factors using Ahn and Horenstein (2013) method and find one factor, which we estimate via the principal component method. We do not report full estimation results, due to the large number of parameters (complete parameter estimation results are available from the authors). Instead in Table 1 we report the estimated mean, standard deviation, Pearson's moment coefficient of skewness, and index of excess kurtosis of the returns distribution using both the 'all NIG' and "all-MJD" estimated model parameters for a selection of assets and the equally-weighted portfolio of all 20 assets. Our objective is to assess the assumption of non-normality of returns by testing if skewness and excess kurtosis are significantly different from 0. For the test we use bootstrap (Efron (1979)) to generate 5,000 re-sampled data sets from our observed one, and we estimate the model parameters on each of the re-sampled data sets. Then, we compute the moments of the distribution of the 20 stocks across the 5,000 data sets according to the model; the α-confidence levels for a given moment are built using the $(1-\alpha)/2$ and $(1+\alpha)/2$ quantiles of that moment over the 5,000 data sets. Based on these confidence intervals, in Table 1 the *, **, and *** represent statistical significance at the 90%, 95%, and 99% levels, respectively.

Results in Table 1 show that in general the skewness is not significantly different from 0, indicating generally symmetric assets' distributions; however, the excess kurtosis is always statistically significant, indicating heavier tails than in the case of normal distributed assets'

[3]If returns are very volatile or the horizon is longer, it becomes essential to work with linear returns. In this case equation (15) no longer holds. For more details, see Meucci (2005).

[4]Apple, Exxon Mobil Corporation, Wal-Mart Stores, Microsoft Corporation, Google, General Electric, IBM, Chevron Corporation, Berkshire Hathaway, AT&T, Procter & Gamble, Pfizer, Johnson & Johnson, Wells Fargo & Co., Coca-Cola, JPMorgan Chase & Co., Oracle, Merck & Co., Verizon Communications, and Amazon.

Table 1: **Estimated Distribution Characteristics**
We report the estimated daily mean (expressed in basis points), standard deviation, Pearson's moment coefficient of skewness, and index of excess kurtosis for the returns of a selection of assets and the equally-weighted portfolio under the 'all NIG' and "all-MJD" estimated models. *, **, and *** represent statistical significance at the 90%, 95%, and 99% levels, respectively (confidence levels obtained by bootstrap resampling techniques with 5,000 iterations). We also report the loadings **a**.

	Apple	Google	AT&T	Coca-Cola	Amazon	Portfolio
Panel A. 'all NIG'						
Mean (bps)	0.0174	9.1666	6.9314	1.8355	7.9301	8.3977
Std. Dev.	0.01887***	0.01537***	0.01037***	0.00977***	0.02147***	0.01387***
Skew.	0.0789	-0.2719	-0.4215**	-0.1067	0.3198	-0.0433
Exc. Kurt.	1.9837***	3.1409***	2.5857***	1.7734***	8.3284***	4.8165***
Panel B. "all-MJD"						
Mean (bps)	0.0175	9.1666	6.9313	1.8355	7.9301	8.3978
Std. Dev.	0.01897***	0.01597***	0.01067***	0.00967***	0.02307***	0.01407***
Skew.	-0.1205	-0.5464	-0.9966**	-0.0147	0.2459	-0.1640
Exc. Kurt.	3.1272***	5.8531***	5.1519***	0.9730***	10.7512***	7.7416***
a	0.0087***	0.0092***	0.0072***	0.0067***	0.0097***	0.0092***

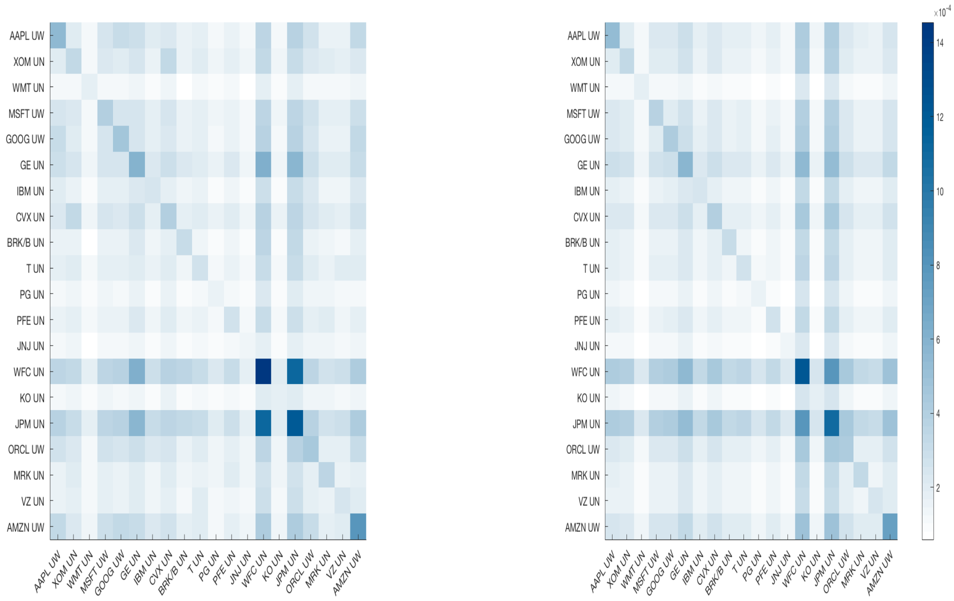
returns. The features of the returns of the equally-weighted portfolio indicate that the non-normality inherited from each asset is persistent regardless of the level of diversification in place. These results confirm the importance of modeling asset returns under realistic distributional assumptions away from normality. The differences in the level of skewness and excess kurtosis between the 'all NIG' and "all-MJD" models reflect the different flexibility offered by the processes in portraying the distribution tails.

Next, we evaluate if our fitted multivariate Lévy model is able to capture the dependence observed in real data. To do so, we compare the sample covariance of the assets in the portfolio with the covariance matrix estimated assuming model (1) for the "all-NIG" case. Figure 1 shows the two covariances, using two color-coded matrices in which each entry is colored according to its value, and the conversion color-code is provided in the lateral color bar. We notice that the "all-NIG" model accurately reproduces the sample covariance among the assets in our data set. Similar results are obtained for the "all-MJD" model (available from the authors).

To further explore if our fitted multivariate Lévy models correctly capture the observed distribution of portfolio returns, we perform a simulation exercise. We randomly generate 1,000 long–only portfolios and 1,000 long–short portfolios using our estimated parameters for the "all-NIG" and "all-MJD" models. For comparison we also generate such portfolios under a Gaussian model. We compare the simulated distributions with the observed sample distribution (as in, e.g., Eberlein and Madan (2009) and Luciano, Marena, and Semeraro (2016)). Long-only weights are generated by drawing an IID sample from a standard normal distribution, taking the absolute value and rescaling by the sum. Long–short portfolio weights are generated similarly, drawing an IID sample from a standard normal distribution

Figure 1: **Sample and Estimated Covariance Matrices**
Figure 1 shows the sample and estimated covariance matrices for the 'all NIG' model. Estimation of the common factor Z is performed via the principal components method. The color-values conversion is provided in the side color bar. Constituents: Apple, Exxon Mobil Corporation, Wal-Mart Stores, Microsoft Corporation, Google, General Electric, IBM, Chevron Corporation, Berkshire Hathaway, AT&T, Procter & Gamble, Pfizer, Johnson & Johnson, Wells Fargo & Co., Coca-Cola, JPMorgan Chase & Co., Oracle, Merck & Co., Verizon Communications, and Amazon.

Graph A. Sample Covariance Matrix *Graph B. Estimated Covariance Matrix*



and rescaling it by the sum of the squares. We perform the Kolmogorov–Smirnov test,[5] with the null hypothesis that the simulated distribution is drawn from the sample distribution. The results, presented in Table 2, show the proportion of portfolios for which the null hypothesis is rejected using 1%, 5%, and 10% significance levels for all different models. We note that both Lévy-based models significantly outperform the Gaussian one; in fact for the case of long–only portfolios the Gaussian model is rejected 100% of the time. Further, the "all-NIG" and "all-MJD" model specifications fit the sample distribution of returns for both long–only and long–short portfolios equally well.

Once we have confirmed that our estimated model is able to successfully replicate the observed distribution of portfolio returns, we proceed to estimate the portfolio VaR and VaR-I following the four steps described in Section IV.A. In Table 3 we report our estimates of 10-day 99% VaR and intra-horizon VaR and the corresponding confidence intervals at the 95% level computed using bootstrap resampling methods. The confidence intervals of the portfolio VaR/VaR-I are calculated using the quantiles of the VaR/VaR-I across the 5,000 instances stemming from the bootstrapped samples. We also report the ratio between the Lévy model estimate and the Gaussian model estimate (Multiples). We observe that VaR-I consistently exceeds the traditional VaR, and that jump risk tends to amplify intra-horizon

[5]We derive the probability density function by inverting the portfolio characteristic function (15) using the COS method of Fang and Oosterlee (2008).

Table 2: **Goodness of Fit Test**
We report the proportion of simulated portfolios for which the null hypothesis of the Kolmogorv–Smirnov test is rejected at the 1%, 5%, and 10% significance levels. The null hypothesis is that the simulated distribution is drawn from the sample distribution. Portfolios are generated under the "all-NIG", "all-MJD", and Gaussian models. Constituents: Apple, Exxon Mobil Corporation, Wal-Mart Stores, Microsoft Corporation, Google, General Electric, IBM, Chevron Corporation, Berkshire Hathaway, AT&T, Procter & Gamble, Pfizer, Johnson & Johnson, Wells Fargo & Co., Coca-Cola, JPMorgan Chase & Co., Oracle, Merck & Co., Verizon Communications, and Amazon.

	Long–Only			**Long–Short**		
Significance Level	"all-NIG"	"all-MJD"	Gaussian	"all-NIG"	"all-MJD"	Gaussian
0.01	0.00%	0.00%	100%	0.00%	0.00%	65.10%
0.05	0.00%	0.00%	100%	0.80%	0.70%	74.20%
0.10	0.00%	0.00%	100%	2.10%	2.00%	79.00%

risk.

These results indicate that the pure-jump "all-NIG" model has the (marginally) thickest tails for both the return and the minimum return distributions, and thus provides the most conservative risk estimates, with a VaR 1.09 times higher than the VaR under the Gaussian model and a VaR-I about 1.11 times higher with respect to the Gaussian one. The VaR and the VaR-I under the jump-diffusion "all-MJD" specification are respectively 1.03 and 1.04 times higher with respect to the corresponding measures under the Gaussian model. These results reflect the slower decay in the distribution tails of the NIG and the MJD processes compared to the Brownian motion, as discussed in Section II.A.

As a final consideration, we observe the following. Although in a few cases Table 1 highlights fairly different estimates for skewness and excess kurtosis in the distribution of each assets between the "all-NIG" and the "all-MJD" models, the relatively similar figures reported in Table 3 for the risk measures of interest show the effect of the allocation in place in the portfolio, which diversifies away the tail risk of the idiosyncratic components but maintains the exposure to the tail risk of the systematic risk factor. This also suggests that a change in the portfolio's weights can potentially generate a significant change in the overall risk measure depending on which asset becomes "predominant" so to speak. This latter effect can only be captured by means of a multivariate model for the returns of each portfolio's components, as opposed to a univariate model for the overall portfolio returns.

Motivated by the previous analysis, we conclude this section with the identification of the risk contribution of each asset to the whole portfolio risk. This is especially relevant for active risk portfolio managers, who require identification of the effects of changes in portfolios' weights on the overall portfolio risk in order to modify the overall risk profile most effectively.

The study is based on a sensitivity analysis of the VaR-I with respect to the portfolio weights, which is performed by finite difference. For illustration purposes, we consider the case of an equally-weighted portfolio; for each asset in the portfolio we perturb its weight

Table 3: **VaR and VaR-I Estimates**
We report the 10-day horizon 99% VaR and VaR-I of an equally-weighted portfolio under the "all-NIG", "all-MJD", and Gaussian models, with confidence intervals at the 95% level. The common factor Z is estimated via the principal component method. Confidence intervals are computed using bootstrap resampling methods (5,000 iterations). Multiples: Lévy model estimate/Gaussian model estimate. Constituents: Apple, Exxon Mobil Corporation, Wal-Mart Stores, Microsoft Corporation, Google, General Electric, IBM, Chevron Corporation, Berkshire Hathaway, AT&T, Procter & Gamble, Pfizer, Johnson & Johnson, Wells Fargo & Co., Coca-Cola, JPMorgan Chase & Co., Oracle, Merck & Co., Verizon Communications, and Amazon.

	VaR				**VaR-I**			
	Estimate	CI(lb)	CI(ub)	Multiples	Estimate	CI(lb)	CI(ub)	Multiples
Gaussian	0.0699	0.0572	0.0831	1.0000	0.0738	0.0614	0.0869	1.0000
"all-MJD"	0.0723	0.0579	0.0888	1.0341	0.0769	0.0630	0.0929	1.0413
"all-NIG"	0.0764	0.0599	0.0961	1.0939	0.0818	0.0652	0.1016	1.1085

Table 4: **Asset's Individual Risk Contribution to the Equally-Weighted Portfolio VaR-I**
We report the component VaR-I, that can be interpreted as the percentage increase in VaR-I for a 1% increase in the weight of the asset in the portfolio. Finite difference is calculated with perturbation 1/100.

Asset	"all-NIG"	"all-MJD"
Apple	4.88%	4.93%
Exxon Mobil Corporation	5.37%	5.38%
Wal-Mart Stores	2.44%	2.24%
Microsoft Corporation	4.88%	4.93%
Google	4.88%	4.93%
General Electric	6.34%	6.73%
IBM	4.39%	4.48%
Chevron Corporation	5.85%	5.83%
Berkshire Hathaway	5.85%	5.83%
AT&T	3.41%	3.59%
Procter & Gamble	2.93%	2.69%
Pfizer	4.39%	4.48%
Johnson & Johnson	2.93%	3.15%
Wells Fargo & Co.	8.29%	8.07%
Coca-Cola	3.41%	3.59%
JPMorgan Chase & Co	8.78%	8.97%
Oracle	7.32%	7.17%
Merck & Co.	4.39%	4.04%
Verizon Communications	2.93%	3.14%
Amazon	6.34%	5.83%

by 1/100 and then recompute the VaR-I. The change in the VaR-I (marginal VaR-I) is the discrete analogue of a derivative. If we multiply marginal VaR-I by the asset weight in the portfolio and divide this product by the original VaR-I, we obtain a percentage measure of the risk contribution of this asset, the so-called Component VaR-I. This measure can be interpreted as the percentage increase in VaR-I for a 1% change in the weight of a given asset in the portfolio. Notice, that given any pre-specified portfolio, risk measures and risk attribution are obtained without re-estimating the underlying multivariate model parameters.

Results on the Component VaR-I of the 20 assets in the portfolio considered in this Section are reported in Table 4. The decomposition highlights the positions that the portfolio is most sensitive to. Interestingly, JPMorgan Chase & Co. and Wells Fargo & Co. are the two assets that contribute the most to the portfolio risk measured by VaR-I. A 1% increase in the portfolio weight of JPMorgan Chase & Co. increases VaR-I by almost 9%. The asset with the smallest Component VaR-I is Wal-Mart stores, with an increase of about 2.5% when its weight in the portfolio is increased by 1%. There are no significant differences between the "all-NIG" and the "all-MJD" models, which suggests that the proposed estimation method is robust with respect to the model choice. Similar results can be obtained for the case of VaR and are available from the authors.

In unreported experiments we repeat the estimation using the S&P 500 index as a proxy of the common factor. The estimated VaR and VaR-I are close to those obtained by estimating the common factor via principal components, displayed in Table 3. However, the ability of the model to fit the portfolio distribution, measured by the Kolmogorov-Smirnov test on the distributions of 1,000 randomly generated long-only portfolios and 1,000 randomly generated long–short portfolios, turns out to be substantially lower than in the case in which the principal component method is adopted for the estimation of the common factor. Results are available from the authors.

V. Conclusions

We propose an estimation procedure for multivariate asset models based on linear transformation of Lévy processes as in Ballotta and Bonfiglioli (2016), allowing for an extension of the use of multivariate Lévy models to risk and portfolio management applications. We note that factor constructions are in line with recommendations from the Basel Committee on Banking Supervision (Basel (2013)) for the development of internal models.

For the case of an N-asset portfolio, the 2-step estimation procedure proposed in this article reduces to the estimation of the common factor Z and the loadings $\boldsymbol{a}$ via principal components, and N univariate estimations, one per each idiosyncratic component; therefore, it is fast to implement and its complexity does not increase with the number of components of the multivariate model. Our simulation study reveals that this approach is almost as accurate as a more traditional direct maximum likelihood estimation of the whole set of parameters, as long as proper univariate estimation methods are used; however, the 2-step procedure proves to be significantly more efficient from the computational point of view.

The proposed approach is flexible with respect to the number of assets included in the portfolio and does not impose any convolution condition on the factors, as it is assumed in other multivariate constructions proposed in the literature. Although in the numerical studies presented in this paper we conveniently assume that all factors are modelled using the same type of process, this assumption can be relaxed as to allow any Lévy process for the idiosyncratic part across all the names included in the portfolio in order to accommodate different tail behaviors.

As an application, we employ the proposed estimation procedure for the calculation of the intra-horizon Value at Risk of a portfolio of assets following the model under consideration by means of the FST algorithm. The numerical study reveals the importance of properly capturing realistic features of asset log-returns, such as skewness and excess kurtosis, by incorporating jumps in the risk dynamic. Results from the empirical study, in fact, highlight the more conservative risk estimates offered by the intra-horizon VaR especially for the case of the NIG, reflecting the different decay behavior of the distribution tails.

Due to the short horizons typical of risk management operations, in this paper we have considered the case of a non-Gaussian multivariate model built on Lévy processes (i.e., processes characterized by independent and stationary increments). For applications aimed at longer horizons, stochastic volatility features are required. Lévy processes can be conveniently equipped with such features by means of suitably constructed time-changes, as proposed by Carr and Wu (2004), (2007), for example, and more recently extended by Ballotta and Rayée (2017). Although multivariate extensions of the time changed Lévy processes framework are currently investigated, in the case in which a similar factor structure is adopted, we envisage the potential of the 2-step methodology proposed in this paper for the estimation under the physical probability measure. Analysis of the validity of conditions ensuring the consistency of the estimated factors and loadings in this context is left to future research. Finally, we observe that the methodology proposed in this paper can also be directly applied to other areas such as portfolio optimization problems based on multivariate Lévy processes (as, e.g., in Loregian (2013)).

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A. Detailed Simulation Procedures and Results

In Appendix A, we provide detailed procedures and complete results of our simulation experiments designed to evaluate the performance of the proposed methodology. We present our results in comparison with the 1-step estimation approach (i.e., via single maximization of the likelihood function (4)). As the detailed simulation study of Bai (2003) shows that the factor estimation method has strong small sample properties, without loss of generality our simulation study assumes that the systematic risk factor is proxied by a well diversified index[6].

Our data generation process is based on daily log-returns of the S&P 500 index and a selection of its constituent stocks; further, we assume that the S&P 500 index is the true driver of the commonality in stocks returns. The observation period ranges from Sept. 10, 2007 to May 20, 2013, for a total of 1,434 observations per series. These data are extracted from Bloomberg and adjusted for dividends. We first estimate the chosen multivariate model using the index log-returns as proxy for the systematic process Z and use the estimated parameters to simulate series of the returns of the assets under consideration. Then the 1-step and 2-step estimation procedures are applied to the generated data to recover the distribution of each parameter.

1 Computation Efficiency

The first objective of our simulation is to evaluate the computation efficiency gains of the 2-step approach versus the 1-step approach. We use estimation errors and estimation time to calculate an efficiency gain index commonly used in Monte Carlo simulation analysis and defined for example in Glasserman (2004). Given a specification of the model (1), characterized by $\bar{k}$ parameters, we compute the efficiency gain, E_{21}, of the 2-step procedure to the 1-step maximum likelihood approach as

$$E_{21} = \frac{\overline{\mathrm{MSE}}_1 \tau_1}{\overline{\mathrm{MSE}}_2 \tau_2}, \tag{A-1}$$

where $\overline{\mathrm{MSE}}$ denotes the average mean square error

$$\overline{\mathrm{MSE}} = \frac{\sum_{k=1}^{\bar{k}} \mathrm{MSE}(\hat{\theta}_k)}{\bar{k}},$$

of the parameters estimated by the 1-step (1) and the 2-step (2) approach. $\mathrm{MSE}(\hat{\theta}_k)$ is the mean square error for S simulation iterations

$$\frac{1}{S} \sum_{s=1}^{S} \left(\hat{\theta}_s - \theta \right)^2,$$

and $\tau_{1\ (2)}$ is the average time needed to estimate the model parameters using the 1 (2) approach. In particular, we compute the efficiency gain index corresponding to the "all-NIG" and "all-MJD" models with $N = 5$ and $N = 15$ components. In the case with 5 assets, for each of the two approaches we consider the mean square errors based on 1,000 simulations; for the case with 15 assets, we rely on 100 simulations only due to the computational cost of the 1-step procedure.

Results are reported in Table A1 for the case in which $T = 500$ (i.e., around 2 years of daily observations): we observe that the 2-step approach is significantly more efficient in terms of com-

[6] We also performed a small simulation study to check the reliability of the factor estimation method; we find that the average correlation between estimated factors and loadings with true factors and loadings is larger than 0.98. Also simulation results using estimated factors and loadings are in magnitude similar to the ones obtained assuming that the factor is observable as reported in this section.

Table A1: **Computation Efficiency Gains**
We report average MSE, computation times (measured in seconds), and efficiency gains of the 2-step approach to the 1-step maximum likelihood. $T = 500$ days and N is the number of assets in the portfolio. Efficiency gains are $E_{21} = \overline{\text{MSE}}_1\tau_1/(\overline{\text{MSE}}_2\tau_2)$ and $\bar{k}$ is the number of model parameters. (Processor: Intel(R) Core(TM) i5-2400 CPU @ 3.10GHz 3.10 GHz; RAM: 4.00 GB).

			2-Step		**1-Step**		
N	Model	$\bar{k}$	$\overline{\text{MSE}}$	**Time**	$\overline{\text{MSE}}$	**Time**	$\mathbf{E}_{21}$
5	**"all-NIG"**	29	0.0857	0.7	0.1407	3,668.5	8,139
	"all-MJD"	35	0.0014	1.0	0.0028	3,756.8	7,295
15	**"all-NIG"**	79	0.1043	1.9	0.0973	10,723.3	5,350
	"all-MJD"	95	0.0016	3.4	0.0017	11,087.5	3,496

putational time. Moreover, for $N = 5$ the average mean square errors attained with the 2-step approach are lower than those given by the 1-step procedure (8.5% vs 14% for the "all-NIG" model, 0.14% vs 0.28% for the "all-MJD" model), whilst they are almost the same for $N = 15$ (about 10% for the all-NIG' model, 0.16% for "all-MJD"). According to the efficiency index (A-1), in our experiment the 2-step procedure performed 3,496 times more efficiently than the 1-step approach in the worst case ("all-MJD", $N = 15$) and 8,139 times more efficiently in the best one ("all-NIG", $N = 5$).

2 Finite Sample Properties

Our second objective is to assess the error, bias, and inefficiency of the parameter estimates obtained by adopting the described estimation procedures. We assess the estimation procedure for four different sample sizes, varying the length of the simulated series from one year up to four years of daily observations and varying the number of components, considering up to 30 assets in the simulated markets.

The assessment is made in terms of root mean square error, bias, and inefficiency, respectively defined as

$$\text{RMSE}(\hat{\theta}) = \sqrt{\frac{1}{S}\sum_{s=1}^{S}\left(\hat{\theta}_s - \theta\right)^2}, \tag{A-2}$$

$$\text{bias}(\hat{\theta}) = \left|\text{E}[\hat{\theta}] - \theta\right|, \tag{A-3}$$

$$\text{ineff}(\hat{\theta}) = \sqrt{\frac{1}{S}\sum_{s=1}^{S}\left\{\left(\hat{\theta}_s - \text{E}[\hat{\theta}]\right)^2\right\}}, \tag{A-4}$$

where $\hat{\theta}$ indicates the estimates of the true parameter set θ used in the simulation step, and $\text{E}[\hat{\theta}] = \sum_{s=1}^{S}\hat{\theta}_s/S$.

We start by analyzing the finite sample properties of the 2-step estimation procedure proposed in Section II.B.

We first estimate the chosen multivariate model using the index log-returns as a proxy for the systematic process Z. Then, we use the estimated parameters to simulate series of the returns of the assets under consideration, which the estimation procedure is re-applied to. This allows us to recover the distribution of each parameter. We assess the estimation procedure in several cases, varying the length of the simulated series from 1 year up to 4 years of daily observations, $T = [250, 500, 750, 1{,}000]$, and varying the number of components, considering up to 30 assets in the simulated markets (N). For each of the 16 cases taken into account we repeat the simulation and estimation $S = 10{,}000$ times, obtaining 10,000 sets of parameters, denoted by $\hat{\boldsymbol{\theta}}_s, s = 1, \ldots, S$.

Given the large number of parameters (if $N = 5$ the total number of parameters is 29 for the "all-NIG" model and 35 for the "all-MJD" model; if $N = 30$ there are 154 parameters for the "all-NIG" model and 185 parameters for the "all-MJD" model), we cannot display detailed results for each asset; hence, for illustrative purposes, we show only the assessment results for the estimation of the common factor Z, the first idiosyncratic factor $Y^{(1)}$, and average results relative to the loadings $a_n, n = 1, \ldots, N$. Complete results are available from the authors. We stress that the focus of our simulation study is on investigating the effectiveness of splitting the estimation procedure of the multivariate model in the two steps presented in Section II.B.

Firstly, we analyze the finite sample properties of the estimated systematic component in our model.

Table A2 displays root mean square error, bias (expressed in percentage terms with respect to the true parameter value), and inefficiency of the estimators for the "all-NIG" and "all-MJD" models, as the length of the simulated series varies in $T = [250, 500, 750, 1{,}000]$. We observe, in general, a low level of bias for all of the estimators, meaning that the maximum likelihood estimators are suitable for the first step of our procedure. Analogous considerations hold in the "all-MJD" case, with estimators obtained by EM. Hence, errors and inefficiency levels can be used as benchmarks to evaluate Step 2.

In some more details, the bias for the "all-NIG" model is generally lower than 2% for sample sizes larger than 500. For smaller sample sizes, we observe only some problems in the estimation of the μ and θ parameters. As previously observed, these parameters control the mean of the process, which is well known to be very difficult to estimate in a reliable way. Concerning the "all-MJD" model, the bias appears to be larger, although still acceptable; the main issues are related to the estimation of the intensity and mean of the jump severities. However, the RMSEs are reasonably small for all parameters.

In general, consistent with the literature (see, e.g., Aït-Sahalia and Jacod (2011) and references therein), infinite activity processes, like the NIG, can be estimated in a more reliable manner than finite activity processes, such as the MJD process.

Then, we focus on the finite sample properties of the estimated idiosyncratic component in our model.

We implement Step 2 by solving first the minimization problem (8) with respect to the loadings $\boldsymbol{a}$; secondly, we use the estimated loadings as starting values to solve the N maximization problems (7) with respect to $\boldsymbol{\theta}_{Y^{(n)}}$ for all $n = 1, \ldots, N$. The minimization procedure (8) is performed by fixing the variance of the common factor equal to the sample variance of the simulated series of the process Z; in this way the assessment of this step turns out to be independent of the model specification.

The results of the estimation of the idiosyncratic process are presented in Table A3 for the case of the first instrument. Results relative to the other assets are available from the authors. In particular, the left-hand side of Table A3 displays root mean square error, bias, and inefficiency of the estimators when the total number of assets is fixed ($N = 30$) and the length of the simulated series varies in $T = [250, 500, 750, 1{,}000]$. On the right-hand side of the same table, we show the assessment results for a fixed $T = 500$, varying the number of assets. Consistently with the results shown above, Table A3 reveals almost similar estimation errors for $N = [5, 10, 15, 30]$, showing that the number of assets has only a minimal impact on the estimation errors of the idiosyncratic terms for both the specifications we tested. Further, results in Table A3 reveal very little bias implying that our estimation procedure performs as expected. Moreover, we observe estimation errors and inefficiency levels in line with those obtained in Step 1; therefore, splitting the estimation procedure

Table A2: **2-Step Procedure Assessment: Common Factor**
Estimation based on S&P daily log-returns. Observation period: Sept. 10, 2007–May 20, 2013. Source: Bloomberg. Estimation errors expressed in absolute terms. RMSE, Bias, Inefficiency: indices computed according to equations (A-2)–(A-4). Bias expressed in percentage terms with respect to the true parameter value.

Z	$T = 250$	$T = 500$	$T = 750$	$T = 1{,}000$
Panel A. "all-NIG" model				
$\mu = 0.0014$				
RMSE	9.85E-04	6.72E-04	5.42E-04	4.65E-04
Bias	3.10%	0.86%	1.32%	0.48%
Inefficiency	9.84E-04	6.71E-04	5.41E-04	4.65E-04
$\theta = -0.0014$				
RMSE	1.47E-03	1.02E-03	8.20E-04	7.12E-04
Bias	2.21%	1.73%	1.33%	0.35%
Inefficiency	1.47E-03	1.02E-03	8.20E-04	7.12E-04
$\sigma = 0.0168$				
RMSE	1.76E-03	1.23E-03	1.01E-03	8.77E-04
Bias	1.05%	0.51%	0.38%	0.28%
Inefficiency	1.75E-03	1.22E-03	1.01E-03	8.75E-04
$k = 3.32$				
RMSE	1.30E+00	8.97E-01	7.26E-01	6.32E-01
Bias	0.58%	0.25%	0.02%	0.17%
Inefficiency	1.30E+00	8.97E-01	7.26E-01	6.32E-01

Z	$T = 250$	$T = 500$	$T = 750$	$T = 1{,}000$
Panel B. "all-MJD" model				
$\mu = 0.0012$				
RMSE	8.24E-04	5.83E-04	4.66E-04	4.05E-04
Bias	2.34%	1.52%	2.07%	2.22%
Inefficiency	8.23E-04	5.83E-04	4.65E-04	4.04E-04
$\sigma = 0.0075$				
RMSE	1.17E-03	8.90E-04	7.41E-04	7.41E-04
Bias	0.97%	1.75%	1.71%	1.96%
Inefficiency	1.17E-03	8.80E-04	7.30E-04	7.27E-04
$\nu = -0.0025$				
RMSE	3.14E-03	1.90E-03	1.51E-03	1.28E-03
Bias	5.39%	6.87%	2.64%	2.65%
Inefficiency	3.13E-03	1.89E-03	1.51E-03	1.28E-03
$\tau = 0.0210$				
RMSE	3.56E-03	2.82E-03	2.39E-03	2.36E-03
Bias	2.53%	2.80%	2.44%	2.66%
Inefficiency	3.52E-03	2.76E-03	2.33E-03	2.30E-03
$\lambda = 0.47$				
RMSE	1.50E-01	1.07E-01	8.72E-02	7.88E-02
Bias	3.14%	4.13%	4.95%	4.28%
Inefficiency	1.49E-01	1.05E-01	8.40E-02	7.62E-02

into two steps, ease of implementation aside, proves to be effective.

As noted above, in this section we only discussed results relative to the first instrument; similar conclusions hold for all assets considered.

We conclude our simulation exercise by comparing the finite sample properties of the 2-step estimation versus the 1-step estimation procedures. Thus we repeat the simulation study for the estimation of the "all-NIG" and "all-MJD" models' parameters using the 1-step ML approach discussed in Section IIB, which represents a useful term of comparison to evaluate the results obtained from the 2-step procedure presented above. Hence, we use the same data set as above, but we relax the assumption that the systematic risk factor $\boldsymbol{Z}$ be observable.

The maximum likelihood estimation consists of maximizing the likelihood function (4); the quadrature of the integral in (4) is performed via the trapezoidal rule.

Due to the computational cost of the procedure highlighted above, we consider a small number of assets ($N = 5$, i.e., 29 parameters to be estimated for the "all-NIG" model, 35 for the "all-MJD" model) repeating the simulation 1,000 times; we then perform 100 simulations to evaluate the estimation for $N = 15$ assets (i.e., 79 parameters for the "all-NIG" model, 95 for the "all-MJD" model). Results relative to the common factor $\boldsymbol{Z}$, the first idiosyncratic component $Y^{(1)}$ and the first loading a_1, are displayed in Table A4. Complete results are available from the authors.

Bearing in mind the different number of simulations performed, we can compare the results of the 2-step procedure assessment with those presented in this section. In particular, for both the "all-NIG" and "all-MJD" models, the results relative to the common factor Z can be compared to those displayed in the second column of Table A2, corresponding to estimates based on $T = 500$ observations, while the results relative to the first idiosyncratic factor can be compared with those in the fifth and seventh columns of Table A3. In particular, we note that in the case of the "all-NIG" model the errors obtained with the 2-step procedure, using ML estimation, are in line with those obtained with the 1-step ML approach, which in principle, computational issues aside, should be the preferred method, exploiting at once all the information contained in the data. On the other hand, in the case of the "all-MJD" model, we observe that the errors of the 2-step procedure are just slightly larger than those obtained with the 1-step ML approach due to the fact that in the

Table A3: **2-Step Procedure Assessment: First Idiosyncratic Component**
Observation period: Sept. 10, 2007–May 20, 2013. Source: Bloomberg. Estimation errors expressed in absolute terms. RMSE, Bias, Inefficiency: indices computed according to equations (A-2)-(A-4). Bias expressed in percentage terms with respect to the true parameter value.

	$N = 30$				$T = 500$			
Y_1	$T = 250$	$T = 500$	$T = 750$	$T = 1{,}000$	$N = 5$	$N = 10$	$N = 15$	$N = 30$
Panel A. "all-NIG" model								
$\mu = 9.92E-04$								
RMSE	2.17E-03	1.13E-03	9.00E-04	7.69E-04	1.13E-03	1.12E-03	1.14E-03	1.13E-03
Bias	1.10%	0.28%	2.60%	0.47%	0.58%	0.01%	1.19%	0.28%
Inefficiency	2.17E-03	1.13E-03	9.00E-04	7.69E-04	1.13E-03	1.12E-03	1.14E-03	1.13E-03
$\theta = 2.15E-04$								
RMSE	2.45E-03	1.37E-03	1.09E-03	9.40E-04	1.39E-03	1.37E-03	1.40E-03	1.37E-03
Bias	4.06%	4.42%	16.08%	4.22%	5.44%	4.08%	8.40%	4.42%
Inefficiency	2.45E-03	1.37E-03	1.09E-03	9.40E-04	1.39E-03	1.37E-03	1.40E-03	1.37E-03
$\sigma = 0.0173$								
RMSE	1.39E-03	9.71E-04	7.97E-04	6.74E-04	9.65E-04	9.60E-04	9.61E-04	9.71E-04
Bias	1.20%	0.59%	0.46%	0.37%	0.61%	0.61%	0.54%	0.59%
Inefficiency	1.37E-03	9.66E-04	7.93E-04	6.71E-04	9.60E-04	9.54E-04	9.56E-04	9.66E-04
$k = 1.483$								
RMSE	6.19E-01	4.31E-01	3.48E-01	3.02E-01	4.29E-01	4.28E-01	4.28E-01	4.31E-01
Bias	1.37%	0.50%	1.01%	0.48%	0.95%	1.03%	0.75%	0.50%
Inefficiency	6.19E-01	4.31E-01	3.47E-01	3.02E-01	4.29E-01	4.27E-01	4.28E-01	4.31E-01
Panel B. "all-MJD" model								
$\mu = 0.00133$								
RMSE	1.10E-03	7.61E-04	6.12E-04	5.33E-04	7.55E-04	7.63E-04	7.57E-04	7.61E-04
Bias	0.70%	0.28%	0.57%	0.01%	0.12%	0.96%	0.18%	0.28%
Inefficiency	1.10E-03	7.61E-04	6.12E-04	5.33E-04	7.55E-04	7.63E-04	7.57E-04	7.61E-04
$\sigma = 0.01113$								
RMSE	1.36E-03	1.01E-03	8.76E-04	8.24E-04	1.03E-03	1.02E-03	1.01E-03	1.01E-03
Bias	0.70%	0.62%	0.44%	0.04%	0.41%	0.50%	0.68%	0.62%
Inefficiency	1.36E-03	1.01E-03	8.74E-04	8.24E-04	1.02E-03	1.02E-03	1.01E-03	1.01E-03
$\nu = -0.0004$								
RMSE	7.78E-03	3.26E-03	2.43E-03	2.03E-03	3.25E-03	3.18E-03	3.09E-03	3.26E-03
Bias	40.44%	8.25%	0.96%	1.40%	10.60%	11.05%	2.75%	8.25%
Inefficiency	7.78E-03	3.26E-03	2.43E-03	2.03E-03	3.25E-03	3.18E-03	3.09E-03	3.26E-03
$\tau = 0.02429$								
RMSE	6.20E-03	4.40E-03	3.81E-03	3.62E-03	4.46E-03	4.40E-03	4.38E-03	4.40E-03
Bias	0.82%	0.93%	0.93%	1.35%	1.14%	1.27%	0.91%	0.93%
Inefficiency	6.20E-03	4.39E-03	3.81E-03	3.60E-03	4.45E-03	4.39E-03	4.38E-03	4.39E-03
$\lambda = 0.29214$								
RMSE	1.61E-01	1.21E-01	1.03E-01	9.06E-02	1.21E-01	1.20E-01	1.21E-01	1.21E-01
Bias	5.53%	5.33%	4.02%	1.86%	5.28%	4.77%	5.55%	5.33%
Inefficiency	1.60E-01	1.20E-01	1.02E-01	9.04E-02	1.20E-01	1.19E-01	1.19E-01	1.20E-01

Table A4: **1-Step Approach Assessment**

Observation period: Sept. 10, 2007–May 20, 2013. Source: Bloomberg. Estimation errors expressed in absolute terms. RMSE, Bias, Inefficiency : indices computed according to equations (A-2)-(A-4). Bias expressed in percentage terms with respect to the true parameter value.

		$N = 5$ **(1,000 sim)**	$N = 15$ **(100 sim)**		$N = 5$ **(1,000 sim)**	$N = 15$ **(100 sim)**		$N = 5$ **(1,000 sim)**	$N = 15$ **(100 sim)**
Panel A. "all-NIG" model									
	Z			$\mathbf{Y}_1$			**First Loading**		
	$\mu = 0.0014$			$\mu = 9.92E-04$			$a_1 = 0.8898$		
RMSE		7.07E-04	5.67E-04		1.19E-03	1.11E-03		4.72E-02	3.85E-02
Bias		4.16%	2.98%		2.53%	15.88%		0.03%	0.18%
Inefficiency		7.04E-04	5.65E-04		1.19E-03	1.10E-03		4.72E-02	3.85E-02
	$\theta = -0.0014$			$\theta = 2.15E-04$					
RMSE		1.11E-03	8.64E-04		1.47E-03	1.52E-03			
Bias		6.89%	4.02%		7.49%	66.61%			
Inefficiency		1.10E-03	8.62E-04		1.47E-03	1.52E-03			
	$\sigma = 0.0168$			$\sigma = 0.0173$					
RMSE		1.24E-03	1.25E-03		9.95E-04	1.12E-03			
Bias		0.21%	0.43%		0.60%	1.14%			
Inefficiency		1.24E-03	1.25E-03		9.90E-04	1.10E-03			
	$k = 3.32$			$k = 1.483$					
RMSE		1.15E+00	8.35E-01		4.87E-01	4.59E-01			
Bias		2.68%	3.89%		0.82%	3.07%			
Inefficiency		1.14E+00	8.25E-01		4.87E-01	4.57E-01			
Panel B. "all-MJD" model									
	Z			$\mathbf{Y}_1$			**First Loading**		
	$\mu = 0.0012$			$\mu = 0.00133$			$a_1 = 0.8898$		
RMSE		6.24E-04	6.23E-04		7.85E-04	7.68E-04		5.44E-02	3.98E-02
Bias		8.04%	8.85%		3.41%	4.60%		1.21%	0.51%
Inefficiency		6.16E-04	6.13E-04		7.84E-04	7.65E-04		5.33E-02	3.95E-02
	$\sigma = 0.0075$			$\sigma = 0.01113$					
RMSE		1.14E-03	6.80E-04		1.12E-03	9.61E-04			
Bias		4.49%	3.16%		0.92%	0.48%			
Inefficiency		1.09E-03	6.38E-04		1.11E-03	9.59E-04			
	$\nu = -0.0025$			$\nu = -0.0004$					
RMSE		1.69E-03	1.85E-03		3.03E-03	3.19E-03			
Bias		3.27%	10.91%		23.86%	17.89%			
Inefficiency		1.69E-03	1.83E-03		3.03E-03	3.19E-03			
	$\tau = 0.0210$			$\tau = 0.02429$					
RMSE		2.72E-03	3.26E-03		4.60E-03	4.27E-03			
Bias		3.73%	7.95%		1.28%	3.59%			
Inefficiency		2.60E-03	2.80E-03		4.59E-03	4.18E-03			
	$\lambda = 0.47$			$\lambda = 0.29214$					
RMSE		2.33E-01	1.49E-01		1.36E-01	1.15E-01			
Bias		34.63%	18.80%		7.35%	4.12%			
Inefficiency		1.66E-01	1.20E-01		1.35E-01	1.14E-01			

2-step procedure the univariate estimations are performed via the less efficient EM algorithm.

This is visually confirmed in Figures A1 and A2, which report the maximum log-likelihood for each simulation (top panel), sorting the simulations by increasing values of the maximum likelihood for better clarity, and the histograms (bottom panel) of the two log-likelihood distributions generated by the two estimation methods. From Figure A1, we note that in the “all-NIG” case the estimates obtained by means of the 1-step and 2-step procedures lead to very close maximum log-likelihoods. Conversely, Figure A2 shows that for the “all-MJD” case the log-likelihoods resulting from the 2-step routine, where the univariate estimations are performed by the EM algorithm, are less close to the ones from the 1-step procedure (i.e., the actual maximum one).

Once more, it is important to notice that the comparison between the 1 and 2-step approaches can only be done for a small number of assets ($N \leq 15$) since the 1-step approach becomes imprecise and non-computable for larger N.

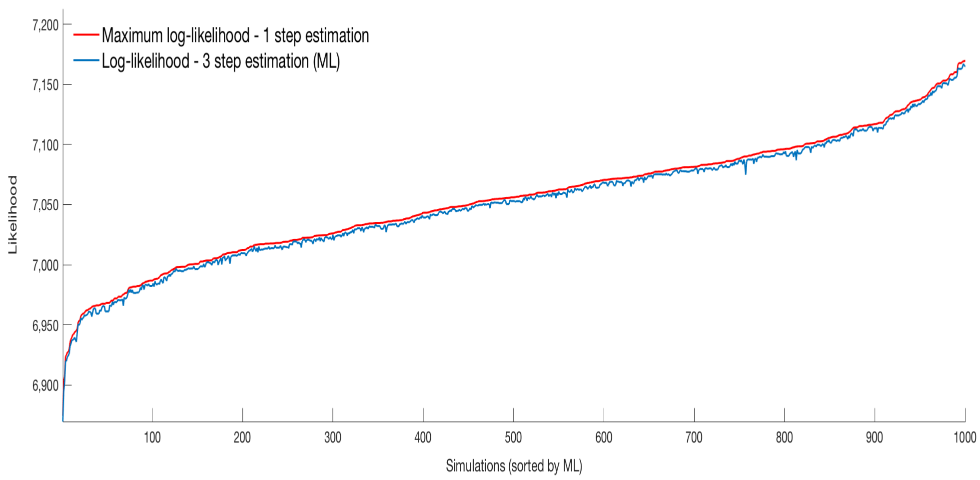
B. Brief Reference to the Literature

In this appendix we present a brief reference to the literature closely related to this paper. Evidence of pure Lévy jump risk representing a large share of uncertainty in stock returns has been put forward for example by Lee and Hannig (2010) and Ornthanalai (2014), among others. In particular, both Lee and Hannig (2010) and Ornthanalai (2014) highlight the role of infinite activity jumps, that is, jumps of small size occurring with high frequency, which in principle could be mis-identified as diffusions. These findings stress the need for hedging and risk management strategies equipped to face not just (rare) crash risk alone but also, and most importantly, risks associated with small and intermediate sized jumps. This issue is particularly relevant when risk management is targeted for short horizons such as the ones applied in the current regulatory risk management framework (10 days - see for example Basel (2010)). Indeed, over such short time horizons the effects of stochastic volatility are in general negligible (mainly due to the diffusive nature of the processes used for the modeling of volatility trends); thus, the jump component of the (log-)returns is relatively more important, as discussed in Aït-Sahalia (2004), for example. This explains our focus on risk management applications. Factor constructions such as the one proposed by Ballotta and Bonfiglioli (2016) have attracted attention mainly due to their simplicity and analytical tractability, which makes them particularly intuitive. Contributions in this direction started with Vašíček (1987) for the case of Brownian motions; alternative constructions for multivariate Lévy processes based on linear transformations have also been put forward by Luciano and Semeraro (2010) and extended in Luciano et al. (2016). However, the focus in these latter contributions is on Lévy processes with explicit representations in terms of subordinated Brownian motions, which are not always available. For a complete literature review, we refer to Ballotta and Bonfiglioli (2016), Luciano et al. (2016) and references therein.

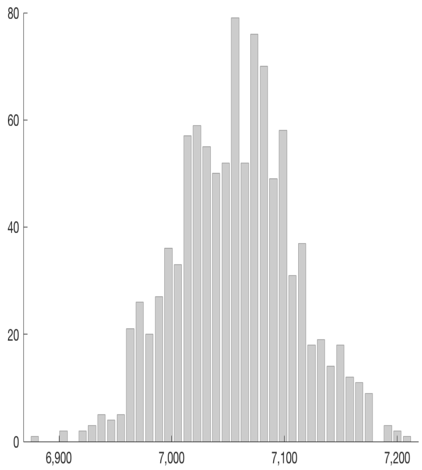
Figure A1: **Maximum Likelihood Comparison: 1-Step versus 2-Step Approach ("all-NIG" model)**

Graph A of Figure A.1 illustrates the maximum log-likelihood for each simulation (simulations sorted by increasing values of the maximum likelihood for better clarity). The bottom graphs display the histograms of the two log-likelihood distributions obtained by the 1-step (Graph B) and the 2-step (Graph C) estimation approaches. Plots are obtained by simulation of 1,000 samples, each made of 500 observations for the "all-NIG" model with 5 components.

Graph A. Log-likelihood comparison ("all-NIG" model)



Graph B. Maximum log-likelihood



Graph C. Log-likelihood – 2-step estimation

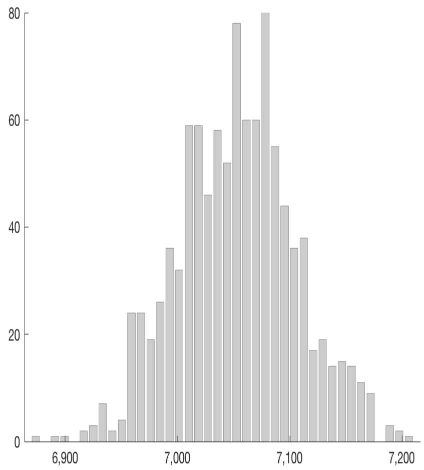
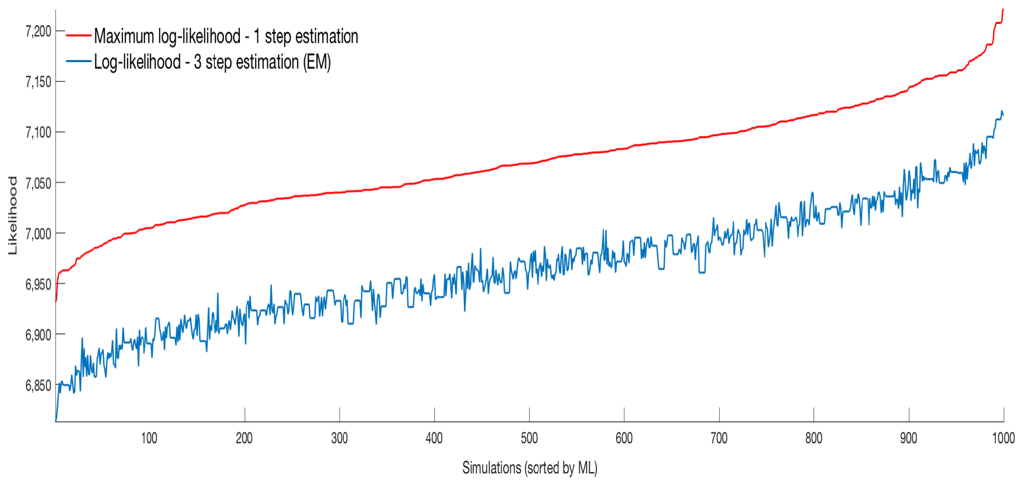
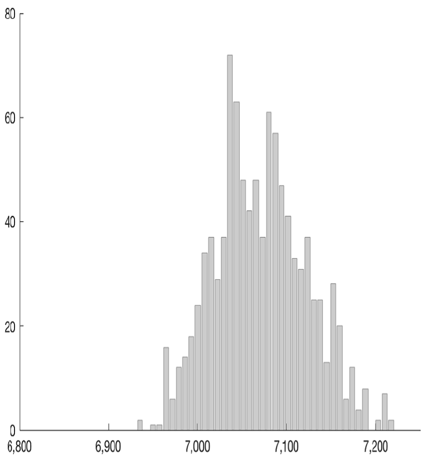


Figure A2: **Maximum Likelihood comparison: 1-Step versus 2-Step Approach ("all-MJD" model)**.
Graph A of Figure A.2 illustrates the maximum log-likelihood for each simulation (simulations sorted by increasing values of the maximum likelihood for better clarity). The bottom graph display the histograms of the two log-likelihood distributions obtained by the 1-step (Graph B) and the 2-step (Graph C) estimation approaches. Plots are obtained by simulation of 1,000 samples, each made of 500 observations for the "all-MJD" model model with 5 components.

Graph A. Log-likelihood comparison ("all-MJD" model)



Graph B. Maximum log-likelihood



Graph C. Log-likelihood – 2-step estimation