Symmetric Equilibrium Strategies in Game Theoretic Real Option Models with Incomplete Information*

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October 5, 2018

Abstract

This paper considers a real options model with incomplete information in a duopoly setting. I show that even in the presence of a first-mover advantage, there are circumstances in which the preemption region is eroded entirely and the only equilibrium strategy is of simultaneous-type: both players should exercise together or not at all. This outcome arises if the price paid by a player from exercising the option simultaneously with his competitor (over exercising alone as a Stackelberg leader) is low. Such an outcome has not been previously recognised in the literature. The underlying information structure has been applied in a one-firm setting to a range of different contexts of late, all of which have competitive pressures in practice. As such, the existence of this equilibrium under certain conditions ought to be recognised.

Keywords: Timing games; Real options; Preemption; Incomplete information.

JEL Classification Numbers: C61; C73; D43; D81.

1 Introduction

This paper applies the Thijssen et al. [11] stochastic environment and complete information extension of Fudenberg and Tirole [6] rent equalisation method for preemption games to a stochastic environment with incomplete information. Much of the real options literature considers decision problems with complete information in the sense that the stochastic processes in these models have stationary increments that are independent of the past. As such, the Thijssen et al. [11] extension is based on the assumption that the uncertainty in the model is driven by a Levy process. In this paper, I consider a real options model with *incomplete* information in that the increments of the stochastic process are not stationary and path dependent because nature determines the state of the world only at the beginning and information arrives stochastically over time to resolve uncertainty.

I derive a preemptive equilibrium strategy for two-player symmetric games in which the uncertainty is resolved over time via the arrival of imperfect signals at irregular intervals. The novel result is that in such a set-up, even when there is a first-mover advantage, the preemption region may be eroded entirely in equilibrium and the outcome is such that it will be optimal for the two players to exercise the option simultaneously or not at all. This outcome arises in situations in which the price paid by a player from investing simultaneously with his competitor (over investing alone as leader) is low. Note that Riedel and Steg [7] also extend the Fudenberg and Tirole [6] method to a stochastic environment with complete

*I thank Jacco Thijssen and Saqib Jafarey for their much appreciated comments and suggestions.

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information. However, their definition of strategies differs from Thijssen et al. [11] in that they are conditional on move history. While I adopt the *unconditional* definition of strategies, this result would still apply if my strategies were conditional as in Riedel and Steg [7].

From a methodological perspective, this paper is most closely related to earlier work by Thijssen et al. [10], who consider the same type of incomplete information structure in a duopoly setting. Their equilibrium results are derived on the basis that the sample path of the underlying stochastic process is continuous; in other words, the equilibrium is defined in terms of beliefs (i.e., probabilities). However, the beliefs are driven by the net number of positive signals about the potential profitability (of the investment) which do not follow continuous sample paths. Therefore, since the net number of signals cannot take any real value, neither can the associated beliefs. As such, the equilibrium is defined in this paper in terms of the net number of positive signals and this gives the equilibrium result described above. It is not, however, a subgame-perfect equilibrium, but even when the equilibrium strategy is of the Stackelberg-type, it is only subgame-perfect subject to a condition which is explained later in the paper. The result that subgame-perfection is satisfied *conditionally* under such an information structure has not been previously recognised in the literature either.

In recent years, the information structure underpinning the model of this paper has been applied in a one-firm setting to a range of different contexts (see, for example, Thijssen et al. [9] (corporate investment); Delaney and Thijssen [3] (corporate disclosure), Delaney and Kovaleva [2] (market microstructure), Elsner et al. [5] (migration), Compernolle et al. [1] (environmental), and Takizawa [8] (private monitoring and punishment), to name but a few). However, in each of those contexts, competitive pressures are a real factor and the applicability of my new result and the importance of recognising the existence of such an equilibrium is important in real options models with incomplete information.

2 The Model

I present a model of incomplete information which has already been applied in a monopoly setting in Thijssen et al. [9], Delaney and Thijssen [3], and Delaney and Kovaleva [2], for different contexts. It has also been applied in a duopoly setting in Thijssen et al. [10], but the simultaneous equilibrium, which is only subgame-perfect subject to a condition I explain later, is not recognised in that paper.

2.1 Set-up

Two identical firms $i = 1, 2$ both have the opportunity to irreversibly invest in some opportunity for which the payoff from investment is uncertain. Time is continuous and the horizon is infinite. The firms are risk neutral and their objectives are to maximise their wealth from investing at discount rate $r > 0$. The payoff can be high or low. A high payoff implies an increase in the investor i's' wealth by an amount $V_i^P > 0$, whereas a low payoff implies a drop in wealth by an amount $V_i^N < 0$. V_i^j for $j = \{P, N\}$ can be seen as an infinite stream of payoffs discounted at rate $r > 0$; i.e., $V_i^j = \int_0^\infty e^{-rt} \left(\pi_i^j - rI\right) dt$, where $I > 0$ is the sunk cost of investing. Once the investment option has been exercised by at least one of the firms, the state of the payoff becomes known to both firms instantaneously; i.e., whether it yields a high or low payoff.

At the time the investment opportunity becomes available, both firms have an identical prior belief in the payoff being high, denoted by $p_0 \in (0, 1)$. Over time, the firms receive a stream of information signals pertaining to the likely payoff from the investment. The arrival of these signals follow a Poisson process with parameter $\mu > 0$ and are correct with probability $\theta \in (1/2, 1]$. A signal is correct if it indicates a high payoff and the true payoff that is achieved when investment is undertaken is indeed high. A correct signal is a binomially distributed random variable with parameters θ and n_t, where n_t

denotes the number of signals received by the firm by time $t > 0$. In my set up, the *net* number of positive over negative signals is a sufficient statistic to derive the firms' investment policies. The net number of signals at time t is denoted by s_t.

Each time a signal is observed, the firms update their prior beliefs of achieving a high investment payoff in Bayesian way. It is shown in Thijssen et al. [9] that, in such a setting, the probability of obtaining a high payoff, conditional on s_t, is given by

$$p_t := p(s_t) = P(H|s_t) = \frac{\theta^{s_t}}{\theta^{s_t} + \zeta(1-\theta)^{s_t}}, \tag{1}$$

where $\zeta := (1 - p_0)/p_0$ is the prior odds ratio and p_t increases monotonically in s_t.

2.2 Strategies and Payoff Functions

From a strategic perspective, there are three possible scenarios: (i) If i is the first to invest, he becomes the Leader and receives $V_L^P > 0$ in the case of a high payoff and $V_L^N < 0$ if the payoff is low. (ii) If the other firm has invested, then i is the Follower and knows the state of the payoff. He will invest if, and only if, the payoff is high i.e., iff the other firm received V_L^P. In this case, the Follower will get V_F^P, where $0 < V_F^P < V_L^P$. (iii) A third possibility is that both firms invest simultaneously at some time $\tau_S > 0$ and each get $V_S^P > 0$ in the case of a high payoff and $V_S^N < 0$ otherwise. The ranking of the payoffs is $V_L^P > V_S^P > V_F^P > 0$ and $V_S^N < V_L^N$ as illustrated by the example provided in Appendix A.

Let p_S denote the belief such that the *ex ante* expected payoff for the Follower is exactly equal to the *ex ante* expected payoff from simultaneous investment.[1]

If the Leader invests at a time $t \geq 0$ where the belief in a high payoff is p_t (where p_t is given by Eq. (1)), the Leader's *ex ante* expected payoff can be written as

$$L(p_t) = \begin{cases} p_t V_L^P + (1-p_t)V_L^N & \text{if } p_t < p_S \\ p_t V_S^P + (1-p_t)V_S^N & \text{if } p_t \geq p_S. \end{cases} \tag{2}$$

The Follower only invests in the case of a high payoff to the Leader. Hence, the *ex ante* expected payoff for the Follower, if the Leader invests when the belief in a high payoff is p_t, is given by

$$F(p_t) = \begin{cases} p_t V_F^P & \text{if } p_t < p_S \\ p_t V_S^P + (1-p_t)V_S^N & \text{if } p_t \geq p_S. \end{cases} \tag{3}$$

Finally, in the case of simultaneous investment (i.e., before the true payoff state is known to both firms) at a point in time when the belief in a high payoff is p, each firm has an *ex ante* expected payoff given by

$$S(p_t) = p_t V_S^P + (1-p_t)V_S^N. \tag{4}$$

2.3 Rent Equalisation

Say that one of the firms is preassigned the Leader role and the other firm is, therefore, the Follower. The Follower must wait until after the Leader has invested before he does so. In this case, the Leader's decision has no effect on the optimal response of the Follower. Thus, the Leader acts as if there is no follower; in other words, as if he were a monopolist. From Delaney and Thijssen [3], it is optimal for a monopolist to exercise a standalone investment option with the characteristics described in Section 2.1

[1] Letting $p_S > p_F$, where p_F denotes the follower's belief in a high payoff, makes no difference to the equilibrium strategies considered in this paper.

for $p_t \geq p_L$, where

$$p_L = \left[1 - \frac{V_L^P}{V_L^N}\Psi\right]^{-1}, \tag{5}$$

for

$$\Psi := \frac{(r+\mu(1-\theta))\left[\beta_1(r+\mu) - \mu\theta(1-\theta)\right] - \mu^2\beta_1\theta(1-\theta)}{(r+\mu\theta)\left[\beta_1(r+\mu) - \mu\theta(1-\theta)\right] - \mu^2\beta_1\theta(1-\theta)} > 0 \tag{6}$$

and $\beta_1 > \theta$ the larger root of the quadratic equation

$$\mathcal{Q}(\beta) = \beta^2 - \left(\frac{r}{\mu} + 1\right)\beta + \theta(1-\theta) = 0.$$

It is easily verified that $L(p_L) > S(p_L)$ and, hence, there is an advantage to being the first and only mover. This implies that by relaxing the assumption that the Leader and Follower roles are preassigned, each firm will try to be the only one to invest at τ_L, where $\tau_L := \inf\{t \geq 0 | p_t = p_L\}$. To do so, a firm will try to preempt its competitor by investing just before him but, in turn, the other firm will try to preempt him by investing just before that. By the principle of rent equalisation for preemption games, formalised by Fudenberg and Tirole [6], this process stops at time τ_P, where $\tau_P := \inf\{t \geq 0 | L(p_t) = F(p_t)\}$, or equivalently, $\tau_P := \inf\{t \geq 0 | p_t = p_P\}$, where

$$\begin{aligned} p_P =& \frac{V_L^N}{V_F^P - V_L^P + V_L^N} \\ <& \frac{V_S^N}{V_F^P - V_S^P + V_S^N} \\ =& p_S. \end{aligned} \tag{7}$$

Note that $p_P < p_L$ iff $V_F^P < (1-\Psi)V_L^P$, which I assume to be true. In other words, I only consider this scenario in which the leader advantage outweighs the information spillover in the remainder of the paper. I do so because the main point I make needs only be explained by examining one scenario, since it pertains in the same way to the war of attrition (i.e., $p_L < p_P$) scenario also.

3 Equilibria of the Game

The equilibrium concepts for timing games in a continuous time framework are outlined in detail in Thijssen et al. [11], so in this section I adopt their notation and point out that the proof of my result uses their definitions of simple strategies, first sequence of atoms, α-consistency, and subgame-perfect equilibrium.

Given the extended mixed strategy $(G_i^0, \alpha_i^0)_{i=1,2}$ for the subgame starting at $t = 0$, the *ex ante* expected present value to firm i (conditional on $\mathcal{F}_0$), denoted by $W_i^0(s_t)$, is given by

$$\begin{aligned} W_i^0(s_t) =& \int_0^{\tau^0} L(p_t)(1 - G_j^0(t))dG_i^0(t) + \int_0^{\tau^0} F(p_t)(1 - G_i^0(t))dG_j^0(t) + \sum_{t\in[0,\tau^0)} \Delta G_i^0(t)\Delta G_j^0(t) S(p_t) \\ &+ (1 - G_i^0(\tau^0-))(1 - G_j^0(\tau^0-))\Big[\frac{\alpha_i^0(\tau^0)(1-\alpha_j^0(\tau^0))}{\alpha_i^0(\tau^0) + \alpha_j^0(\tau^0) - \alpha_i^0(\tau^0)\alpha_j^0(\tau^0)} L(p_{\tau^0}) \\ &+ \frac{\alpha_j^0(\tau^0)(1-\alpha_i^0(\tau^0))}{\alpha_i^0(\tau^0) + \alpha_j^0(\tau^0) - \alpha_i^0(\tau^0)\alpha_j^0(\tau^0)} F(p_{\tau^0}) + \frac{\alpha_i^0(\tau^0)\alpha_j^0(\tau^0)}{\alpha_i^0(\tau^0) + \alpha_j^0(\tau^0) - \alpha_i^0(\tau^0)\alpha_j^0(\tau^0)} S(p_{\tau^0})\Big], \end{aligned} \tag{8}$$

where $\tau_i^0 := \inf\{t \geq 0 \big| \alpha_i^0(t,\omega) > 0\}$ for $i = 1,2$. $\tau^0 := \tau_1^0 \wedge \tau_2^0$, but only symmetric strategies are considered so $\tau^0 = \tau_1^0 = \tau_2^0$. Furthermore, $\frac{\alpha_i^0(\tau^0)(1-\alpha_j^0(\tau^0))}{\alpha_i^0(\tau^0)+\alpha_j^0(\tau^0)-\alpha_i^0(\tau^0)\alpha_j^0(\tau^0)}$ is the probability that *only firm i* invests first in the interval $[t, t+dt)$, while $\frac{\alpha_i^0(\tau^0)\alpha_j^0(\tau^0)}{\alpha_i^0(\tau^0)+\alpha_j^0(\tau^0)-\alpha_i^0(\tau^0)\alpha_j^0(\tau^0)}$ is the probability that both firms invest together, and $\Delta G_i^0(t) := G_i^0(t) - G_i^0(t-)$.

The underlying state variable p_t is driven by the net number of positive signals which cannot take any real value and, as such, the levels of p_t are such that $p_t := p(s_t)$ for $s_t \in \mathbb{Z}$. However, if we consider that the preemption and simultaneous investment beliefs, both defined in Eq. (7), do not depend on the net number of signals, and that the inverse function of p_t is given by

$$s_t = \frac{\log\left(\frac{1-p_t}{p_t}\right) - \log\zeta}{\log\left(\frac{1-\theta}{\theta}\right)}, \tag{9}$$

it is clear that the *ex ante* expected payoff levels associated with the levels of p_P and p_S will not be realisable if the levels of s_P and s_S associated with these respective beliefs are not integers. Therefore, for $z = \{P, S\}$, and $s_z \in \mathbb{R}$, I let $\lceil s_z \rceil := \min\{s_t \in \mathbb{Z} | s_t \geq s_z\}$. Moreover, $T_P := \inf\{t \geq 0 | s_t \geq \lceil s_P \rceil\}$ and $T_S := \inf\{t \geq 0 | s_t \geq \lceil s_S \rceil\}$.

Now, since s_t is monotonic and increasing in p_t (cf. Eq. (9)), $p_P < p_S \implies s_P < s_S$. However, if $s_P \notin \mathbb{Z}$ and $\lceil s_P \rceil - 1 < s_P < s_S \leq \lceil s_P \rceil$, then $\lceil s_P \rceil \equiv \lceil s_S \rceil$. In the following proposition I show that this implies that even when there is a first-mover advantage, the preemption region may be eroded entirely yielding an equilibrium outcome in which both players invest simultaneously or not at all.

Theorem 1. *In a signalling model played in continuous time, the tuple of closed-loop strategies $\left((G_i^0, \alpha_i^0)\right)$ for $i = 1, 2$ given by*

1. *If $\lceil s_P \rceil < \lceil s_S \rceil$*

$$G_i^0(s_t) = \begin{cases} 0 & \text{if } t < T_P \\ \frac{L(p_{T_P}) - S(p_{T_P})}{L(p_{T_P}) + F(p_{T_P}) - 2S(p_{T_P})} & \text{if } T_P \leq t < T_S \\ 1 & \text{if } t \geq T_S, \end{cases} \tag{10}$$

and

$$\alpha_i^0(s_t) = \begin{cases} 0 & \text{if } t < T_P \\ \frac{L(p_t) - F(p_{T_P})}{L(p_{T_P}) - S(p_{T_P})} & \text{if } T_P \leq t < T_S \\ 1 & \text{if } t \geq T_S. \end{cases} \tag{11}$$

2. *If $\lceil s_P \rceil = \lceil s_S \rceil$*

$$G_i^0(s_t) = \begin{cases} 0 & \text{if } t < T_S \\ 1 & \text{if } t \geq T_S, \end{cases} \tag{12}$$

and

$$\alpha_i^0(s_t) = \begin{cases} 0 & \text{if } t < T_S \\ 1 & \text{if } t \geq T_S \end{cases} \tag{13}$$

are symmetric α-consistent equilibria for the subgame starting at $t = 0$.

In both cases, the resulting payoffs for all $t \geq 0$ are given by $W_i^0(s_t) = F(p_{\tau^0})$, where p_t is given by Eq. (1).

Furthermore, these equilibria will be subgame-perfect if, and only if, $s(p_P) = \lceil s_P \rceil \in \mathbb{Z}$.

Proof. See Appendix B. ■

The first case in this proposition mirrors that in Thijssen et al. [10] in which a signalling model with an identical information structure to the one in this paper is considered in a duopoly setting. However, what is not considered in Thijssen et al. [10] (or indeed would be covered by the more recent subgame-perfect equilibrium methods for dynamic games proposed by Thijssen et al. [11] and Riedel and Steg [7], whose equilibrium results also mirror the first case in Theorem 1 even though the uncertainty in those models is driven by stationary and path-dependent Levy processes), is the second case in which it may be that $\lceil s_P \rceil \equiv \lceil s_S \rceil$ and the resulting equilibrium is of simultaneous-type with no preemption region.

The preemption region will be eroded entirely if p_P is not significantly lower than p_S or, equivalently, if $V_L^P - V_S^P$ is not much larger (or indeed, less than) than $V_L^N - V_S^N$. In other words, if the relative price that the investor pays for investing simultaneously with his competitor (over investing alone as the leader) in the case of a profitable project is not much larger (or even lower) than it would be in the case of a loss on the investment. This will be true if such costs are very low. The following example highlights one of the many contexts in which this result could apply.

Example 1. *Elsner et al. [5] use the information structure to model the decision problem of one potential migrant facing uncertainty about her job prospects abroad. An earlier version of that paper (Elsner et al. [4]) describes the model and information structure in detail. They assume that the potential migrant has an initial prior belief p_0 that she gets a good job abroad. This could, for example, be the fraction of previous migrants that get a good job. A good job increases her earnings relative to what she would earn if she stayed by an amount $w_G > 0$ (equivalent to V_L^P in this paper). A bad job reduces her earnings by an amount $w_B < 0$ (equivalent to V_L^N). Note that they also assume there is a fixed cost to move M, which is an unncessary assumption that I ignore since the payoff from a bad outcome is lower than the status quo. Over time, she receives information from the diaspora network in the form of signals which update her belief about her job prospects in the receiving country. k_t denotes the* net *number of positive signals at time t and is equivalent to s_t in my paper. The quality of the signal is dependent on the degree of integration between the host and diaspora networks. A more integrated network implies a more accurate signal. This parameter, denoted by them as λ, is represented by θ in my paper. Thus, they show that her expected payoff from migrating is given by $p(k)w^G + (1 - p(k))w^B$ (for $p(k) \geq p_L$) which corresponds directly with the expected payoff from exercising in the monopoly case (or preassigned leader case) in this paper since their $p(k)$ and p_L are given by the above Eqs. (1) and (5), respectively.*

Extending this example to two potential migrants: If, say, the first to migrate were to have an advantage in that there is greater earnings potential in the host network available to her as she can pick the best paid job without competition; i.e., $V_L^P > V_S^P > V^F$ and $V_S^N < V_L^N$. Consider the following parameter values (which correspond closely with those chosen for the numerical example provided by Elsner et al. [4]): $V_L^P = 20$; $V_L^N = -20$, $V_S^P = 12$; $V_S^N = -25$; $V_F^P = 8$; $\theta = 0.75$ and $p_0 = 0.5$ which give $p_P = 0.625$; $s_P = 0.465$ ($\lceil s_P \rceil = 1$); $p_S = 0.862$ and $s_S = 1.667$ (or $\lceil s_S \rceil = 2$). Hence $s_P < \lceil s_P \rceil < s_S < \lceil s_S \rceil$. These results correspond with the first (preemption) case in Theorem 1 and is depicted in Fig. 1 where p_t, s_t and $\lceil s_t \rceil$ are plotted against the payoff functions for the Leader, Follower, and Simultaneous cases, given by Eqs. (2), (3) and (4), respectively.

However, if there are plenty of job opportunities available in the host network, and if two migrants were migrate and enter that network at the same time, their potential earnings would not be significantly lower than the first mover's. For example, let $V_S^P = 18$ (keeping all other values as previously defined). In this case, the cost from not being the first and only migrant is lower than in the previous case ($V_L^P - V_S^P = 2$ versus 8) and the simultaneous equilibrium described above emerges; i.e., $p_S = 0.714$; $s_S = 0.834$ and $\lceil s_S \rceil = 1$. Thus, $s_P < s_S < \lceil s_P \rceil = \lceil s_S \rceil$, which is depicted in Fig. 2.

Moreover, the condition that the equilibria will be subgame-perfect if, and only if, $s_P \equiv \lceil s_P \rceil$ arises because of the nature of the information structure and, in particular, because $p_P \leq p_{T_P} = p(\lceil s_P \rceil)$. The region for which subgame-perfection may not hold would be for some $u \in (t^0, T_P]$ in which $s_u \notin \mathbb{Z}$. While such a scenario cannot be realised in this model, it is also not possible to claim that the ex ante expected payoff to player i is less than it would be if he played the equilibrium strategy *for every* $t < T_P$ when time is continuous.

Therefore, Theorem 1 provides a complete description of the equilibrium outcomes for continuous-

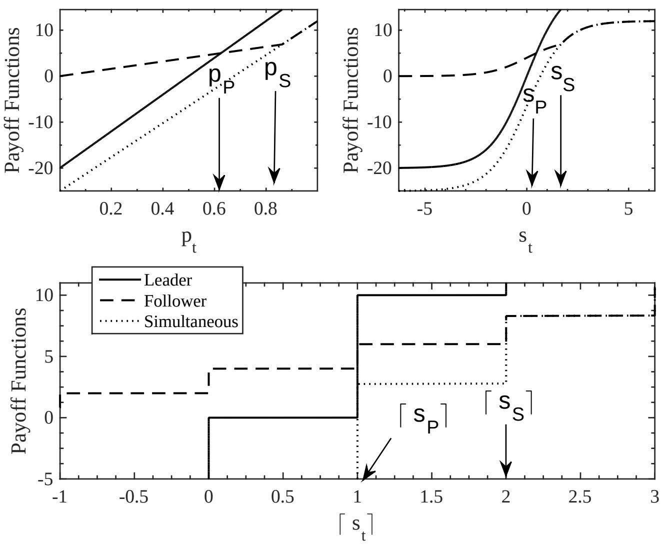

Figure 1: Preemption Case

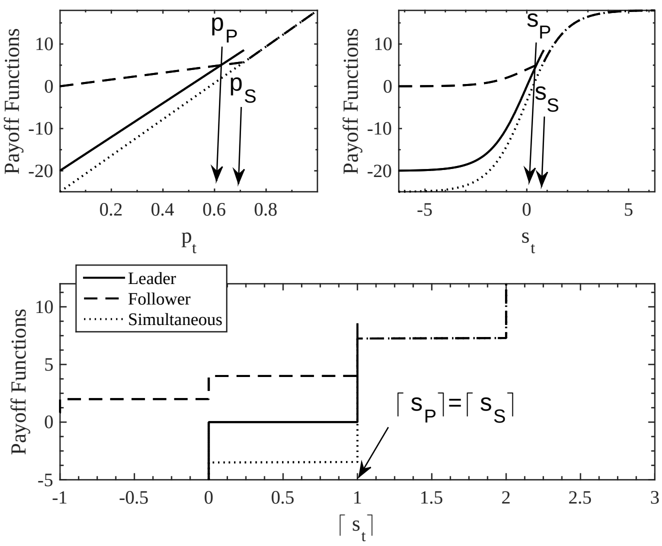

Figure 2: Simultaneous Case

time real option games in a stochastic environment, but with incomplete information driven by the signalling model described in Section 2.

Appendix

A Example of the Framework

Two firms have the opportunity to invest in a homogeneous good whose payoff can be High or Low. Let the inverse demand function be given by

$$P(Q) = \begin{cases} Y - Q & \text{if payoff is High} \\ \frac{1}{2}Y - Q & \text{if payoff is Low,} \end{cases} \tag{A.1}$$

for $Y > 0$.

If the firms engage in quantity competition, in the case of a High payoff, standard computations for a Stackelberg game give $q_L^P = \frac{Y}{2}$ and $q_F^P = \frac{Y}{4}$. If the firms invest simultaneously, the Cournot outcome gives $q_S^P = \frac{Y}{3}$. Therefore, if $V_i^j = \int_0^\infty e^{-rt}\pi_i^j dt - I$, for $j = \{P, N\}$ and $\pi_i^j = q_j^i p_j^i$, and where I is the cost of investing, we get that $V_L^P > V_S^P > V_F^P > 0$.

In the case of a low payoff, the follower never invests so $V_F^N = 0$, but $V_L^N = \frac{Y^2}{16r} - I < 0$ and $V_S^N = \frac{Y^2}{36r} - I < 0$. Hence, $V_S^N < V_L^N < 0$.

B Proof of Theorem 1

1. $\lceil s_P \rceil \leq \lceil s_S \rceil$

   (a) If $s_t < \lceil s_P \rceil$, then since $p(\lceil s_P \rceil - 1) < p_P \leq p(\lceil s_P \rceil)$, (i) if $p_t \leq p(\lceil s_P \rceil - 1)$, then $p_t < p_P$, clearly. However, it cannot be that $p_t \in (p(\lceil s_P \rceil - 1), p(\lceil s_P \rceil))$ since $p_t = p(s_t)$ for $s_t \in \mathbb{Z}$. Then, $s_t < \lceil s_P \rceil \implies p_t < p_P$ and $t < \tau^0 \leq T_P$. Therefore, $G_i^0(t) = 0$ for $i = 1, 2$ because $L(p_t) < F(p_t)$ for $p_t < p_P$ and, by the definition of α-consistency in Thijssen et al. [11], $\alpha_i^0(t) = 0$ also. Hence

   $$\begin{aligned} W_i^0(s_t) =& \frac{\alpha_i^0(\tau^0)(1 - \alpha_j^0(\tau^0))}{\alpha_i^0(\tau^0) + \alpha_j^0(\tau^0) - \alpha_i^0(\tau^0)\alpha_j^0(\tau^0)} L(p_{\tau^0}) + \frac{\alpha_j^0(\tau^0)(1 - \alpha_i^0(\tau^0))}{\alpha_i^0(\tau^0) + \alpha_j^0(\tau^0) - \alpha_i^0(\tau^0)\alpha_j^0(\tau^0)} F(p_{\tau^0}) \\ &+ \frac{\alpha_i^0(\tau^0)\alpha_j^0(\tau^0)}{\alpha_i^0(\tau^0) + \alpha_j^0(\tau^0) - \alpha_i^0(\tau^0)\alpha_j^0(\tau^0)} S(p_{\tau^0}), \end{aligned} \tag{B.1}$$

   where $\alpha^0(\tau^0) \in [0, 1]$. Maximising this equation with respect to $\alpha_i^0(\tau^0)$, we get $\partial W_i^0(s_t)/\partial \alpha_i^0(\tau^0) = 0$ for $\alpha_j^0(\tau^0) = (L(p_{\tau^0}) - F(p_{\tau^0})) \,/\, (L(p_{\tau^0}) - S(p_{\tau^0}))$. But in this region $\tau^0 = \inf\{t' > t | p_{\tau'} = p_P\}$. Therefore, $p_{\tau^0} = p_P \implies L(p_{\tau^0}) = F(p_{\tau^0})$. Hence, the expected value for player i is maximised for $\alpha_j^0(\tau^0) = 0$ giving $W_i^0(s_t) = L(p_{\tau^0}) \equiv F(p_{\tau^0})$.

   (b) If $s_t \geq \lceil s_S \rceil$, then $p_t \geq p_S$ and it is optimal for player i to stop at time $t = \tau^0 = T_S = T_P$ since $L(p_t) = F(p_t) = S(p_t)$ for $p_t \geq p_S$. Therefore, $G_i^0(t) = 1$ implying $\alpha_i^0(t) = 1$ (α-consistency) and by Eq. (8)

   $$\begin{aligned} W_i^0(s_t) =& (1 - \alpha_j^0(\tau^0))L(p_{\tau^0}) + \alpha_j^0(\tau^0)S(p_{\tau^0}) \\ =& L(p_{\tau^0}) = F(p_{\tau^0}). \end{aligned}$$

2. $\lceil s_P \rceil < \lceil s_S \rceil$:

   (a) If $\lceil s_P \rceil \le s_t < \lceil s_S \rceil$, then we have that $p_P \le p(\lceil s_P \rceil) \le p_t < p_S < p(\lceil s_S \rceil)$. Note that by the same reasoning as above, $p_t \notin (p_S, p(\lceil s_s \rceil))$ for $s_t \in \mathbb{Z}$. Moreover, $t = \tau^0 = T_P < T_S$ so that $G^0(u) = \alpha^0(u) = 0$ for all $u < t = \tau^0$. Thus, $W_i^0(s_t)$ is given by Eq. (B.1) and maximised for $\alpha_j^0(\tau^0) = \frac{L(p_{\tau^0}) - F(p_{\tau^0})}{L(p_{\tau^0}) - S(p_{\tau^0})}$. However, the strategies are symmetric so $\alpha_j^0(\tau^0) = \alpha_i^0(\tau^0)$ and, by substitution into (B.1), we get $W_i^0(s_t) = F(p_{\tau^0})$.

   Finally, by the definition of α-consistency in Thijssen et al. [11],

$$G_i^0(t) = \frac{1}{2 - \alpha_i^0(\tau^0)} = \frac{L(p_{\tau^0}) - S(p_{\tau^0})}{L(p_{\tau^0}) + F(p_{\tau^0}) - 2S(p_{\tau^0})}.$$

3. $\lceil s_P \rceil = \lceil s_S \rceil$, then $p(\lceil s_P \rceil - 1) < p_P < p_S \le p(\lceil s_P \rceil)$ and the only possible equilibria are those described in 1(a) and 1(b), respectively because $p_t \notin (p(\lceil s_P \rceil - 1), p(\lceil s_P \rceil))$ for $s_t \in \mathbb{Z}$.

Next we need to check whether these equilibria are subgame-perfect. Let player i deviate from the equilibrium strategy defined above, but let player j abide by it. Thus $\tau_j^0 = \tau^0$, but $\tau_i^0 \ne \tau^0$. Then for $s_t < s_P \le \lceil s_P \rceil$, $G_j^0(t) = 0$ for all $t < \tau^0$ and

$$\begin{aligned}\widetilde{W}_i^0(s_t) = &\int_0^{\tau^0} L(p_t) dG_i^0(t) + (1 - G_i^0(\tau^0-))\Big[\frac{\alpha_i^0(\tau^0)(1-\alpha_j^0(\tau^0))}{\alpha_i^0(\tau^0) + \alpha_j^0(\tau^0) - \alpha_i^0(\tau^0)\alpha_j^0(\tau^0)} L(p_{\tau^0}) \\ &+ \frac{\alpha_j^0(\tau^0)(1-\alpha_i^0(\tau^0))}{\alpha_i^0(\tau^0) + \alpha_j^0(\tau^0) - \alpha_i^0(\tau^0)\alpha_j^0(\tau^0)} F(p_{\tau^0}) + \frac{\alpha_i^0(\tau^0)\alpha_j^0(\tau^0)}{\alpha_i^0(\tau^0) + \alpha_j^0(\tau^0) - \alpha_i^0(\tau^0)\alpha_j^0(\tau^0)} S(p_{\tau^0})\Big]\end{aligned} \tag{B.2}$$

(cf. Eq. (8)).

Now consider the following cases

1. If $\tau_i^0 < \tau_j^0$, then $G_i^0(u) = 0$ for all $u < \tau_i^0$. Moreover, τ^0 in Eq. (8) is defined as $\tau^0 = \tau_i^0 \wedge \tau_j^0$. Thus, in this case $\tau^0 = \tau_i^0$ in Eq. (B.2) and, hence, $\alpha_j^0(\tau_i^0) = 0$ according to his equilibrium strategy in this region. This gives $\widetilde{W}_i^0(s_t) = L(p_{\tau_i^0})$. However, the equilibrium strategy defined above describes stopping at $\tau^0 = \tau_j^0$, and $F(p_u) > L(p_u)$ for all $u < \tau^0$. Therefore,

$$\widetilde{W}_i^0(s_t) = L(p_{\tau_i^0}) < F(p_{\tau^0}) = W_i^0(s_t).$$

2. If $\tau_i^0 > \tau_j^0 = \tau^0$, then $\alpha_j^0(\tau^0) = 0$ from Point 1(a) above, and

$$\begin{aligned}\widetilde{W}_i^0(s_t) = &\int_0^{\tau^0} L(p_t) dG_i^0(t) + (1 - G_i^0(\tau^0-))L(p_{\tau^0}) \\ \le &F(p_{\tau_i^0}) = W_i^0(s_t)\end{aligned}$$

   because $L(p_u) < F(p_u)$ for all $u < \tau^0$ and $L(p_{\tau^0}) = F(p_{\tau^0})$.

3. If $\tau_i^0 = \tau_j^0 = \tau^0$, but $\alpha_i^0(\tau^0) \ne \alpha_j^0(\tau^0) (= 0)$, then

$$\widetilde{W}_i^0(s_t) = \int_0^{\tau^0} L(p_t) dG_i^0(t) + (1 - G_i^0(\tau^0-))L(p_{\tau^0}) \le W_i^0(s_t).$$

Thus, in all cases $\widetilde{W}_i^0(s_t) \le W_i^0(s_t)$ which is the condition for subgame-perfection defined by Thijssen et al. [11].

Now, for $s_t < \lceil s_P \rceil$, it is argued in 1(a) that $t < \tau^0$ since $s_t \in \mathbb{Z}$, but $\tau^0 \le T_P$. If $s_P = \lceil s_P \rceil$, then $\tau^0 = T_P$, and the equilibrium is subgame-perfect. However, if $t^0 < T_P$, then one cannot argue that unilateral deviations from the equilibrium strategy would not make the player better off in terms of his expected value for all $u \in (\tau^0, T_P]$ because in this region $L(p_u) > F(p_u)$.

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