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A Comparative Study of Parametric Mortality Projection Models

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Abstract

The relative merits of different parametric models for making life expectancy and annuity value predictions at both pensioner and adult ages are investigated. This study builds on current published research and considers recent model enhancements and the extent to which these enhancements address the deficiencies that have been identified of some of the models. The England & Wales male mortality experience is used to conduct detailed comparisons at pensioner ages, having first established a common basis for comparison across all models. The model comparison is then extended to include the England & Wales female experience and both the male and female USA mortality experiences over a wider age range, encompassing also the working ages.

Key words and phrases: Mortality forecasting; binomial response models; age-period effects; age-period-cohort effects; forecast statistics; model and forecast comparison; back-fitting

1. Introduction

In this paper, we contribute to the debate on the relative merits of various extrapolation models used as a means of projecting future mortality rates. We focus, in particular, on comparing the key indices of life expectancy and annuity value predictions, as computed by the cohort method. In formulating our approach, we establish a common basis for comparison across models, and this means that noteworthy differences in the predicted indices of interest may be directly attributable to the choice of model predictor structure.

The details of the models and methodology are set out systematically in Section 2, which is supported by technical Appendices, A & B, for completeness. The models include a group of 4 parametric predictor models based on, and including the Lee & Carter (1992) bilinear structure, with the optional inclusion of a second pair of age-period components and the capture of cohort effects; together with a further group of 8 linear parametric predictors based on Cairns *et al.* (2009) and including extensions due to Plat (2009).

A comparative study of the models, using the England & Wales 1961-2007 male mortality experience, restricted to pensioner ages is reported in Section 3. The age restriction is imposed in order to accommodate the models due to Cairns *et al.* (2009), denoted by *M*5-*M*8, and which are designed for use at pensioner ages only. Results based on the different stages of model building are set out in Section 2 and are presented pictorially. Diagnostic checks on each model and the accompanying random walk period index model are conducted by monitoring residual plots. Life expectancy and annuity model predictions are examined for robustness by systematically truncating the time span of the data at the two extremities, before repeated modelling.

In Section 4, the age restriction imposed in Section 3 is lifted and further comparative studies are reported. These are again conducted following the different stages set out in Section 2, using both the England & Wales and USA mortality experiences for each gender and involving a wider age span that includes the working ages as well as pensioner ages.

A detailed discussion of the issues arising is presented in Section 5, followed by a summary in Section 6.

2. Methodology

2.1 Data array

We denote a rectangular mortality data array, partitioned into unit square cells of size one year by

$$\left(d_{xt}, e_{xt}, \omega_{xt}\right): \text{ age } x = x_1, x_2, ...x_k, \text{ period } t = t_0, t_1, ..., t_n$$

where

d_{xt} - reported number of deaths

e_{xt} - matching *initial* exposures to the risk of death

ω_{xt} - 0/1 weights to indicate empty or omitted data cells

When initial exposures are required for analysis and only central exposures are available, as in this paper, we approximate the initial exposures to the risk of death by adding half the matching reported numbers of deaths to the central exposures (e.g. Section 2.2, Forfar *et al.* 1988).

2.2 Model structures

We target and project the probability of death q_{xt} throughout. A common basis for comparison across all models is established by using the log-odds function to link q_{xt} to the parametric predictor structure η_{xt} in all cases, so that, typically, for any model *H*

$$H: \log\left(\frac{q_{xt}}{1-q_{xt}}\right) = \eta_{xt}.$$

The log-odds function is also chosen because of the historical ties with the early actuarial work of Perks (1932).

The following predictor structures are compared

$$LC: \eta_{xt} = \alpha_x + \beta_x \kappa_t$$

$$H_1: \eta_{xt} = \alpha_x + \beta_x \kappa_t + \iota_{t-x}$$

$$M: \eta_{xt} = \alpha_x + \beta_x \kappa_t + \beta_x^{(0)} \iota_{t-x}$$

$$LC2: \eta_{xt} = \alpha_x + \beta_x^{(1)} \kappa_t^{(1)} + \beta_x^{(2)} \kappa_t^{(2)}$$

$$M5: \eta_{xt} = \kappa_t^{(1)} + \left(x - \bar{x}\right)\kappa_t^{(2)}$$

$$M6: \eta_{xt} = \kappa_t^{(1)} + \left(x - \bar{x}\right)\kappa_t^{(2)} + \iota_{t-x}$$

$$M7: \eta_{xt} = \kappa_t^{(1)} + \left(x - \bar{x}\right)\kappa_t^{(2)} + b\left(x\right)\kappa_t^{(3)} + \iota_{t-x}$$

$$M8: \eta_{xt} = \kappa_t^{(1)} + \left(x - \bar{x}\right)\kappa_t^{(2)} + \left(x_c - x\right)\iota_{t-x}$$

$$M5^*: \eta_{xt} = \alpha_x + \kappa_t^{(1)} + \left(\bar{x} - x\right)\kappa_t^{(2)} + \left(\bar{x} - x\right)^+ \kappa_t^{(3)}$$

$$M6^*: \eta_{xt} = \alpha_x + \kappa_t^{(1)} + \left(\bar{x} - x\right)\kappa_t^{(2)} + \left(\bar{x} - x\right)^+ \kappa_t^{(3)} + \iota_{t-x}$$

$$M7^*: \eta_{xt} = \alpha_x + \kappa_t^{(1)} + \left(\bar{x} - x\right)\kappa_t^{(2)} + \left(\bar{x} - x\right)^+ \kappa_t^{(3)} + b\left(x\right)\kappa_t^{(4)} + \iota_{t-x}$$

$$M8^*: \eta_{xt} = \alpha_x + \kappa_t^{(1)} + \left(\bar{x} - x\right)\kappa_t^{(2)} + \left(\bar{x} - x\right)^+ \kappa_t^{(3)} + \left(x_c - x\right)\iota_{t-x}$$

where

$$\left(\bar{x} - x\right)^+ = \max\left\{\left(\bar{x} - x\right), 0\right\},\ b(x) = \left\{\left(x - \bar{x}\right)^2 - \frac{1}{k}\sum_{i=x_1}^{x_k}\left(i - \bar{x}\right)^2\right\},\ \bar{x} = \frac{1}{k}\sum_{i=x_1}^{x_k} i$$

and x_c is a pre-determined constant.

For a discussion of the first 3 predictor structures, *LC*, H_1 and *M*, together with

$$H_0: \eta_{xt} = \alpha_x + \kappa_t + \iota_{t-x} \tag{1}$$

which we shall have occasion to refer to, see Renshaw and Haberman (2006). The structures are nested in the sense that

$$LC \subset H_1 \subset M\ ;\ H_0 \subset H_1 \subset M\ .$$

We note that various studies using the *LC* model applied to the probability of death q_{xt} have been presented by Cossette *et al.* (2007) and by Haberman and Renshaw (2008).

The predictor *LC*2 is a natural extension to *LC* and has been discussed by Lee (2000) and by Renshaw and Haberman (2003). For these two structures

$$LC \subset LC2\ .$$

The 4 structures *M*5–*M*8 are due to Cairns *et al.* (2008a,b,c; 2009). We note that a re-parameterised version of *M*5 features prominently in Cairns *et al.* (2006). The 4 extensions, *M*5*–*M*8* are motivated by Plat (2009) who specifically introduces *M*6*. We note that the Cairns *et al.* (2009) structures, denoted by *M*5–*M*8, are retrieved by

eliminating the terms in α_x and $(\bar{x}-x)^+$, and reversing the sign of $(\bar{x}-x)$. We also note that these structures satisfy the relationship

$$M5^* \subset M6^* \subset M7^*.$$

2.3 Model fitting

For consistency across models and in common with standard generalised linear modelling (GLM) practice, we choose to fit each structure by minimising the binomial deviance

$$Dev(\boldsymbol{y},\boldsymbol{q}) = 2\sum_{x,t} w_{xt} \int_{q_{xt}}^{y_{xt}} \frac{y_{xt}-u}{u(1-u)}\,du = 2\sum_{x,t} w_{xt} \left\{ y_{xt} \log\left(\frac{y_{xt}}{q_{xt}}\right) + (1-y_{xt}) \log\left(\frac{1-y_{xt}}{1-q_{xt}}\right)\right\}$$

where

$$Y_{xt} = \frac{D_{xt}}{e_{xt}},\ E(Y_{xt}) = q_{xt},\ Var(Y_{xt}) = \frac{q_{xt}(1-q_{xt})}{e_{xt}},\ w_{xt} = \omega_{xt} e_{xt}.$$

This is equivalent to maximising the log-likelihood function based on the assumption

$$D_{xt} \sim bin(e_{xt}, q_{xt}),\ i.i.d.$$

subject to the application of the 0/1 prior weights ω_{xt}.

Initial estimates for α_x, ensuring that the parameter set $\{\alpha_x\}$ represents a static life table (on the log-odds scale), are given by

$$\hat{\alpha}_x = \frac{1}{\sum_t \omega_{xt}} \sum_t \omega_{xt} \log\left(\frac{d_{xt}}{e_{xt}-d_{xt}}\right).$$

While the structure of the reference model H_0, expression (1), is linear in the parameters, fitting this model is complicated by the relationship

cohort = period – age.

For this reason, and noting that

$$H_0 \subset H_1,\ H_0 \subset M,\ H_0 \subset M6^*,\ H_0 \subset M7^*,\ H_0 \subset M8^*$$

we apply a two stage-fitting strategy when fitting the models H_1, M, $M6^*$–$M8^*$: treating $\hat{\alpha}_x$ as an offset when estimating the other parameters. For a fuller discussion of the issues involved, we refer the reader to Renshaw and Haberman (2009).

The first 3 predictors LC, H_1 and M, which are non-linear in the parameters, are fitted using the algorithm reported in Renshaw and Haberman (2006), subject to modified updating relationships based on the binomial deviance. Similarly, the $LC2$ predictor, which is also non-linear in the parameters, is fitted using an obvious expansion of the suitably adjusted *LC* fitting algorithm (Brouhns *et al.* (2002), Renshaw and Haberman (2003)). Finally, given the linear nature of the remaining 4 predictors $M5^*$–$M8^*$, these

may be fitted using the GLM facilities in standard statistical packages, subject to the declaration of $\hat{\alpha}_x$ as an offset, when required.

2.4 Parameter constraints

For the 4 nonlinear structures, we set the following constraints

$$LC: \sum_x \beta_x = 1,\ \kappa_{t_1} = 0$$

$$H_1: \sum_x \beta_x = 1,\ \kappa_{t_1} = 0$$

$$M: \sum_x \beta_x = \sum_x \beta_x^{(0)} = 1,\ \kappa_{t_1} = 0$$

$$LC2: \sum_x \beta_x^{(1)} = \sum_x \beta_x^{(2)} = 1,\ \kappa_{t_1}^{(1)} = \kappa_{t_1}^{(2)} = 0.$$

Other choices are possible without affecting the subsequent model projections. Also as a precaution, we set the additional constraint $\beta_x > 0 \forall x$ for H_1 and *M*, (see Renshaw and Haberman (2009)).

For models *M*5–*M*8 and *M*5*–*M*8* the GLM constraints are listed in Appendix A and are adjusted to conform to the Cairns *et al.* (2009) constraints as detailed in Appendix A.

2.5 Model diagnostics

We denote the optimum value of the deviance

$$Dev(\boldsymbol{y}, \hat{\boldsymbol{q}}) = \sum_{x,t} \hat{d}_{xt}$$, with scale parameter estimate $\hat{\phi} = \sum_{x,t} \hat{d}_{xt} \Big/ \nu$

where ν is the number of degrees of freedom supported by the model structure and data. In assessing the goodness of fit of the different models, we make extensive use of diagnostic residual plots on fitting the various model structures, using the scaled deviance residuals

$$r_{xt} = sign(y_{xt} - \hat{q}_{xt}) \sqrt{\hat{d}_{xt} / \hat{\phi}}.$$

2.6 Model dynamics

For models *LC*2, *M*5–*M*8 and *M*5*–*M*8*, we follow Cairns *et al.* (2006; 2008a,b,c) in using a multivariate random walk, with a vector of drift parameters $\boldsymbol{\theta}$, to drive the dynamics of the multiple period indices, so that

$$\boldsymbol{\kappa}_t = \boldsymbol{\theta} + \boldsymbol{\kappa}_{t-1} + \boldsymbol{\varepsilon}_t,\ t = t_1, t_2, ..., t_n;\ \boldsymbol{\varepsilon}_t \sim N(\boldsymbol{0}, \boldsymbol{\Omega}),\ \boldsymbol{\Omega} = \boldsymbol{C}\boldsymbol{C}'$$

where $\boldsymbol{\kappa}_t$ is the vector of period indices and $\boldsymbol{C}$ the Cholesky factorisation matrix of the variance-covariance matrix $\boldsymbol{\Omega}$ (see Appendix B). For LC, H_1 and M we apply the univariate version of the random walk, thereby establishing consistency with the other models.

The parameters $\boldsymbol{\theta}$ and $\boldsymbol{\Omega}$ are estimated by ordinary least squares (OLS), and the component residuals $\hat{\boldsymbol{\varepsilon}}_t$ are plotted against t as a diagnostic check on the goodness of fit. The residuals are standardised before presentation.

2.7 Indices of interest

In terms of the key indices of interest, we consider life expectancies and level immediate annuities, computed by the cohort method, thereby allowing for the future evolution of mortality rates. We focus on the most recent period t_n, for which data are available, and so consider individuals aged x at that time. The indices are computed, respectively, as

$$e_x(t_n) = \frac{\sum_{j\geq 0} l_{x+j}(t_n+j)\left\{1-\frac{1}{2}q_{x+j,t_n+j}\right\}}{l_x(t_n)},\ a_x(t_n) = \frac{\sum_{j\geq 1} l_{x+j}(t_n+j)\,\mathrm{v}^j}{l_x(t_n)} = \sum_{j\geq 1} S_{x,t_n}(j)\,\mathrm{v}^j$$

where

$$l_{x+1}(t+1) = (1-q_{xt})\,l_x(t)$$

with a discount factor v and survivor index (representing the probability of survival from age x to age $x+t$ on the basis of the mortality experience of the cohort aged x in year t_n) $S_{x,t_n}(t): t\geq 0,\ S_{x,t_n}(0)=1$. We note that both of these indices do not require the time series modelling of the main cohort index, when it is present in the model being considered.

2.8 Prediction intervals

Based on the findings reported in Renshaw and Haberman (2009), we simulate prediction intervals (fan charts) for the indices of interest using the following algorithm, which makes full allowance for the forecast error generated by the multivariate random walk (Appendix B)

Algorithm

For simulation m = 1,2,…,M

1. randomly sample $\boldsymbol{z}_m^*$ from $mulN(\boldsymbol{0}, \boldsymbol{I})$

For j = 1,2,…,J

2. compute $\boldsymbol{\kappa}^*_{t_n+j} = \boldsymbol{\kappa}_{t_n} + j\hat{\boldsymbol{\theta}} + \sqrt{j}\hat{\boldsymbol{C}}\boldsymbol{z}_m^*$
3. compute $q^*_{x+j,t_n+j,m}$

4. Compute the indices of interest.

For example, in Step 3, for *M5–M8*: on taking into account the reversal in the sign of the prescribed 2nd period index age modulating coefficient, we use

$$M5: \log\left(\frac{q^{*}_{x+j,t_n+j}}{1-q^{*}_{x+j,t_n+j}}\right) = \boldsymbol{B}'_{x+j}\boldsymbol{\kappa}^{*}_{t_n+j}$$

$$M6,\ M7: \log\left(\frac{q^{*}_{x+j,t_n+j}}{1-q^{*}_{x+j,t_n+j}}\right) = \boldsymbol{B}'_{x+j}\boldsymbol{\kappa}^{*}_{t_n+j} + \hat{\imath}_{t_n-x}$$

$$M8: \log\left(\frac{q^{*}_{x+j,t_n+j}}{1-q^{*}_{x+j,t_n+j}}\right) = \boldsymbol{B}'_{x+j}\boldsymbol{\kappa}^{*}_{t_n+j} + \left\{x_c-(x+j)\right\}\hat{\imath}_{t_n-x}$$

where

$$M5, M6, M8: \boldsymbol{B}_{x+j} = \begin{bmatrix} 1 \\ (x+j-\overline{x}) \end{bmatrix}, \boldsymbol{\kappa}^{*}_{t_n+j} = \begin{bmatrix} \kappa^{(1)*}_{t_n+j} \\ \kappa^{(2)*}_{t_n+j} \end{bmatrix},$$

$$M7: \boldsymbol{B}_{x+j} = \begin{bmatrix} 1 \\ (x+j-\overline{x}) \\ b(x+j) \end{bmatrix}, \boldsymbol{\kappa}^{*}_{t_n+j} = \begin{bmatrix} \kappa^{(1)*}_{t_n+j} \\ \kappa^{(2)*}_{t_n+j} \\ \kappa^{(3)*}_{t_n+j} \end{bmatrix};$$

with equivalent, although more complex expressions for *M*5*–*M*8*.

2.9 Topping-out by age

Projected mortality rates

$$q_{x+j,t_n+j} : j = 1,2,...,x_k - x \ \left(x < x_k\right),$$

restricted (above) by x_k, are available for the computation of the indices of interest by cohort trajectory. In order to implement topping-out by age, projected log mortality rates are extrapolated further along the age axis up to age ω $(> x_k)$, using the differencing formula

$$u_j = \log\left(q_{x+j,t_n+j}\right) = a + bj + cj\left(j+1\right);\ \ j = x_k - x - 1, x_k - x, x_k - x + 1, ..., \omega - x,$$

which requires u_{x_k-x-1}, u_{x_k-x} and the specification of ω, $u_{\omega-x}$. This technique has been proposed by Renshaw and Haberman (2009), as a variant of the widely used demographic method introduced by Coale and Kisker (1990).

2.10 A basis for comparison

The following common features provide a basis for comparing model predictions:

- Model fitting is on the basis of optimising the binomial deviance (likelihood).
- The mapping of the predictor structure η_{xt} to q_{xt} involves the same link function.
- The period indices, however many, are modelled as a multivariate random walk with drift with the appropriate number of components.
- Topping out by age, is applied consistently, by choosing ω, $q_{\omega,t_n+\omega-x}$ at the outset.
- A common approach (across models) is used to simulate prediction intervals for the indices of interest.

3. Study: England & Wales 1961-2007 male mortality experience, ages 55-89

In this section, we conduct a comparative study between the models, following the key elements presented in Section 2 and highlighting the salient features to emerge. A more detailed discussion is reserved for Section 5.

3.1 The data

We focus on the numbers of recorded deaths and matching population sizes exposed to the risk of death, as compiled by the UK Government Actuary's Department (GAD) for the England & Wales male mortality experience. The data are cross-classified by individual calendar year 1961-2007 and individual age last birthday 0-89. The data are truncated at age 55 from below so that, for this study, we consider models *M*5–*M*8 to be appropriate (rather than *M*5*–*M*8*).

3.2 Objectives

The purpose of this study is twofold: (1) to compare predictions of the indices of interest (Section 2.7), that are dependent on future mortality rates, using the different model structures (Section 2.2) under the basis for comparison outlined in Section 2.10; and (2) to examine the predictions for robustness. We aim to achieve these objectives by comparing predictions for individuals aged 60, 65, 70, 75 (abbreviated 60(05)75) subject to systematic biennial data deletions at either end of the period span. Thus, in the first of two exercises, data covering the periods 1961 to 1985, 1987, …, 2007 (abbreviated 85(02)07) are successively retained, and predictions made for the most recent retained year t_n (which is thus successively 1985, 1987, …, 2007). In the second exercise, 2007 predictions are compared having truncated the data at the front-end on a successive biennial basis, prior to modelling: so that we model the data for the periods 1961-2007, 1963-2007, …, 1983-2007, (abbreviated 61(02)83).

3.3 Model diagnostics

For the full period 1961 to 2007 only, model diagnostics in the form of residual plots are presented separately, for each model, in Figs 1-8. The layout of each figure is similar, with the deviance residuals plotted respectively against period, age and cohort

presented in the upper row of panels. (The remaining panels in these figures relate to the respective model dynamics, which we shall discuss in Section 3.5). In respect of these residual plots, we note the following:

- The capture of period effects by all 8 models (upper LH panels, Figs 1-8).
- The capture of cohort effects by models H_1, *M* and *M*6-*M*8 (upper RH panels, Figs 3, 4, 6-8). Models *LC*, *LC*2 and *M5* fail to capture this appreciable effect (upper RH panels, Figs 1, 2, 5). It is of interest to note that the discontinuity in these plots coincides with the influenza pandemic following the Great War. Also, in common with other studies involving these data, we have zero weighted the data cells for the cohort year of birth 1886: see Renshaw and Haberman (2006).
- The capture of age effects by *LC*, *LC*2, H_1 and *M* (centre panels, Figs 1-4), the failure of *M*5 adequately to capture all of the age effects (centre panel Fig 5), and the mild residual ripple age effect which suggests that not all of the age effects are captured by *M*6-*M*8 (centre panels Figs 6-8).

The accompanying Akaike Information Criteria (*AIC*), Bayes Information Criteria (*BIC*), and Hannan-Quinn Criteria (*HQC*), together with their respective rankings across models (in brackets), read as follows:

	LC	***LC*2**	***H*1**	***M***	***M*5**	***M*6**	**M7**	***M*8**
AIC	-12679(7)	-11446(6)	-10608(3)	-10588(2)	-15245(8)	-10805(5)	-10380(1)	-10733(4)
BIC	-12989(7)	-11973(6)	-11132(3)	-11111(2)	-15498(8)	-11270(5)	-10968(1)	-11197(4)
HQC	-12794(7)	-11642(6)	-10802(3)	-10782(2)	-15339(8)	-10977(5)	-10598(1)	-10905(4)

The indices take the form $\ell - g(d)$ where ℓ denotes the maximum log-likelihood, *d* denotes the dimension of the parameter space, with respective penalty functions $g(d) = d$, $g(d) = 0.5d \log\left(\sum_{x,t} \omega_{xt}\right)$ and $g(d) = d \log\left\{\log\left(\sum_{x,t} \omega_{xt}\right)\right\}$. We note the closeness in value of the matching H_1 and *M* statistics. We note also that each of these 3 Criteria leads to the same ranking of models – with the best 3 fitting models being *M*7, *M* and H_1.

3.4 Further model details

Additional details associated with the fitting of *LC*, *LC*2, H_1 and *M*, are presented in Fig 9. These include the respective age modulating indices, and the H_1 and *M* fitted cohort indices (centre and lower RH panels, Fig 9). In the case of model *M*, it is also apparent from the plotted cohort index, that zero weights have been allocated to the first and last 3 cohort years prior to modelling: a feature applied consistently throughout when fitting this model. For these models, it is possible to smooth the age modulating indices as illustrated (using the S-plus super smoother), in order to avoid the possibility of any localised age induced anomalies being carried forward when projecting mortality rates. In a pilot study, smoothing was not found to make a material difference when depicting the simulated indices of interest reported in this paper, and hence, it has not been applied.

Additional details associated with the fitting of *M*5-*M*8 are presented in Fig 10. These include graphs of the prescribed age modulating functions forming the upper row

of panels, the fitted cohort indices ι_{t-x} for *M*6-*M*8 in the second row of panels, and the deviance profile used to determine in value of the constant x_c for *M*8. We note the following:

- For models *M*6-*M*8, the close agreement between cohort index patterns (Fig 10) and the matching patterns reported in Cairns *et al*. (2009) based on a slightly reduced data set.
- For *M*8, the choice of $x_c = 89$ is sensitive to the choice of age range and, in this case, it has been restricted to the age range of the data.

3.5 Model dynamics

Details of the components of the period index random walk time series for the respective models, including time series residual plots and forecasts, are depicted in the second and subsequent rows of Figs 1-8, with a separate row for each component. The number of components involved is as follows:

Model	***LC***	***LC*2**	***H*1**	***M***	***M*5**	***M*6**	***M*7**	***M*8**
# components	1	2	1	1	2	2	3	2

Irrespective of the number of components, it is convenient to refer to the first component as the primary component. For models *M*5-*M*8, which are characterised by more than one component, comparison of the ordinate scales of the component time series forecasts (RH panels, Figs 5-8), indicates the dominant role played by the primary component over the secondary component in the construction of mortality rate forecasts. This feature extends to the dominance of the secondary component over the tertiary component in the case of *M*7. In the case of period indices with multiple components, the ranking of the components in this way does not extend to the case of *LC*2. With respect to these panels, we note the following important features:

- The close agreement between period index patterns (Figs 1, 4-8) and the matching patterns reported in Cairns *et al*. (2009), in all cases.
- The characteristic feature of random walk (component) projections, which are generated by extrapolating the straight line drawn through the first and last (component) values of the time series.
- The downwards trend in the dominant primary period component (2^{nd} row RH panel, Figs 1-8), across all models.
- The pattern of irregularities in the primary component residuals (2^{nd} row LH panel, Figs 1-8) which, with the possible exception of *LC*2 (Fig 2), is the same for all models, subject to differences in the orientation of this pattern, relative to the abscissa.
- Supporting quantile-quantile (Q-Q) time series component residual plots, for each model, checking for normally distributed residuals, which are presented in Appendix C.
- The pattern in primary period component is linear in the case of H_1, *M* and *M*6 (Figs 3, 4, 6), but, in all of the other cases, exhibits curvature: a feature reflected in the orientation of the matching residual patterns. We observe that these

features are preserved when the data are truncated at either age extremity prior to repeat analysis.

- For models *M*5-*M*8, the forecast trend in the secondary period component (3rd row RH panel, Figs 5-8) is variable in direction across the models. Further, the nature of the associated prescribed age modulating function (upper LH panel, Fig 10) with its change of sign means that the contribution to the forecast mortality rate switches direction at the mid-point of the age range.
- For model *M*7, the tertiary period component forecast is controlled by the age modulating function (upper central panel, Fig 10) which changes sign twice and hence switches direction twice, at points which depend on the width of the age range.

3.6 Predictions

First, we present the evolving biennial 85(02)07 life expectancy (Fig 11a) and 4% fixed rate annuity value (Fig 11b) predictions and prediction intervals, computed using a cohort trajectory, for individuals aged 60(05)75, and the 8 different models. The plotted points are generated by computing and recording the 5, 50 (median) and 95 percentiles (cumulative probabilities) of the simulated predicted values of the index in question, under the $12 \times 4 \times 8 = 384$ different period-age-model combinations. The predicted values relate to the cohorts at the selected ages in years 1985, 1987, …, 2007. The simulated percentiles are plotted against the abscissa and arranged systematically, (here the ordinate scale should be ignored). Given the nature of this back-fitting exercise, involving the modelling of data from periods 1961-1985, 1961-1987, …, 1961-2007, the associated forecasts are derived from time series of increasing length. Topping-out by age is conducted by setting $q_{99} = 0.5$ throughout. Referring to Figs 11a&b, we note the following important features:

- The similarity of matching *LC* and *LC*2 predictions, subject to comparatively wider prediction intervals in the case of *LC*2. Measures of relative dispersion in the prediction intervals across models are discussed at greater length in Section 5.
- The similarity of matching H_1 and *M* predictions, noting the greater irregularity of prediction in the case of model *M* at age 60.
- The marked differences in location between matching H_1 and *M* predictions on the one hand and the *LC* and *LC*2 predictions on the other hand: this is directly attributable to the capture (or non-capture) of demonstrable cohort effects present in the data, compounded by the mild curvature in the period index in the case of the *LC* and *LC*2 models.
- The general pattern of alignment of the matching H_1, *M* and *M*6 predictions, all 3 of which are associated with linear primary period indices.
- The general pattern of alignment of matching *M*5, *M*7 and *M*8 predictions, all 3 of which are associated with a mild degree of curvature in the respective primary period index. Although *M*7 and *M*8 are seen to capture the known strong cohort effects present in the data (residual plots Figs 7&8), we note that this is not reflected in the predictions when compared with *M*5 which does not capture the known cohort effects.

- The contribution from the 2nd component forecast under *M*6 is opposite in trend direction to that of the 2nd component forecasts under *M*5 and *M*7 (Figs 5-7), and was found to remain so throughout the back fitting exercise. The 2nd component forecast under *M*8 (Fig 8) was found to change trend direction during the course of the back fitting exercise.

Secondly, we present the respective 2007 simulated life expectancy and 4% fixed rate annuity value 5, 50 and 95 percentile predictions (Figs 12a&b), computed along cohort trajectories, for individuals aged 60(05)75, and the different models, having first subjected the data to the biennial front-end truncations 61(02)83. These predictions relate to the cohorts at the selected ages in 2007, based on models fitted to data for the periods 1961-2007, 1963-2007, …, 1983-2007. Referring to Figs 12a&b we note the following important features:

- The static nature (vertical alignment) of the median predictions for *M* and *M*6, contrasting with the mild steady reduction in median predictions with reducing front-end data spans for *LC*, *LC*2, *M*5, *M*7, and *M*8 together with the reversal of this trend for H_1.
- The mild tapering of prediction intervals with decreasing period span, in a counter intuitive direction, for all models.
- The similarity of the matching *LC* and *LC*2 predictions which contrast with the marked increased matching H_1 and *M* predictions, attributable to the known strong cohort effects which are present in these data (as noted earlier).
- The approximate alignment of matching *M*5, *M*7 and *M*8 predictions, with no apparent increase in the median predictions as a consequence of capturing the cohort effect under *M*7 and *M*8 when contrasted with *M*5.

3.7 Prediction error in retrospective study

This section is motivated by Booth *et al.* (2006). In a further investigation, all 8 models are fitted to a truncated version of the England & Wales male mortality experience, restricted to the period 1961-1982 (ages 55-89), and the 1982 predicted life expectancies and 4% annuity values are calculated for individual ages 65(01)80 by the cohort method (using the same time series models and same topping out procedure by age). Then, using the actual raw mortality rates for the period 1983-2007, the same calculations are repeated and the errors (predicted – actual) in the life expectancy and annuity indices of interest calculated. We have chosen 1982 as the pivotal year in these calculations because it is approximately half-way through the period for which we have data available. The errors, plotted against an individual's age are displayed in the various panels in Fig 13a, with life expectancy represented in the LH panels and annuities in the RH panels. We note the following

- The positive nature of these errors for *M*6, with the implication of overstated 1982 predictions. Otherwise, we note that the 1982 predictions are understated to varying degrees by the remaining 7 models: with *LC*2 and *M*5 being the worst performing of the 8 models being considered.

We have also calculated the errors in the log death rates (predicted – actual) for the domain bounded by ages 60-89, period 1983-2007 and year-of-birth 1894-1923, inclusive of the log death rates used to construct Fig 13a. The resulting mean errors, calculated by averaging the errors respectively for age, period and year-of-birth for all 8 models are displayed systematically in Fig 13b. We note the following

- The negative nature of the average errors for *M*6, with the implication of understated 1982 predictions (consistent with the findings above). Otherwise, we note that the average errors are overstated to varying degrees for the other 7 models.
- Model *LC*2 and *M*5 are the worst performing models under these criteria.

4. Extending the age range, ages 20-89

In this section, we conduct similar comparative studies between models using a wider age range. In order to accommodate this, we switch from models *M*5–*M*8 to the respective age augmented versions *M*5*–*M*8* as defined in Section 2 (which are motivated by the work of Plat (2009)). In addition, we choose to omit our findings for models *M* and *M*8*. On using model *M*, we find that the pattern in the simulated life expectancy and annuity predictions under biennial back-fitting becomes increasingly less cohesive as the age of the individual in question is reduced: evidence of the start of this process is to be found in Figs 11a&b at age 60. In the case of model *M*8* we find that the deviance profile used to determine the age modulating index $(x_c - x)$ sets the value of x_c at the lower extremity of the age range (in the cases investigated) leading to predictions which are out of kilter with all of the other models. We comment that these two models are the only ones where the cohort index is multiplied by an age modulating factor.

4.1 Objectives, scope of reporting

For the comparative studies reported next, we again follow the detailed modelling procedures laid out in Section 2. We restrict the detailed reporting of our findings (out of practical necessity due to the lack of space) to the depiction of life expectancy and 4% annuity values predictions under the systematic deletion of the most recent data available; however, full details are available from the authors on request. In order to broaden the investigation, we analyse each of following data sets in turn

Country	***Gender***	***Period***	***Ages***
E&W	male/female	1961-2007	20-89
USA	male/female	1961-2006	20-89

In all 4 (country – gender) studies, the diagnostic marginal residual plots against calendar year, age and year of birth, consistently indicate that all 6 models adequately capture both period and age effects but that there are appreciable cohort effects associated with all 4 data sets, which remain un-captured on using *LC*, *LC*2 and *M*5* modelling. The *AIC*, *BIC* and *HQC* statistics using the full age ranges for all 4 studies are reported individually below.

The dynamics are again conducted by modelling the period indices as a multivariate random walk time series with the requisite number of components, listed as follows

Model	***LC***	***LC2***	***H1***	***M5****	***M6****	***M7****
# components	1	2	1	3	3	4

The dominance of certain period index components over others, in the sense described in Section 3.5, continues to apply, as follows

Classification	***Component***	***Models***
Primary	1^{st}	*All*
Secondary	2^{nd} and 3^{rd}	*M*5*, *M*6*, *M*7*
Tertiary	4^{th}	*M*7*

Other aspects of the time series, such as the linearity or otherwise of the primary time series components, are reported below.

Model predictions are compared and reported by depicting the statistics of interest for a range of ages subject to the systematic biennial deletion of the most recently available crude mortality rates in all 4 cases. The same topping-out criteria as reported in Section 3.6 are set.

4.2 England & Wales mortality experiences

Here, we present predictions for the life expectancy and 4% annuity values for individuals aged 40(05)75, subject to the systematic biennial deletion of the most recently available data involving the time span 1993(02)07, for both males (Fig 14a) and females (Fig14b). The accompanying Akaike Information Criteria (*AIC*), Bayes Information Criteria (*BIC*), and Hannan-Quinn Criteria (*HQC*), together with their respective rankings across models (in brackets), computed on the basis of the full period span 1961-2007 read as follows:

	LC	***LC2***	***H1***	***M5****	***M6****	***M7****
AIC	-23218(6)	-20790(4)	-19383(3)	-23074(5)	-18945(2)	-18272(1)
BIC	-23782(6)	-21704(4)	-20294(3)	-23711(5)	-19722(2)	-19189(1)
HQC	-23420(6)	-21117(4)	-19710(3)	-23302(5)	-19223(2)	-18600(1)

E&W males

	LC	***LC2***	***H1***	***M5****	***M6****	***M7****
AIC	-22498(6)	-19309(4)	-19022(3)	-20625(5)	-18205(2)	-17453(1)
BIC	-23062(6)	-20224(4)	-19933(3)	-21262(5)	-18982(2)	-18370(1)
HQC	-22700(6)	-19637(4)	-19348(3)	-20853(5)	-18483(2)	-17782(1)

E&W females

We note that each of these 3 Criteria leads to the same ranking of the models – with the best 3 fitting models being *M*7*, *M*6* and H_1.

Before commenting in detail on Figs 14a&b, we note that, as a general feature of these and other similar figures, the outline slope of the prediction intervals, stacked according to the data period used, is indicative of the predicted rates of improvement in mortality by age (over the period concerned). This feature highlights the differential rates of mortality improvement predicted across ages.

For the male mortality experience (Fig 14a), we have observed that the primary period index is linear throughout the back-fitting exercise for H_1 but exhibits curvature, to varying degrees, for other 5 models. We note the following:

- The close agreement between matching *LC* and *LC*2 predictions.
- The shifted location of H_1 predictions, capturing the strong cohort effects, compared with the matching *LC*/*LC*2 predictions which are not designed to capture this effect.
- The seemingly regressive effect of the capture of cohort effects on predictions under *M*6* and *M*7* compared with the non-capture of cohort effects under *M*5*.
- The conservative nature of the predictions at pensioner ages 60(05)75 using *M*6* compared with using *M*6 (Figs 11a&b): this feature is associated with the appreciable curvature in the primary period index using *M*6* compared with the linear nature of this time series component using *M*6 (Fig 6).

For the female mortality experience (Fig 14b), we have observed that the primary period index is linear throughout the back-fitting exercises for *LC*, *LC*2 and H_1 but exhibits curvature for the other 3 models. We note the following:

- The close agreement between matching *LC* and *LC*2 predictions.
- The generally small shift in location of the H_1 predictions relative to matching *LC*/*LC*2 predictions with the exception of the earlier 1993(02)99 predictions at pensioner ages 65(05)75.
- The seemingly regressive effect of the capture of cohort effects on *M*6* predictions at younger ages 40(05)55 compared with matching *M*5, *LC* and *LC*2 predictions, coupled with an almost zero rate of mortality improvement at ages 40(05)55.
- The seemingly regressive effect of the capture of cohort effects on *M*7* predictions for ages 40(05)50.
- A comparison of matching predictions between Fig 14a and Fig 14b is indicative of the differences between male and female mortality predictions (noting that the same abscissa scales apply to matching panels).

4.3 USA mortality experiences

In this section, we present the equivalent USA predictions for the life expectancy and 4% annuity values, subject to the systematic biennial deletion of the most recently available data involving the time span 1993(02)05,06, for males (Fig 15a) and females (Fig15b). Note data for the year 2007 were not currently available for inclusion in the studies. The accompanying Akaike Information Criteria (*AIC*), Bayes Information Criteria (*BIC*), and Hannan-Quinn Criteria (*HQC*), together with their respective rankings across models (in brackets), computed on the basis of the full period span 1961-2006 read as follows:

	LC	***LC*2**	***H*1**	***M*5***	***M*6***	***M*7***
AIC	-60289(6)	-43599(4)	-40035(3)	-52240(5)	-29720(2)	-29406(1)
BIC	-60848(6)	-44505(4)	-40938(3)	-52866(5)	-30482(2)	-30305(1)

HQC	-60489(6)	-43923(4)	-40359(3)	-52464(5)	-29993(2)	-29728(1)

USA males

	LC	***LC*2**	***H*1**	***M*5***	***M*6***	***M*7***
AIC	-37012(5)	-32290(4)	-29235(3)	-38453(6)	-28084(2)	-26694(1)
BIC	-37571(5)	-33196(4)	-30138(3)	-39079(6)	-28846(2)	-27593(1)
HQC	-37213(5)	-32615(4)	-29559(3)	-38678(6)	-28357(2)	-27016(1)

USA females

We note that each of these 3 Criteria leads to the same ranking of the models – with the best 3 fitting models being *M*7*, *M*6* and H_1.

In presenting predictions for the USA male mortality experience (Fig 15a), we have allocated zero weights to all of the time series components prior to 1968, thereby ensuring that the primary period component is essentially linear (with negative slope) throughout the back-fitting exercises for each model. On assessing the situation in the case of the USA female experience (Fig 15b), we have decided not to allocate the zero weights. Then, the patterns in the primary period components, across models, are best described as mildly meandering about a negative linear slope. There are no strong curvilinear patterns of the type encountered when modelling the England and Wales male mortality experience (e.g. Figs 1, 5).

For the male experience (Fig 15a) we note the following features

- The near identical patterns in matching *LC* and *LC*2 predictions and prediction intervals for all ages.
- Similarly the near identical patterns in matching *M*6* and *M*7* predictions and prediction intervals for all ages.
- The shifted location of H_1 predictions compared with matching *LC*/*LC*2 predictions for pensioner ages 60-75, meaning that H_1 predictions are higher.

For the female experience (Fig 15b) we note the following feature

- The striking similarity of matching prediction patterns across models, largely irrespective of the capture or non-capture of cohort effects, with the possible exception of ages 40 and 45 (e.g. H_1 relative to *LC* etc.).

By comparing matching predictions for males and females (Fig 15a vs. Fig 15b), we note the following features

- The differential locations of matching predictions by gender with their obvious interpretation.
- The relative shallower stacking angle of predictions for males relative to females (each age) which is indicative of a faster rate of mortality improvement for males than for females.

5. General discussion

In this section, we look at a number of more general issues arising from these investigations.

5.1 Under the basis for comparison established in Section 2.9, differences in model predictions are directly attributable to the differences in the parametric predictor structures η_{xt}. While other such bases for comparison are possible: for example, by using Poisson response models with central exposures and a log link function throughout, (targeting the force of mortality in the first instance), or by simply replacing the log-odds link with the complementary log-log link function throughout, such possibilities (although worthy of exploration) do not affect the comparative aspects of our specific findings.

5.2 Classification of predictor structures is possible based on the presence (H_1, *M*, *M*6–*M*8 or *M*6*–*M*8*) or absence (*LC*, *LC*2, *M*5 or *M*5*) of the provision for capturing cohort effects. Then given the capture of demonstrable cohort effects (reflected in the relevant residual plots), we contend that this should feed through in the form of differential median predictions in the indices of interest across models, especially when anticipated on the basis of external evidence (e.g. Willets (2004) and Renshaw and Haberman (2006) in the case of the England and Wales mortality experiences). We suggest that this should be added to the list of criteria for assessing model performance.

5.3 Classification of predictor structures is also possible based on the type of provision given to the incorporation of age effects into the predictor structure. Thus, whereas the *LC*, *LC*2, H_1 and model *M* structures include specific provision for capturing the main age effect (α_x), the *M*5-*M*8 structures do not. Instead, for these structures, age effects are determined solely through the pre-specified age modulating functions and, as such are designed to be applied to data sets with relatively short age spans. Then, the consequences of not fully allowing for age effects are reflected in the patterns of residuals when plotted against age (Figs 5-8).

For short age spans, it is possible to correct for this effect by moving to a set of models, intermediate to *M*5-*M*8 and *M*5*–*M*8*, by including the main age effects term α_x but not the accompanying period effects term, controlled by the age modulating function $\left(\bar{x} - x\right)^+$, as suggested by Plat (2009). The effect of doing just this on the simulated predictions depicted in Figs 11a&b is to leave the respective *M*5-*M*8 prediction patterns visually unchanged, with the exception of *M*6, when the median predictions become more conservative and more comparable with the depicted *M*5 median predictions and patterns. The detailed plots are omitted.

5.4 The question of how best to generate time series forecasts in the presence of mild curvature in the primary period index remains an issue for further consideration. Given the unsatisfactory performance of a 2nd order differencing *ARIMA* process in this role despite providing a good fit for a period index with mild curvature (Renshaw and Haberman (2009) Section 4.12), the retention of 1st order differencing *ARIMA* in combination with a rolling 'optimum fitting period', of the type investigated in Cairns *et al.* (2008c), would appear to be the best option. For further discussion of these issues in the context of *LC* modelling and the England & Wales male mortality experience, see

Renshaw and Haberman (2009) Section 3.10, and, for a more general discussion Denuit and Goderniaux (2005).

5.5 The life expectancy and level immediate annuity predictions reported in these studies are computed exclusively by the cohort method, and, as such, only require mortality rate predictions restricted to the upper triangular region bounded by the latest period t_n, the upper age $x_k (= 89)$ and the limiting cohort $t_n - x_1$ (subject to any prior triangular data deletion in the lower RH corner of the data array). As such, the computation of these predictions does not require the extrapolation of the cohort index ι_{t-x} (where this is present in the model).

If mortality rate predictions are required in the lower triangular region bounded by the limiting cohort $t_n - t_1$, the lower age x_1, and the outer period $t_n + x_k - x_1$, it becomes necessary to extrapolate the cohort index (where applicable). This is done by Cairns *et al.* (2008a,b) for *M*6-*M*8 and *M* under the assumption that the dynamic of the cohort index is independent of the dynamic of the period components. By design, for models *M*6 and *M*7, the mapping of the cohort index as the independent residuals of a regression model (Appendix A) can be used as justification for doing this. However there is no equivalent justification for treating the cohort index independently of the period index for model *M* (or H_1) and so we suggest that this should be avoided.

5.6 For the evolving 1985(02)07 life expectancy prediction intervals (as in Fig 11a), we depict the matching measures of relative dispersion in Fig 16, where

$$\text{relative disperion} = \frac{\text{95th.percentile - 5th.percentile}}{\text{50th.percentile}}.$$

Here the values are displayed in descending sequence for each age 60(05)75 and each model separately: matching the detail of Fig 11a. We note the following

- The individual profile (reduction) trends in relative dispersion as further data become available for analysis, for all ages and models with little exception.
- The degree of relative dispersion and the ordering of the profile trends by age are relatively similar for *M*5-*M*8 and H_1.
- The degree of relative dispersion is smaller under *LC* while the ordering of the profile trends by age is largely the reverse of that under *M*5-*M*8 and H_1.
- The two sets of outliers for *LC*2, which are consistent with the narrowing of the two most recent *LC*2 matching sets of prediction intervals in Fig 11a.
- There is little separation of the relative dispersion profile trends by age under *M* and the individual profile patterns by age are more diffuse than for the other models examined.

5.7 The reason for the mild tapering of the simulated prediction intervals subject to the biennial front-end prior data deletions 1962(02)83 (Figs 12a&b) in a counter intuitive direction, in all cases, remains unexplained. One hypothesis is that where there is curvature in the time trends this could be regarded as a switch from one underlying downward linear slope to a different downward linear slope. For predictions based on the most recent data (and hence shorter time series), the model estimates are based on data

that are less distorted by the earlier trend, leading to less uncertainty in the predictions. A comparison with the pattern of theoretically constructed prediction intervals subject to the same pattern of front-end data deletions, possible for *LC* modelling in combination with a random walk time series (Denuit (2007) would be of interest in this respect. Such a theoretical study, requiring mortality data above age 89, lies outside our present remit.

5.8 Cairns *et al* (2009), (2008a) make a number of criticisms of model *M* as part of their wider comparative study using the England & Wales male mortality with the shorter age range. These points have also been made by Plat (2009). We discuss each in turn.

1. Reported situations in which the age modulating factor $\hat{\beta}_x$ of the (primary) period index in model *M* changes sign part way along the age range leading to possible unwelcome consequences for mortality rate predictions. This possibility is trivially avoided by imposing the constraint $\beta_x^{(1)} > 0 \ \forall \ x$ on the fitting algorithm. See Renshaw and Haberman (2009) for further discussion. The potential for this to happen is also present when fitting *LC* or H_1 and can be countered in the same way.
2. The rate of convergence when fitting *M* is slow, indicating the determination of a stationary point in a flat region of the likelihood (deviance). We note that our two-stage approach to model fitting, differs from that assumed by Cairns *et al.* (2009) and, as such, converges more rapidly since we do not update α_x in the core of the fitting algorithm at each cycle. Likewise, the rate of convergence improves still further when either the cohort index age modulating factor $\beta_x^{(0)}$ is predetermined, as for H_1, or the period index age modulating factors β_x is predetermined as discussed in Renshaw and Haberman (2009).
3. Reported situations in which 'the parameter values jump to a set of values that is qualitatively quite different from the pervious year's estimates' (Cairns *et al* (2008a)). We have not encountered this phenomenon during our extensive data trimming exercises involving the England and Wales male mortality experience. However, it has been necessary to restrict the period index age modulating factor post 2005 in our analysis of these data (Renshaw and Haberman (2009)).
4. A lack of stability in the cohort index predictions associated with data trimming exercises. This concern is superseded since we do not believe that the modelling of the cohort index as a time series, separate from the period index time series, can be justified.
5. A lack of stability in mortality rate predictions associated with data trimming exercises. We have not experience this effect in our analysis of the England and Wales male mortality experience (ages 55-89).

Finally, as reported at the beginning of Section 4, we can add to this list the observation that the pattern in the simulated life expectancy and annuity predictions under biennial back-fitting becomes increasingly less consistent as the age of the individual in question is reduced below age 60. As a consequence of this and some of the other findings reported above, we propose that the model H_1 rather than *M* should be used for making predictions. However, we note that the parameter patterns for model *M*

which are obtained on fitting the England and Wales mortality data using the full age range 0-89, and are depicted in Renshaw and Haberman (2006), are particularly informative.

6. Summary

In the literature, it has been pointed out that, in order to assess whether any stochastic mortality model is a good model or not, it is important to consider certain criteria against which the model can be tested. Following Cairns *et al*. (2008a), we consider the following key criteria:

1. *The model should be consistent with historical data*; we have shown that some of the models investigated provide a good fit with historical data.
2. *Parameter estimates and model forecasts should be robust relative to the period of data and range of ages employed*; the empirical studies presented in this paper support this criterion.
3. *Forecast levels of uncertainty and central trajectories should be plausible and consistent with historical trends and variability in mortality data*; the methodology leads to smooth estimates which are plausible and consistent with historical trends.
4. *The model should be straightforward to implement using analytical methods or fast numerical algorithms*; all the computations have been implemented using standard GLM software packages.
5. *The model should be relatively parsimonious*; in the presence of cohort effects, the models have a relatively simple model structure.
6. *It should be possible to use the model to generate sample paths and calculate prediction intervals*; the proposed method utilizes parametric bootstrapping and gives bootstrap prediction intervals for future mortality rates and life expectancy (and annuity values) that are calculated on a cohort basis.
7. *The structure of the model should make it possible to incorporate parameter uncertainty in simulations*; the proposed method makes it possible to use parametric bootstrapping and for each bootstrap sample, the model parameters can be estimated.
8. *At least for some countries, the model should incorporate a stochastic cohort effect*; the model structures readily incorporate a stochastic cohort effect.
9. *The model should have a non-trivial correlation structure*; there is a non-trivial correlation structure.
10. *The model is applicable for the full age range*; a subset of the models has been applied to the England and Wales and US mortality experiences over a wide range of adult ages.

Our conclusions can be summarised as follows:

- We have focused on life expectancy and annuity value interval predictions: comparing results using various parameterised predictor structures. The predictions are computed by cohort trajectory and do not require the extrapolation

of the cohort index when it is present in the model structure. The interval predictions are generated by simulating the error in the period index time series.

- Models H_1 and *M* do not support the independent time series forecasting of the cohort index.
- By design, the representation of the cohort index in models *M*6/*M*6* and *M*7/*M*7* as the *i.i.d.* normally distributed residuals of a regression model (albeit subject to residual patterns in practice), can be used as justification for using a random walk time series with zero drift in order to forecast future values of the index: as such, the index should necessarily be trend free.
- The presence of a mild degree of curvature in the primary period index time series poses projection problems for some of the models.
- Structures building on H_0, which comprise all three age-period-cohort main effects, are fitted in two stages.
- Certain problems that have been identified with using model *M* as the means of making predictions are resolved by using the simpler structured model H_1.
- Model *M*5 would appear to be lacking in structure when it comes to the capture of age effects.
- The *LC*2 life expectancy and annuity value predictions are essentially the same as the matching *LC* predictions.
- The investigations with the England & Wales and US mortality experiences for the full adult age range indicate that the best 3 fitting models are *M*7*, *M*6* and H_1.
- For the indices of interest, the capture of demonstrable cohort effects present in the data should be reflected in the level of their predicted values, thereby adding to the list of criteria above.

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Appendix A

(Parameter constraints for *M*5–*M*8 / *M*5*–*M*8*)

Structure	*# constraints*	*Fitting constraints*	*Cairns et al. constraints*
*M*5	0	-	-
*M*5*	3	$\kappa_{\min}^{(1)} = \kappa_{\max}^{(2)} = \kappa_{\max}^{(3)} = 0$	-
*M*6/*M*6*	2	$\iota_{\min} = \kappa_{\max}^{(2)} = 0$	See below
*M*7/*M*7*	3	$\iota_{\min} = \kappa_{\max}^{(2)} = \kappa_{\max}^{(3)} = 0$	See below
*M*8	2	$\iota_{\min} = \iota_{\max} = 0$	$\sum_{t-x} \iota_{t-x} = 0$
*M*8*	2	$\iota_{\max} = \iota_{\max-1} = 0$	$\sum_{t-x} \iota_{t-x} = 0$

*M*6: Regress ι_{t-x} on $(t-x)$, so that

$$\iota_{t-x} = \phi_1 + \phi_2 (t-x) + \varepsilon_{t-x};\ \varepsilon_{t-x} \sim N(0, \sigma^2)\ i.i.d$$

thereby estimating $\phi_1,\ \phi_2,\ \{\varepsilon_{t-x}\}$, followed by the mapping

$$\iota_{t-x} \mapsto \varepsilon_{t-x},\ \ \kappa_t^{(2)} \mapsto \kappa_t^{(2)} - \phi_2,\ \ \kappa_t^{(1)} \mapsto \kappa_t^{(1)} + \phi_1 + \phi_2 (t - \bar{x}).$$

For *M*6*, as above, subject to the change $\kappa_t^{(2)} \mapsto \kappa_t^{(2)} + \phi_2$.

*M*7: Regress ι_{t-x} on $(t-x)$ and $(t-x)^2$, so that

$$\iota_{t-x} = \phi_1 + \phi_2 (t-x) + \phi_3 (t-x)^2 + \varepsilon_{t-x};\ \varepsilon_{t-x} \sim N(0, \sigma^2)\ i.i.d$$

thereby estimating $\phi_1,\ \phi_2,\ \phi_3\ \{\varepsilon_{t-x}\}$, followed by the mapping

$$\iota_{t-x} \mapsto \varepsilon_{t-x},\ \ \kappa_t^{(3)} \mapsto \kappa_t^{(3)} + \phi_3,\ \ \kappa_t^{(2)} \mapsto \kappa_t^{(2)} - \phi_2 - 2\phi_3 (t - \bar{x})$$

$$\kappa_t^{(1)} \mapsto \kappa_t^{(1)} + \phi_1 + \phi_2 (t - \bar{x}) + \phi_3 \left\{ (t - \bar{x})^2 + \frac{1}{k} \sum_{i=x_1}^{x_k} (i - \bar{x})^2 \right\}.$$

For *M*7*, as above, subject to the change $\kappa_t^{(2)} \mapsto \kappa_t^{(2)} + \phi_2 + 2\phi_3 (t - \bar{x})$.

Under the transformation of ι_{t-x} into the residuals of a linear regression model, by construction,

$$\sum_{t-x} \iota_{t-x} = 0,\ \sum_{t-x} (t-x) \iota_{t-x} = 0$$

for all 4 cases, with additionally

$$\sum_{t-x} (t-x)^2 \iota_{t-x} = 0 \ \text{ for } M7 \text{ and } M7^*.$$

For *M*8 & *M*8* the mapping is obvious.

Appendix B
(Multivariate random walk with drift)

Denote a multivariate time series

$$\boldsymbol{\kappa}_t : t = 0,1,2,...,n$$

with first order differences

$$\boldsymbol{y}_t = \boldsymbol{\kappa}_t - \boldsymbol{\kappa}_{t-1} : t = 1,2,...,n\,.$$

For the random walk with drift

$$\boldsymbol{y}_t = \boldsymbol{\theta} + \boldsymbol{\varepsilon}_t\,. \qquad \left(\boldsymbol{\kappa}_t, \boldsymbol{y}_t, \boldsymbol{\theta}, \boldsymbol{\varepsilon}_t, \text{ all } 1\times \mathrm{m}\right).$$

Refer to the multivariate Gaussian model

$$\mathbf{Y} = \mathbf{GA} + \boldsymbol{\varepsilon} \qquad \left(\mathbf{Y}, \boldsymbol{\varepsilon} \text{ are } \mathrm{n}\times\mathrm{m}, \mathbf{G} \text{ is } \mathrm{n}\times 1, \mathbf{A} \text{ is } 1\times\mathrm{m}\right)$$

for which

$$\mathbf{Y} = \begin{bmatrix} y_{11} & y_{21} & - \\ y_{12} & y_{22} & - \\ - & - & - \\ y_{1n} & y_{2n} & - \end{bmatrix},\ \mathbf{G} = \begin{bmatrix} 1 \\ 1 \\ - \\ 1 \end{bmatrix},\ \mathbf{A} = \begin{bmatrix} \theta_1 & \theta_2 & - \end{bmatrix}$$

and $\boldsymbol{\varepsilon} \sim N\left(\mathbf{0}, \boldsymbol{\Omega}\right)$, so that

$$\mathbf{G}'\mathbf{G} = n,\ \ \mathbf{G}'\mathbf{Y} = \begin{bmatrix} y_{1+} & y_{2+} & - \end{bmatrix},\quad \left(y_{i+} = \sum_{t=1}^{n} y_{it} = \kappa_{in} - \kappa_{i0}\right).$$

Then the OLS estimates

$$\hat{\mathbf{A}} = \left(\mathbf{G}'\mathbf{G}\right)^{-1}\left(\mathbf{G}'\mathbf{Y}\right) = \left[\left(\kappa_{in} - \kappa_{i0}\right)/n\right], \text{ with } \hat{\boldsymbol{\Omega}} = \hat{\boldsymbol{\varepsilon}}'\hat{\boldsymbol{\varepsilon}}/\left(n-1\right).$$

The matrix of residuals

$$\hat{\boldsymbol{\varepsilon}} = \mathbf{Y} - \mathbf{G}\hat{\mathbf{A}} = \left[y_{it} - \hat{\boldsymbol{\theta}}_i\right] = \left[r_{it}\right], \text{ so that } \hat{\boldsymbol{\varepsilon}}'\hat{\boldsymbol{\varepsilon}} = \begin{bmatrix} \sum_t r_{1t}^2 & \sum_t r_{1t}r_{2t} & \sum_t r_{1t}r_{3t} & - \\ \sum_t r_{1t}r_{2t} & \sum_t r_{2t}^2 & \sum_t r_{2t}r_{3t} & - \\ \sum_t r_{1t}r_{3t} & \sum_t r_{2t}r_{3t} & \sum_t r_{3t}^2 & - \\ - & - & - & - \end{bmatrix}.$$

Forecasting: successive substitution gives

$$\boldsymbol{\kappa}_{t_n+j} = \boldsymbol{\kappa}_{t_n} + j\boldsymbol{\theta} + \boldsymbol{\varepsilon}_{t_n+j} + \boldsymbol{\varepsilon}_{t_n+j-1} + ... + \boldsymbol{\varepsilon}_{t_n+1}\,.$$

Then, taking expected values, the *j*-step ahead forecast, from t_n (= *n*), is

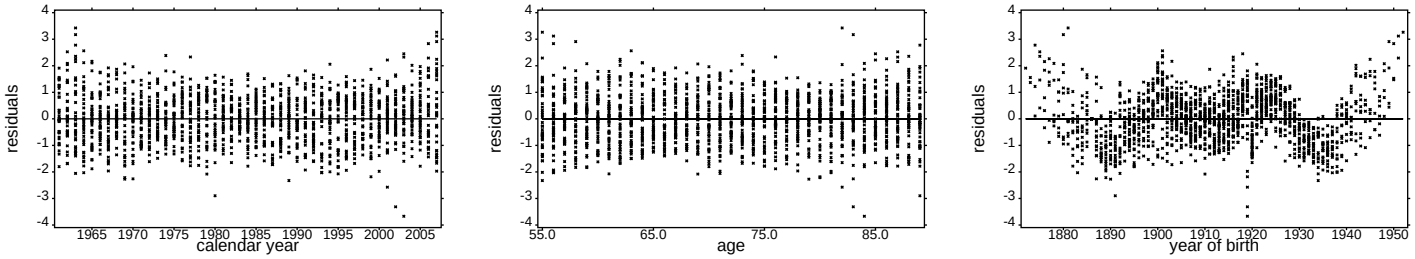
$$\hat{\boldsymbol{\kappa}}_{t_n+j|t_n} = \boldsymbol{\kappa}_{t_n} + j\boldsymbol{\theta}\,.$$

For the mean square error forecast:

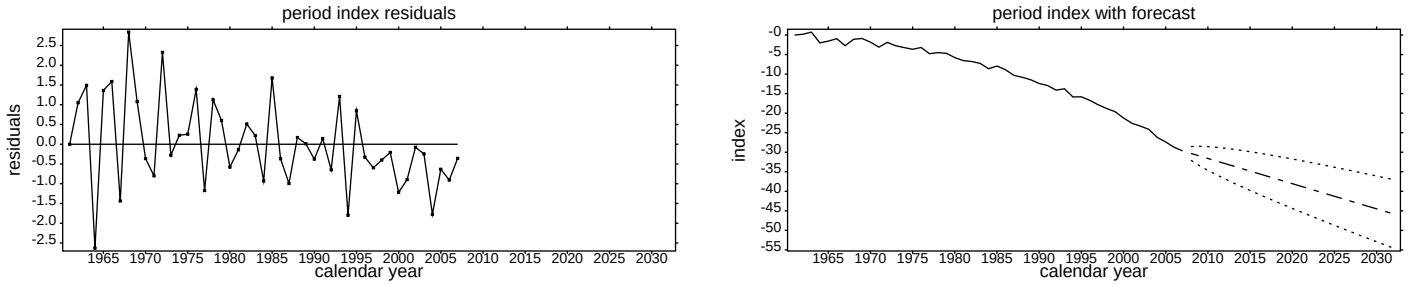
$$\boldsymbol{\kappa}_{t_n+j} - \hat{\boldsymbol{\kappa}}_{t_n+j|t_n} = \boldsymbol{\varepsilon}_{t+j} + \boldsymbol{\varepsilon}_{t+j-1} + ... + \boldsymbol{\varepsilon}_{t+1}$$

and

$$MSEF\left(\hat{\boldsymbol{\kappa}}_{t_n+j|t_n}\right) = E\left\{\left(\boldsymbol{\kappa}_{t_n+j} - \hat{\boldsymbol{\kappa}}_{t_n+j|t_n}\right)\left(\boldsymbol{\kappa}_{t_n+j} - \hat{\boldsymbol{\kappa}}_{t_n+j|t_n}\right)'\right\} = j\boldsymbol{\Omega}\,.$$

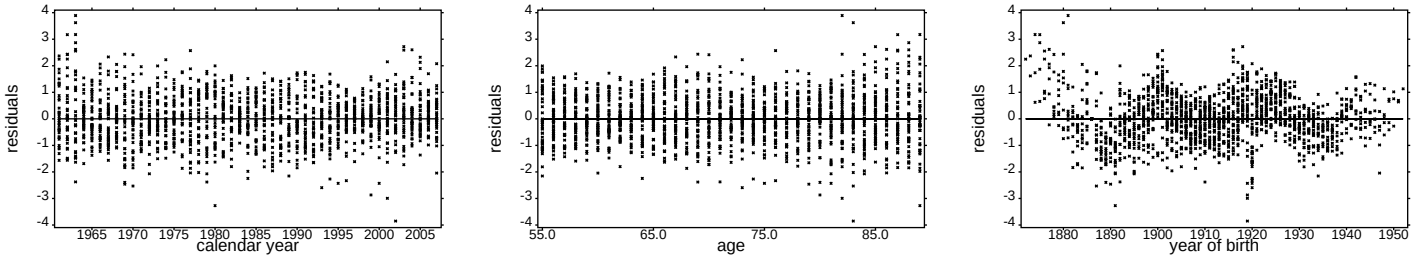


Model LC: deviance residual plots

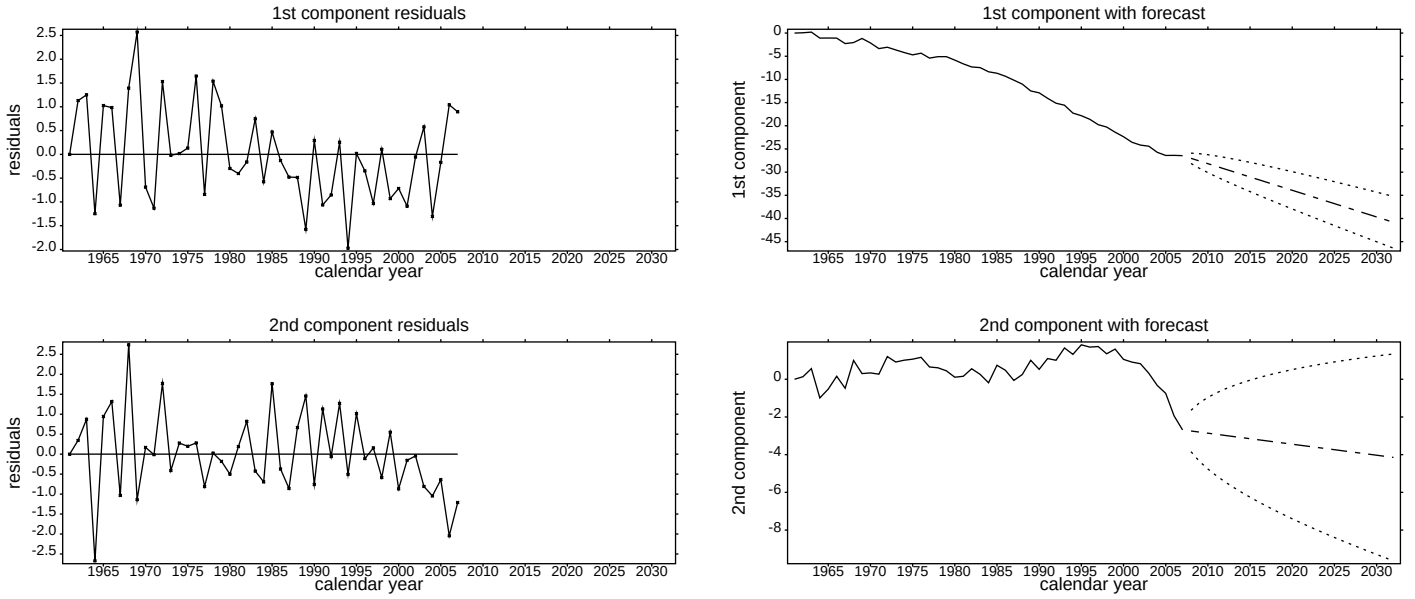


Model LC: period index random walk

Fig 1. England & Wales 1961-2007 male mortality experience, ages 55-89.
Binomial responses, log-odds link, target q(x,t), predictor LC.

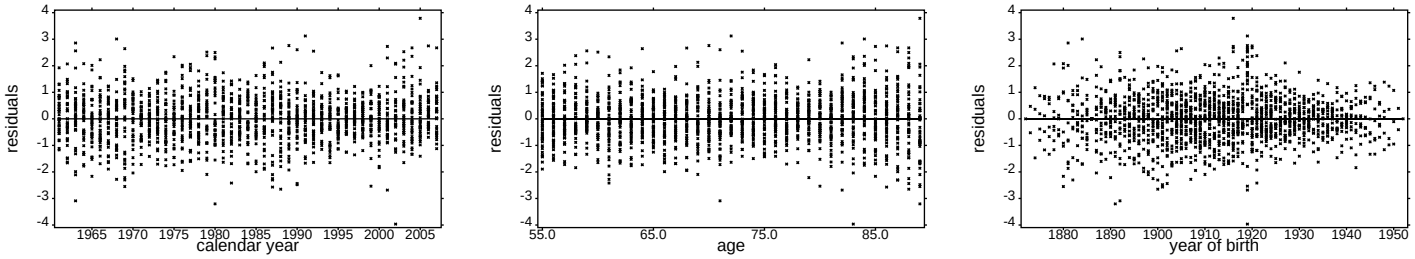


Model LC2: deviance residual plots

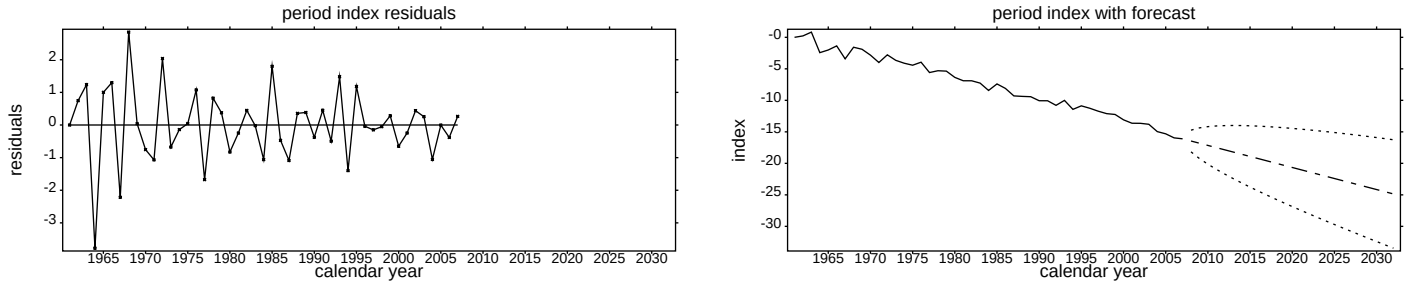


Model LC2: period indices bi-variate random walk

Fig 2. England & Wales 1961-2007 male mortality experience, ages 55-89.
Binomial responses, log-odds link, target q(x,t), predictor LC2.

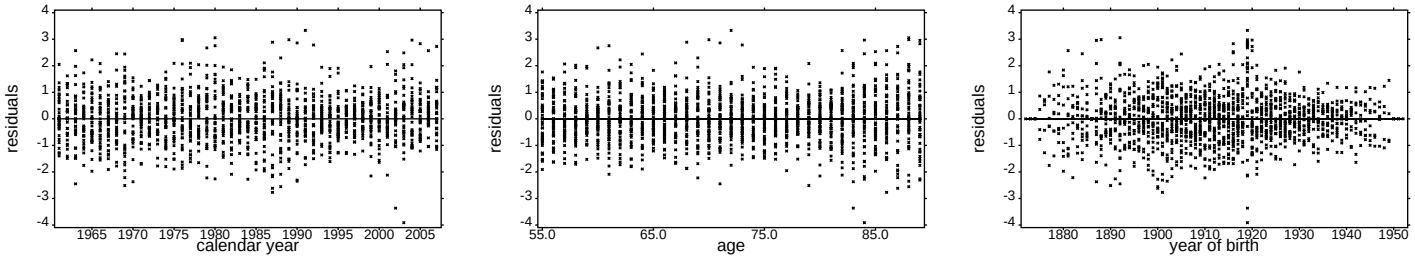


Model H1: deviance residual plots

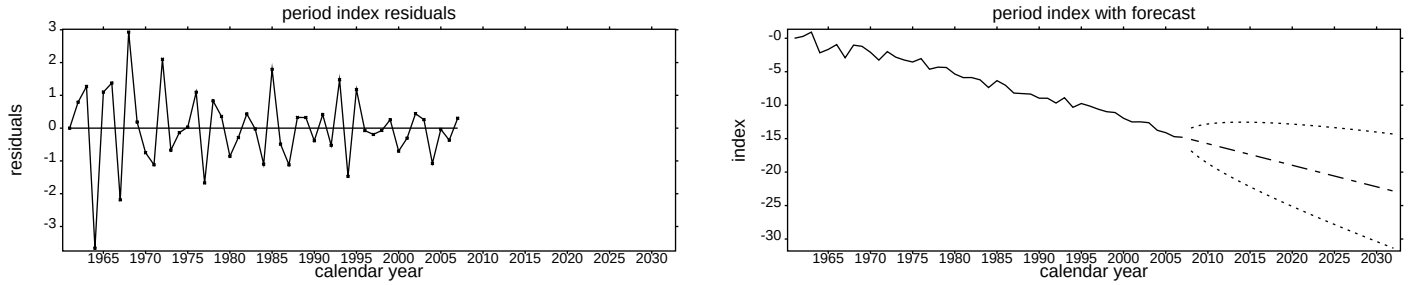


Model H1: period index random walk

Fig 3. England & Wales 1961-2007 male mortality experience, ages 55-89.
Binomial responses, log-odds link, target q(x,t), predictor H1.

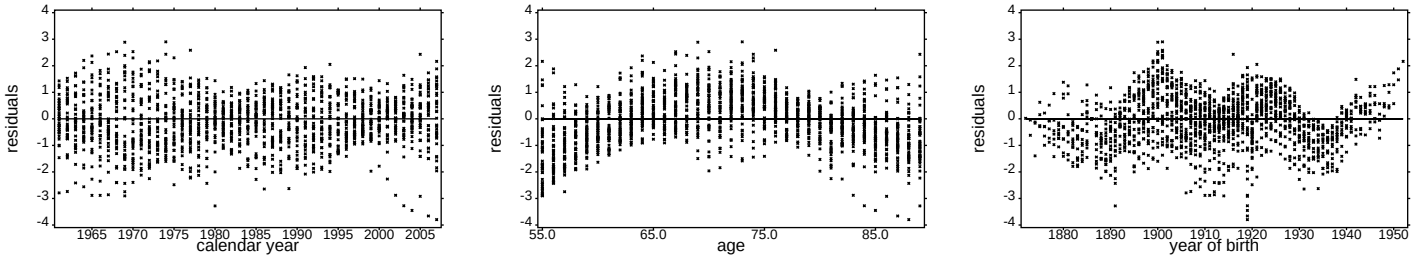


Model M: deviance residual plots

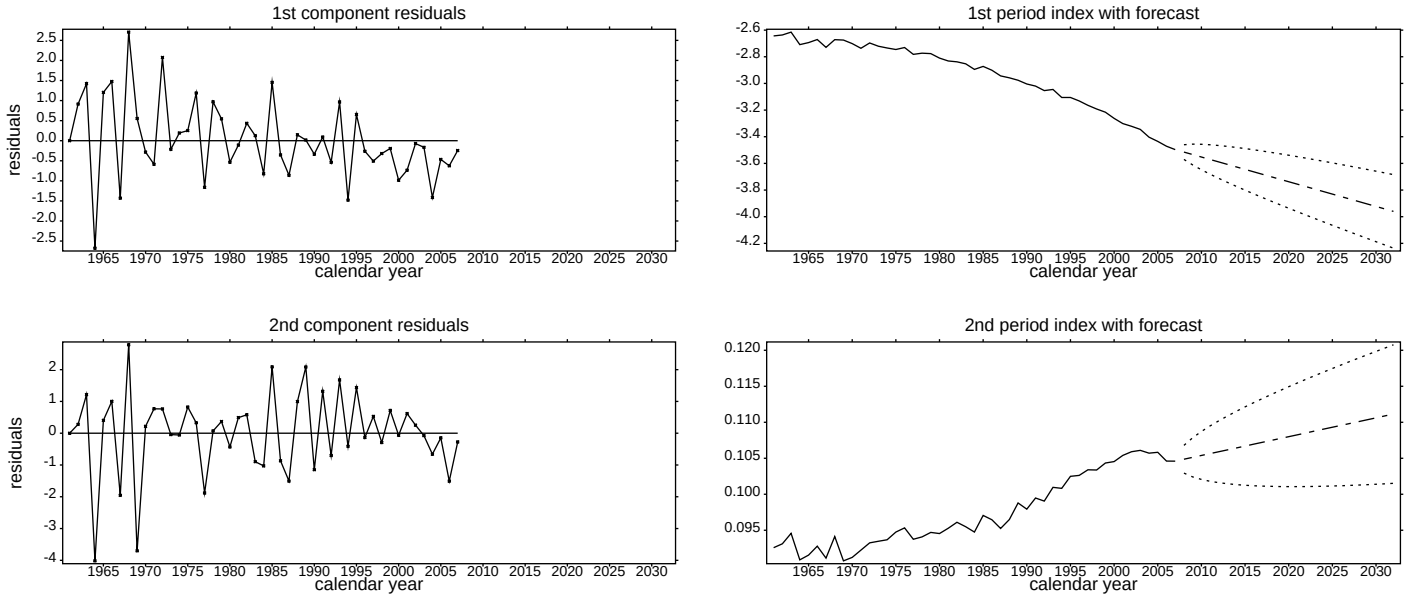


Model M: period index random walk

Fig 4. England & Wales 1961-2007 male mortality experience, ages 55-89.
Binomial responses, log-odds link, target q(x,t), predictor M.

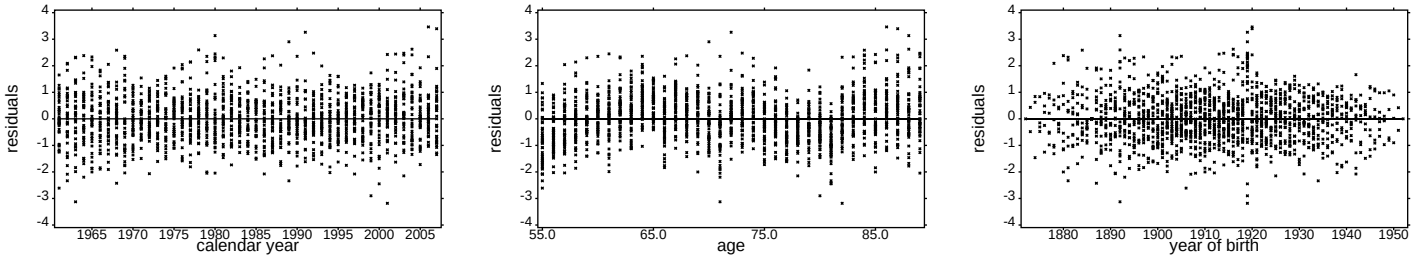


Model M5: deviance residual plots

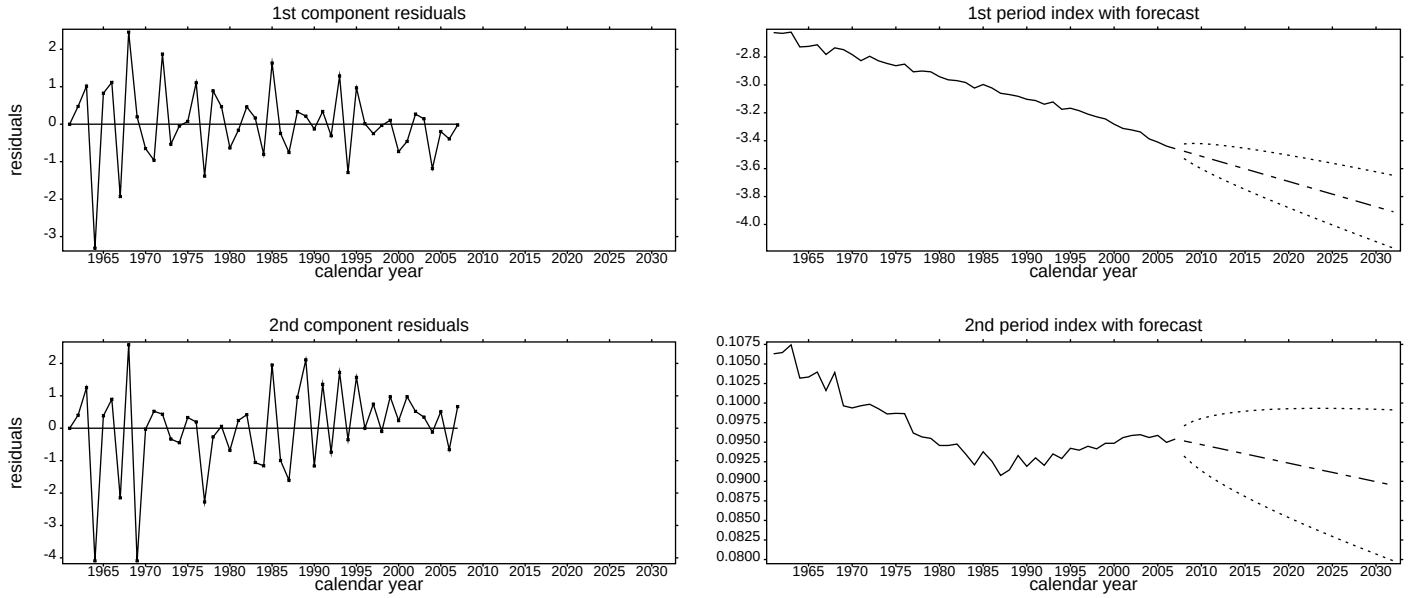


Model M5: period indices bi-variate random walk

Fig 5. England & Wales 1961-2007 male mortality experience, ages 55-89.
Binomial responses, log-odds link, target q(x,t), predictor M5.

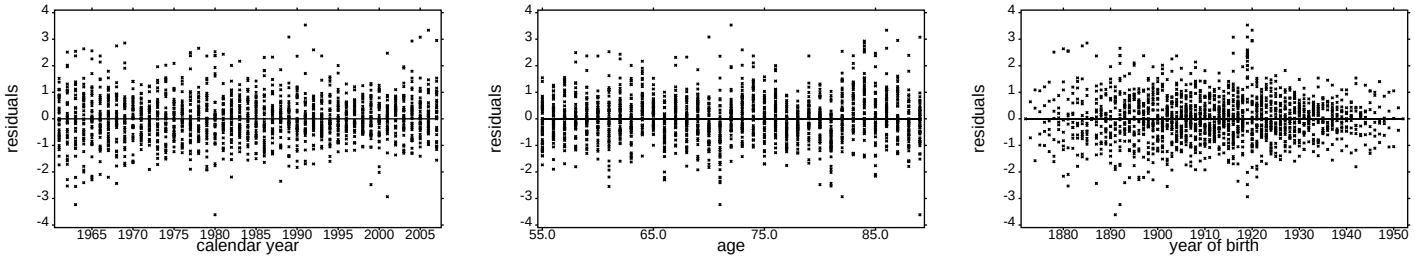


Model M6: deviance residual plots

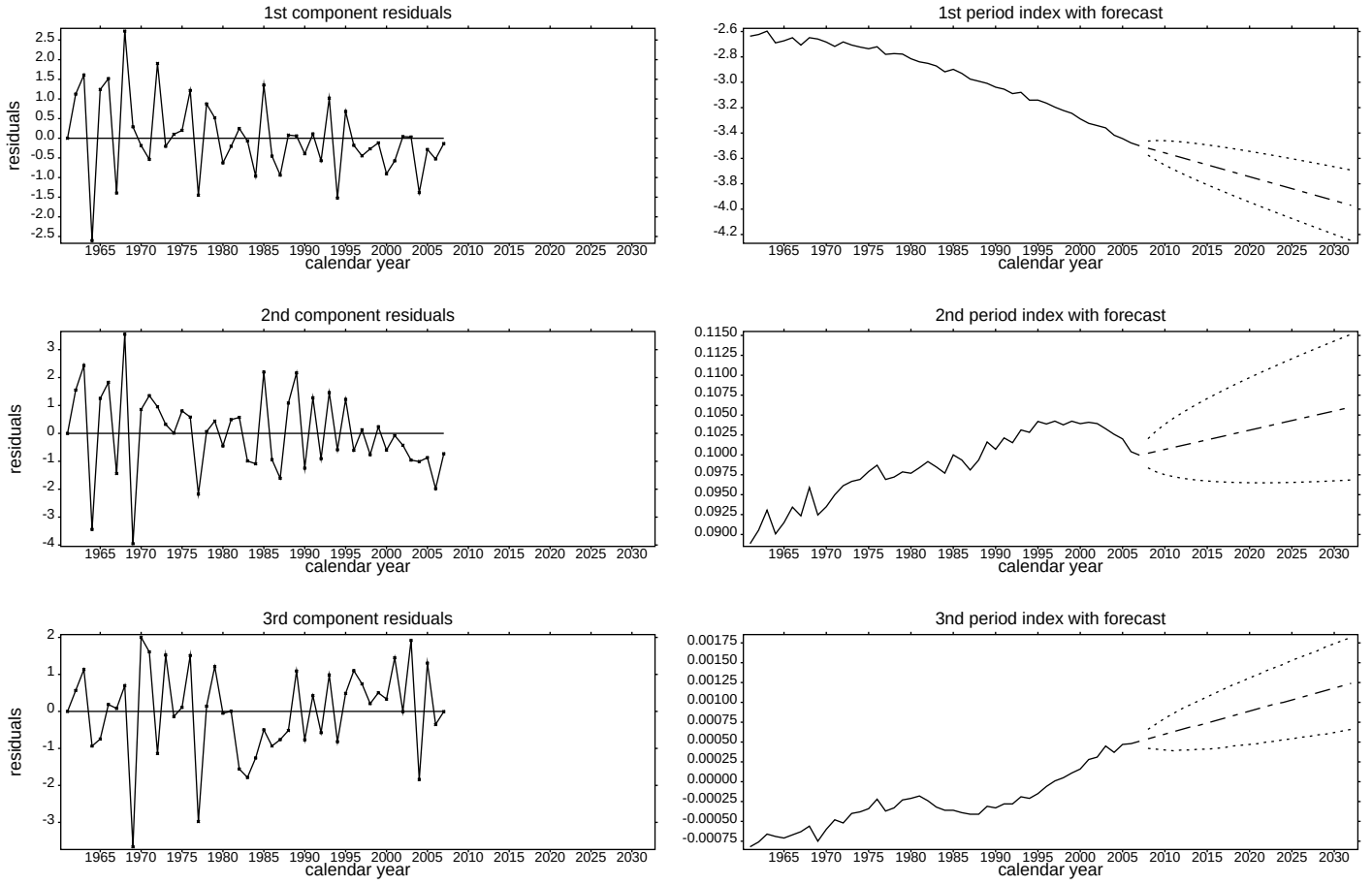


Model M6: period indices bi-variate random walk

Fig 6. England & Wales 1961-2007 male mortality experience, ages 55-89.
Binomial responses, log-odds link, target q(x,t), predictor M6.

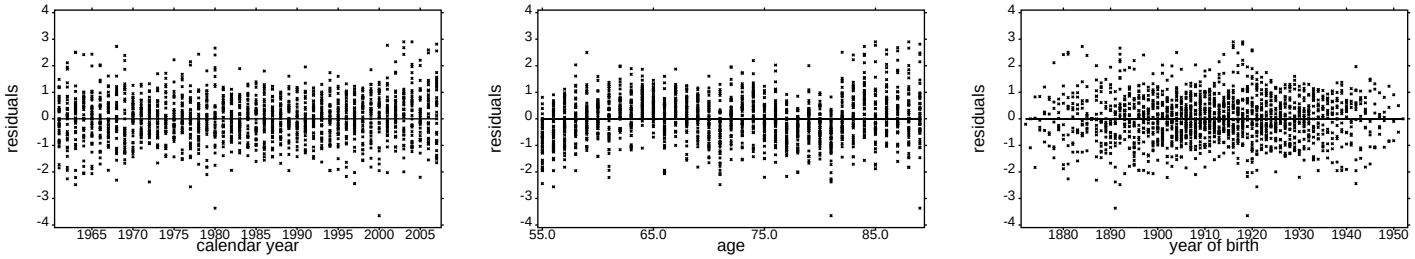


Model M7: deviance residual plots

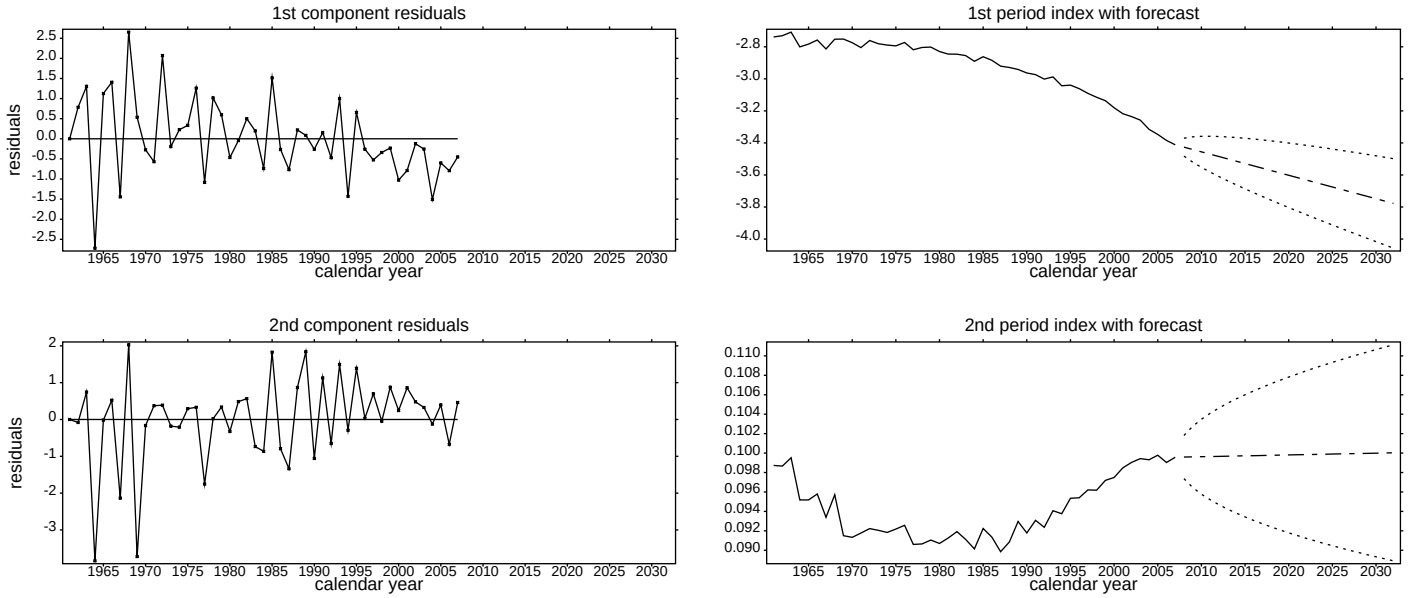


Model M7: period indices tri-variate random walk

Fig 7. England & Wales 1961-2007 male mortality experience, ages 55-89.
Binomial responses, log-odds link, target q(x,t), predictor M7.

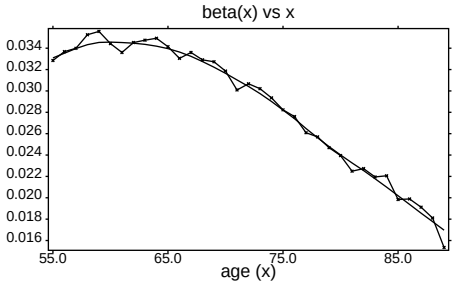


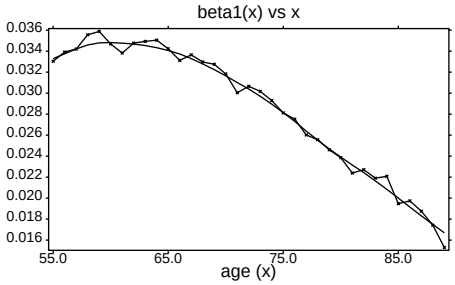
Model M8: deviance residual plots

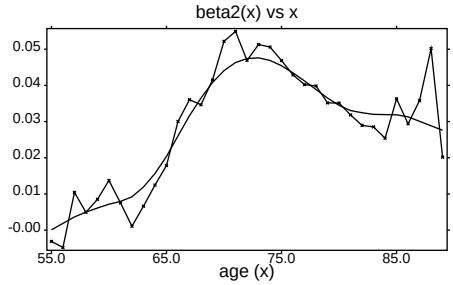


Model M8: period indices bi-variate random walk

Fig 8. England & Wales 1961-2007 male mortality experience, ages 55-89.
Binomial responses, log-odds link, target q(x,t), predictor M8.

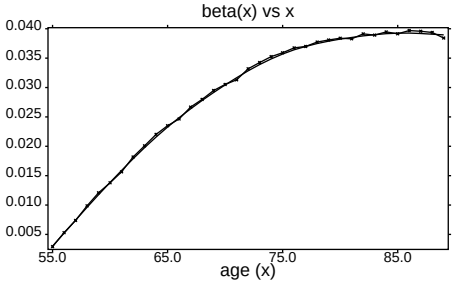


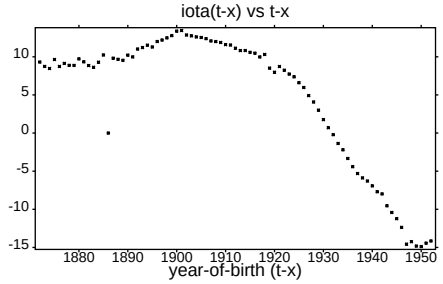




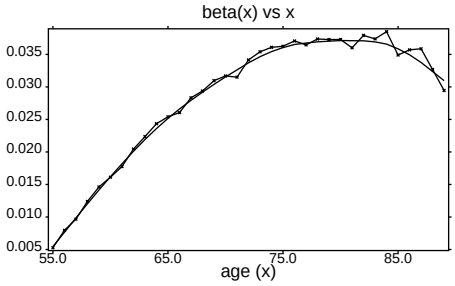
LC: age modulating index

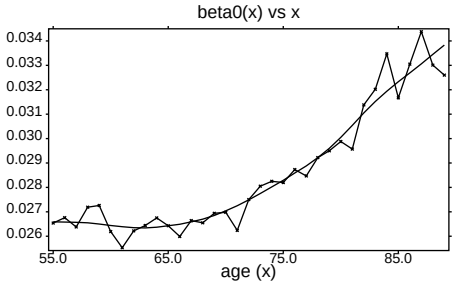
LC2: age modulating indices

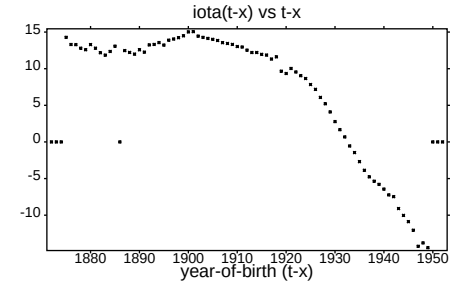




H1: age modulating index; year-of-birth index

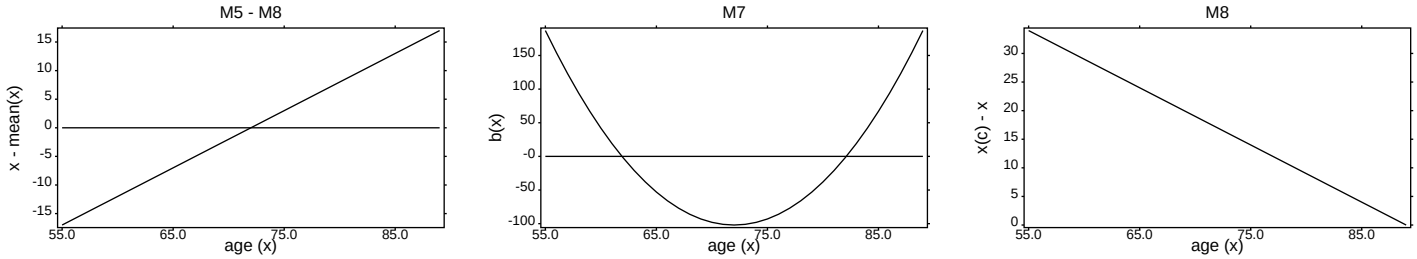




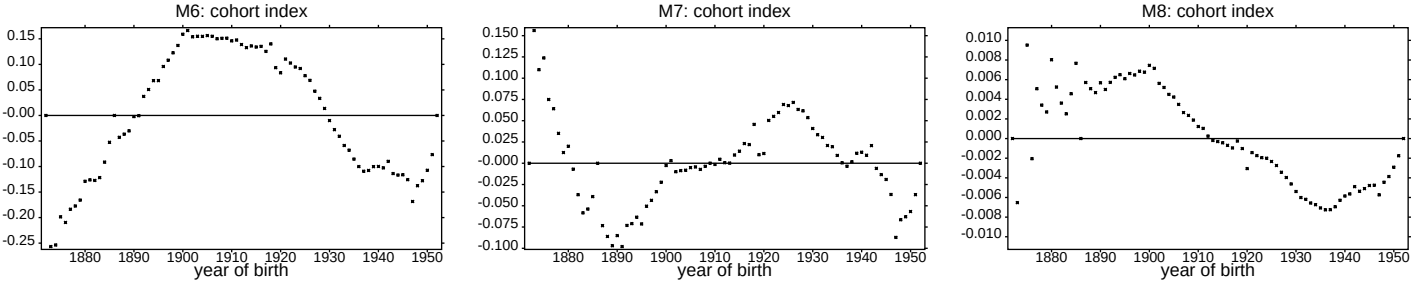


M: age modulating indices; year-of-birth index

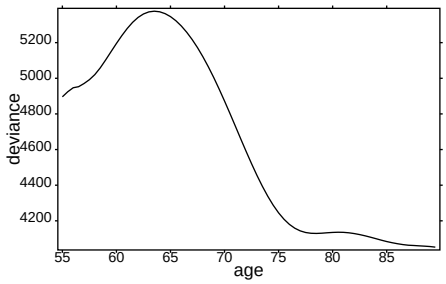
Fig 9. England & Wales 1961-2007 male mortality experience, ages 55-89.
Binomial responses, log-odds link, target q(x,t), predictors LC, LC2, H1, M.



Models M5-M8: age modulating functions



Models M6-M8: year-of-birth indices



Model M8: deviance profile

Fig 10. England & Wales 1961-2007 male mortality experience, ages 55-89. Binomial responses, log-odds link, target q(x,t), predictors M5-M8.

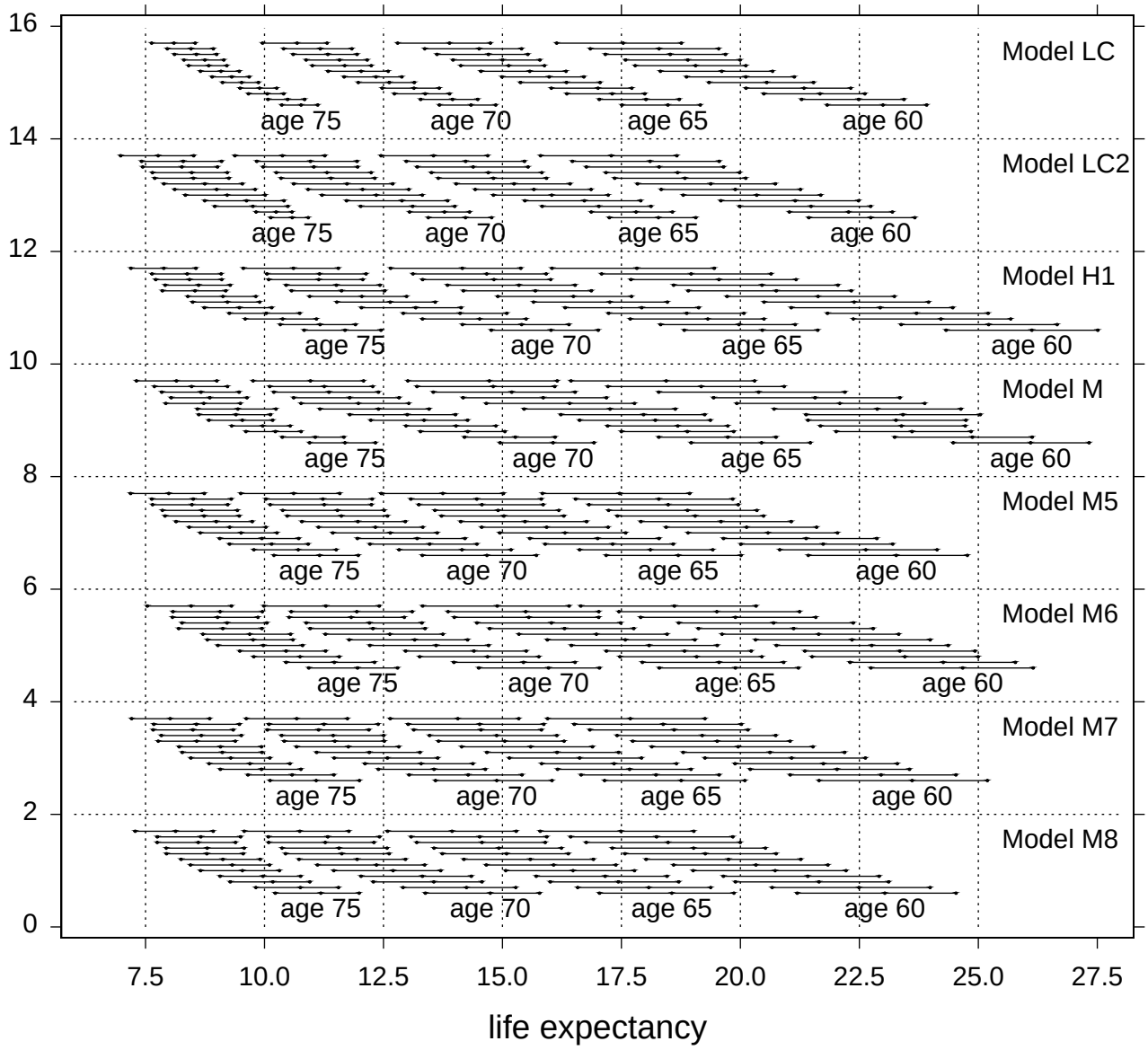


Fig 11a. E&W male 1961-2007 mortality experience, age range 55-89.
Evolving 1985(02)07 life expectancy predictions (5, 50, 95 percentiles),
presented in descending sequence, for individuals aged 60(05)75.
Predictions by the cohort method.

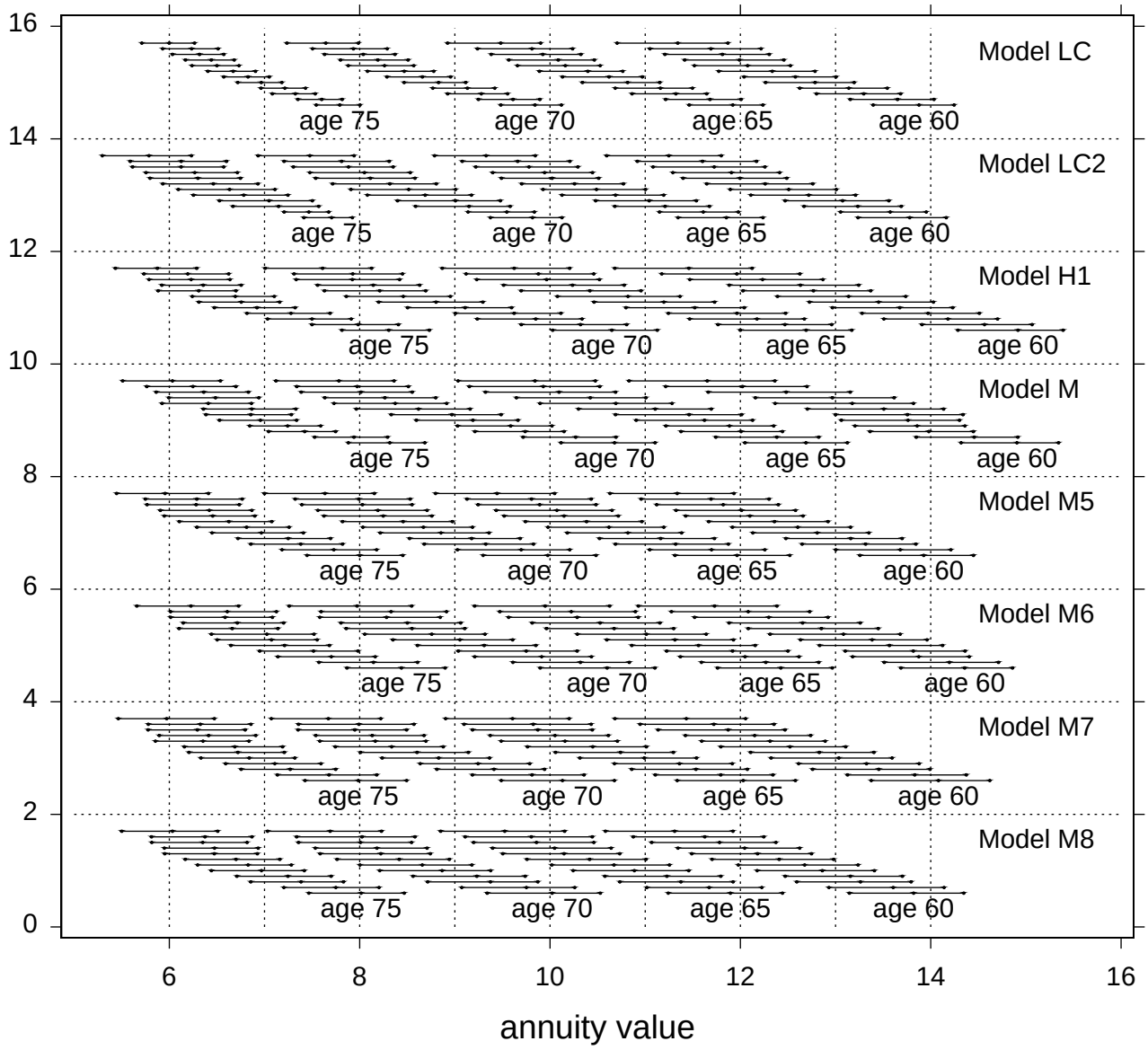


Fig 11b. E&W male 1961-2007 mortality experience, age range 55-89.
Evolving 1985(02)07 4% annuity predictions (5, 50, 95 percentiles),
presented in descending sequence, for individuals aged 60(05)75.
Predictions by the cohort method.

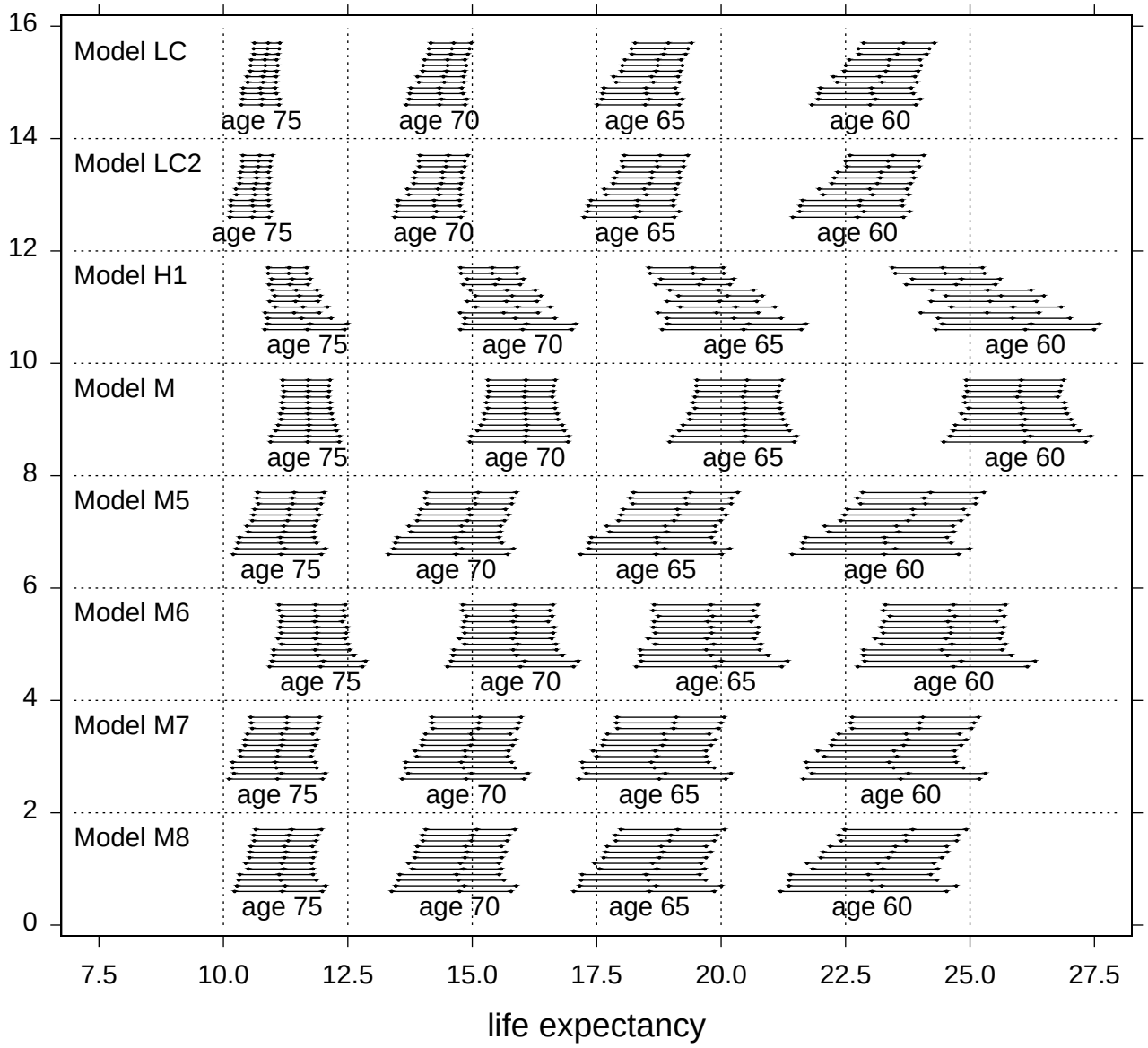


Fig 12a. E&W male 1961-2007 mortality experience, age range 55-89.
2007 life expectancy predictions (5, 50, 95 percentiles), subject to
biennial front-end data deletion 1961(02)83, shown in ascending sequence,
for individuals aged 60(05)75. Predictions by cohort method.

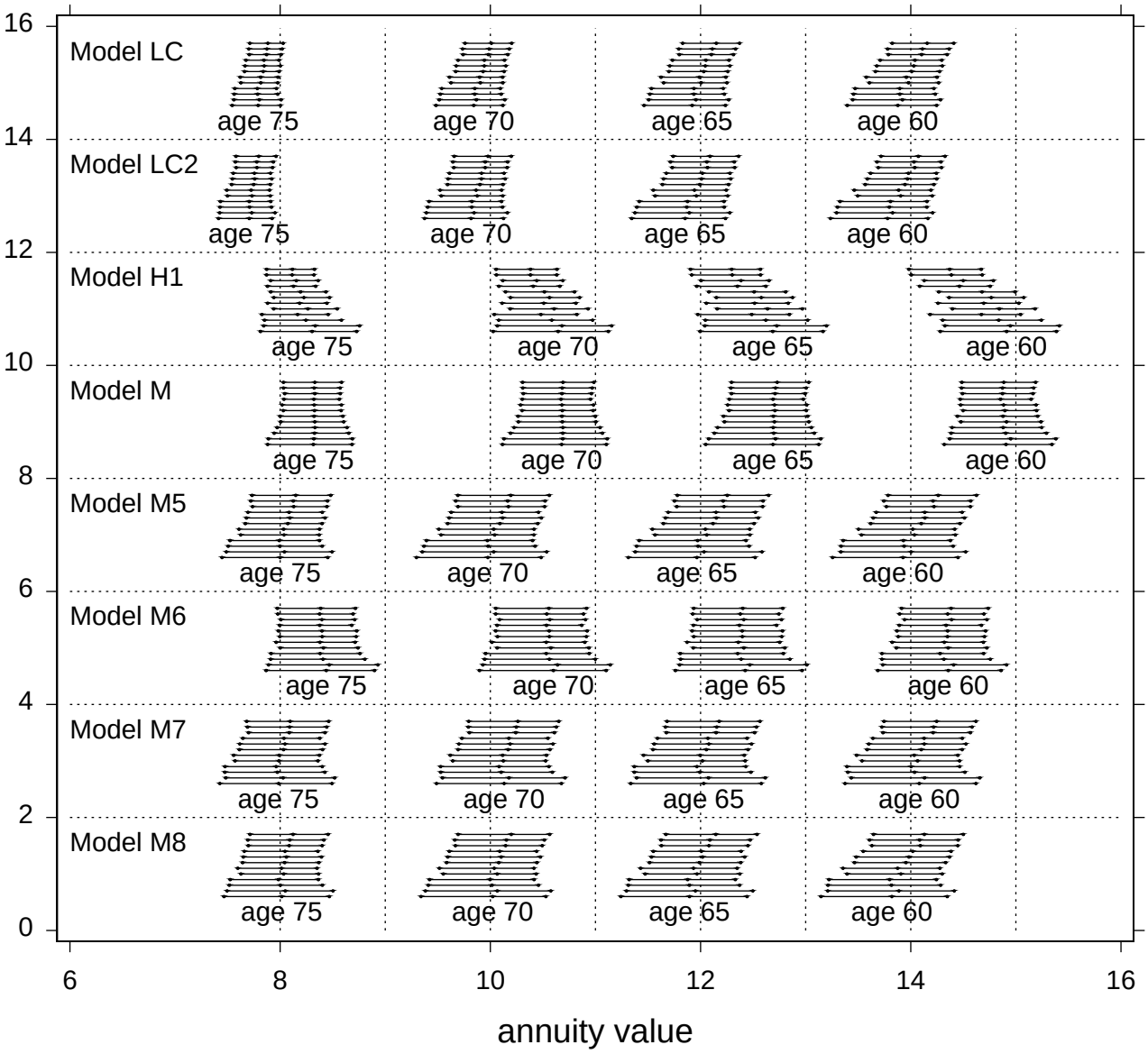


Fig 12b. E&W male 1961-2007 mortality experience, age range 55-89. 2007 present value 4% annuity predictions (5, 50, 95 percentiles), subject to biennial front-end data deletion 1961(02)83, shown in ascending sequence, for individuals aged 60(05)75. Predictions by cohort method.

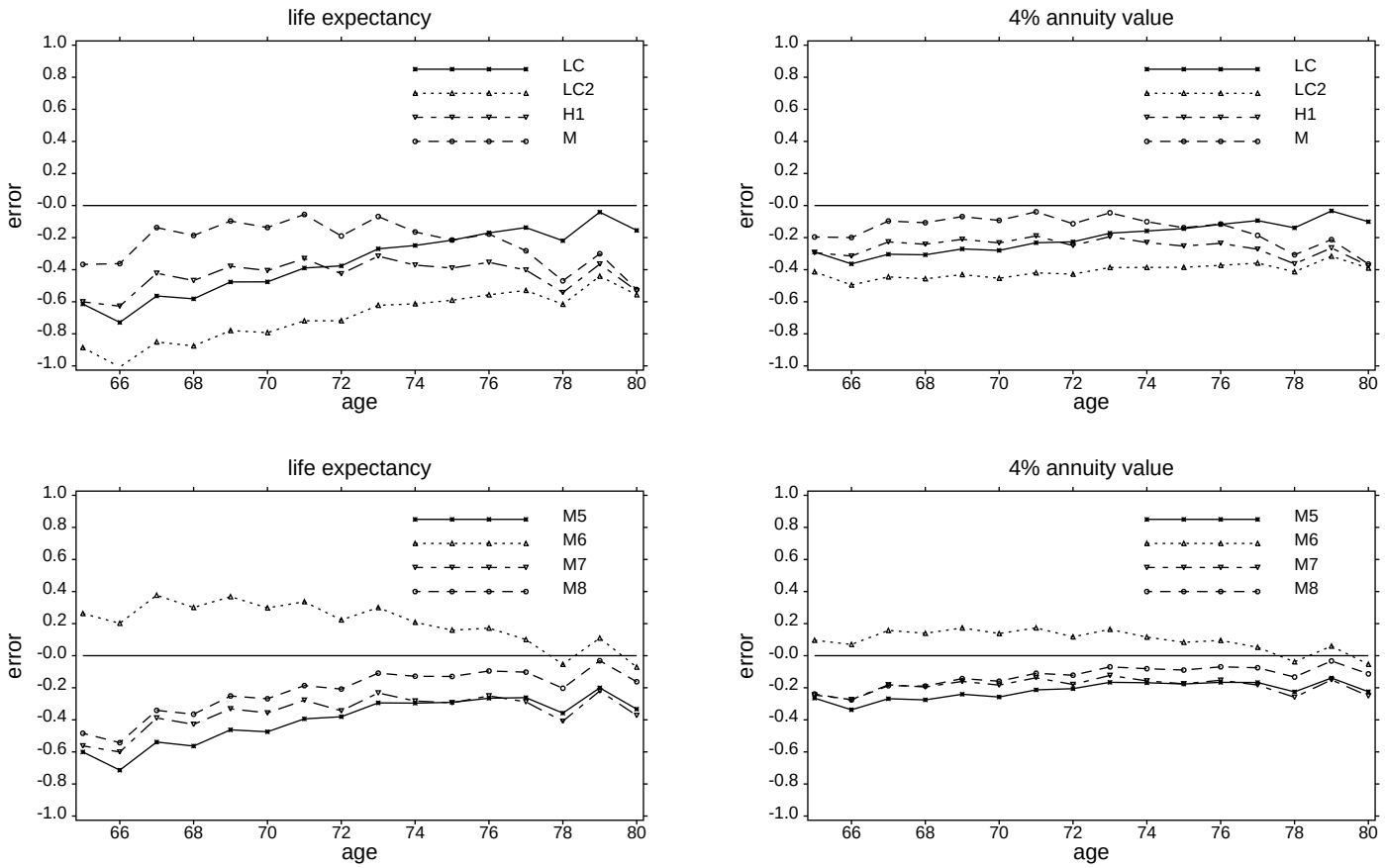


Fig 13a. Retrospective error in 1982 predicted life expectancies (LH panels) and annuity values (RH panels): ages 65-80.

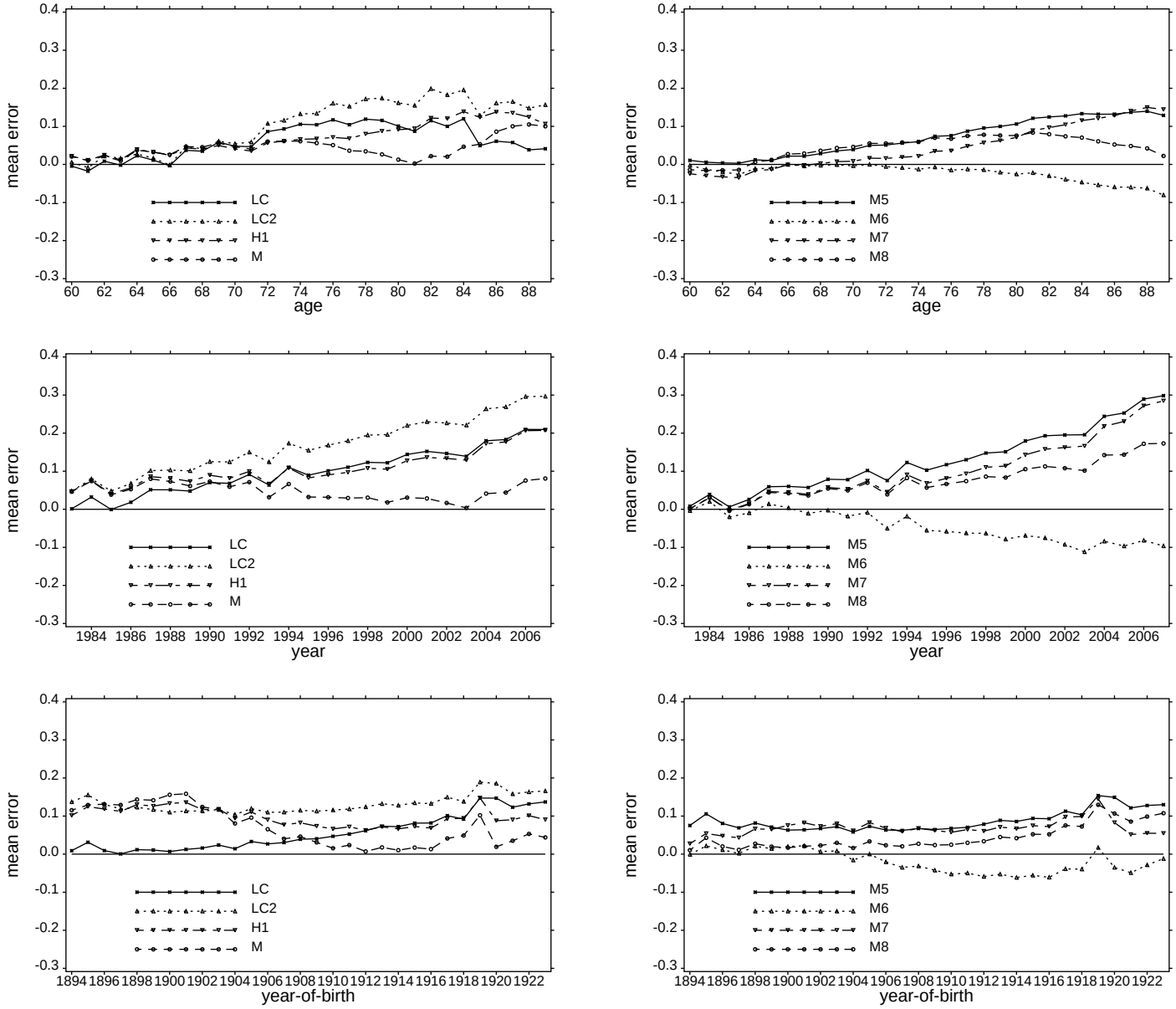


Fig 13b. Retrospective error in 1982 predicted log death rates, averaged over ages, years, cohorts respective (1st to 3rd row of panels): based on the region bounded by ages 60-89, years 1983-2007, cohorts 1894-1923.

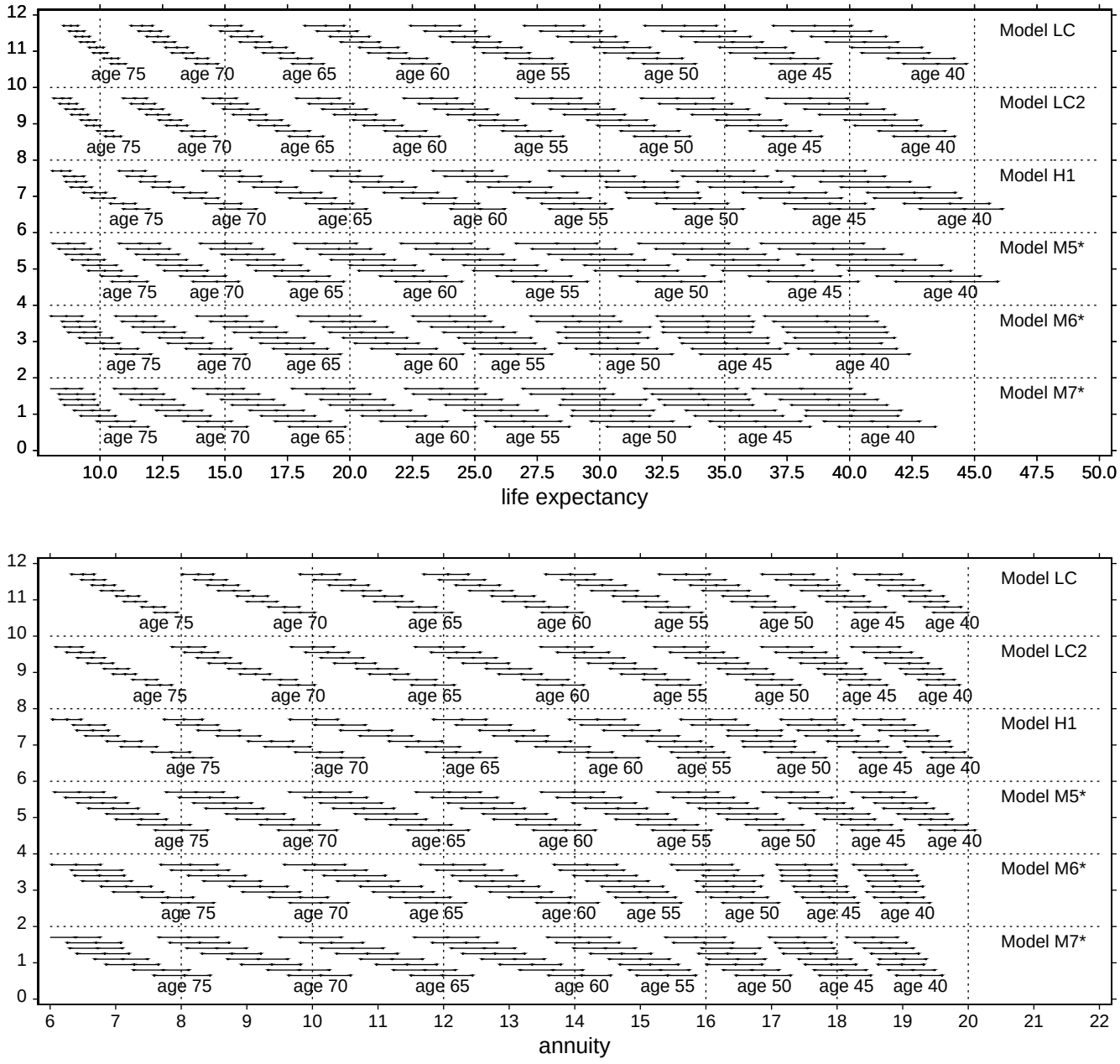


Fig 14a. EW male 1961-2007 mortality experience, age range 20-89.
Evolving 1993(02)07 life expectancy & 4% annuity prediction intervals.

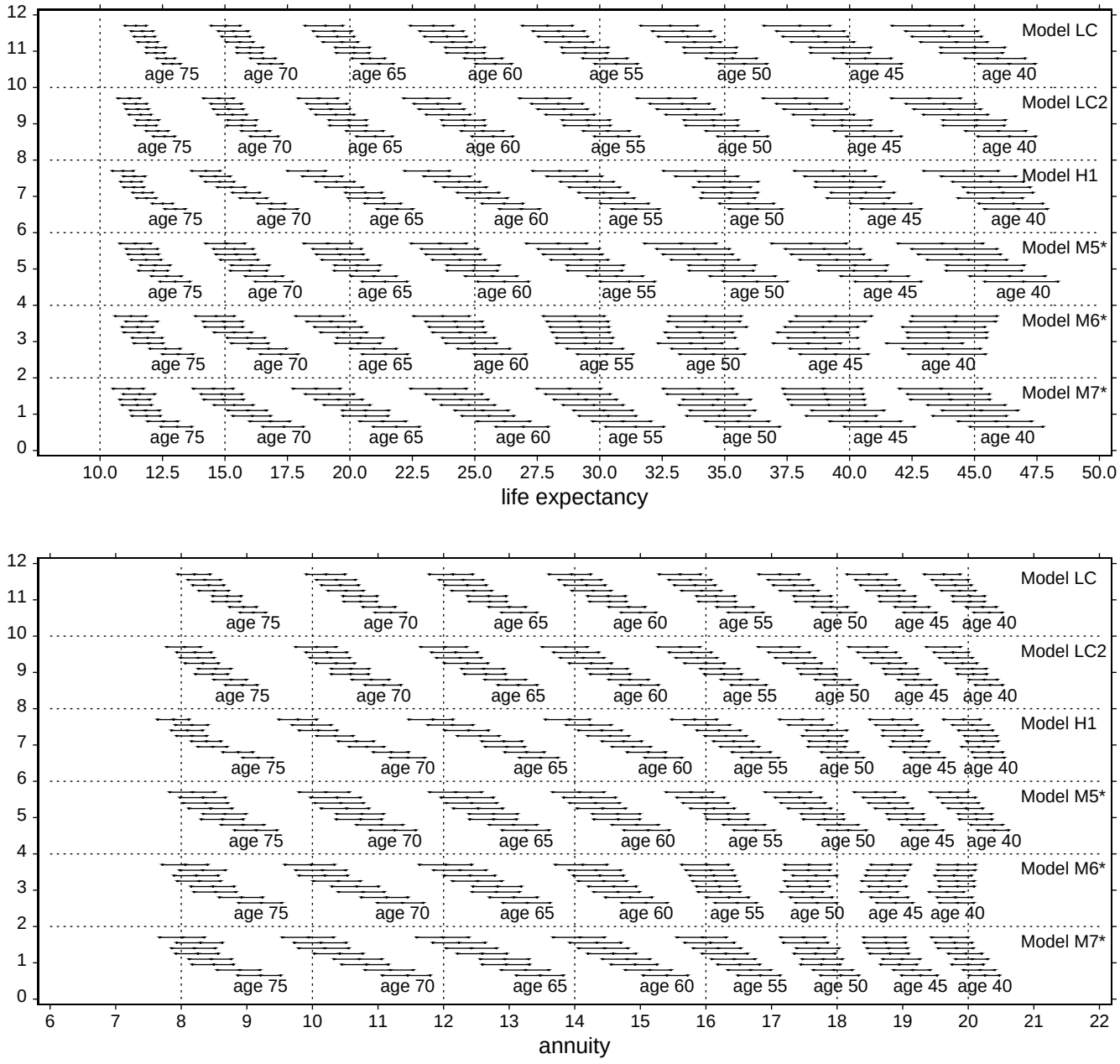


Fig 14b. EW female 1961-2007 mortality experience, age range 20-89.
Evolving 1993(02)07 life expectancy & 4% annuity prediction intervals.

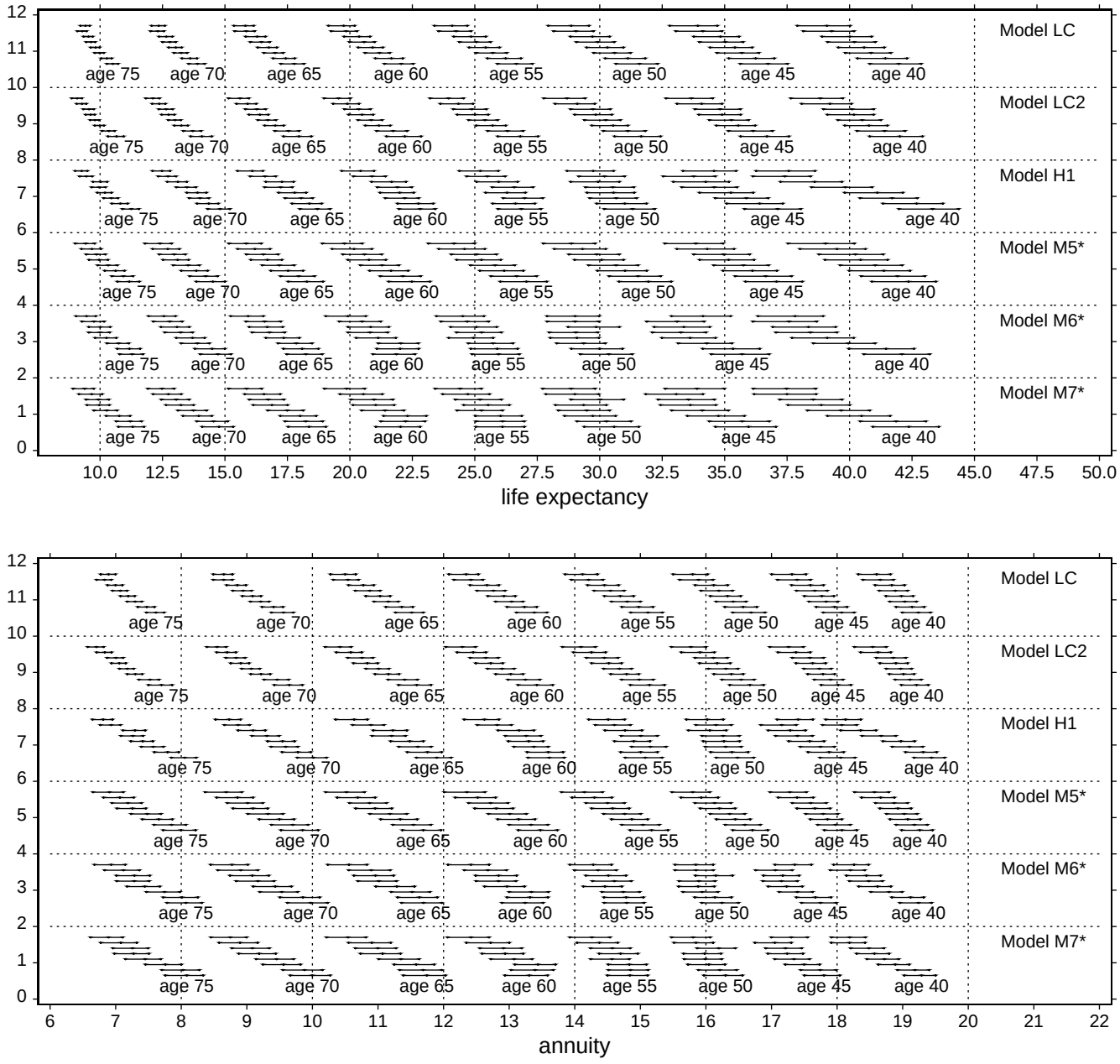
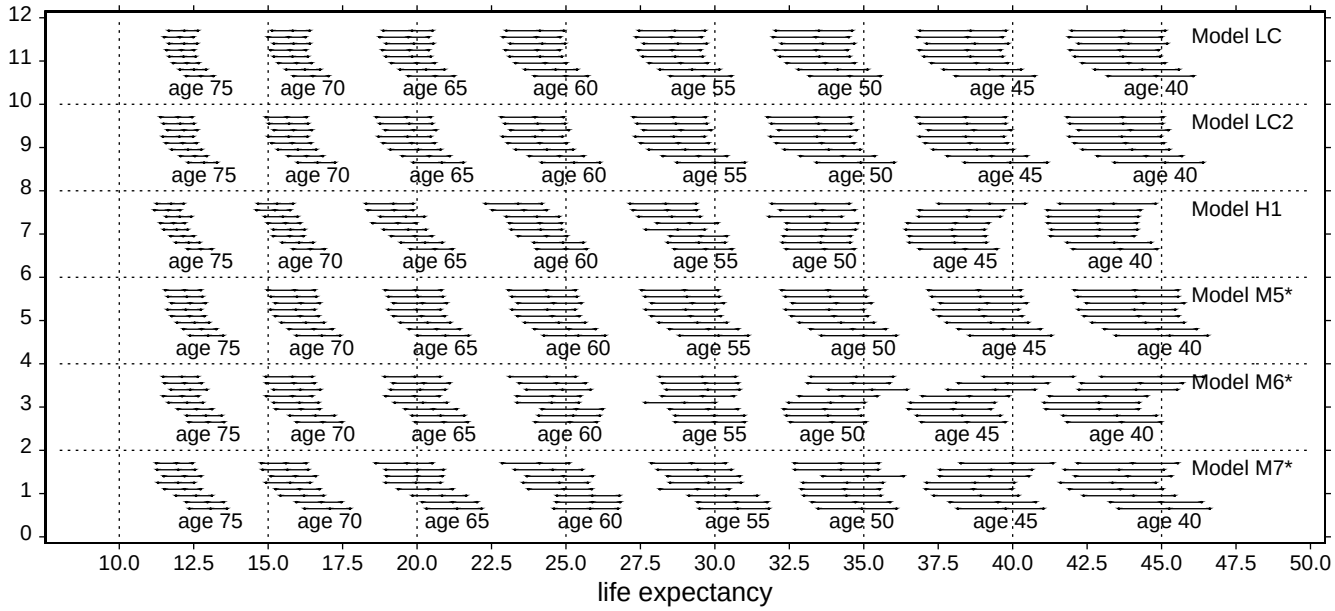


Fig 15a. USA male 1961-2006 mortality experience, age range 20-89.
Evolving 1993(02)05,06 life expectancy & 4% annuity prediction intervals.



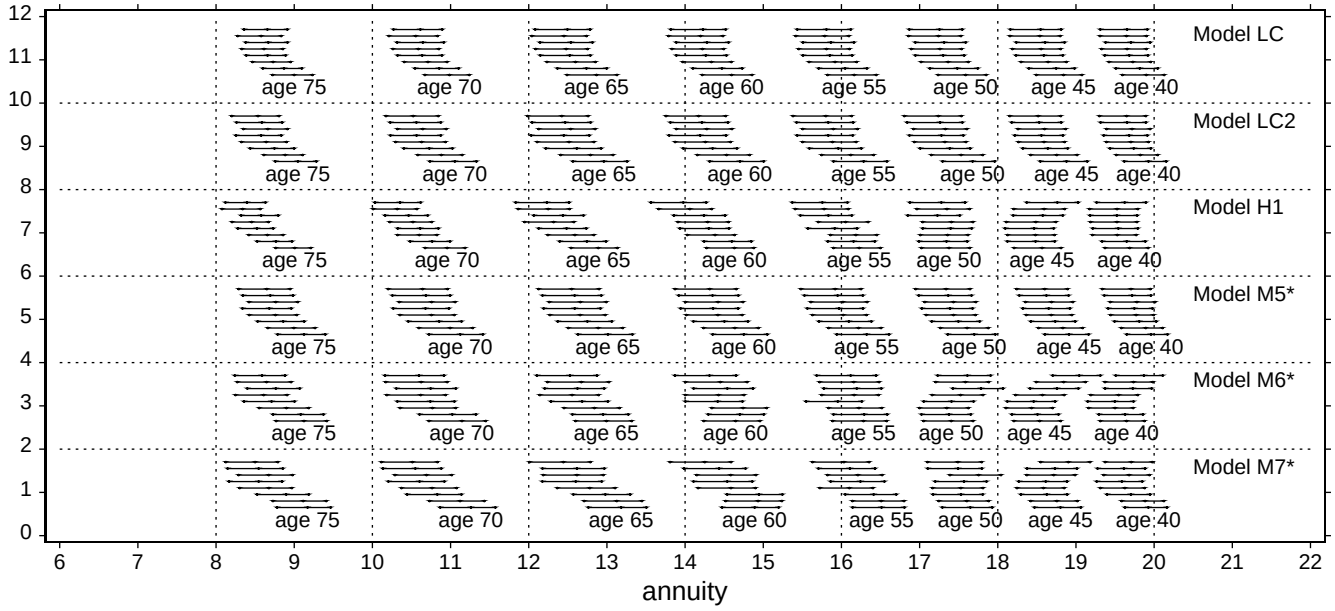


Fig 15b. USA female 1961-2006 mortality experience, age range 20-89.
Evolving 1993(02)05,06 life expectancy & 4% annuity prediction intervals.

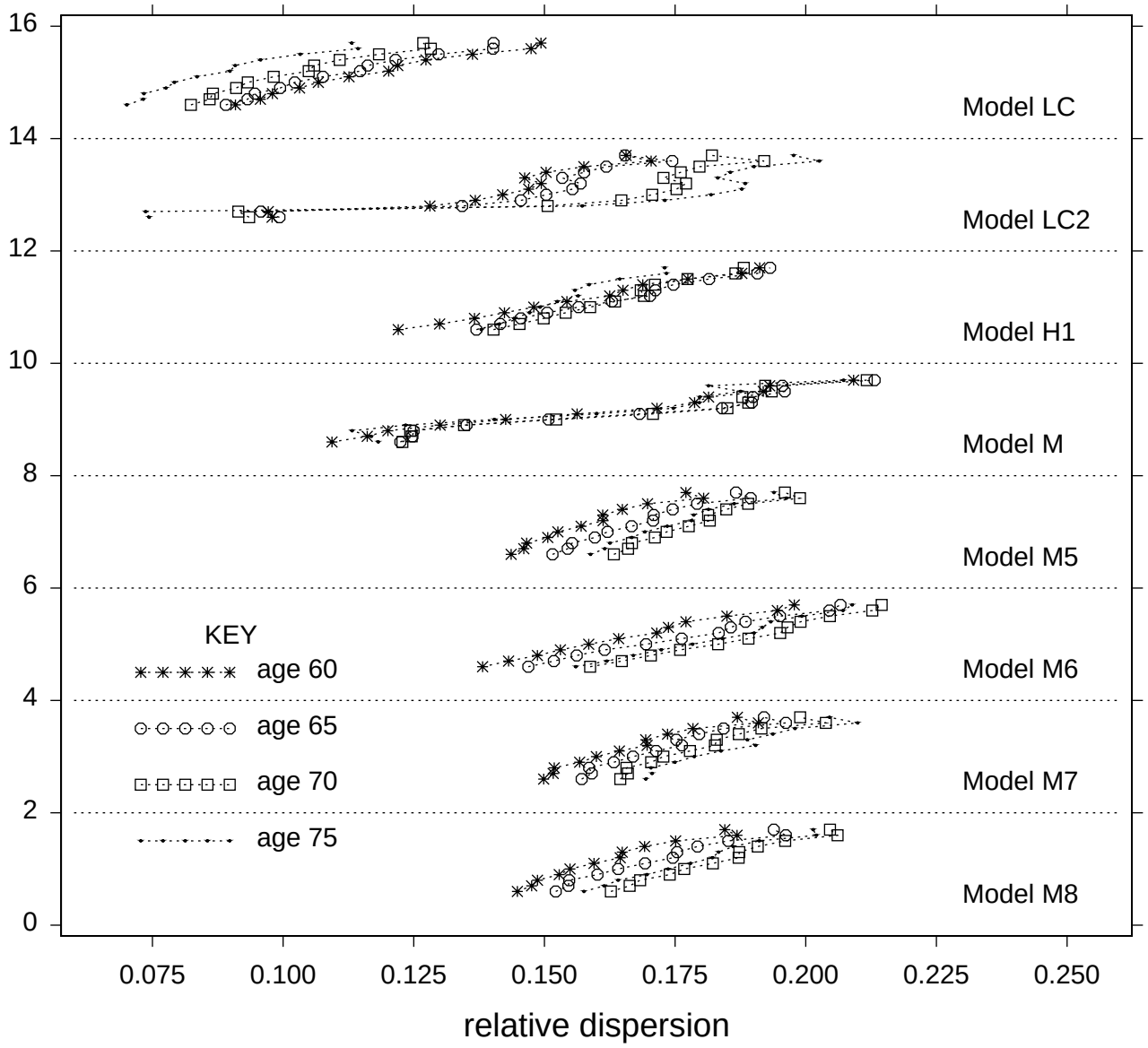


Fig 16. England & Wales male mortality experience, age range 55-89. Evolving biennial 1985(02)07 measures of relative dispersion (r.d.) for life expectancy predictions in Fig 11a: descending sequence, individuals aged 60(05)75. [r.d. = (95th percentile - 5th. percentile)/median]

Appendix C

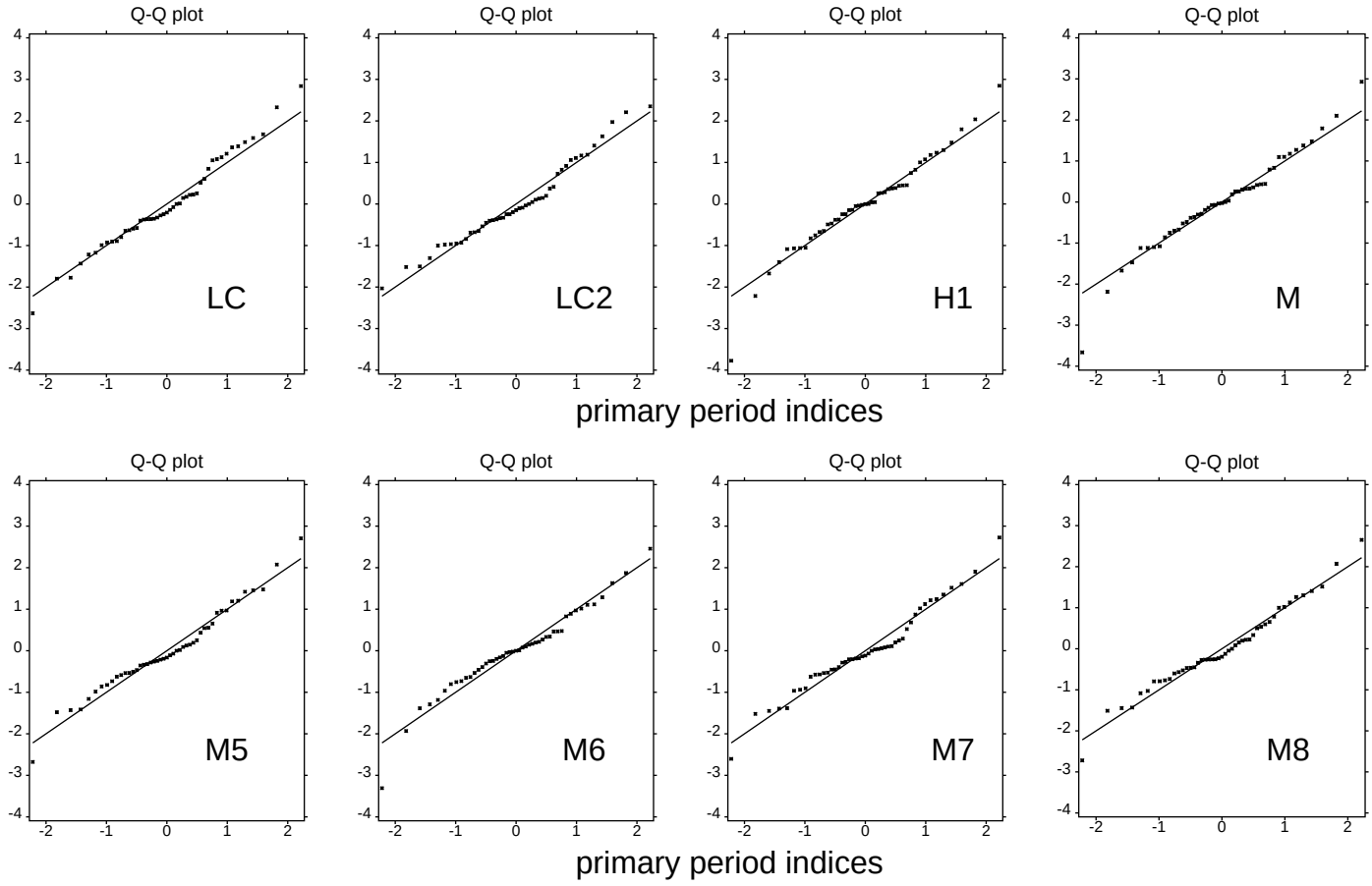


Fig C1. England & Wales 1961-2007 male mortality experience, ages 55-89. Quantile-Quantile plots: random walk primary period component time series standardised residuals (checking for Normal residuals).

Appendix C

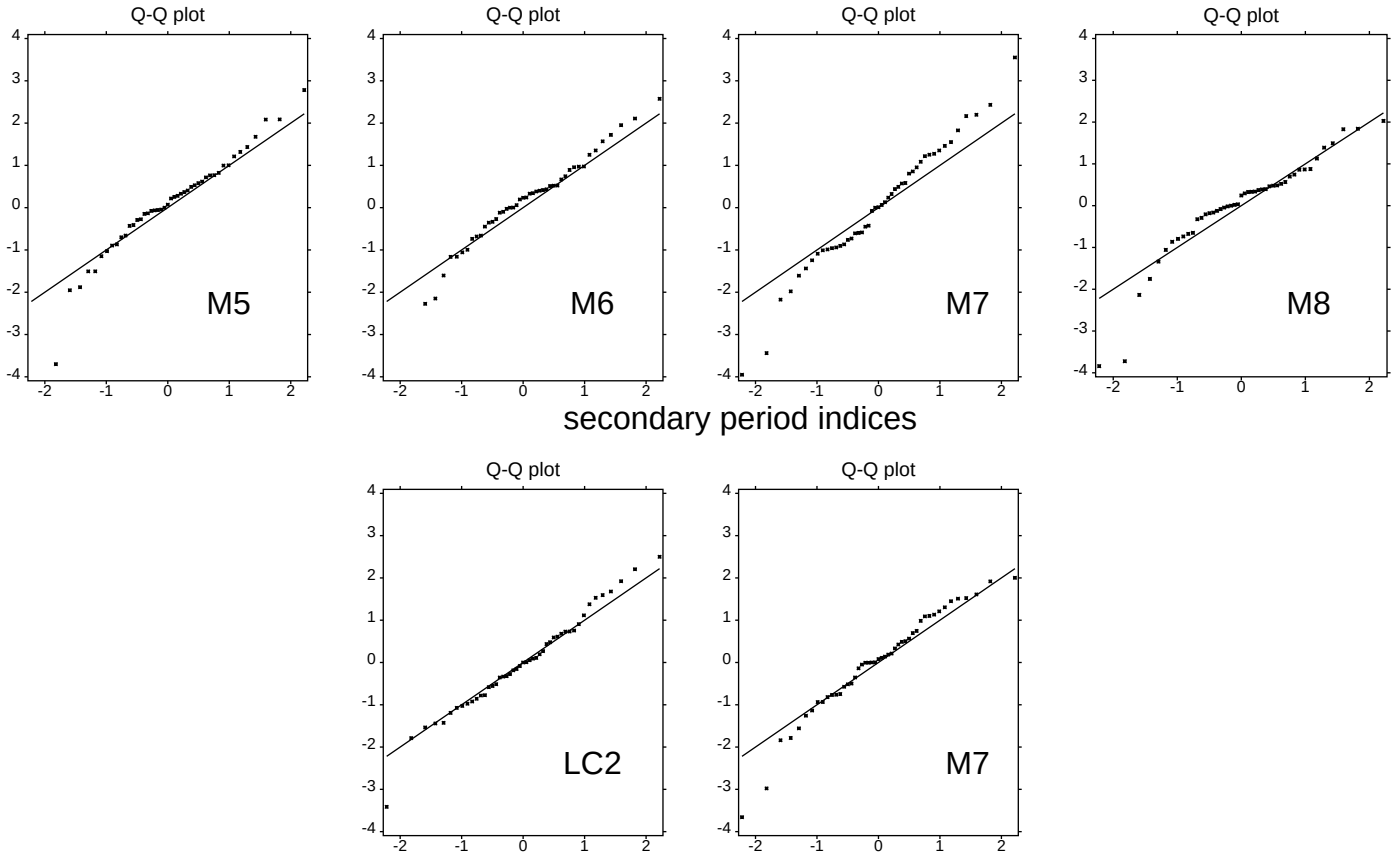


Fig C2. England & Wales 1961-2007 male mortality experience, ages 55-89. Quantile-Quantile plots: random walk secondary (M5-M8, LC2) and tertiary (M7) period component time series standardised residuals (checking for Normal residuals).

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