THEORETICAL AND EMPIRICAL STUDY ON OPTIMAL INSURANCE AND REINSURANCE DESIGN

by

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A thesis submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy

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CERTIFICATE OF EXAMINATION

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The thesis by

Junlei Hu

entitled:

Theoretical and Empirical Study on Optimal Insurance and Reinsurance Design

is accepted in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

Date ______________________   ______________________
Chair of the Thesis Examination Board

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Chapters 2-5 of this thesis contain published articles. The full text of these has been redacted from this version of the thesis for copyright reasons.

Acknowledgements

Firstly, I would like to thank Dr. Alexandru V. Asimit for introducing me to the world of research and encouraging me to find out the research topic that I am truly enthusiastic about. I am specially thankful for his patience and support in overcoming numerous obstacles I have been facing through my research.

I also owe many thanks to Dr. Andreas Tsanakas for his practical advice and constant support from stage to stage during my PhD research.

I am especially grateful to my parents, Yuehua and Qinfang, and my brother, Shenggang, for their loving support and encouragement to achieve my goals.

Last but not the least, I owe many thanks to my, husband Tao, our son, Gabriel, and our dogs, Yuki and Penny, whose spiritual support helped me to smoothly graduate from this programme.

Co-Authorship Statement

Each of the chapters in this thesis are my own original idea and work. The works have been obtained through collaborations with my supervisor, Dr. Alexandru V. Asimit.

Chapter 2 has been published by *Insurance: Economics and Mathematics*, and is a joint work with Dr. Alexandru V. Asimit and Dr. Yichun Chi. Chapter 3 has been accepted and published online by *North American Actuarial Journal*, and is a joint work with Dr. Alexandru V. Asimit, Dr. Tao Gao and Dr. Kim Eun-Seok. Chapter 4 has been published by *European Journal of Operational Research*, and is a joint work with Dr. Alexandru V. Asimit, Dr. Valeria Bignozzi, Dr. Ka Chun Cheung and Dr. Eun-Seok Kim. Chapter 5 has been submitted for publication, and is a joint work with Dr. Alexandru V. Asimit and Professor Yuantao Xie.

Abstract

Insurance and reinsurance are important tools of risk management. A well-designed (re)insurance strategy can help individuals and institutions to effectively adjust its risk position to match its risk appetite while meeting other targets such as profitability. Thus, optimal (re)insurance design has been a popular research area during the last fifty years.

The first contribution investigates the optimal reinsurance contract from the perspective of an insurer who would like to minimise its risk exposure under Solvency II. Under this regulatory framework, the insurer is exposed to the retained risk, reinsurance premium and change in the risk margin requirement as a result of reinsurance. Depending on how the risk margin corresponding to the reserve risk is calculated, two optimal reinsurance problems are formulated. We show that the optimal reinsurance policy can be in the form of two layers. Further, numerical examples illustrate that the optimal two-layer reinsurance contracts are only slightly different under these two methodologies.

In the second contribution, numerical optimisation methods that are practically implementable and solvable are discussed with actuarial applications. The efficiency of these methods is extremely good for some well-behaved convex problems, such as the Second-Order Conic Problems. Specific numerical solutions are provided in order to better explain the advantages of appropriate numerical optimisation methods chosen to solve various risk transfer problems. The stability issues are also investigated together with a case study performed for an insurance group that aims capital efficiency across the entire organisation.

The next two contributions aim to identify a robust optimal insurance contract that is not sensitive to the chosen risk distribution. The first of the two contributions focuses on the classical robust optimisation models, namely the *worst-case* and the *worst-regret* model, which have been already investigated in literature relating to optimal investment portfolio problems, while Bayesian type robust optimisation models are discussed in the second contribution. A caveat of robust optimisation is that the optimal solution may not be unique, and therefore, it may not be economically acceptable, i.e. not Pareto optimal. This issue is numerically addressed and simple numerical methods are found for constructing insurance contracts that are both Pareto and robust optimal.

Keywords: *General Premium Principle, Linear Programming, Optimal Reinsurance, Risk Margin, Risk Measure, Risk Transfer, Robust optimisation, Robust/Pareto* *optimal insurance, Second-Order Conic Programming, Solvency II, Technical Provision, Uncertainty modelling.*

Dedicated to Tao and Gabriel.

Chapter 1

Introduction

The topic of optimal insurance/reinsurance design has attracted particular interests from both the academics and practitioners since the pioneering work of Borch (1960). A well-constructed (re)insurance strategy is an efficient risk management tool, and hence, better reinsurance strategies are being constantly sought. Closed-form solution of optimal (re)insurance problem, including risk measure minimisation, expected utility maximisation and employment of various reinsurance premium principle, has been widely discussed by, for example, Arrow (1963), Gajek and Zagrodny (2000), Kaluszka (2001), Cai *et al.* (2008) and Chi and Tan (2011, 2013). Although some carefully constructed optimal (re)insurance models, which are usually built with simplifying assumptions, can be solved theoretically for closed-form solutions, the majority of the problems can only be solved numerically. A good reference is given by Tan and Weng (2014).

Most of the existing literature on optimal (re)insurance assumes that the underlying risk distribution is completely known, i.e. the parameter and model risks are ignored. However, whenever such risks are present, it is prudent to identify a robust optimal contract that is not sensitive to the chosen risk distribution, which is precisely what *robust optimisation* does. It is a vast area of research with applications in various fields and a standard reference is Ben-Tal *et al.* (2009), while comprehensive surveys can be found in Ben-Tal and Nemirovski (2008), Bertsimas *et al.* (2011) and Gabrel *et al.* (2014).

Chapter 2 investigates the optimal reinsurance strategies when new regulations on capital requirement introduced by Solvency II are adopted. Two different optimisation models are formulated depending on how risk margin of the reserve risk is measured. Closed-form solutions are found when expected-value premium principle is employed, while the problem is solved numerically under a wide class of premium principle known as the Wang's principle. We find that insurer demands for reinsurance cover in a more conservative manner when capital requirements in Solvency II are included, which is in line with the principle of Solvency II Regime.

Chapter 3 focuses on numerical optimisation methods with actuarial applications. Various optimal risk transfer problems are discussed to demonstrate the computational efficiency of the methods, which can also be easily extended to other actuarial problems. It also shows how one can take the advantage of computational methods when the underlying risk distribution is unknown. That is, rather than focusing on model-specific closed-form solutions, it is possible to search for robust decisions using efficient numerical methods. The stability issues are also investigated together with a case study performed for an insurance group that aims capital efficiency across the entire organisation, which demonstrates how a practical problem may be implemented via existing optimisation techniques.

In Chapters 4 and 5, optimal insurance problems are studied by taking into account the presence of parameter and model risks, i.e. the decision-maker aims to identify the optimal insurance contract that is robust to the choice of risk distribution. Chapter 4 considers two robust optimisation models, namely the *Worst-case* model and the *Worst-regret* model, which have been already used in robust optimisation literature related to the investment portfolio problem. Closed-form solutions are obtained for the VaR Worst-case scenario, while *Linear Programming (LP)* formulations are provided for all other cases. Bayesian type robust optimisation models, such as the *Additive* model, the *Weighted Average* model and the *Weighted Worst-case* model, are discussed in Chapter 5, which could be efficiently solved using numerical methods. Extensive numerical experiments have been carried out under various risk preference choices and various sample sizes of data. We found that, with relatively large sample size, the modeller should focus on finding the best possible fit for the unknown probability model in order to achieve the most robust decision. When only small samples are available, the modeller should consider either the Weighted Average Model or the Weighted Worst-case Model depending on how much interest the modeller puts on the tail risk when defining its objective function. A caveat of robust optimisation is that the optimal solution may not be unique, and therefore, it may lead to Pareto inefficient solutions. This issue has been numerically addressed in both Chapters by proposing simple numerical methods of identifying insurance contracts that are both Pareto and robust optimal.

Finally, Chapter 6 discusses my considerations on future research in two parts. The first part considers some possible extensions of works discussed in Chapters 4 and 5, while the second part outlines the idea of a new project relating to data visualisation with actuarial applications.

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Chapter 2

Optimal Non-life Reinsurance under Solvency II Regime

2.1 Introduction

A standard reinsurance contract is usually reached between two parties: the *insurer*, also known as *cedent*, *insurance buyer*, or even simpler, *buyer*, who has an interest in transferring part of its risk to the *reinsurer*, also known as *insurance seller*, or even simpler, *seller*. Mathematically, let $X \geq 0$ be the total risk that the insurer faces during a fixed period, with distribution function denoted by $F(\cdot)$ and survival function $\bar{F}(\cdot) = 1 - F(\cdot)$. In addition, the right end-point of $F(\cdot)$ is denoted by $x_F := \inf\{z \in \Re : F(z) = 1\}$, where $\inf \emptyset = +\infty$ by convention. The reinsurance seller agrees to pay, $R[X]$, the amount by which the entire loss exceeds the insurer's amount, $I[X]$, and therefore $I[X] + R[X] = X$. Two most common reinsurance contracts are the *Quota-share* and *Stop-loss*, where $I[X] = cX$ (with $0 \leq c \leq 1$) and $I[X] = X \wedge M := \min\{X, M\}$ (with $0 \leq M \leq x_F$), respectively. In order to avoid potential moral hazard issues arising from the reinsurance arrangement, the set of feasible contracts is usually given by

$$\mathcal{F} := \{0 \leq R[x] \leq x : R[x] \text{ and } x - R[x] \text{ are non-decreasing functions}\}. \tag{2.1.1}$$

[1] A version of this chapter is published: *Insurance: Mathematics and Economics*, 65, 227–237.

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Alexandru V. Asimit, Yichu Chi, Junlei Hu (2015), Optimal non-life reinsurance under Solvency II Regime, *Insurance: Mathematics and Economics*, 65, 227-237.
DOI: 10.1016/j.insmatheco.2015.09.006

Chapter 3

Optimal Risk Transfer: A Numerical Optimisation Approach

3.1 Introduction

Various actuarial problems involve decision-making procedures that evaluate the most favourable risk position of an insurance company. For example, capital efficiency and asset/liability management are part of the Enterprise Risk Management Process of any insurance/reinsurance conglomerate and serve as quantitative methods to fulfill the strategic planning within the organisation. The decision-makers are prone to combine expert judgement with core quantitative methods, which involve numerical optimisation and often, intensive computing skills. Therefore, many optimisation problems are not practically implementable in a straightforward manner to practitioners and academics that are not operation research inclined. Unfortunately, numerical issues are anecdotally disregarded and for this reason, we aim to implement optimisation algorithms that are hardly accessible to non-specialists in this field. In order to better communicate the advantages and caveats of possible solutions, we plan to focus on optimal risk transfer problems, but the numerical methods are transferable skills when implementing other actuarial problems.

Consider a two-player insurance setting where the first player is the risk holder who transfers a portion of its risk to the second player. At the same time, the second player charges the first player to cover its cost of transfer. This setting in-

[2] A version of this chapter is accepted and published online by: *North American Actuarial Journal*.

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Alexandru V. Asimit, Tao Gao, Junlei Hu & Eun-Seok Kim (2018) Optimal Risk Transfer: A Numerical Optimization Approach, *North American Actuarial Journal*, 22:3, 341-364, DOI: 10.1080/10920277.2017.1421472

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Chapter 4

Robust and Pareto Optimality of Insurance Contracts

4.1 Introduction

Finding the optimal insurance contract has represented a topic of interest in the actuarial science and insurance literature for more than 50 years. The seminal papers of Borch (1960) and Arrow (1963) had opened this field of research and since then, many papers discussed this problem under various assumptions on the risk preferences of the insurance players involved in the contract and how the cost of insurance (known as *premium*) is quantified. Specifically, the optimal contracts in the context of Expected Utility Theory are investigated amongst others in Kaluszka (2005), Kaluszka and Okolewski (2008) and Cai and Wei (2012). Extensive research has been carried out when the preferences are made via coherent risk measures (as defined in Artzner *et al.*, 1999; recall that CVaR is an element of this class) and VaR; for example, see Cai and Tan (2007), Balbás *et al.* (2009 and 2011), Asimit *et al.* (2013b), Cheung *et al.* (2014) and Cai and Weng (2016) among others.

The choice of a risk measure is usually subjective, but VaR and CVaR represent the most known risk measures used in the insurance industry. Solvency II and Swiss Solvency Test are the regulatory regimes for all (re)insurance companies that operate within the European Union and Switzerland, respectively, and their capital

[4]A version of this chapter is published: *European Journal of Operational Research*, 262(2), 720–732.

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Alexandru V. Asimit, Valeria Bignozzi, Ka Chun Cheung, Junlei Hu & Eun-Seok Kim (2017), Robust and Pareto optimality of insurance contracts, *European Journal of Operational Research*, 262:2, 720-732. DOI: 10.1016/j.ejor.2017.04.029

[redacted]

Chapter 5

Optimal Robust Insurance with a Finite Uncertainty Set

5.1 Introduction

The seminal works by Borch (1960) and Arrow (1963) mark the beginning of the theory of optimal insurance/reinsurance in the field of actuarial science, but the same problem is known as the insurance demand problem in insurance economics field. In the last 50 years, many research outputs have contributed into these fields of research by identifying the optimal insurance/reinsurance contracts under various risk preferences. Examples outside the Expected Utility Theory are numerous; for example, risk measure-based models have been studied by Cai *et al.* (2008), Balbás *et al.* (2009 and 2011), Chi and Tan (2011), Asimit *et al.* (2013 and 2015), Cheung *et al.* (2014), Lu *et al.* (2014) and Cai and Weng (2016), where *Value-at-Risk (VaR)* and *Conditional-Value-at-Risk (CVaR)* based decisions are the focal interest, since these particular risk preferences are easy to interpret and are the most common in the insurance sector.

The majority of the contributions from the existing literature assumes that the model specifications are completely known, which purposely removes the model and parameter risks – the risk of choosing a "wrong" model or the risk of choosing the "right" parametric model with the "wrong" parameter values/estimates. Such risks

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Chapter 6

Future Research

Some extensions of the present work might be considered for future research. In the last two Chapters, the robust optimal insurance problem focuses only on the non-life business sector. It will be of great interest to extend the research to the life sector, as uncertainties in the mortality risk is considered as a major factor that drives the realisation of loss away from its expected value. The nature of life insurance problem that often lasts for more than one time period may bring extra complexity, and therefore, closed-form solution will be difficult to obtain, while numerical approach may still be feasible to seek.

Another idea for future research focuses on data visualisation methods with applications to actuarial problems. This data mining subfield has been a very hot topic for some time and enables to detect interpretable patterns and gain information from massive data sets. The visualisation tool has its obvious great advantage of reducing the mathematical complexity for the end-user, which explains why such methods are so popular amongst practitioners. In particular, a group of projection methods in data visualisation aiming dimensionality reduction, known as *Multidimensional Scaling* (MDS), is one of the most popular tool of visualising high-dimensional data. It is a technique of analysing (dis)similarity data on a set of n-dimensional objects by representing them as points in the space of lower-dimensionality d, $d < n$, such that the (dis)similarities, usually measured as distances among points in a geometric space, are preserved as much as possible through the projection. The objective data may be performance measurement of test items, credit ratings of insurance companies or macroeconomic indices of countries, and hence, MDS has very wide application areas, e.g. business analysis, psychometrics, pharmacology etc. Although MDS enables a graphical display of the structure of high-dimensional data that is much easier to understand than an array of numbers as noises are smoothed out, its descriptive solution is usually derived upon a particular observed sample without any assessment on issues such as stability or sampling error. Therefore, finding possible remedies to address such inference issues in MDS applications is a popular topic to investigate. In fact, inference strategies such as the Maximum Likelihood and Bayesian approaches have been proposed for both parametric and non-parametric MDS models in the last 50 years. Another potential candidate of inference strategy that may be worth investigating is bootstrapping, which is a well known resampling method of estimating statistical properties such as the variance.