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Dade's ordinary conjecture implies the Alperin–McKay conjecture

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Abstract. We show that Dade's ordinary conjecture implies the Alperin–McKay conjecture. We remark that some of the methods can be used to identify a canonical height zero character in a nilpotent block.

Mathematics Subject Classification. 20C20.

Keywords. Dade's ordinary conjecture, Alperin–McKay conjecture, Height zero characters.

Dade proved in [4] that his projective conjecture [4, 15.5] implies the Alperin–McKay conjecture. Navarro showed in [11, Theorem 9.27] that the group version of Dade's ordinary conjecture implies the McKay conjecture. We show here that Dade's ordinary conjecture [3, 6.3] implies the Alperin–McKay conjecture. Let p be a prime number.

Theorem 1. *If Dade's ordinary conjecture holds for all p -blocks of finite groups, then the Alperin–McKay conjecture holds for all p -blocks of finite groups.*

The proof combines arguments from Sambale [17] and formal properties of chains of subgroups in fusion systems from [7]. Let (K, \mathcal{O}, k) be a p -modular system. We assume that k is algebraically closed, and let \bar{K} be an algebraic closure of K . By a character of a finite group, we will mean a \bar{K} -valued character. For a finite group G and a block B of $\mathcal{O}G$, let $\text{Irr}(B)$ denote the set of irreducible characters of G in the block B , and let $\text{Irr}_0(B)$ denote the set of irreducible height zero characters of G in B . For a central p -subgroup Z of G and a character η of Z , let $\text{Irr}_0(B|\eta)$ denote the subset of $\text{Irr}_0(B)$ consisting of

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those height zero characters which cover the character η . The following lemma is implicit in [17].

Lemma 2. *Let P be a finite p -group, let \mathcal{F} be a saturated fusion system on P , and let $Z \leq Z(\mathcal{F})$. Suppose that η is a linear character of P . There exists a linear character $\hat{\eta}$ of P such that $\hat{\eta}|_Z = \eta|_Z$ and $\text{foc}(\mathcal{F}) \leq \text{Ker}(\hat{\eta})$.*

Proof. First consider the case that $\eta|_Z$ is faithful. Then $Z \cap [P, P] = 1$. Hence by [5, Lemma 4.3], $\text{foc}(\mathcal{F}) \cap Z = 1$. The result is now immediate. Now suppose $Z_0 = \text{Ker}(\eta|_Z)$ and let $\bar{\mathcal{F}} = \mathcal{F}/Z_0$. By the previous argument, applied to P/Z_0 and $\bar{\mathcal{F}}$, there exists a character $\hat{\eta}$ of P/Z_0 such that $\hat{\eta}|_{Z/Z_0} = \eta|_{Z/Z_0}$ and $\text{foc}(\bar{\mathcal{F}}) \leq \text{Ker}(\hat{\eta})$. Denote also by $\hat{\eta}$ the inflation of $\hat{\eta}$ to P . Then $\hat{\eta}$ has the required properties since $\text{foc}(\bar{\mathcal{F}}) = \text{foc}(\mathcal{F})Z_0/Z_0$. \square

The following result is a special case of a result due to Murai; we include a proof for convenience.

Lemma 3 (cf. [9, Theorem 4.4]). *Let G be a finite group, B be a block of $\mathcal{O}G$, and P a defect group of B . Let Z be a central p -subgroup of G and let η be an irreducible character of Z such that $\text{Irr}_0(B|\eta) \neq \emptyset$. Then η extends to P .*

Proof. By replacing K by a suitable finite extension, we may assume that K is a splitting field for all subgroups of G . Let $i \in B^P$ be a source idempotent of B and let V be a KG -module affording an element of $\text{Irr}_0(B|\eta)$. Then $n := \dim_K(iV)$ is prime to p (see [13]). Since i commutes with P , iV is a KP -module via $x \cdot iv = ixv$, where $x \in P, v \in V$. Let $\rho : P \rightarrow \text{GL}_n(K)$ be a corresponding representation and let $\delta : P \rightarrow K^\times$ be the determinantal character of ρ . Then $\delta|_Z = \eta^n$. The result follows since n is prime to p . \square

Lemma 4. *Let G be a finite group, let B be a block of $\mathcal{O}G$ with a defect group P , and let Z be a central p -subgroup of G . Then $|\text{Irr}_0(B)|$ equals the product of $|\text{Irr}_0(B|1_Z)|$ with the number of distinct linear characters η of Z which extend to P .*

Proof. Let $\mathcal{F} = \mathcal{F}_{(P, e_P)}(G, B)$ be the fusion system of B with respect to a maximal B -Brauer pair (P, e_P) , and let η be a linear character of Z which extends to P . Since $Z \leq Z(\mathcal{F})$, by Lemma 2 there exists a linear character $\hat{\eta}$ of P such that $\hat{\eta}|_Z = \eta$ and $\text{foc}(\mathcal{F}) \leq \text{Ker}(\hat{\eta})$. By the properties of the Broué-Puig $*$ -construction [1, 16] the map $\chi \mapsto \hat{\eta} * \chi$ is a bijection between $\text{Irr}_0(B|1_Z)$ and $\text{Irr}_0(B|\eta)$. The result follows by Lemma 3. \square

Slightly strengthening the terminology in [10], we say that a pair (G, B) consisting of a finite group G and a block B of $\mathcal{O}G$ is a *minimal counterexample to the Alperin–McKay conjecture* if B is a counterexample to the Alperin–McKay conjecture and if G is such that first $|G : Z(G)|$ is smallest possible and then $|G|$ is smallest possible.

Proposition 5. *Let (G, B) be a minimal counterexample to the Alperin–McKay conjecture. Then $O_p(G) = 1$.*

Proof. By a result of Murai [10], we have that $Z := O_p(G)$ is central in G . Let P be a defect group of B and let C be the block of $\mathcal{O}N_G(P)$ in Brauer correspondence with B . By Lemma 4, $|\text{Irr}_0(B)| = |\text{Irr}_0(C)|$ if and only if $|\text{Irr}_0(\bar{B})| = |\text{Irr}_0(\bar{C})|$ where \bar{B} (resp. \bar{C}) is the block of $\mathcal{O}G/Z$ (resp. $\mathcal{O}N_G(P)/Z$) dominated by B (resp. C). The result follows since $N_{G/Z}(P/Z) = N_G(P)/Z$ and \bar{B} and \bar{C} are in Brauer correspondence. \square

Let \mathcal{F} be a saturated fusion system on a finite p -group P , and let \mathcal{C} be a full subcategory of \mathcal{F} which is upwardly closed; that is, if Q, R are subgroups of P such that Q belongs to \mathcal{C} and if $\text{Hom}_{\mathcal{F}}(Q, R)$ is nonempty, then also R belongs to \mathcal{C} . Drawing upon notation and facts from [7, §5], $S_{\triangleleft}(\mathcal{C})$ is the category having as objects nonempty chains $\sigma = Q_0 < Q_1 < \cdots < Q_m$ of subgroups Q_i of P belonging to \mathcal{C} such that $m \geq 0$ and Q_i is normal in Q_m , for $0 \leq i \leq m$. Morphisms in $S_{\triangleleft}(\mathcal{C})$ are given by certain ‘obvious’ commutative diagrams of morphisms in \mathcal{F} ; see [7, 2.1, 4.1] for details. With this notation, the *length* of a chain σ in $S_{\triangleleft}(\mathcal{C})$ is the integer $|\sigma| = m$. The chain σ is called *fully normalised* if Q_0 is fully \mathcal{F} -normalised and if either $m = 0$ or the chain $\sigma_{\geq 1} = Q_1 < Q_2 < \cdots < Q_m$ is fully $N_{\mathcal{F}}(Q_0)$ -normalised. Every chain in $S_{\triangleleft}(\mathcal{C})$ is isomorphic (in the category $S_{\triangleleft}(\mathcal{C})$) to a fully normalised chain. There is an involution n on the set of fully normalised chains which fixes the chain of length zero P and which sends any other fully normalised chain σ to a fully normalised chain $n(\sigma)$ of length $|\sigma| \pm 1$. This involution is defined as follows. If $\sigma = P$, then set $n(\sigma) = \sigma$. If $\sigma = Q_0 < Q_1 < \cdots < Q_m$ is a fully normalised chain different from P such that $Q_m = N_P(\sigma)$, then define $n(\sigma)$ by removing the last term Q_m ; if $Q_m < N_P(\sigma)$, then define $n(\sigma)$ by adding $N_P(\sigma)$ as last term to the chain. Then $n(\sigma)$ is fully normalised, and $n(n(\sigma)) = \sigma$. Denote by $[S_{\triangleleft}(\mathcal{C})]$ the partially ordered set of isomorphism classes of chains in $S_{\triangleleft}(\mathcal{C})$, and for each chain σ by $[\sigma]$ its isomorphism class. We have a partition

$$[S_{\triangleleft}(\mathcal{C})] = \{[P]\} \cup \mathcal{B} \cup n(\mathcal{B}),$$

where \mathcal{B} is the set of isomorphism classes of fully normalised chains σ satisfying $|n(\sigma)| = |\sigma| + 1$. The following Lemma is a very special case of a functor cohomological statement [7, Theorem 5.11].

Lemma 6. *With the notation above, let $f : [S_{\triangleleft}(\mathcal{C})] \rightarrow \mathbb{Z}$ be a function on the set of isomorphism classes of chains in $S_{\triangleleft}(\mathcal{C})$ satisfying $f([\sigma]) = f([n(\sigma)])$ for any fully normalised chain σ in $S_{\triangleleft}(\mathcal{C})$. Then*

$$\sum_{[\sigma] \in [S_{\triangleleft}(\mathcal{C})]} (-1)^{|\sigma|} f([\sigma]) = f([P]).$$

Proof. The hypothesis on f implies that the contributions from chains in \mathcal{B} cancel those from chains in $n(\mathcal{B})$, whence the result. \square

Proposition 7. *Let G be a finite group such that $O_p(G) = 1$, and let B be a block of $\mathcal{O}G$ with nontrivial defect groups. Suppose that Dade's ordinary conjecture holds for B and that the Alperin–McKay conjecture holds for any block of any proper subgroup of G . Then the Alperin–McKay conjecture holds for the block B .*

Proof. Let (P, e) be a maximal B -Brauer pair, and denote by \mathcal{F} the associated fusion system on P . For d a positive integer, denote by $\mathbf{k}_d(G, B)$ the number of ordinary irreducible characters in B of defect d . If $p^d = |P|$, then $\mathbf{k}_d(G, B)$ is the number of height zero characters, and if $p^d > |P|$, then $\mathbf{k}_d(G, B) = 0$.

Let \mathcal{C} be the full subcategory of \mathcal{F} consisting of all nontrivial subgroups of P . We briefly describe the standard translation process between chains in a fusion system of a block and the associated chains of Brauer pairs. The map sending a chain $\sigma = Q_0 < Q_1 < \dots < Q_m$ in $S_{\triangleleft}(\mathcal{C})$ to the unique chain of nontrivial B -Brauer pairs $\tau = (Q_0, e_0) < (Q_1, e_1) < \dots < (Q_m, e_m)$ contained in (P, e) induces a bijection between isomorphism classes of chains in $S_{\triangleleft}(\mathcal{C})$ and the set of G -conjugacy classes of normal chains of nontrivial B -Brauer pairs (cf. [7, 2.5]). If σ is fully normalised, then the corresponding chain of Brauer pairs $\tau = (Q_0, e_0) < (Q_1, e_1) < \dots < (Q_m, e_m)$ has the property that $e_\tau = e_m$ remains a block of $N_G(\tau)$, and by [7, 5.14], $N_P(\sigma) = N_P(\tau)$ is a defect group of e_τ as a block of $N_G(\tau)$. Denote by $n(\sigma)$ the chain of Brauer pairs corresponding to $n(\sigma)$.

Let $d > 0$ such that $p^d = |P|$. Define a function f on $S_{\triangleleft}(\mathcal{C})$ by setting

$$f([\sigma]) = \mathbf{k}_d(N_G(\tau), e_\tau)$$

for any fully normalised chain σ and corresponding chain τ of Brauer pairs. If $N_P(\sigma)$ is a proper subgroup of P , then $f([\sigma]) = 0$, and if $N_P(\sigma) = P$, then $f([\sigma])$ is the number of height zero characters of the block e_τ of $N_G(\tau)$. Dade’s ordinary conjecture for B , reformulated here in terms of chains of Brauer pairs, asserts that $\mathbf{k}_d(G, B)$ is equal to the alternating sum

$$\sum_{[\sigma] \in S_{\triangleleft}(\mathcal{C})} (-1)^{|\sigma|} f([\sigma]).$$

The passage between formulations in terms of normalisers of chains of Brauer pairs rather than normalisers of chains of p -subgroups is well known; see e.g. [6, 4.5], [15].

If $|n(\sigma)| = |\sigma| + 1$, then, setting $H = N_G(\tau)$, we have $N_G(n(\tau)) = N_H(N_P(\tau), e_{n(\tau)})$; that is, $(N_P(\tau), e_{n(\tau)})$ is a maximal (H, e_τ) -Brauer pair. By the assumptions, the Alperin–McKay conjecture holds for the block e_τ of H . This translates to the equality $f([\sigma]) = f([n(\sigma)])$. That is, the function f satisfies the hypotheses of Lemma 6. Thus the above alternating sum is equal to $f([P])$, which by definition is $\mathbf{k}_d(N_G(P, e), e)$, and thus the Alperin–McKay conjecture holds for B . \square

Theorem 1 follows now immediately from combining Propositions 5 and 7.

Remark 8. By work of Dade [2] and Okuyama and Wajima [12], the Alperin–McKay conjecture holds for blocks of finite p -solvable groups. G. R. Robinson pointed out that Proposition 5 yields another short proof of this fact.

Remark 9. Let G be a finite group, B a block algebra of $\mathcal{O}G$, (P, e_P) a maximal (G, B) -Brauer pair with associated fusion system \mathcal{F} on P , and let Z be a central p -subgroup of G . Let η be a linear character of Z , and suppose that η extends to a linear character $\hat{\eta}$ of P satisfying $\text{foc}(\mathcal{F}) \leq \text{Ker}(\hat{\eta})$. The proof of

Lemma 4 is based on the fact that the $*$ -construction $\chi \mapsto \hat{\eta} * \chi$ yields a bijection $\text{Irr}(B|1_Z) \rightarrow \text{Irr}(B|\eta)$. There is some slightly more structural background to this. For $\chi \in \text{Irr}(B)$, denote by $e(\chi)$ the corresponding central primitive idempotent in $K \otimes_{\mathcal{O}} B$. Set

$$e_1 = \sum_{\chi \in \text{Irr}_0(B|1_Z)} e(\chi), \quad e_\eta = \sum_{\chi \in \text{Irr}_0(B|\eta)} e(\chi).$$

Identify B with its image in $K \otimes_{\mathcal{O}} B$. Multiplying B by the central idempotents e_1 and e_η in $K \otimes_{\mathcal{O}} B$ yields the two \mathcal{O} -free \mathcal{O} -algebra quotients Be_1 and Be_η of B . By [8, Theorem 1.1], there is an \mathcal{O} -algebra automorphism α of B which induces the identity on $k \otimes_{\mathcal{O}} B$ and which acts on $\text{Irr}(B)$ as the map $\chi \rightarrow \hat{\eta} * \chi$. Thus the extension of α to $K \otimes_{\mathcal{O}} B$ sends e_1 to e_η , and hence induces an \mathcal{O} -algebra isomorphism

$$Be_1 \cong Be_\eta.$$

We conclude this note with an observation regarding canonical height zero characters in nilpotent blocks, based in part on some of the above methods.

Let G be a finite group, B a block algebra of $\mathcal{O}G$, P a defect group of B , and $i \in B^P$ a source idempotent of B . Denote by \mathcal{F} the fusion system of B on P determined by the choice of i . Suppose that K is a splitting field for all subgroups of G . For V a finitely generated \mathcal{O} -free B -module, denote by

$$\Delta_{V,P,i} : P \rightarrow \mathcal{O}^\times$$

the map sending $u \in P$ to the determinant of the \mathcal{O} -linear automorphism of iV induced by the action of u on V (this makes sense since all elements in P commute with i). By standard properties of determinants, this map depends only on the $(B^P)^\times$ -conjugacy class of i and the isomorphism class of the $K \otimes_{\mathcal{O}} B$ -module $K \otimes_{\mathcal{O}} V$. Thus if V affords a character $\chi \in \text{Irr}(B)$, we write $\Delta_{\chi,P,i}$ instead of $\Delta_{V,P,i}$.

Proposition 10. *With the notation above, let $\chi \in \text{Irr}(B)$ and $\eta \in \text{Irr}(P/\text{foc}(P))$. Regard η as a linear character of P . We have*

$$\Delta_{\eta * \chi, P, i} = \eta^{\chi(i)} \Delta_{\chi, P, i}.$$

Proof. The statement makes sense as the value of χ on an idempotent is a positive integer. Let V be an \mathcal{O} -free $\mathcal{O}G$ -module affording χ . By [8, Theorem 1.1] there exists an \mathcal{O} -algebra automorphism α of B such that the module V^α (obtained from twisting V by α) affords $\eta * \chi$ and such that $\alpha(ui) = \eta(u)ui$ for all $u \in P$. Since in particular $\alpha(i) = i$, it follows that

$$\Delta_{V^\alpha, P, i}(u) = \Delta_{V, P, i}(\eta(u)u)$$

for all $u \in P$. The result follows as $\text{rank}_{\mathcal{O}}(iV) = \chi(i)$. \square

Denote by $\text{Irr}'(B)$ the set of all $\chi \in \text{Irr}(B)$ such that $\Delta_{\chi, P, i}$ is the trivial map (sending all elements in P to 1). Set $\text{Irr}'_0(B) = \text{Irr}'(B) \cap \text{Irr}_0(B)$. The maximal local pointed groups on B are G -conjugate. Thus if P' is any other defect group of B and $i' \in B^{P'}$ a source idempotent, then there exist $g \in G$ and $c \in (B^{P'})^\times$ such that $P' = gPg^{-1}$ and $i' = c g i g^{-1} c^{-1}$. Therefore the map

$\Delta_{V,P,i}$ is trivial if and only if the map $\Delta_{V,P',i'}$ is trivial, and hence the sets $\text{Irr}'(B)$ and $\text{Irr}'_0(B)$ are independent of the choice of P and i . The following is immediate.

Proposition 11. *The sets $\text{Irr}'(B)$ and $\text{Irr}'_0(B)$ are invariant under any automorphism of G which stabilises B .*

The next result shows that if B is nilpotent, then $\text{Irr}'_0(B)$ consists of a single element.

Proposition 12. *Suppose that B is nilpotent. Then $|\text{Irr}'_0(B)| = 1$. Moreover, if p is odd, then the unique element of $\text{Irr}'_0(B)$ is the unique p -rational height zero character in B .*

Proof. Let $\chi \in \text{Irr}_0(B)$. Since i is a source idempotent of B , $\chi(i)$ is prime to p . Hence if η, ζ are linear characters of P , then $\eta^{\chi(i)} = \zeta^{\chi(i)}$ implies that $\eta = \zeta$. Since B is nilpotent, we have that $\text{foc}(\mathcal{F}) = [P, P]$ and $|\text{Irr}_0(B)| = |P : [P, P]|$. Thus, by Proposition 10, the map $\chi \mapsto \Delta_{\chi,P,i}$ is a bijection from $\text{Irr}_0(B)$ to $\text{Irr}(P/[P, P])$. This proves the first assertion.

Suppose that p is odd. Let χ_0 be the unique p -rational character in $\text{Irr}_0(B)$. Let $W(k)$ be the ring of Witt vectors in \mathcal{O} . By the structure theory of nilpotent blocks (see [14]), there exists a $W(k)G$ -module V affording χ_0 . Since the source idempotent i can be chosen to be in $W(k)G$, we have that $\Delta_{\chi,P,i}$ takes values in $W(k)$. Since p is odd, it follows that the trivial character of P is the unique linear character of P which takes values in $W(k)$. \square

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