BLOCK ALGEBRAS WITH HH^1 A SIMPLE LIE ALGEBRA

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Abstract. The purpose of this note is to add to the evidence that the algebra structure of a p-block of a finite group is closely related to the Lie algebra structure of its first Hochschild cohomology group. We show that if B is a block of a finite group algebra kG over an algebraically closed field k of prime characteristic p such that $HH^1(B)$ is a simple Lie algebra and such that B has a unique isomorphism class of simple modules, then B is nilpotent with an elementary abelian defect group P of order at least 3, and $HH^1(B)$ is in that case isomorphic to the Witt algebra $HH^1(kP)$. In particular, no other simple modular Lie algebras arise as $HH^1(B)$ of a block B with a single isomorphism class of simple modules.

1. Introduction

Let p be a prime and k an algebraically closed field of characteristic p. For G a finite group, a *block of* kG is an indecomposable direct factor of the group algebra kG. There is an abundance of stable equivalences in block theory, but it is notoriously difficult to pin down even the most basic numerical invariants - such as the number of isomorphism classes of simple modules - through stable equivalences.

The main motivation for the present note is that on the one hand, the Lie algebra structure of $HH^1(B)$ of a block B of a finite group algebra kG is invariant under stable equivalences of Morita type (cf. [4, Theorem 10.7]), and on the other hand, there is evidence for close structural connections between the algebra structure of B and the Lie algebra structure of $HH^1(B)$ (cf. [1]). Understanding those connections might therefore ultimately contribute towards determining block invariants in some cases.

We describe some of the structural connections between B and $HH^1(B)$ in two extreme cases for blocks with a single isomorphism class of simple modules.

Theorem 1.1. *Let G be a finite group and let B be a block algebra of kG having a unique isomorphism class of simple modules. Then $HH^1(B)$ is a simple Lie algebra if and only if B is nilpotent with an elementary abelian defect group P of order at least 3. In that case, we have a Lie algebra isomorphism $HH^1(B) \cong HH^1(kP)$.*

Theorem 1.1 implies in particular that none of the other simple modular Lie algebras occur as $HH^1(B)$ of some block algebra of a finite group with the property that B has a single isomorphism class of simple modules. See [7], [8] for details and further references on the classification of simple Lie algebras in positive characteristic. We do not know whether the hypothesis on B to have a single isomorphism class of simple modules is necessary in Theorem 1.1.

Theorem 1.2. *Let G be a finite group and let B be a block algebra of kG having a nontrivial defect group and a unique isomorphism class of simple modules. Then $\dim_k(HH^1(B)) \geq 2$.*

2010 *Mathematics Subject Classification.* 16E40, 16G30, 16D90.

The hypothesis that B has a single isomorphism class of simple modules is necessary in Theorem 1.2. For instance, if P is cyclic of order $p \geq 3$ and if E is the cyclic automorphism group of order $p-1$ of P, then $HH^1(k(P \rtimes E))$ has dimension one. This follows immediately from the centraliser decomposition of Hochschild cohomology; see [3, Theorem 1.4] for a more general result.

2. Quoted results

We collect in this section results needed for the proof of Theorem 1.1.

Theorem 2.1 (Okuyama and Tsushima [5]). *Let G be a finite group and B a block algebra of kG. Then B is a nilpotent block with an abelian defect group if and only if $J(B) = J(Z(B))B$.*

Let A be a finite-dimensional (associative and unital) k-algebra. A derivation on A is a k-linear map $f : A \to A$ satisfying $f(ab) = f(a)b + af(b)$ for all a, $b \in A$. The set $\mathrm{Der}(A)$ of derivations on A is a Lie subalgebra of $\mathrm{End}_k(A)$, with respect to the Lie bracket $[f, g] = f \circ g - g \circ f$, for any f, $g \in \mathrm{End}_k(A)$. For $c \in A$, the map sending $a \in A$ to the additive commutator $[c, a] = ca - ac$ is a derivation on A; any derivation arising this way is called an *inner derivation on A*. The set $\mathrm{IDer}(A)$ of inner derivations is a Lie ideal in $\mathrm{Der}(A)$, and we have a canonical identification $HH^1(A) \cong \mathrm{Der}(A)/\mathrm{IDer}(A)$. See [9, Chapter 9] for more details on Hochschild cohomology. If A is commutative, then $HH^1(A) \cong \mathrm{Der}(A)$. A k-algebra A is *symmetric* if A is isomorphic to its k-dual A^* as an A-A-bimodule; this implies that A is finite-dimensional.

Theorem 2.2 ([1, Theorem 3.1]). *Let A be a symmetric k-algebra and let E be a maximal semisimple subalgebra. Let $f \colon A \to A$ be an E-E-bimodule homomorphism satisfying $E + J(A)^2 \subseteq \ker(f)$ and $\mathrm{Im}(f) \subseteq \mathrm{soc}(A)$. Then f is a derivation on A in $\mathrm{soc}_{Z(A)}(\mathrm{Der}(A))$, and if $f \neq 0$, then f is an outer derivation of A. In particular, we have*

$$\sum_S \dim_k(\mathrm{Ext}^1_A(S, S)) \leq \dim_k(\mathrm{soc}_{Z(A)}(HH^1(A)))$$

where in the sum S runs over a set of representatives of the isomorphism classes of simple A-modules.

Corollary 2.3 ([1, Corollary 3.2]). *Let A be a local symmetric k-algebra. Let $f \colon A \to A$ be a k-linear map satisfying $k \cdot 1 + J(A)^2 \subseteq \ker(f)$ and $\mathrm{Im}(f) \subseteq \mathrm{soc}(A)$. Then f is a derivation on A in $\mathrm{soc}_{Z(A)}(\mathrm{Der}(A))$, and if $f \neq 0$, then f is an outer derivation of A. In particular, we have*

$$\dim_k(J(A)/J(A)^2) \leq \dim_k(\mathrm{soc}_{Z(A)}(HH^1(A))) \ .$$

Theorem 2.4 (Jacobson [2, Theorem 1]). *Let P be a finite elementary abelian p-group of order at least 3. Then $HH^1(kP)$ is a simple Lie algebra.*

The converse to this theorem holds as well.

Proposition 2.5. *Let P be a finite abelian p-group. If $HH^1(kP)$ is a simple Lie algebra, then P is elementary abelian of order at least 3.*

Proof. Suppose that P is not elementary abelian; that is, its Frattini subgroup $Q = \Phi(P)$ is nontrivial. Since P is abelian, we have $HH^1(kP) = \mathrm{Der}(kP)$. We will show that the set of derivations with image contained in $I(kQ)kP = \ker(kP \to kP/Q)$ is a nonzero Lie ideal in $\mathrm{Der}(kP)$, where $I(kQ)$ is the augmentation ideal of kQ. Indeed, every element in Q is equal to x^p for some $x \in P$, and hence every element in $I(kQ)$ is a linear combination of elements of the form

$(x-1)^p$, where $x \in P$. Every derivation on kP annihilates all elements of this form (using the fact that k has characteristic p), and hence every derivation on kP preserves $I(kQ)kP$. Thus there is a canonical Lie algebra homomorphism $\mathrm{Der}(kP) \to \mathrm{Der}(kP/Q)$. This homomorphism is nonzero; indeed, it is an isomorphism on the components of Hochschild cohomology corresponding to $H^1(P;k) \cong H^1(P/Q;k)$ under the centraliser decomposition. The kernel of this canonical Lie algebra homomorphism contains all derivations with image in $\mathrm{soc}(kP)$, so this kernel is nonzero by Corollary 2.3. Thus $HH^1(kP)$ is not simple as a Lie algebra, whence the result. □

Remark 2.6. Theorem 1.1 implies that the hypothesis on P being abelian is not necessary in the statement of 2.5.

3. Auxiliary results

In order to exploit the hypothesis on HH^1 being simple in the statement of Theorem 1.1, we consider Lie algebra homomorphisms into the HH^1 of subalgebras and quotients.

Lemma 3.1. *Let A be a finite-dimensional k-algebra and f a derivation on A. Then f sends $Z(A)$ to $Z(A)$, and the map sending f to the induced derivation on $Z(A)$ induces a Lie algebra homomorphism $HH^1(A) \to HH^1(Z(A))$.*

Proof. Let $z \in Z(A)$. For any $a \in A$ we have $az = za$, hence $f(az) = f(a)z + af(z) = f(z)a + zf(a) = f(za)$. Comparing the two expressions, using $zf(a) = f(a)z$, yields $af(z) = f(z)a$, and hence $f(z) \in Z(A)$. The result follows. □

Lemma 3.2. *Let A be a local symmetric k-algebra such that $J(Z(A))A \neq J(A)$. Then the canonical Lie algebra homomorphism $HH^1(A) \to HH^1(Z(A))$ is not injective.*

Proof. Since $J(Z(A))A < J(A)$, it follows from Nakayama's lemma that $J(Z(A))A + J(A)^2 < J(A)$. Thus there is a nonzero linear endomorphism f of A which vanishes on $J(Z(A))A + J(A)^2$ and on $k \cdot 1_A$, with image contained in $\mathrm{soc}(A)$. In particular, f vanishes on $Z(A) = k \cdot 1_A + J(Z(A))$. By 2.3, the map f is an outer derivation on A. Thus the class of f in $HH^1(A)$ is nonzero, and its image in $HH^1(Z(A))$ is zero, whence the result. □

Lemma 3.3. *Let A be a local symmetric k-algebra and let f be a derivation on A such that $Z(A) \subseteq \ker(f)$. Then $f(J(A)) \subseteq J(A)$.*

Proof. Since A is local and symmetric, we have $\mathrm{soc}(A) \subseteq Z(A)$, and $J(A)$ is the annihilator of $\mathrm{soc}(A)$. Let $x \in J(A)$ and $y \in \mathrm{soc}(A)$. Then $xy = 0$, hence $0 = f(xy) = f(x)y + xf(y)$. Since $y \in \mathrm{soc}(A) \subseteq Z(A)$, it follows that $f(y) = 0$, hence $f(x)y = 0$. This shows that $f(x)$ annihilates $\mathrm{soc}(A)$, and hence that $f(x) \in J(A)$. □

Lemma 3.4. *Let A be a finite-dimensional k-algebra and J an ideal in A.*

(i) Let f be a derivation on A such that $f(J) \subseteq J$. Then $f(J^n) \subseteq J^n$ for any positive integer n.

(ii) Let f, g be derivations on A and let m, n be positive integers such that $f(J) \subseteq J^m$ and $g(J) \subseteq J^n$. Then $[f,g](J) \subseteq J^{m+n-1}$.

Proof. In order to prove (i), we argue by induction over n. For $n = 1$ there is nothing to prove. If $n > 1$, then $f(J^n) \subseteq f(J)J^{n-1} + Jf(J^{n-1})$. Both terms are in J^n, the first by the assumptions, and the second by the induction hypothesis $f(J^{n-1}) \subseteq J^{n-1}$. Let $y \in J$. Then $[f,g](y) = f(g(y)) - g(f(y))$. We have $g(y) \in J^n$; that is, $g(y)$ is a sum of products of n elements in J.

Applying f to any such product shows that the image is in J^{m+n-1}. A similar argument applied to $g(f(y))$ implies (ii). □

Proposition 3.5. *Let A be a finite-dimensional k-algebra. For any positive integer m denote by $\mathrm{Der}_{(m)}(A)$ the k-subspace of derivations f on A satisfying $f(J(A)) \subseteq J(A)^m$.*

(i) For any two positive integers m and n we have $[\mathrm{Der}_{(m)}(A), \mathrm{Der}_{(n)}(A)] \subseteq \mathrm{Der}_{(m+n-1)}(A)$.

(ii) The space $\mathrm{Der}_{(1)}(A)$ is a Lie subalgebra of $\mathrm{Der}(A)$.

(iii) For any positive integer m, the space $\mathrm{Der}_{(m)}(A)$ is an ideal in $\mathrm{Der}_{(1)}(A)$.

(iv) Suppose that A is local. The space $\mathrm{Der}_{(2)}(A)$ is a nilpotent Lie subalgebra of $\mathrm{Der}(A)$.

Proof. Statement (i) follows from 3.4 (ii). The statements (ii) and (iii) are immediate consequences of (i). Since A is local and since 1 is annihilated by any derivation on A, statement (iii) follows from (i) and the fact that $J(A)$ is nilpotent. □

4. Proof of Theorems 1.1 and 1.2

Let G be a finite group and B a block of kG. Suppose that B has a single isomorphism class of simple modules. If B is nilpotent and P a defect group of B, then by [6], B is Morita equivalent to kP, and hence there is a Lie algebra isomorphism $HH^1(B) \cong HH^1(kP)$. Thus if B is nilpotent with an elementary abelian defect group P of order at least 3, then $HH^1(B)$ is a simple Lie algebra by 2.4.

Suppose conversely that $HH^1(B)$ is a simple Lie algebra. If $J(B) = J(Z(B))B$, then B is nilpotent with an abelian defect group P by 2.1. As before, we have $HH^1(B) \cong HH^1(kP)$, and hence 2.5 implies that P is elementary abelian of order at least 3.

Suppose that $J(Z(B))B \neq J(B)$. Let A be a basic algebra of B. Then $J(Z(A))A \neq J(A)$. Moreover, A is local symmetric, since B has a single isomorphism class of simple modules. Thus $\mathrm{soc}(A)$ is the unique minimal ideal of A. We have $J(A)^2 \neq \{0\}$. Indeed, if $J(A)^2 = \{0\}$, then $\mathrm{soc}(A)$ contains $J(A)$, and hence $J(A)$ has dimension 1, implying that A has dimension 2. In that case B is a block with defect group of order 2. But then $HH^1(A) \cong HH^1(kC_2)$ is not simple, a contradiction. Thus $J(A)^2 \neq \{0\}$, and hence $\mathrm{soc}(A) \subseteq J(A)^2$. By 3.2, the canonical Lie algebra homomorphism $HH^1(A) \to HH^1(Z(A))$ is not injective. Since $HH^1(A)$ is a simple Lie algebra, it follows that this homomorphism is zero. In other words, every derivation on A has $Z(A)$ in its kernel. It follows from 3.3 that every derivation on A sends $J(A)$ to $J(A)$. Thus, by 3.4, every derivation on A sends $J(A)^2$ to $J(A)^2$. This implies that the canonical surjection $A \to A/J(A)^2$ induces a Lie algebra homomorphism $HH^1(A) \to HH^1(A/J(A)^2)$. Note that the algebra $A/J(A)^2$ is commutative, and hence $HH^1(A/J(A)^2) = \mathrm{Der}(A/J(A)^2)$. Since $J(A)^2$ contains $\mathrm{soc}(A)$, it follows that the kernel of the canonical map $HH^1(A) \to HH^1(A/J(A)^2)$ contains the classes of all derivations with image in $\mathrm{soc}(A)$. Since there are outer derivations with this property (cf. 2.3), it follows from the simplicity of $HH^1(A)$ that the canonical map $HH^1(A) \to HH^1(A/J(A)^2) = \mathrm{Der}(A/J(A)^2)$ is zero. Thus every derivation on A has image in $J(A)^2$. But then 3.5 implies that $\mathrm{Der}(A) = \mathrm{Der}_{(2)}(A)$ is a nilpotent Lie algebra. Thus $HH^1(A)$ is nilpotent, contradicting the simplicity of $HH^1(A)$. The proof of Theorem 1.1 is complete.

Proof of Theorem 1.2. Denote by A a basic algebra of B. Since B has a unique isomorphism class of simple modules and a nontrivial defect group, it follows that A is a local symmetric algebra of dimension at least 2. By 2.3 we have $\dim_k(HH^1(A))) \geq \dim_k(J(A)/J(A)^2)$. Thus

$\dim_k(HH^1(A)) \geq 1$. Moreover, if $\dim_k(HH^1(A)) = 1$, then $\dim_k(J(A)/J(A)^2) = 1$, and hence A is a uniserial algebra. In that case B is a block with a cyclic defect group P and a unique isomorphism class of simple modules, and hence B is a nilpotent block. Thus $A \cong kP$. We have $\dim_k(HH^1(kP)) = |P|$, a contradiction. The result follows. □

Remark 4.1. All finite-dimensional algebras in this paper are split thanks to the assumption that k is algebraically closed. It is not hard to see that one could replace this by an assumption requiring k to be a splitting field for the relevant algebras. The statements 3.1 and 3.4 do not require any hypothesis on k.

Acknowledgement. The present paper was partially funded by EPSRC grant EP/M02525X/1 of the first author.

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