HOMOMORPHISMS BETWEEN BUBBLE ALGEBRA MODULES

By

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Abstract

In this thesis, we study the representation theory of the bubble algebras. We focus on determining the homomorphisms between cell modules for these algebras.

We show that the bubble algebras are cellular, and form a tower of recollement. By decomposing Gram matrices we are able to determine homomorphisms in the one arc case. We determine a generating set for the algebra (and its cardinality), and use this with a hypercuboid approach to determine homomorphisms in the general case. We end with a conjectural alternative approach to the general case involving matrix methods.

Acknowledgements

I am deeply indebted to Dr.Anton G Cox, my supervisor, who has continually guided me throughout my doctorate research. His insight and enthusiasm towards my work has been tremendous. From him I have gained an incredible depth of understanding on the subject area which has contributed significantly to my research.

In addition, I would like to thank School of Engineering and Mathematical Science for their financial support during my first three years of degree. I also would like to thank the staff in the Center of Mathematics by providing support and the provision of a structured learning environment has helped me greatly during my research.

Finally I am extremely indebted to my wife, Bamathy, my little son, Harien and my family who have always believed in me and shown me great support throughout my research.

I also thank God who has given me the strength, knowledge and wisdom to successfully complete my research and also want to thank my maths teachers R.Arulchelvam and late P.Velautham (vector master).

City University London, Northampton Square, London EC1 OHB Mahadevan Jegan
April 19, 2013

Dedication

I dedicate this to my family, who have endured me for the longest.
To my parents, my wife Bamathy and my little son Harien, this is for you.

Introduction

In this thesis, we are concerned with the problem of finding homomorphisms between two cell modules for the bubble algebra. This algebra was defined by Grimm and Martin [30] in 2003. We shall start with a brief review in order to motivate this study.

From the study of dilute lattice models [56, 64], Grimm and Pearce [29] came up with certain generalisations of some diagram algebras (algebras with a diagrammatic formulation [45]), such as the Temperley-Lieb [60] and BMW [53, 6] algebras. These algebras play a significant role in the theory of solvable lattice models of two dimensional statistical mechanics [3] and are related to link and knot invariants [62]. The idea behind the generalisation arises on the diagram level by introducing diagrams with different colour lines. Each of the algebras was then described by generators by the requirement of solving the Yang-Baxter equations. However, the topological underpinning was not precisely formalised.

The discovery of the solvable lattice models called dilute lattice models [56, 64, 40] stimulated their generalisation. This has very strong connection to models of dilute loops on a lattice [2, 54]. These models include the solvable companion of the two-dimensional Ising model in a magnetic field [64, 31, 32], which is one of the unsolved problems in statistical mechanics. The idea was to consider two colours and regard the second colour as the dilution of the first.

Grimm and Pearce's design of the relations perfectly fulfilled the requirement of solving the Yang-Baxter equations in the two-colour case. These equations are enough to make sure of solvability in the sense of commuting transfer matrices [3, 62]. The representation theory of this kind of algebra facilitated the formation of the solvable dilute and two-colour lattice models. Many different representations and related models are considered in [29, 24, 25, 34, 35, 26].

The new representation of the algebra contained the previously known lattice models [24, 25]. It also led to a new series of solvable lattice models [34, 34, 26]. They had generators and relations, and enough representations to show that these relations do not imply a trivial algebra. However at that time they did not have any knowledge of dimensions or even of finiteness, and also had no idea of the irreducible representations. This was unlike the representation theory of the Temperley-Lieb algebra itself, which is very well studied and understood, in part because of its importance in several areas of mathematics and physics [18, 43, 41, 45, 37, 38, 42].

Grimm and Martin [30] defined the bubble algebra entirely diagrammatically, in such a way that it satisfied the general framework of [45, 49, section 9.5]. After that they have shown it provides a diagrammatic realisation of the Grimm-Pearce multi-colour Temperley-Lieb algebra. They used the general method to find the generic representation theory of these algebras. They came up with the machinery to investigate their representation theory (analogous to that of ordinary Hecke algebras at q a root of unity). They also showed how irreducible representations may be associated with physical observables in the corresponding lattice models. They concluded in their paper with the discussion of their results for the Bethe ansatz on models derived using this algebra.

In this paper we consider the algebra considered by Grimm and Martin. Their paper mainly considered the two colour case. In our thesis we have considered the bubble algebra with h colours.

There are various generalisations of the Temperley-Lieb algebra [49, 50, 7, 58, 23], so it is natural to ask why the bubble algebra in particular should be studied. There are a number of important reasons that can be given.

First, the diagram form of the Temperley-Lieb algebra is a deep and powerful property [43, 41, 45], and the bubble algebra realisation provides a natural generalisation on the diagram level.

Second, it provides solutions of the Yang-Baxter equation. The Temperley-Lieb algebra is also related to the blob algebra, which has been shown [17] to be useful in solving the reflection equation [39]. It is useful in constructing integrable boundary conditions for certain solvable lattice models, including conformal-twisted boundary conditions [33, 4, 55, 27, 28]. It is quite important to boundary conformal field theory. A bubble algebra analogue of these relationships would be very interesting.

Third, it is a part of a class of algebras amenable to the methods of [49]. It is quite relevant for growth of the statistical mechanics (cf [63, 65, 61]); see for example [36]. It looks like it should be useful for circuit design and even transport network design. There are also some similarities with Murakami Birman Wenzl algebras [53, 6] and Fuss-Catalan algebras [15]. Both these algebras have been used to construct integrable systems.

The final reason, which is the key motivation for this thesis, is that the bubble algebras have some useful technical features of interest in representation theory. There is a general programme of abstract algebraic Lie theory in diagram algebras which has been introduced by Cox, Martin, Parker and Xi [13] as towers of recollement, and the bubble algebra also fits into this framework (as we will prove in this thesis).

We end this introduction with a brief survey of the rest of this thesis. The first chapter is devoted to the general theory of cellular algebras, and introduces the Temperley-Lieb algebra and bubble algebra. In the first section of this chapter we will start by giving the definition of Graham and Lehrer [20] of a cellular algebra. In the next section we will introduce the Temperley-Lieb algebra and review the proof that it is a cellular algebra. After that we will introduce the bubble algebra and show in a similar way that it is a cellular algebra. This have not been done explicitly before. We will provide the complete proof for this in this chapter.

In the second chapter, we turn our attention to a paper of Cox, Martin, Parker and Xi [13]. This chapter is about towers of recollement, which form an axiomatic framework for studying the representation theory of towers of algebra. If a family of algebras is a tower of recollement, then we can apply Theorem 2.1.27 in Chapter 2. This theorem helps us to know whether we have a non-zero homomorphism between two standard (which in our case are also cell) modules by reducing to the case where one is simple. This allows us to restrict attention in this thesis to the problem of determining the non-zero homomorphisms from a simple cell module.

The third chapter is devoted to certain special idempotents, and considers the Gram matrix associated to a module. We show how certain idempotent subalgebras of the bubble algebra correspond to tensor products of Temperley-Lieb algebras, and relate the cell modules of these two types of algebra. We will also show that the Gram matrix in general has a similar decomposition into products.

From this point onwards we start to concentrate on finding homomorphisms between cell modules. The fourth chapter is devoted to the special case where the second module contains a single arc in its basis elements. By finding the matrix corresponding to the homomorphism between the modules we can find the homomorphism.

For our convenience, we consider a certain matrix R_n as in (4.1.10). Later we will see that it is a matrix corresponding to the homomorphism between two Temperley-Lieb algebra modules. The determinant of this matrix R_n satisfies a difference equation (4.1.11). This helps us to find the homomorphism between cell modules for different values of δ as in Proposition 4.2.4 and the families of non-zero homomorphism between two modules for the same values of δ as in Proposition **??** equation (**??**). At the end of the Chapter 4, we will show that the homomorphism we found is unique.

The fifth chapter is devoted to considering certain generators of the bubble algebra. We will classify all the generators into four cases. By finding the generators in each case, we will give a formula for the total number of generators for the bubble algebra with h colours. It is given by the Proposition 5.3.1. We use the generators to prove the important Theorem in Chapter 6.

In Chapter 6 we will find the non-zero homomorphism between cell modules in the general case. In Chapter 6, we will introduce the idea of the hypercuboid to help us find the homomorphism between two given modules. Here, we find the homomorphism between each colour module separately. By gluing each colour shape and by looking at the colour shape change, we find the homomorphism between h colour modules. This method is the best way to find the homomorphism between the given two modules. Theorem 6.2.2 displays the main result in Chapter 6.

Chapter 1

Cellular algebras, Temperley-Lieb algebras, and bubble algebras

1.1 Cellular algebras

Cellular algebras were introduced by Graham and Lehrer [20] in 1996. In general terms, a cellular algebra A is an algebra with a very special basis which helps us to study the representation theory of A. This section discusses the theory of cellular algebras, our motivation being that both Temperley-Lieb algebras and bubble algebras are cellular algebras.

A cellular algebra A have two main properties. The first property is that there is a cellular basis which gives a filtration of A, and defines certain special modules (called cell modules) of A. The second property is that there are associated bilinear forms on each of the cell modules. Further, the quotient of a cell module by the radical of its bilinear form is either zero or absolutely irreducible, and every irreducible (up to isomorphism) arises in this way. However, it is difficult to determine when the quotients are zero and non-zero.

A basic question in any branch of representation theory is to determine the number of non-isomorphic simple modules. One of the strengths of the theory of cellular algebras is that it provides (in principle) a complete list of absolutely irreducible modules for the algebra.

We can also define a decomposition matrix for a given cellular algebra. One of the nice properties of celluar algebras is that this decomposition matrix is always unitrangular and A is semisimple if and only if the decomposition matrix is the identity.

1.1.1 Formulating the cellular algebra model

We begin by recalling the basic definition of Graham and Lehrer [20].

Definition 1.1.1. Let A be an algebra over a ring R. Suppose that we have a finite partially ordered set $(\Lambda, \geq)$, and for each $\lambda \in \Lambda$ a finite set $T(\lambda)$ such that there exists a basis of A of the form

$$\mathcal{C} = \{C^{\lambda}_{st} : \lambda \in \Lambda \text{ and } s, t \in T(\lambda)\}. \tag{1.1.1}$$

For each $\lambda \in \Lambda$, let $\check{A}^{\lambda}$ be the R-submodule of A with basis

$$\{C^{\mu}_{uv} : \mu \in \Lambda, \mu > \lambda \text{ and } u, v \in T(\mu)\} \tag{1.1.2}$$

and A^{λ} be the R-submodule of A with basis

$$\{C^{\mu}_{uv} : \mu \in \Lambda, \mu \geq \lambda \text{ and } u, v \in T(\mu)\}. \tag{1.1.3}$$

The pair $(\mathcal{C}, \Lambda)$ is called a **cellular basis** of A if it satisfies the following two conditions.

(i) There should be an algebra anti-isomorphism "$*$" of A such that

$$C^{\lambda^*}_{st} = C^{\lambda}_{ts} \tag{1.1.4}$$

for all $\lambda \in \Lambda$ and all $s, t \in T(\lambda)$.

(ii) For any $\lambda \in \Lambda, t \in T(\lambda)$ and $a \in A$ there exist $r_v \in R$ such that for all $s \in T(\lambda)$ we have

$$C^\lambda_{st} a \equiv \sum_{v \in T(\lambda)} r_v C^\lambda_{sv} \mod \breve{A}^\lambda. \tag{1.1.5}$$

If A has a cellular basis we say that A is a **cellular algebra**.

Note that in part (ii), we should write $r_v = r^a_{vt}$ since r_v depends on v, t and a; what is really important is that r_v does not depend on s.

The following Lemma summarizes some basic properties of a cellular algebra (see for example [52], which will be our main reference for standard results about cellular algebras).

Lemma 1.1.2. *Let λ be an element of Λ.*

(i) If $s \in T(\lambda)$ and $a \in A$, then for all $t \in T(\lambda)$ we have

$$a^* C^\lambda_{st} \equiv \sum_{u \in T(\lambda)} r_u C^\lambda_{ut} \mod \breve{A}^\lambda, \tag{1.1.6}$$

where for each u, r_u is the element of R determined by (1.1.5).

(ii) The R-modules A^λ and $\breve{A}^\lambda$ are two-sided ideals of A.

(iii) If s and t are elements of $T(\lambda)$, then there exists an element r_{st} of R such that for any $u, v \in T(\lambda)$ we have

$$C^\lambda_{us} C^\lambda_{tv} \equiv r_{st} C^\lambda_{uv} \mod \breve{A}^\lambda. \tag{1.1.7}$$

1.1.2 Formulating the cell module

Let A be an algebra with basis as in (1.1.1). Then A^λ is an subalgebra of A with basis C^μ_{st}, where $\mu \geq \lambda$, and $\breve{A}^\lambda$ is an ideal in A^λ with basis C^μ_{st}, where $\mu > \lambda$. Therefore, $A^\lambda / \breve{A}^\lambda$ is an algebra with basis $C^\lambda_{st} + \breve{A}^\lambda$, where $s, t \in T(\lambda)$.

Fix an element λ of Λ. If $s \in T(\lambda)$ define $C^\lambda(s)$ to be the R-submodule of $A^\lambda/\breve{A}^\lambda$ with basis

$$\{C_{st}^\lambda + \breve{A}^\lambda : t \in T(\lambda)\}. \tag{1.1.8}$$

$C^\lambda(s)$ is a right A-module and the action of A on $C^\lambda(s)$ is completely independent of s by (1.1.5). That is

$$C^\lambda(s) \cong C^\lambda(t) \tag{1.1.9}$$

for all $s, t \in T(\lambda)$. This allows to define the right cell module C^λ to be the right A-module which is free as an R-module with basis $\{C_t^\lambda : t \in T(\lambda)\}$. For each $a \in A$ we have

$$C_t^\lambda a = \sum_{v \in T(\lambda)} r_v C_v^\lambda, \tag{1.1.10}$$

where r_v is the element of R determined by (1.1.5). Then

$$C^\lambda \cong C^\lambda(s), \tag{1.1.11}$$

for all $s \in T(\lambda)$, via the canonical R-linear map which sends C_t^λ to $C_{st}^\lambda + \breve{A}^\lambda$, for all $t \in T(\lambda)$.

Definition 1.1.3. By Lemma 1.1.2(iii), there is a unique bilinear map
$\langle \quad , \quad \rangle : C^\lambda \times C^\lambda \to R$ such that $\langle C_s^\lambda, C_t^\lambda \rangle$, for $s, t \in T(\lambda)$, is given by

$$\langle C_s^\lambda, C_t^\lambda \rangle C_{uv}^\lambda \equiv C_{us}^\lambda C_{tv}^\lambda \mod \breve{A}^\lambda, \tag{1.1.12}$$

where u and v are elements of $T(\lambda)$.

We have

Proposition 1.1.4. *[52] If $\lambda \in \Lambda$ and $x, y \in C^\lambda$, then*

(i) $\langle x, y \rangle = \langle y, x \rangle$

(ii) $\langle xa, y\rangle = \langle x, ya^*\rangle$ *for all* $a \in A$

(iii) $xC^\lambda_{uv} = \langle x, C^\lambda_u\rangle C^\lambda_v$ *for all* $u, v \in T(\lambda)$.

Hence $\langle \quad , \quad \rangle$ is both symmetric and associative.

1.1.3 The radical of a cell module

Given a cellular algebra we can make the following definition.

Definition 1.1.5. The radical of the module C^λ is given by

$$\mathrm{rad}C^\lambda = \{x \in C^\lambda : \langle x, y\rangle = 0 \text{ for all } y \in C^\lambda\}. \tag{1.1.13}$$

By proposition 1.1.4(ii), radC^λ is an A-submodule of C^λ.

Recall that the Jacobson radical of a module M is the intersection of the maximal ideals of M. There will be no confusion over terminology because of the following proposition.

Proposition 1.1.6. *[52] Suppose that R is a field and let μ be any element of Λ.*

(i) The Jacobson radical of C^μ is equal to $\mathrm{rad}C^\mu$.

(ii) The right A-module

$$L^\mu = C^\mu/\mathrm{rad}C^\mu \tag{1.1.14}$$

is irreducible.

Corollary 1.1.7. *Suppose that R is a field and let μ and λ be elements of Λ such that $L^\mu \neq 0$ and $L^\mu \cong L^\lambda$. Then $\mu = \lambda$.*

One of the main results of Graham and Lehrer [20] is

Theorem 1.1.8. *Suppose that R is a field and let Λ be finite. Set*

$$\Lambda_0 = \{\mu \in \Lambda : L^\mu \neq 0\}.$$

Then $\{L^\mu : \mu \in \Lambda_0\}$ is a complete set of pairwise inequivalent irreducible A-modules.

Theorem 1.1.8 classifies the simple A-modules; however it is often difficult to determine the set Λ_0.

Definition 1.1.9. Let $\mu \in \Lambda_0$ and $\lambda \in \Lambda$. Define

$$d_{\lambda\mu} = [C^\lambda : L^\mu] \tag{1.1.15}$$

to be the composition multiplicity of the irreducible module L^μ in C^λ. By the Jordan-Hölder theorem, $d_{\lambda\mu}$ is well-defined. The so-called decomposition matrix D of A is given by

$$\mathrm{D} = (d_{\lambda\mu}), \tag{1.1.16}$$

where $\lambda \in \Lambda$ and $\mu \in \Lambda_0$.

The decomposition matrix has the following special form.

Corollary 1.1.10. *[52] Suppose that R is a field. Then the decomposition matrix D of A is unitriangular; that is, if $\mu \in \Lambda_0$ and $\lambda \in \Lambda$ then $d_{\mu\mu} = 1$ and $d_{\lambda\mu} \neq 0$ only if $\lambda \geq \mu$.*

Decomposition matrices can also be used to determine when a cellular algebra is semisimple.

Corollary 1.1.11. *[52] Suppose that R is a field. Then the following are equivalent.*

(i) A is (split) semisimple.

(ii) $C^\lambda = L^\lambda$ for all $\lambda \in \Lambda$.

(iii) $\mathrm{rad}(C^\lambda) = 0$ *for all* $\lambda \in \Lambda$.

(iv) $d_{\lambda\mu} = \delta_{\lambda\mu}$ *for all* λ *and* μ *in* Λ.

1.2 The Temperley-Lieb algebra

The Temperley-Lieb algebras were first introduced in 1971 by Temperley and Lieb [60]. They were used to study the single bond transfer matrices for the Ising model in statistical mechanics. Some time after this, they were independently found by Jones [37] when he characterized the algebras arising from the tower construction of semisimple algebras in the study of subfactors in mathematics. Their connection with knot theory comes from their role in the definition of Jones polynomial.

We first define the Temperley-Lieb algebra as in [8].

Definition 1.2.1. The **Temperley-Lieb algebra** $TL_n(\delta)$ is the associative algebra over R with generators 1 (the identity), $e_1, \ldots, e_{n-1}$ subject to the following conditions:

(1) $e_ie_je_i = e_i$ if $|j-i| = 1$,

(2) $e_ie_j = e_je_i$ if $|j-i| > 1$,

(3) $e_i^2 = \delta e_i$ for $1 \leq i \leq n-1$.

By using this definition it is quite hard to understand the nature of the algebra. The algebra $TL_n(\delta)$ can be easily described by diagrams in the plane. Here e_i is the diagram of the form as in Figure 1.1. We discuss this in more detail in the next section.

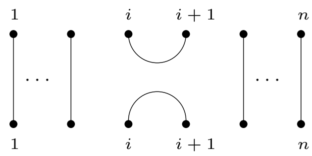


Figure 1.1:

1.2.1 Formulating the Temperley-Lieb algebra model

We want to describe the Temperley-Lieb algebra as a diagram algebra — an algebra with a diagrammatic formulation. The basis of the algebra will be rectangular diagrams with n nodes at the northern edge and n nodes at the southern edge, decorated with lines which connect the nodes in pairs without crossing the other lines and with no internal loops. A line in a diagram with one endpoint in the northern edge and one in the southern edge is called a propagating line and one with both endpoints in the same edge is called an arc. The identity element is the unique diagram all of whose lines are propagating.

We can form the product of any two diagrams a, b by concatenating them, writing a above b, and the southern endpoints of lines in a coincide with the northern endpoints of lines in b (NB. This requires only that the number of nodes matches up). Each node of coincidence may then be regarded as an interior point of a continuous line passing though the concatenated $a|b$. The multiplication ab is the new diagram of the combined region which results from this. In this multiplication, if we get any diagram with closed loop in the middle, then closed loop is removed and replaced with the loop replacement scalar δ times the same diagram without the closed loops. If several loops are removed, the scalar is a power of δ raise to the number of loops. At the beginning of Example 1.2.3, we will see the multiplication of two diagrams under this multiplication rule.

1.2.2 The Temperley-Lieb algebra is a cellular algebra

One of the easiest examples of a cellular algebra is the Temperley-Lieb Algebra. We will review the proof of this result, as the corresponding proof for the bubble algebra will be based on this.

Proposition 1.2.2. *The Temperley-Lieb algebra $TL_n(\delta)$ is a cellular algebra.*

Proof. Our finite partially ordered set Λ takes the values of the number of propagating lines from northern edge to southern edge. Our order on Λ is the opposite of the usual order on natural numbers. It is always possible to cut every decorated diagram from eastern edge to western edge in such a way that only propagating lines are cut. These upper halves and lower halves of diagrams are called **half diagrams**. The finite indexing set $T(\lambda)$ is given by

$$T(\lambda) = \{\text{half diagrams with } \lambda \text{ free lines}\}$$

for each λ in Λ.

We will define a set of basis elements of $TL_n(\delta)$ which we will denote by

$$\mathcal{C} = \{\ C^{\lambda}_{st} : \lambda \in \Lambda \text{ and } s, t \in T(\lambda)\ \}.$$

We denote the algebra $TL_n(\delta)$ by A for convenience. Let us define an element C^{λ}_{st} of $\mathcal{C}$ which is a basis element of A with $s, t \in T(\lambda)$. Here, s is the upper half and t is the lower half diagram of the basis element, where upper half diagram has been flipped and drawn above the lower half, and propagating lines from the two halves are connected in the unique possible way. Recall that we have defined (1.1.2)

$$\check{A}^{\lambda} = \text{span}\{\ C^{\mu}_{st} : \mu \in \Lambda \text{ and } \mu > \lambda\ \}. \tag{1.2.1}$$

We will continue proving $TL_n(\delta)$ is a cellular algebra after looking at the following example.

Example 1.2.3. Let us consider the algebra $TL_4(\delta)$, which has basis elements those diagrams with four nodes at the northern edge and four nodes at the southern edge and non-crossing lines connecting them in pairs. Figure 1.2 is a basis element of $TL_4(\delta)$.

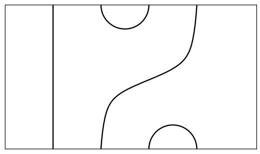


Figure 1.2:

Multiplication rule

Figure 1.3 shows the multiplication of two basis elements. This is equivalent to δ times the Figure 1.4.

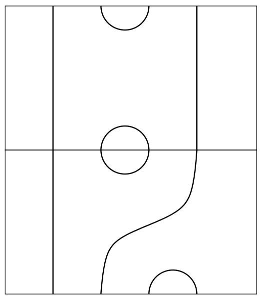


Figure 1.3:

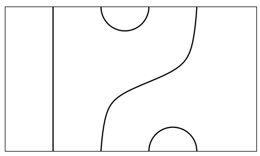


Figure 1.4:

We analyse the algebra $TL_4(\delta)$ by finding Λ, $T(\lambda)$, the order of Λ and the basis elements of the algebra of $TL_4(\delta)$. Arcs are formed by connecting two nodes in an edge. Basis elements of $TL_4(\delta)$ can have no arcs or one arc or two arcs. Therefore, Λ can be given by

$$\Lambda = \{0, 2, 4\}.$$

A finite indexing set $T(\lambda)$ is given by Figure 1.5, Figure 1.6 and Figure 1.7.

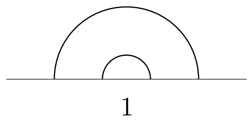


Figure 1.5: Half diagrams in $T(0)$

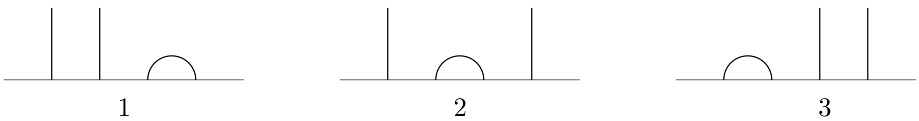


Figure 1.6: Half diagrams in $T(2)$

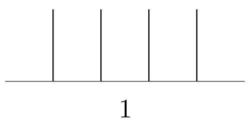


Figure 1.7: Half diagrams in $T(4)$

The basis element in Figure 1.2 has been constructed by drawing the second half diagram in the northern edge and the first half diagram in the southern edge of the figures in Figure 1.6. Let us construct all basis elements of $TL_4(\delta)$.

Let us name the basis elements in the form C'^{λ}_{st}, where s and t are half diagrams in $T(\lambda)$ at the northern and southern edge respectively. Diagrams in Figure 1.8, Figure 1.9, and

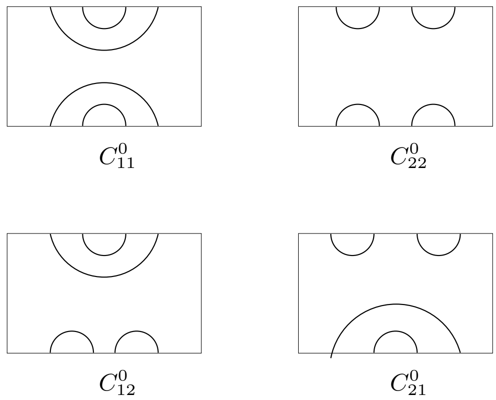


Figure 1.8: Basis elements of $TL_4(\delta)$ with 0 lines

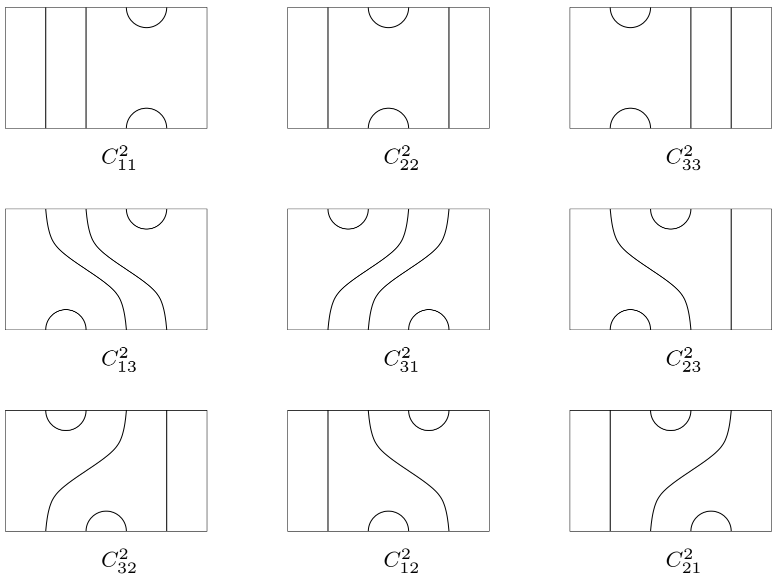


Figure 1.9: Basis elements of $TL_4(\delta)$ with 2 lines

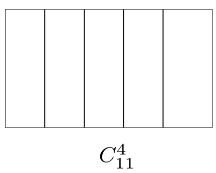


Figure 1.10: Basis elements of $TL_4(\delta)$ with 4 lines

Figure 1.10 are the basis elements of $TL_4(\delta)$. Therefore, set of basis elements of $TL_4(\delta)$ can be written as

$$\begin{aligned}\mathcal{C} = \{&C^0_{11}, C^0_{22}, C^0_{12}, C^0_{21},\\ &C^2_{11}, C^2_{22}, C^2_{33}, C^2_{12}, C^2_{21}, C^2_{13}, C^2_{31}, C^2_{23}, C^2_{32},\\ &C^4_{11}\}.\end{aligned}$$

The order on the elements of Λ is as follows:

$$0 \quad \geq \quad 2 \quad \geq \quad 4.$$

Let us find the basis of $\breve{A}^0$, $\breve{A}^2$ and $\breve{A}^4$.

$$\begin{aligned}\text{Basis of } \breve{A}^0 =& \{C^\mu_{st} : s, t \in T(\mu) \text{ and } \mu > 0\}\\ =& \emptyset\\ \text{Basis of } \breve{A}^2 =& \{C^\mu_{st} : s, t \in T(\mu) \text{ and } \mu > 2\}\\ =& \{C^0_{11}, C^0_{22}, C^0_{12}, C^0_{21}\}\\ \text{Basis of } \breve{A}^4 =& \{C^\mu_{st} : s, t \in T(\mu) \text{ and } \mu > 4\}\\ =& \{C^0_{11}, C^0_{22}, C^0_{12}, C^0_{21},\\ & C^2_{11}, C^2_{22}, C^2_{33}, C^2_{13}, C^2_{31}, C^2_{23}, C^2_{32}, C^2_{12}, C^2_{21}\}\end{aligned}$$

This example helped us to understand the basis elements of algebra A and its ideal $\breve{A}^\lambda$ for each $\lambda \in \Lambda$. This understanding will help us to continue proving $TL_n(\delta)$ is a cellular algebra.

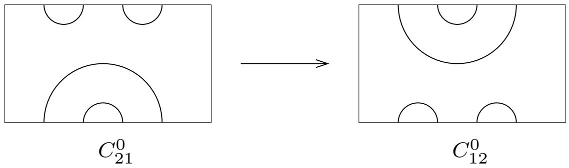


Figure 1.11: Basis to basis map

First condition: Anti-isomorphism

Define the map $*$ of A as follows

$$* : A \to A$$

$$C_{st}^{\lambda^*} = C_{ts}^{\lambda}.$$

This mapping reflects diagrams upside down as shown in Figure 1.11 for the $n = 4$ case. Let us explain why $*$ is an anti-homomorphism. We need to show

$$(ma)^* = a^*(m)^*.$$

The left-hand side of the above equation says find the multiplication ma then flip upside down. On the other hand, the right-hand side says first flip a then flip m then multiply. Obviously both are the same. From this we can say $*$ is an anti-homomorphism. $*$ is injective by the way it is defined (it takes a basis to a basis). Now we will show $*$ is surjective by using the rank nullity-theorem.

$$\text{dim(A)} = \text{dim(Ker}(*)) + \text{dim(Im}(*))$$

We know that

$$\dim(\ker(*)) = 0.$$

Therefore, this implies that

$$\dim(A) = \dim(\text{Im}(*))$$

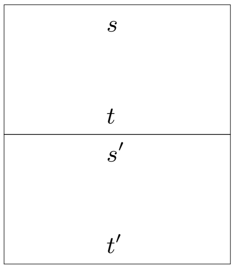


Figure 1.12:

and so $*$ is surjective. We have shown $*$ is anti-homomorphism, injective and surjective therefore, $*$ is anti-isomorphism.

Second condition:

Now we will check the second condition of the cellular basis. Let C^{λ}_{st} be a basis element of A and $a \in A$. We can write a as the linear combination of the basis elements of A. Therefore, a can be written as follows

$$a = \sum d_{\lambda' s' t'} C^{\lambda'}_{s' t'},$$

where $d_{\lambda' s' t'} \in \mathbb{C}$. If we find $C^{\lambda}_{st} a$ we will get

$$\begin{aligned} C^{\lambda}_{st} a &= C^{\lambda}_{st} \sum d_{\lambda' s' t'} C^{\lambda'}_{s' t'} \\ &= \sum d_{\lambda' s' t'} C^{\lambda}_{st} C^{\lambda'}_{s' t'}. \end{aligned}$$

Figure 1.12 illustrates the multiplication of $C^{\lambda}_{st} C^{\lambda'}_{s' t'}$. The product of these two diagrams is a new diagram whose number λ'' of propagating lines is less than or equal to the minimum of the two numbers λ and λ'. That is in the order on Λ we have $\lambda'' \geq \max(\lambda, \lambda')$.

Case (i) $\lambda = \lambda'$.

In this case multiplication gives us the same number of propagating lines or less than λ

(order on Λ is not being used). First we look into the situation that multiplication gives the same number of propagating lines. In this situation, we may get loops at the middle of the diagrams when southern edge half diagram t of C^{λ}_{st} and northern edge half diagram s' of $C^{\lambda'}_{s't'}$ meet each other. If the number of propagating lines is unchanged then the multiplication does not affect the northern edge s of C^{λ}_{st} and the southern edge of $C^{\lambda'}_{s't'}$. Therefore, we can say

$$C^{\lambda}_{st}C^{\lambda'}_{s't'} = rC^{\lambda}_{st'}.$$

Here r is dependent on t and s' and most importantly not dependent on s.

Now we look into the situation where multiplication gives fewer propagating lines than λ and λ' (order on Λ is not being used). Let us say, we get λ'' number of propagating lines. According to our order λ'' is greater than λ. Therefore,

$$C^{\lambda}_{st}C^{\lambda'}_{s't'} = \alpha C^{\lambda''}_{s''t''},$$

where $C^{\lambda''}_{s''t''}$ is a basis element constructed from the half diagrams in $T(\lambda'')$ and $\alpha \in \mathbb{C}$.

Case (ii) $\lambda < \lambda'$.

In this case multiplication gives us fewer propagating lines than λ(order on Λ is not being used). Let us say we get λ'' propagating lines. According to our order, λ'' is greater than λ. Therefore, this case is very similar to the latter part of Case(i). This implies that,

$$C^{\lambda}_{st}C^{\lambda'}_{s't'} = \alpha C^{\lambda''}_{s''t''},$$

where $C^{\lambda''}_{s''t''}$ is a basis element constructed from the half diagrams in $T(\lambda'')$, $\alpha \in \mathbb{C}$ and $\lambda'' > \lambda$.

Case (iii) $\lambda > \lambda'$

In this case multiplication gives us a number of propagating lines less than or equal to λ'

(order on Λ is not being used). Let us say we get λ'' number of propagating lines. According to our order λ'' greater than or equal to λ. If $\lambda'' = \lambda$ then

$$C^{\lambda}_{st}C^{\lambda'}_{s't'} = rC^{\lambda}_{st'}.$$

Here r is dependent on t and s' and most importantly not dependent on s.

If $\lambda'' > \lambda$ then,

$$C^{\lambda}_{st}C^{\lambda'}_{s't'} = \alpha C^{\lambda''}_{s''t''},$$

where $C^{\lambda''}_{s''t''}$ is a basis element constructed from the half diagrams in $T(\lambda'')$, $\alpha \in \mathbb{C}$ and $\lambda'' > \lambda$.

From Cases (i–iii),we see that

$$C^{\lambda}_{st}a = \sum rC^{\lambda}_{st'} + \sum \alpha C^{\lambda''}_{s''t''}.$$

This can be written as

$$C^{\lambda}_{st}a \equiv \sum rC^{\lambda}_{st'} \mod \check{A}^{\lambda}.$$

Hence $(\mathcal{C}, \Lambda)$ is a cellular basis of A. Therefore, $A = TL_n(\delta)$ is a cellular algebra. □

Example 1.2.4. Let the algebra A be $TL_4(\delta)$ and $\lambda = 2$. By (1.1.3) we have that A^2 is a subalgebra of A with basis elements of the form C^{μ}_{st}, where $\mu \geq \lambda$. Similarly, by (1.1.2) and Lemma 1.1.2(ii) we have that $\check{A}^2 \subset A^2$ is an ideal in A^2 with basis elements of the form C^{μ}_{st}, where $\mu > \lambda$. Therefore, $A^2/\check{A}^2$ is an algebra with basis elements of the form $C^{2}_{st} + \check{A}^2$.

$A^2/\check{A}^2 \cong$ algebra with basis elements have exactly 2 propagating lines and if a is a diagram and b is a diagram and $a.b$ has 0 number of propagating lines then define $a.b = 0$.

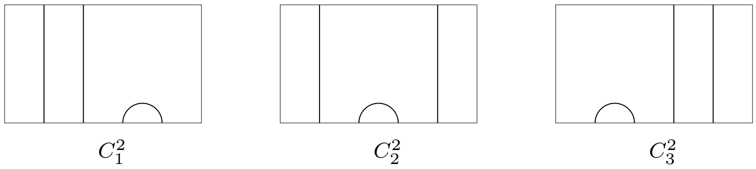


Figure 1.13:

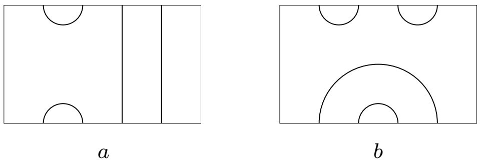


Figure 1.14:

The right cell module C^2 is a right A-module with basis as in Figure 1.13. The set of basis elements of C^2 is

$$\{C_t^2 : t \in T(2)\} = \{C_1^2, C_2^2, C_3^2\},$$

where $T(2)$, the set of labels with two propagating lines, is given by $\{1, 2, 3\}$. These three basis elements have been constructed by fixing the northern edge half diagram and choosing the southern edge half diagram from $T(2)$.

For each $a \in A$ we have

$$C_t^2 a = \sum_{v \in T(2)} r_v C_v^2 \mod \breve{A}^2. \tag{1.2.2}$$

If we take algebra elements a and b as in Figure 1.14, then $C_2^2 a$ and $C_2^2 b$ are illustrated in Figure 1.15. From these Figures we can say, $C_2^2 a$ is C_3^2 and $C_2^2 b$ is 0. This helps us to understand the multiplication in (1.2.2).

We will work out the inner product $\langle \quad , \quad \rangle$ of the cell module C^2 basis elements in the following example.

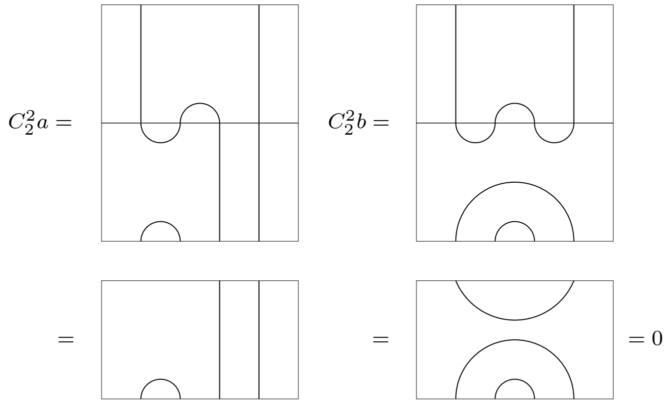


Figure 1.15:

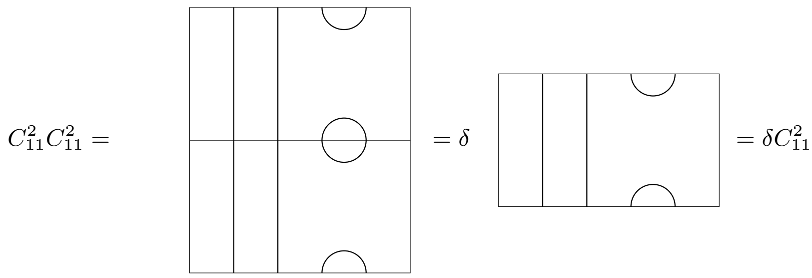


Figure 1.16:

Example 1.2.5. Let us consider the algebra $TL_4(\delta)$ and the associated cell module C^2. Figure 1.6 shows the half diagrams of $T(2)$. Therefore, the basis elements of C^2 are

$$\{C_1^2, C_2^2, C_3^2\}.$$

By using (1.1.12) we can find the following.

i) $\langle C_1^2, C_1^2 \rangle C_{11}^2 = C_{11}^2 C_{11}^2$

From Figure 1.16, we can say multiplication of C_{11}^2 and C_{11}^2 gives us δC_{11}^2. From this we can say,

$$\langle C_1^2, C_1^2 \rangle C_{11}^2 = \delta C_{11}^2.$$

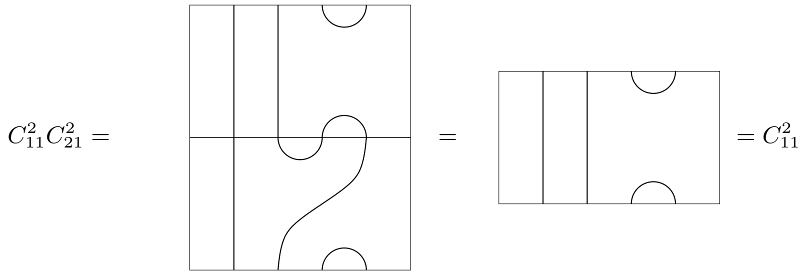


Figure 1.17:

This implies that

$$\langle C^2_1, C^2_1 \rangle = \delta.$$

ii) $\langle C^2_1, C^2_2 \rangle C^2_{11} = C^2_{11}C^2_{21}$

From Figure 1.17, we can say $C^2_{11}C^2_{21}$ gives us C^2_{11}. From this we can say

$$\langle C^2_1, C^2_2 \rangle C^2_{11} = C^2_{11}.$$

Therefore,

$$\langle C^2_1, C^2_2 \rangle = 1.$$

iii) $\langle C^2_1, C^2_3 \rangle C^2_{11} = C^2_{11}C^2_{31}$

From Figure 1.18, we can say $C^2_{11}C^2_{31}$ is C^0_{22}. From this we can say

$$\langle C^2_1, C^2_3 \rangle C^2_{11} = C^0_{22}.$$

Therefore,

$$\langle C^2_1, C^2_3 \rangle = 0.$$

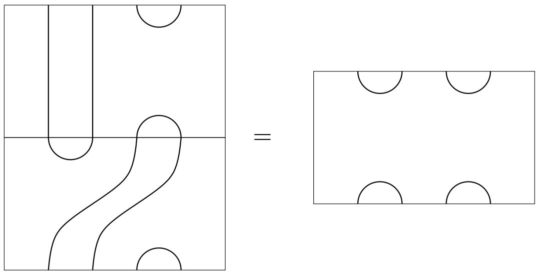
$C^2_{11}C^2_{31} =$  $= C^0_{22}$

Figure 1.18:

From these and similar calculations, we get

$$\langle C^2_1, C^2_1\rangle = \langle C^2_2, C^2_2\rangle = \langle C^2_3, C^2_3\rangle = \delta$$
$$\langle C^2_1, C^2_2\rangle = \langle C^2_2, C^2_3\rangle = 1 \tag{1.2.3}$$
$$\langle C^2_1, C^2_3\rangle = 0.$$

Example 1.2.6. Let us find radC^2 of the cell module of the algebra $TL_4(\delta)$. According to (1.1.13) we can say

$$\text{rad}C^2 = \{x \in C^2 : \langle x, y\rangle = 0 \text{ for all } y \in C^2\}.$$

Suppose $\langle x, y\rangle = 0$ with x and y in C^2. Therefore, we can write x and y as the linear combination of the basis elements of C^2. That is

$$x = \alpha_1 C^2_1 + \alpha_2 C^2_2 + \alpha_3 C^2_3,$$

$$y = \beta_1 C^2_1 + \beta_2 C^2_2 + \beta_3 C^2_3.$$

Substituting for x, y and by solving $\langle x, y\rangle = 0$ with the help of (1.2.3) and the proposition 1.1.4 we get

$$\alpha_1\beta_1\delta + \alpha_1\beta_2 + \alpha_2\beta_1 + \alpha_2\beta_2\delta + \alpha_2\beta_3 + \alpha_3\beta_2 + \alpha_3\beta_3\delta = 0.$$

However, this equation can be written as

$$(\alpha_1\delta + \alpha_2)\beta_1 + (\alpha_1 + \alpha_2\delta + \alpha_3)\beta_2 + (\alpha_2 + \alpha_3\delta)\beta_3 = 0.$$

The above equation should be true for all values of β_1, β_2 and β_3. Therefore, we can say

$$\alpha_1\delta + \alpha_2 = 0, \tag{1.2.4}$$

$$\alpha_1 + \alpha_2\delta + \alpha_3 = 0, \tag{1.2.5}$$

$$\alpha_2 + \alpha_3\delta = 0. \tag{1.2.6}$$

Equations (1.2.4), (1.2.5) and (1.2.6) can be written as the matrix equation

$$\begin{pmatrix} \delta & 1 & 0 \\ 1 & \delta & 1 \\ 0 & 1 & \delta \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \underline{\mathbf{0}}. \tag{1.2.7}$$

We get non zero solutions to (1.2.7) only if

$$\det \begin{pmatrix} \delta & 1 & 0 \\ 1 & \delta & 1 \\ 0 & 1 & \delta \end{pmatrix} = 0.$$

This implies

$$\delta(\delta^2 - 2) = 0.$$

From this we can say $\delta = 0$ or $\delta = \pm\sqrt{2}$.

If $\delta \neq 0$ and $\delta \neq \pm\sqrt{2}$ then $\alpha_1 = \alpha_2 = \alpha_3 = 0$. Therefore, $x = 0$, which implies that

$$\text{rad}C^2 = \{\underline{0}\}.$$

Therefore, C^2 is a simple module for almost every value of δ.

When $\delta = 0$ equations (1.2.4), (1.2.5) and (1.2.6) give us $\alpha_2 = 0$ and $\alpha_3 = -\alpha_1$. This implies that

$$\text{rad}C^2 = \{\alpha_1(C_1^2 - C_3^2) : \alpha_1 \in \mathbb{C}\},$$

which is a one dimensional space with basis $(C_1^2 - C_3^2)$. In this case C^2 is not a simple module.

When $\delta = \pm\sqrt{2}$ equations (1.2.4), (1.2.5) and (1.2.6) give us $\alpha_2 = \mp\sqrt{2}\alpha_1$ and $\alpha_3 = \alpha_1$. Therefore,

$$\text{rad}C^2 = \{\alpha_1(C_1^2 \mp \sqrt{2}C_2^2 + C_3^2) : \alpha_1 \in \mathbb{C}\},$$

which is a one dimensional space with basis $C_1^2 \mp \sqrt{2}C_2^2 + C_3^2$. In this case C^2 is not a simple module.

We have shown that C^2 is a simple module if and only if $\delta \neq 0$ and $\delta \neq \pm\sqrt{2}$. Similarly we can show that C^0 is a simple module if and only if $\delta \neq 0$ and $\delta \neq \pm 1$ and C^4 is a simple module for all $\delta \in \mathbb{C}$.

1.3 The bubble algebra

Bubble algebras were first introduced in 2003 by Grimm and Martin [30]. These are diagram algebras which provide multiparameter generalizations of the Temperley-Lieb algebra [60, 44]. They can be used to help solve the Yang-Baxter equation [30, section 3, equation (5)].

In this section, we will define the bubble algebra and show that it is a cellular algebra. Thereafter, we will discuss the cell modules of the bubble algebra and their reducibility. Figure 4.1 denotes the labeling for the red, green and black propagating lines and arc colour of the figures in the remaining Chapters.

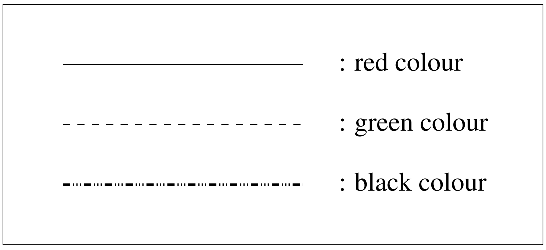


Figure 1.19: Colour labeling

1.3.1 Formulating the bubble algebra model with two colours

Just as for the Templerley-Lieb algebra, we shall first define a basis using diagrams, and them introduce a multiplication rule on diagrams. The basis of this algebra will consist of rectangular diagrams with n nodes at the northern edge and n nodes at the southern edge which connect the nodes in pairs with two different colours red and green, without any crossings of strings of the same colour and with no internal loops. Different colour strings can cross each other, but we exclude crossing occurring on the frame of the rectangle. For example, look at Figures 1.20 and 1.21. (These figures have been taken from [30, Section 2].)

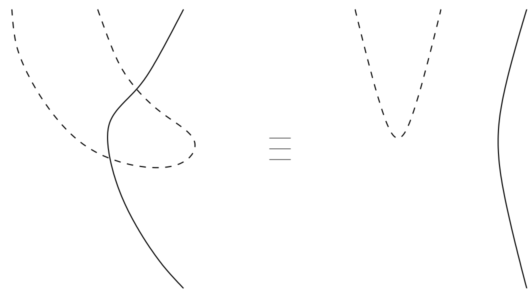


Figure 1.20: Both are equivalent

Let us think of this, as Grimm and Martin [30] did, as lines embedded not in a rectangle, but in a sheet of bubble wrap. (Bubble wrap is made from two sheets of polythene welded

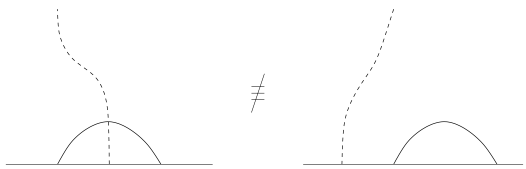


Figure 1.21: Both are not equivalent

together along certain lines to trap bubbles). We allow red lines on the weld and the back of the sheet; green lines are allowed on the welds and the front sheet.

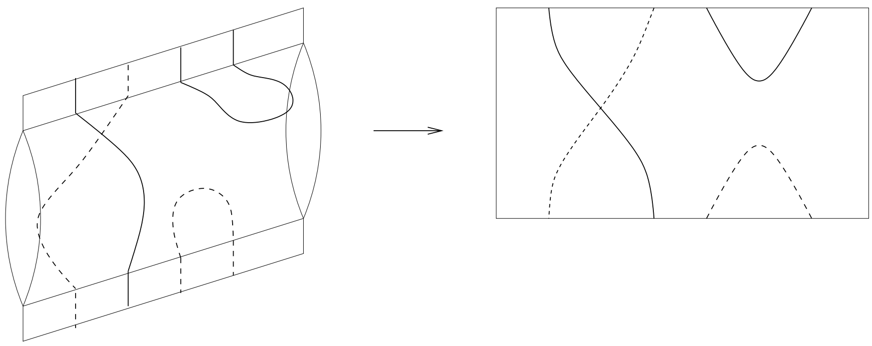


Figure 1.22: Bubble wrap and the equivalent rectangular diagram

In this realization, lines on the same sheet(or on the weld) are not allowed to touch. Figure 1.22 shows this and the equivalent rectangular diagram. (This figure has been taken from [30, section 2].)

We define the multiplication of the diagrams whenever the number of end points match up. We call the match up precise if the colours match up precisely. The composite is zero unless the match up is precise. If the match up is precise, then we concatenate the diagrams just as for the Temperley-Lieb algebra. When we multiply two diagrams, if we get any loop inside then that diagram can be replaced by an appropriate loop replacement scalar times the rest of the diagram. If the loop is red (respectively green) then the loop replacement scalar is δ_R (respectively δ_G).

We denote a bubble algebra with n nodes and 2 colours red and green by $TL_n^2(\delta_R, \delta_G)$,

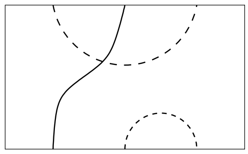


Figure 1.23: A basis element of $TL_3^2(\delta_R, \delta_G)$

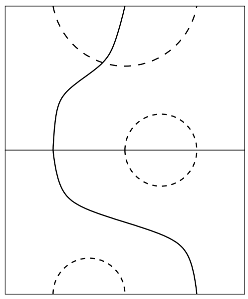


Figure 1.24: Loop form at the middle

where δ_R, δ_G are the red and green loop replacement parameters. We can define an h colour generalization as $TL_n^h(\delta_{C_1}, \dots, \delta_{C_h})$, where δ_{C_i} is the loop replacement scalar for the colour C_i loop for each $i \in \{1, \dots, h\}$.

Our modules and algebras have more than one colour. Instead of using actual colours, we have used different types of lines to denote the label of the colours.

Example 1.3.1. Let us consider the algebra $TL_3^2(\delta_R, \delta_G)$. It has three nodes at the northern and southern edges with two colours red and green. Figure 1.23 is a basis element of $TL_3^2(\delta_R, \delta_G)$. Figure 1.24 is an example of the multiplication of two diagrams. This is equivalent to δ_G times Figure 1.25.

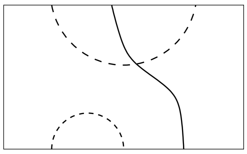


Figure 1.25: Loop removed

1.3.2 The bubble algebra is a cellular algebra

We will show that the bubble algebra is a cellular algebra. The full proof of this has not previously been given. Therefore, it is quite an important matter to discuss.

Proposition 1.3.2. *The bubble algebra $TL_n^h(\delta_{C_1}, \dots, \delta_{C_h})$ is a cellular algebra.*

Proof. We consider the finite partially ordered set

$$\Lambda = \left\{ (c_1, \dots, c_h) : 0 \leq c_i \leq n,\ 1 \leq i \leq h \text{ and } \sum_{i=1}^{h} c_i = n - 2t \text{ for some } t \geq 0 \right\}.$$

In our diagrams c_i will be the number of colour C_i propagating lines and t will denote the total number of arcs. Let t_{C_i} denote the number of C_i colour arcs. Therefore, t can be given by

$$t = \sum_{i=1}^{h} t_{C_i}.$$

An arc can be constructed by connecting two nodes. Therefore, the total number of propagating lines will be $n - 2t$ for some $0 \leq t \leq \frac{n}{2}$. Therefore, we can say

$$\sum_{i=1}^{h} c_i = n - 2t$$

and so each diagram is associated to an element in Λ. Define the order on Λ by

$$(c_1, \dots, c_h) \geq (c_1', \dots, c_h') \text{ if and only if } c_1 \leq c_1', \dots, c_h \leq c_h' \tag{1.3.1}$$

Let $\lambda \in \Lambda$. Therefore, λ can be given by

$$\lambda = (c_1, \dots, c_h)$$

for some $c_i \in \{1, \dots, n\}$. We define the finite indexing set $T(\lambda)$ to contain half diagrams with c_i colour C_i propagating lines, where $i = 1, 2 \dots, h$. (Note that there is no condition on the colour of any arcs.) As for the Temperley-Lieb algebra, we can form a unique bubble algebra diagram from a pair of half diagrams in $T(\lambda)$ by inverting the first and concatenating. It is clear that the set of all possible basis elements of the bubble algebra $TL_n^h(\delta_{C_1}, \dots, \delta_{C_h})$ arises as we allow λ to vary.

Let us call the algebra A and set

$$\mathcal{C} = \{\ C_{st}^{\lambda} : \lambda \in \Lambda \text{ and } s, t \in T(\lambda)\ \}.$$

Here C_{st}^{λ} is the basis element of A with $s, t \in T(\lambda)$, where s is the upper half and t is the lower half diagram of the basis element. Following (1.1.2) we set

$$\breve{A}^{\lambda} = \text{span}\{\ C_{st}^{\mu}\ : \mu \in \Lambda \text{ and } \mu > \lambda\ \} \tag{1.3.2}$$

Now we define the anti-isomorphism "$*$" as follows.

$$* : A \to A$$
$$C_{st}^{\lambda^*} = C_{ts}^{\lambda}$$

which corresponds to reflecting a diagram in the horizontal axis.

The verification that

$$(ma)^* = a^* m^*. \tag{1.3.3}$$

is exactly as for the Temperley-Lieb algebra in Proposition 1.2.2 .

Now we check the second condition. Let $C^\lambda_{st} \in \mathcal{C}$ and $a \in A$. An element a can be written as a linear combination of the basis elements in A. Therefore we can give a as

$$a = \sum d_{\lambda' s' t'} C^{\lambda'}_{s' t'},$$

where $d_{\lambda' s' t'}$ is some scalar in $\mathbb{C}$. If we find $C^\lambda_{st} a$ we will get

$$\begin{aligned} C^\lambda_{st} a &= C^\lambda_{st} \sum d_{\lambda' s' t'} C^{\lambda'}_{s' t'} \\ &= \sum d_{\lambda' s' t'} C^\lambda_{st} C^{\lambda'}_{s' t'}. \end{aligned}$$

Now we analyze the possible values of $C^\lambda_{st} C^{\lambda'}_{s' t'}$

Case (i) $\lambda = \lambda'$.

In this situation if the colour sequences match and the number of propagating lines does not change then

$$C^\lambda_{st} C^{\lambda'}_{s' t'} = r C^\lambda_{st'},$$

where $r = 1$ or a monomial in $\delta_{C_1} \ldots, \delta_{C_h}$. This r is not dependent on s, since any loops are not dependent on s. If the colours in t and s' did not match up then the product will be zero.

If the colour sequences match and the number of propagating lines does change, say to λ'', then we get less propagating lines than λ. Therefore, λ'' greater than λ and

$$C^\lambda_{st} C^{\lambda'}_{s' t'} = r'' C^{\lambda''}_{s'' t''} \in \check{A}^\lambda.$$

Case (ii) $\lambda \neq \lambda'$.

In this situation two things can happen. If the colors do not match up then

$$C^\lambda_{st} C^{\lambda'}_{s' t'} = 0.$$

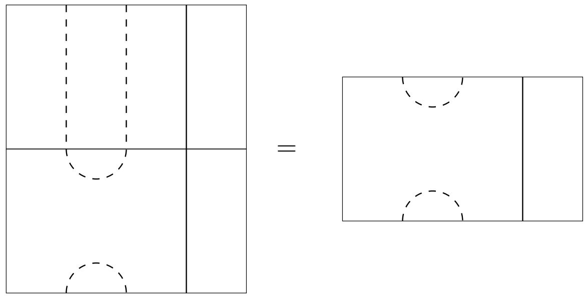


Figure 1.26:

On the other hand, if the colours match up let us denote by λ'' the number of propagating lines in the product. In this situation we get less than or equal to λ propagating lines. Therefore, according to our order in (1.3.1) we can say $\lambda'' > \lambda$ or $\lambda'' = \lambda$.

When $\lambda'' > \lambda$ we can say

$$C^{\lambda}_{st}C^{\lambda'}_{s't'} = r''C^{\lambda''}_{s''t''} \in \breve{A}^{\lambda}.$$

When $\lambda'' = \lambda$ we have by the same argument as in case (i) that

$$C^{\lambda}_{st}C^{\lambda'}_{s't'} = rC^{\lambda}_{st'},$$

From case(i) and case (ii) we can say

$$\begin{aligned} C^{\lambda}_{st}a &= \sum rC^{\lambda}_{st'} + \sum r''C^{\lambda''}_{s''t''} \\ &= \sum rC^{\lambda}_{st'} \text{ mod } \breve{A}^{\lambda} \end{aligned}$$

Hence we have shown that $(\mathcal{C}, \Lambda)$ is a cellular basis of A, and so $A = TL^h_n(\delta_1, \ldots, \delta_h)$ is a cellular algebra. $\square$

Example 1.3.3. From the above proof we know that $TL^2_3(\delta_R, \delta_G)$ is a cellular algebra. We begin by enumerating the basis elements.

λ	Half diagrams in $T(\lambda)$
$(3,0)$	Figure 1.27
$(0,3)$	Figure 1.28
$(2,1)$	Figure 1.29
$(1,2)$	Figure 1.30
$(1,0)$	Figure 1.31
$(0,1)$	Figure 1.32

Table 1.1:

Let Λ have elements as ordered pairs with first part denoting the number of red propagating lines and second part denoting the number of green propagating lines. If we take any half-diagram in a diagram in TL_3^2, it can either have all propagating lines or one propagating line and one arc. If we take $\lambda \in \Lambda$ it will be of the form

$$\lambda = (m, n),$$

where m denotes the number of red propagating lines and n denotes the number of green propagating lines. Therefore, we can say

$$m + n = 3 \text{ or } 1.$$

From this we can say

$$\Lambda = \{(3,0), (2,1), (1,0), (0,3), (1,2), (0,1)\}.$$

For each $\lambda \in \Lambda$ the indexing set $T(\lambda)$ gives half diagrams with m being the number of red propagating lines and n being the number of green propagating lines. Let us find the half diagrams for each value of λ. We illustrate these in the Figures indicated in Table 1.1.

The order in Λ is

$$(a, b) \geq (c, d) \text{ if and only if } a \leq c \text{ and } b \leq d.$$

Figure 1.33 shows the order in Λ.

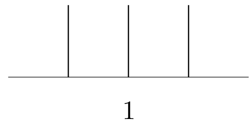


Figure 1.27: Half diagram in $T((3,0))$

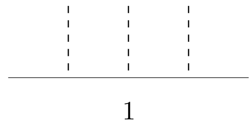


Figure 1.28: Half diagram in $T((0,3))$

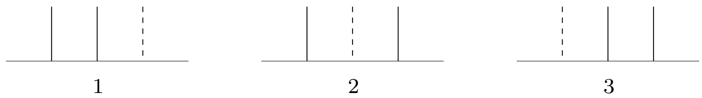


Figure 1.29: Half diagrams in $T((2,1))$

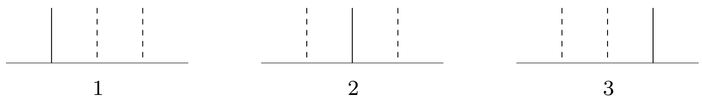


Figure 1.30: Half diagrams in $T((1,2))$

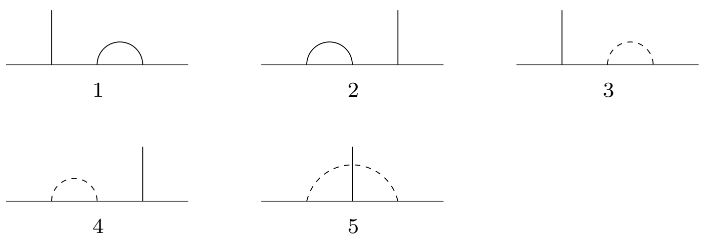


Figure 1.31: Half diagrams in $T((1,0))$

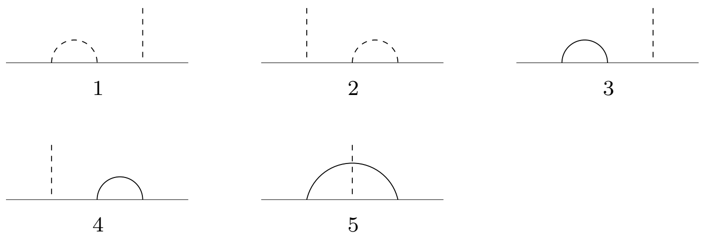


Figure 1.32: Half diagrams in $T((0,1))$

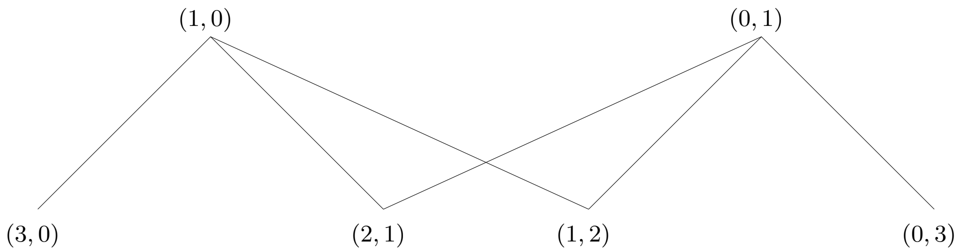


Figure 1.33: Partial order in Λ

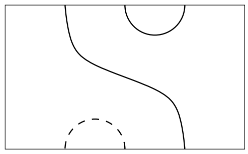


Figure 1.34: Basis element of $TL_3^2(\delta_R, \delta_G)$

In general this order is very important to show TL_3^2 is a cellular algebra even though it is not important in this example.

Let us pick the half diagrams denoted by 1 and 4 from $T((1,0))$. Draw the half diagram 1 at the northern edge and 4 at the southern edge of the rectangular box. We will get the diagram in Figure 1.34. This is a basis element of $TL_3^2(\delta_R, \delta_G)$. We name this $C_{14}^{(1,0)}$.

Example 1.3.4. Let algebra A be $TL_3^2(\delta_R, \delta_G)$. We find the basis elements of $C_3^{(1,0)}$. This is an A-submodule of $A^{(2,1)}/\breve{A}^{(2,1)}$. Half diagrams in $T((1,0))$ are in Figure 1.31. There-

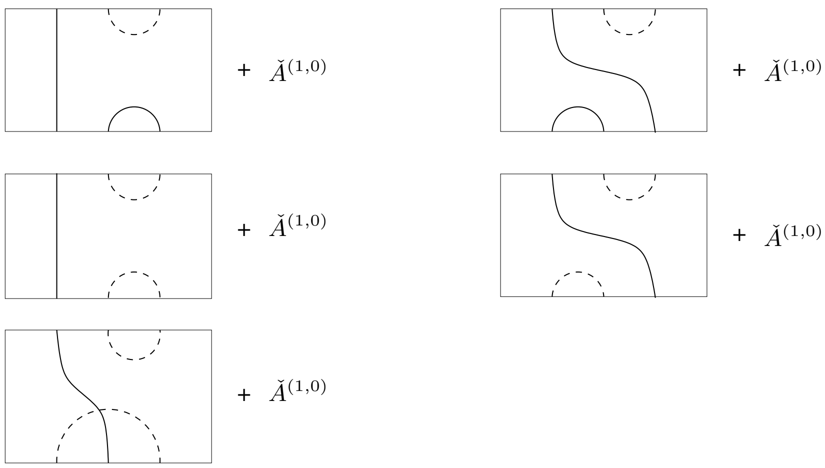


Figure 1.35:

fore, basis of $C_3^{(1,0)}$ are $C_{3t}^{(1,0)} + \breve{A}^{(1,0)}$ where $t \in T((1,0))$. These basis are given by the

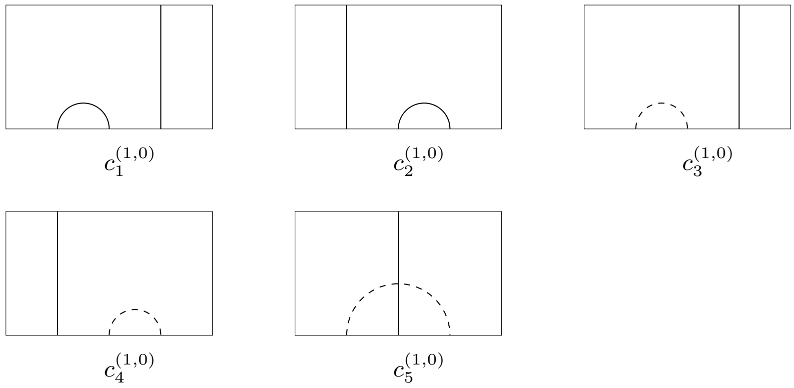


Figure 1.36:

Figure 1.35. (We normally ignore the final term $\check{A}^{(1,0)}$.) If we look at each diagram, all have the same northern edge half diagram.

Basis elements of the cell module $C^{(1,0)}$ are $C_1^{(1,0)}$, $C_2^{(1,0)}$, $C_3^{(1,0)}$, $C_4^{(1,0)}$ and $C_5^{(1,0)}$ which are in Figure 1.36.

Let see what will happen if we multiply $C_4^{(1,0)}$ in Figure 1.36 by the algebra element a from TL_3^2 which is illustrated in the Figure 1.37. From this we can say

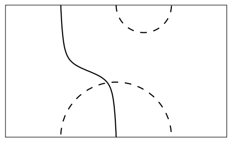
$$C_4^{(1,0)}a = \delta_G c_5^{(1,0)}.$$

Example 1.3.5. Let algebra A be $TL_3^2(\delta_R, \delta_G)$ and consider the cell module $C^{(1,0)}$. Now we find the inner product between the basis elements of the cell module. Figure 1.38 shows the half diagrams of the basis element of the cell module $C^{(1,0)}$. Basis of $C^{(1,0)}$ is given by

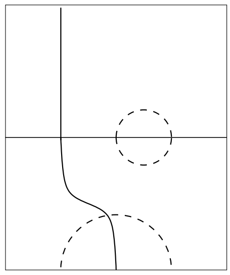
$$C^{(1,0)} = \{C_1^{(1,0)}, C_2^{(1,0)}, C_3^{(1,0)}, C_4^{(1,0)}, C_5^{(1,0)}\}.$$

Let us apply the rule

$$\langle C_s^\lambda, C_t^\lambda \rangle C_{uv}^\lambda \equiv C_{us}^\lambda C_{tv}^\lambda \mod \check{A}^\lambda,$$

$a \;=$ 

When we multiply $c_4^{(1,0)}$ and a we will get

$c_4^{(1,0)}a \;=$ 

By replacing the loop by δ_G we can get

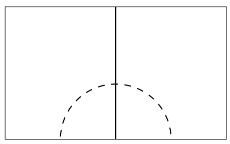
$= \delta_G$ 

Figure 1.37:

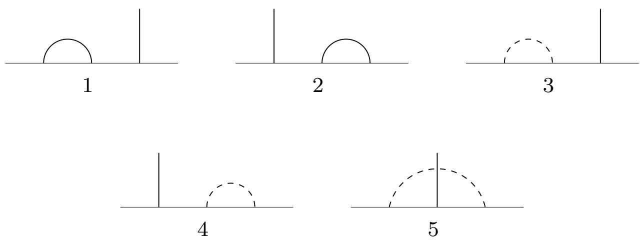


Figure 1.38: Half diagrams in $T((1,0))$

in (1.1.12) All the changes happen at the middle of the two diagrams. Half diagrams correspond to the basis in $< \, , \, >$ responsible for those changes. This observation makes the working out $< \, , \, >$ very easy.

$$
\begin{array}{ll}
< C_1^{(1,0)}, C_4^{(1,0)} > = 0 & < C_3^{(1,0)}, C_3^{(1,0)} > = \delta_G \\
< C_1^{(1,0)}, C_5^{(1,0)} > = 0 & < C_3^{(1,0)}, C_4^{(1,0)} > = 0 \\
< C_2^{(1,0)}, C_2^{(1,0)} > = \delta_R & < C_3^{(1,0)}, C_5^{(1,0)} > = 0 \\
< C_2^{(1,0)}, C_3^{(1,0)} > = 0 & < C_4^{(1,0)}, C_4^{(1,0)} > = \delta_G \\
< C_2^{(1,0)}, C_4^{(1,0)} > = 0 & < C_4^{(1,0)}, C_5^{(1,0)} > = 0 \\
< C_2^{(1,0)}, C_5^{(1,0)} > = 0 & < C_5^{(1,0)}, C_5^{(1,0)} > = \delta_G
\end{array}
$$

Example 1.3.6. Let us find rad$C^{(1,0)}$

$$\text{rad}C^{(1,0)} = \{ \, x \in C^{(1,0)} :< x, y >= 0 \text{ for all } y \in C^{(1,0)} \, \}.$$

Let x and y be elements in $C^{(1,0)}$. Therefore, we can write x and y as linear combinations of the basis elements of $C^{(1,0)}$. That is

$$x = \alpha_1 C_1^{(1,0)} + \alpha_2 C_2^{(1,0)} + \alpha_3 C_3^{(1,0)} + \alpha_4 C_4^{(1,0)} + \alpha_5 C_5^{(1,0)},$$

$$y = \beta_1 C_1^{(1,0)} + \beta_2 C_2^{(1,0)} + \beta_3 C_3^{(1,0)} + \beta_4 C_4^{(1,0)} + \beta_5 C_5^{(1,0)}.$$

Suppose $< x, y >= 0$ for all y. By substituting for x and y we obtain the following.

$$\alpha_1 \delta_R + \alpha_2 = 0 \tag{1.3.4}$$

$$\alpha_1 + \alpha_2 \delta_R = 0 \tag{1.3.5}$$

$$\alpha_3 \delta_G = 0 \tag{1.3.6}$$

$$\alpha_4 \delta_G = 0 \tag{1.3.7}$$

$$\alpha_5 \delta_G = 0 \tag{1.3.8}$$

If we solve (1.3.4) and (1.3.5) we obtain

$$\alpha_1(\delta_R^2 - 1) = 0. \tag{1.3.9}$$

When $\delta_R \neq \pm 1$ and $\delta_G \neq 0$ then equations (1.3.4), (1.3.5), (1.3.6), (1.3.7), (1.3.8) and (1.3.9) imply

$$\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = 0.$$

Therefore,

$$\text{rad } C^{(1,0)} = \{\underline{0}\}.$$

From this we can say $C^{(1,0)}$ is a simple module for almost every value of δ.

When $\delta_R = \pm 1$ and $\delta_G \neq 0$ we have

$$\text{rad } C^{(1,0)} = \{\alpha_1(C_1^{(1,0)} \mp C_2^{(1,0)}) | \alpha_1 \in \mathbb{C}\}.$$

This is a one dimensional vector space with basis $C_1^{(1,0)} \mp C_2^{(1,0)}$.

When $\delta_R \neq \pm 1$ and $\delta_G = 0$ we have

$$\text{rad } C^{(1,0)} = \{\alpha_3 C_3^{(1,0)} + \alpha_4 C_4^{(1,0)} + \alpha_5 C_5^{(1,0)} | \alpha_3, \alpha_4, \alpha_5 \in \mathbb{C}\}.$$

This is a three dimensional vector space with basis $C_3^{(1,0)}$, $C_4^{(1,0)}$ and $C_5^{(1,0)}$.

When $\delta_R = \pm 1$ and $\delta_G = 0$ we have

$$\text{rad } C^{(1,0)} = \{\alpha_1(C_1^{(1,0)} \mp C_2^{(1,0)}) + \alpha_3 C_3^{(1,0)} + \alpha_4 C_4^{(1,0)} + \alpha_5 C_5^{(1,0)} | \alpha_1, \alpha_3, \alpha_4, \alpha_5 \in \mathbb{C}\}.$$

This is a four dimensional vector space with basis $C_1^{(1,0)} \mp C_2^{(1,0)}, C_3^{(1,0)}, C_4^{(1,0)}$and $C_5^{(1,0)}$. The module $C^{(1,0)}$ is not a simple module in the last three cases.

If we repeat this calculation for $C^{(0,1)}, C^{(2,1)}, C^{(1,2)}, C^{(3,0)}$ and $C^{(0,3)}$ we obtain the following results:

$C^{(1,0)}$ is a simple module if $\delta_R \neq \pm 1$ and $\delta_G \neq 0$.

$C^{(0,1)}$ is a simple module if $\delta_G \neq \pm 1$ and $\delta_R \neq 0$.

$C^{(2,1)}, C^{(1,2)}, C^{(3,0)}$ and $C^{(0,3)}$ are simple modules for all δ_R, δ_G.

Chapter 2

Towers of recollement

In this chapter we are going to discuss an axiomatic framework for studying the representation theory of towers of algebras. This has been introduced by Cox, Martin, Parker and Xi [13] in 2006. They introduced a new class of algebras called contour algebras, and proved that they satisfy the axiomatic framework of towers of recollement. Brauer and walled Brauer algebras also form towers of recollement [10].

Let A_n (with $n \in \mathbb{N}$) be a family of finite dimensional algebras, with idempotents e_n in A_n, defined over an algebraically closed field K. Such a family of algebras which satisfies the axioms $(A1)$ to $(A6)$ which we are going to discuss soon is called a tower of recollement.

If a family of algebras is a tower of recollement, then we can apply Theorem 2.1.27 which we are going to discuss later in this chapter. This Theorem helps us to know whether we have a non-zero homomorphism between two standard modules by reducing to the case when one is simple.

2.1 Bubble algebras satisfies the axiomatic framework

We are going to show that our family of algebras TL_n^h (with $n \in \mathbb{N}$) is a tower of recollement when all of the δ_{C_i} are non-zero. We introduce each axiom for a tower of recollement followed by the proof that the bubble algebra satisfies it.

In chapters 4 and 6, we will discuss how to find the non-zero homomorphisms (if they exist) between two given cell modules where the first has no arcs. If we consider two modules with the first having some arcs, then by using the Theorem 2.1.27, we can reduce the size of the modules until the first module has no arcs in it. Therefore, it is enough to consider whether a homomorphism between the given two modules exists or not in this special case.

Let $A_n = TL_n^h(\delta_{C_1}, \ldots, \delta_{C_h})$. We want to define idempotents e_n in A_n. Let e_n be the sum over all possible colourings of strings and arcs (with the two arcs coloured the same) of diagrams as in Figure 2.1, where each such diagram is multiplied by the scalar $\frac{1}{\delta_{C_i}}$ which corresponds to the colour of the arc at the northern edge and southern edge of the diagram. As the arcs at the northern edge and southern edge of the diagram are the same in colour they can be coloured in h ways, and the rest of the $n-2$ lines can each be coloured by h different colours. Therefore, e_n is a sum of h^{n-1} diagrams.

To be an idempotent element e_n should satisfy the condition

$$e_n^2 = e_n.$$

When we find $e_n \times e_n$, we get zero for the southern edge colour sequence of the diagrams comes from the first e_n, and the northern edge colour sequence of the diagram come from the second e_n do not match. However, we get a loop at the middle of the diagram if the southern edge colour sequence of the diagram coming from the first e_n and the northern edge colour sequence of the diagram coming from the second e_n are the same. For this reason we defined e_n as the sum of diagrams as in Figure 2.1 multiplied by the scalar $\frac{1}{\delta_{C_i}}$, as this gives the desired idempotent property.

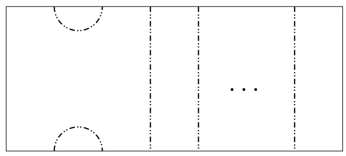


Figure 2.1:

We will show our algebras $A_n = TL_n^h$ satisfies the tower of recollement axioms (A1) to (A6) in [13].

Definition 2.1.1. Let Λ_n be an indexing set for the simple A_n-modules and Λ^n be an indexing set for the simple $A_n/A_ne_nA_n$-modules.

For any algebra A with idempotent $e \in A$ we have the following Theorem.

Theorem 2.1.2. *(Green [22]) Let $\{L(\lambda) : \lambda \in \Lambda\}$ be a full set of simple A-modules, and set $\Lambda^e = \{\lambda \in \Lambda : L(\lambda)e \neq 0\}$. Then $\{L(\lambda)e : \lambda \in \Lambda^e\}$ is a full set of simple eAe-modules. Further, the simple modules $L(\lambda)$ with $\lambda \in \Lambda \setminus \Lambda^e$ are a full set of simple A/AeA-modules.*

If we take the algebra A to be A_n, the indexing set Λ to be Λ_n and the idempotent e to be e_n in this Theorem, then Λ^e is Λ_{n-2} and Λ^n is $\Lambda_n \setminus \Lambda_{n-2}$.

Axiom 2.1.3. *(A1) For each $n \geq 2$ we have an isomorphism*

$$\phi_n : A_{n-2} \to e_nA_ne_n.$$

Algebra satisfies the first Axiom (A1)

Let us prove our TL_n^h satisfies the Axiom 2.1.3. It is convenient to define the linear map from $e_n TL_n^h e_n$ to TL_{n-2}^h and show that the map is an isomorphism.

Define a linear map

We will define the linear map

$$\theta : e_n TL_n^h e_n \to TL_{n-2}^h$$

on diagrams as in Figure 2.2. Here D is a diagram in TL_n^h. By removing the arc at the

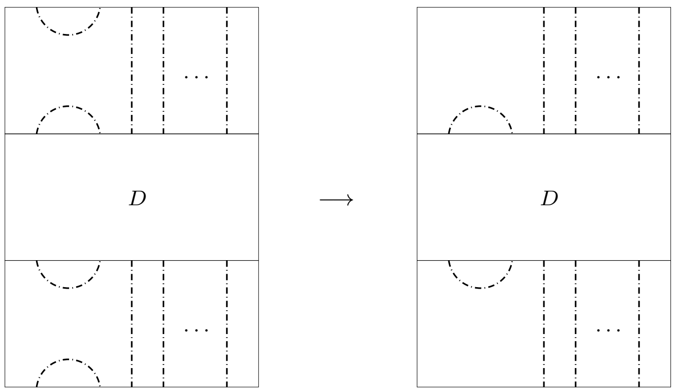


Figure 2.2:

northern edge of e_n and southern edge of e_n at the left-hand side of the diagram in $e_n TL_n^h e_n$; we map to the right-hand side diagram in TL_{n-2}^h.

As we have shown in Figure 2.3, $e_n D e_n$ is the sum of diagrams over the colour of arcs and propagating lines of e_n at the northern edge and southern edge of D. Each e_n is the sum of h^{n-1} diagrams. Therefore, $e_n D e_n$ is a sum of $h^{2(n-1)}$ diagrams. However, most of the diagrams do not survive because the northern edge and southern edge colour sequence of D do not match with the e_n in the northern edge and southern edge of D. If we apply θ to the diagram on the left-hand side of the Figure 2.3 we are left with only one diagram as in the right-hand side of Figure 2.3. Therefore, the linear map θ is actually a map from $h^{2(n-1)}$ diagrams to a single diagram.

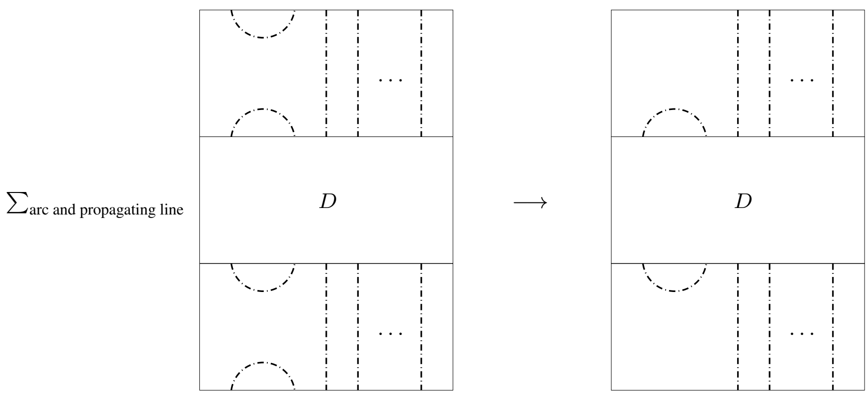


Figure 2.3:

Map is a homomorphism

Now we will show that θ is a homomorphism. We need to check the condition

$$\theta(e_n D e_n)\theta(e_n E e_n) = \theta(e_n D e_n e_n E e_n).$$

However, we can simplify the bit inside the right hand side as

$$e_n D e_n e_n E e_n = e_n D e_n E e_n.$$

Therefore, it is enough to check

$$\theta(e_n D e_n)\theta(e_n E e_n) = \theta(e_n D e_n E e_n).$$

$e_nDe_n =$

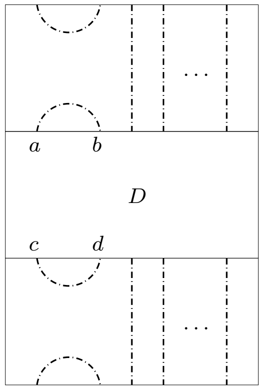


Figure 2.4:

$e_nEe_n =$

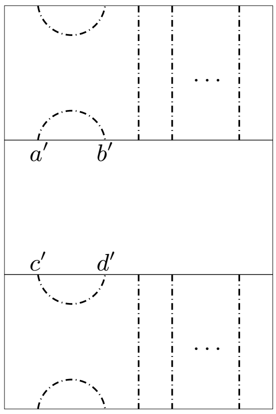


Figure 2.5:

Let e_nDe_n and e_nEe_n be the diagrams in $e_nTL_n^he_n$. Here D and E are the diagrams in TL_n^h. If we consider the diagram e_nDe_n, it will be of the form as in Figure 2.4. Here a, b, c and d are the positions of the nodes of D and e_nDe_n is a sum of $h^{2(n-1)}$ diagrams. According to the Figure a and b are connected by the arc coming from the southern edge of e_n and c and d are connected by the arc coming from the northern edge of e_n. We get e_nDe_n equals 0 if the colour of nodes a and b differ or the colour of nodes c and d differ. If the colour of the nodes a and b are the same and c and d are the same then e_nDe_n is a sum of $h^{2(n-1)}$ diagrams. However, only one of the diagrams in this sum will survive and the others will become zero because the colour sequence of northern edge and southern edge of D do not match with the colour sequence of e_n. Similarly, e_nEe_n can be given by the Figure 2.5. Here a', b', c' and d' are the positions of the nodes, and arguments as for e_nDe_n above also apply.

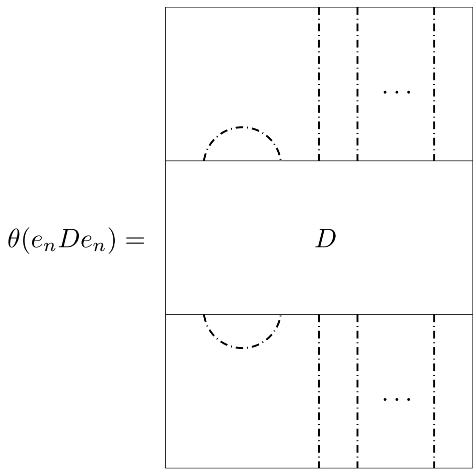


Figure 2.6:

Now we will find $\theta(e_nDe_n)$ and $\theta(e_nEe_n)$. These are given by Figures 2.6 and Figure 2.7. However, $\theta(e_nDe_n)$ is 0 if e_nDe_n is 0. Similarly, $\theta(e_nEe_n)$ is 0 if e_nEe_n is 0. If

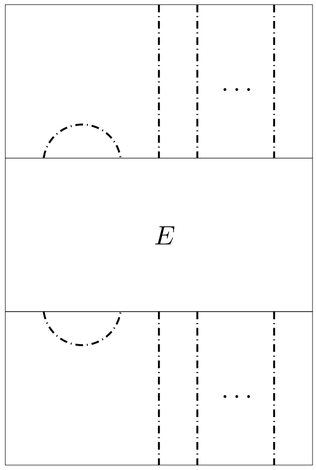
$\theta(e_n E e_n) =$ 

Figure 2.7:

we find $\theta(e_n D e_n)\theta(e_n E e_n)$ we will get Figure 2.8. This may be zero if the southern edge of $e_n D e_n$ and northern edge of $e_n E e_n$ do not match or $e_n D e_n$ is 0 or $e_n E e_n$ is 0. We know the idempotent element satisfies the condition

$$e_n^2 = e_n.$$

Therefore, we can say

$$e_n D e_n . e_n E e_n = e_n D e_n E e_n.$$

The diagram of $e_n D e_n E e_n$ is given by Figure 2.9. This is actually the sum of diagrams over the colour of an arc and the propagating lines of e_n. However, this may be zero if the bottom of D and the northern edge of E do not match. In this situation, the southern edge of $e_n D e_n$ and the northern edge of $e_n E e_n$ do not match. If they do match only one diagram will survive in the sum. Therefore $\theta(e_n D e_n E e_n)$ is given by 0 or by the Figure 2.10. From these we can say

$$\theta(e_n D e_n)\theta(e_n E e_n) = \theta(e_n D e_n E e_n)$$

$\theta(e_n D e_n)\theta(e_n E e_n) =$

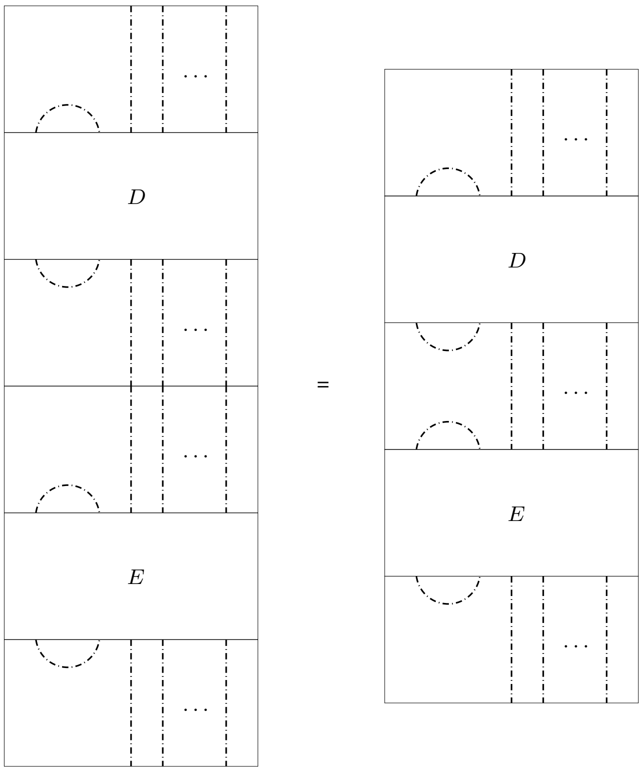


Figure 2.8:

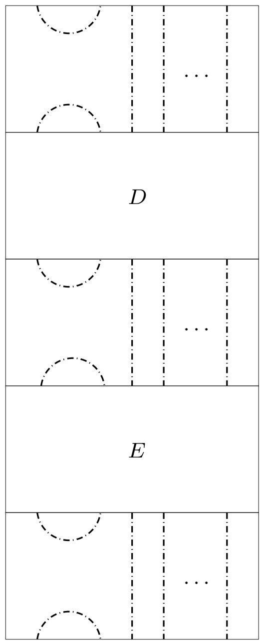
$e_n D e_n E e_n =$ 

Figure 2.9:

because $\theta(e_nDe_n)\theta(e_nEe_n)$ and $\theta(e_nDe_nEe_n)$ both give ous 0, when southern edge of e_nDe_n and northern edge of e_nEe_n do not match, or are the same diagram as in Figure 2.8 and Figure 2.10. Therefore θ is a homomorphism.

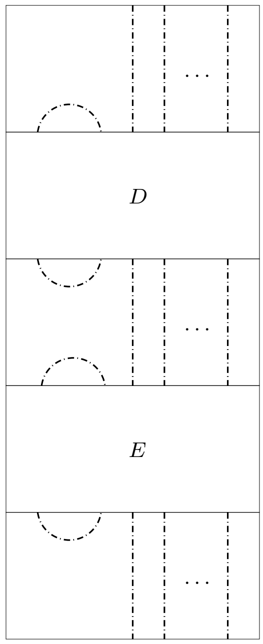
$\theta(e_nDe_nEe_n) =$ 

Figure 2.10:

Map is injective

Now we will show θ is injective. Suppose that

$$\theta(e_nDe_n) = \theta(e_nEe_n).$$

This is given by Figure 2.11. For θ to be injective we require that

$$e_nDe_n = e_nEe_n$$

as in Figure 2.12. This is obvious because if the diagrams in Figure 2.12 are different we

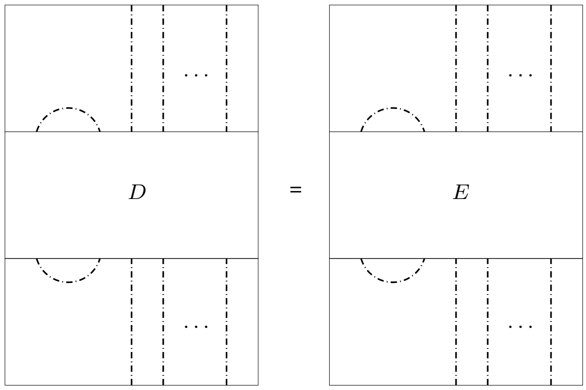


Figure 2.11:

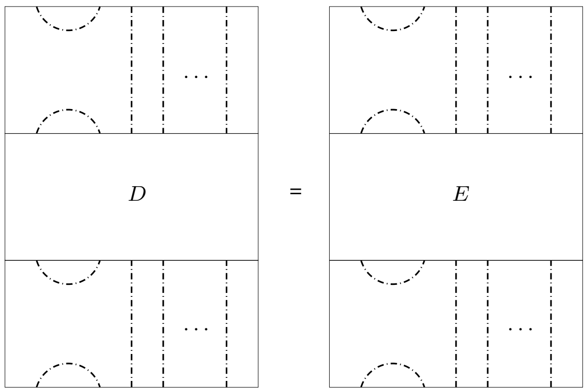


Figure 2.12:

can not get the diagram in Figure 2.11. From this we can say θ is injective.

Map is surjective

Now we will show that θ is surjective. Suppose $X \in TL^h_{n-2}$. We want an element e_nYe_n of $e_nTL^h_ne_n$, where $Y \in TL^h_n$ such that

$$\theta(e_nYe_n) = X.$$

We will see how to make $Y \in TL^2_7$ from $X \in TL^2_5$ in the following cases. If X as in

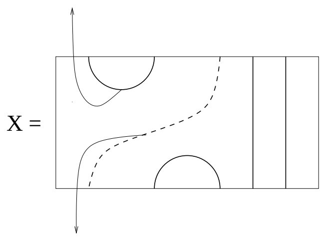


Figure 2.13:

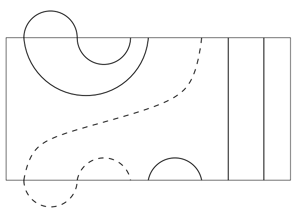


Figure 2.14:

Figure 2.13 pull the arc and the propagating line in the direction of the arrow. We will get the Figure 2.14. Diagram inside the box is Y. If X is of the form as in Figure 2.15 then pull the line in the shown direction. We will get Figure 2.16. In this case, the diagram inside the box is Y. If we look at Figure 2.17, the diagram on the left-hand side of Figure X can be drawn as the multiplication of 3 diagrams in the middle of the Figure. If we call

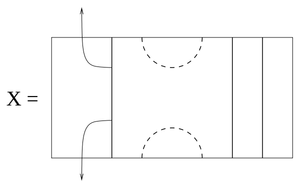


Figure 2.15:

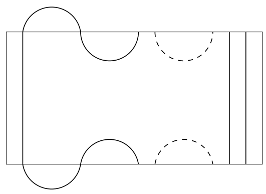


Figure 2.16:

the middle diagram Y we can say

$$\theta(e_7 Y e_7) = X$$

If we use this technique for any $X \in TL^h_{n-2}$ then there is an element $e_n Y e_n \in e_n TL^h_n e_n$, where $Y \in TL^h_n$ such that

$$\theta(e_n Y e_n) = X.$$

Therefore, θ is surjective.

We have shown θ is injective and surjective. Therefore, we have an isomorphism

$$TL^h_{n-2} \to e_n TL^h_n e_n.$$

Definition 2.1.4. Suppose that we have algebras A_n and idempotents e_n satisfying (A1). We define a pair of families of functors

$$F_n : A_n\text{-mod} \to A_{n-2}\text{-mod}$$

$$M \mapsto Me_n$$

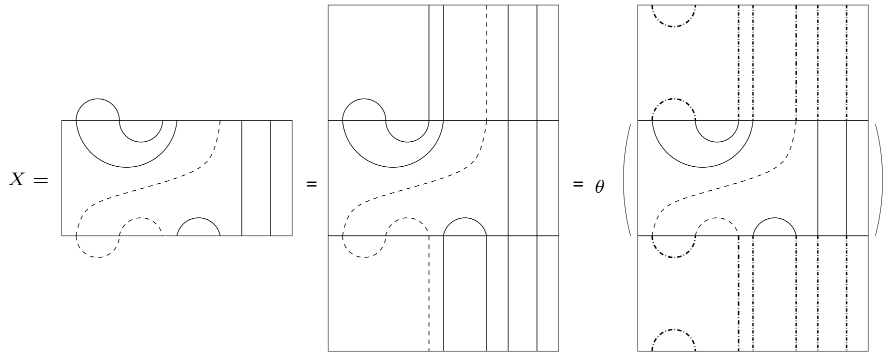


Figure 2.17:

and

$$G_n : A_n\text{-mod} \to A_{n+2}\text{-mod}$$
$$G_{n-2}(N) = N \otimes_{e_nA_ne_n} e_nA_n$$

via the isomorphism in 2.1.3. The right inverse to F_n is G_{n-2}.

From Axiom 2.1.3 and Theorem 2.1.2 we have

$$\Lambda_n = \Lambda^n \sqcup \Lambda_{n-2}. \tag{2.1.1}$$

If we find F_nG_{n-2} we will get

$$\begin{aligned} F_nG_{n-2}(N) &= N \otimes_{e_nA_ne_n} e_nA_ne_n \\ &= Ne_nA_ne_n \otimes_{e_nA_ne_n} 1 \\ &= N \otimes_{e_nA_ne_n} 1 \\ &\cong N. \end{aligned}$$

Cline, Parshall and Scott [9] use this idea to provide examples of recollement [5] in the context of quasi-heredity and highest weight categories.

Set $e_{n,0} = 1$ in A_n, and for $1 \leq i \leq n/2$ define new idempotents in A_n by setting

$$e_{n,i} = \phi_n(e_{n-2,i-1}).$$

There are associated quotients $A_{n,i} = A_n/(A_n e_{n,i+1} A_n)$.

Axiom 2.1.5. *(A2) (i) The algebra $A_n/A_n e_n A_n$ is semisimple.*
(ii) For each $n \geq 0$ and $0 \leq i \leq n/2$, setting $e = e_{n,i}$ and $A = A_{n,i}$ the surjective multiplication map $Ae \otimes_{eAe} eA \to AeA$ is a bijection.

Algebra satisfies the second Axiom (A2)

Let us prove that our TL_n^h satisfies the first statement of Axiom 2.1.5. To prove this we should know the following Claim.

Claim 2.1.6. $TL_n^h e_n TL_n^h$ has as a basis all diagrams with at most $n-2$ propagating lines.

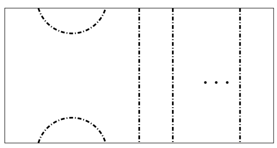


Figure 2.18:

For simplicity we just represent e_n by the Figure 2.18. Let us prove the Claim.

Proof. Suppose y has $(n-2)$ propagating lines or fewer. From this we can say it has at least one arc on the northern edge and one arc on the southern edge as in Figure 2.19. This Figure we choose for y to cover all the possible cases. If the first two nodes from the western edge are in an arc we can use that to construct the middle diagram. Other arcs (not in the first two nodes) will not come in the construction of the middle diagram. We can

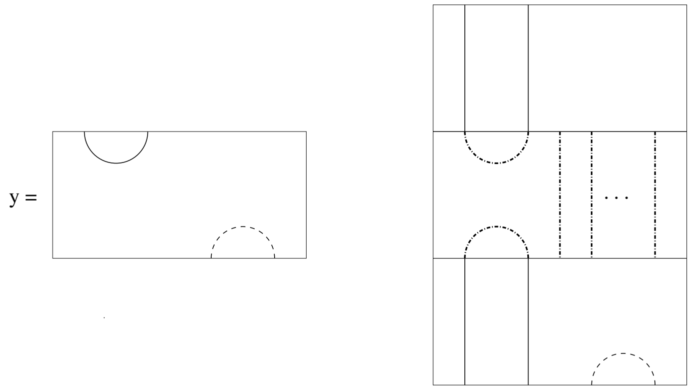


Figure 2.19:

see this in Figure 2.19. We can get the first diagram in the left-hand side of Figure 2.19 by doing the concatenation of the three diagrams as in the right-hand side of the Figure 2.19. The middle diagram is one of the diagrams of e_n, and we call the diagram above D_1 and below D_2. Both diagrams are in TL_n^h. When we do the concatenation $D_1e_nD_2$, only one diagram in e_n gives the above diagram. Others give zero because the colours did not match up. Therefore, we can say

$$y = D_1e_nD_2 \in TL_n^h e_n TL_n^h.$$

From this we can say that any diagram y with $n-2$ lines or fewer will be in $TL_n^h e_n TL_n^h$.

It is obvious that diagrams in $TL_n^h e_n TL_n^h$ has at most $n-2$ lines or fewer because the middle e_n does not allow to have diagram with all lines propagating. This proves the converse. $\square$

Now we show that algebra TL_n^h satisfies the first statement of the Axiom $(A2)$. According to Claim 2.1.6 we can say that all diagrams in $TL_n^h e_n TL_n^h$ have at most $n-2$ propagating lines. Therefore, the quotient of TL_n^h by $TL_n^h e_n TL_n^h$ has a basis of diagrams containing only propagating lines. However, this is isomorphic to the semisimple algebra

$K^{(h^n)}$. Therefore $TL_n^h/TL_n^h e_n TL_n^h$ is semisimple.

The proof of the second part of Axiom (A2) is now very similar to the corresponding proof for the contour algebras in [13, Proposition 2.10].

Definition 2.1.7. For $m \leq n$ we define Λ_n^m to be those elements in Λ_n which first appeared in the indexing set Λ^m.

Example 2.1.8. If we take the algebra as TL_4^2 then the indexing set Λ_4 and the Λ_4^2 are given by

$$\Lambda_4 = \{(4,0),(2,2),(0,4),(2,0),(1,1),(0,2),(0,0)\},$$
$$\Lambda_4^2 = \{(2,0),(1,1),(0,2)\}.$$

Example 2.1.9. For our algebra TL_n^h the indexing sets Λ_n and Λ_n^m are given by

$$\Lambda_n = \{(a_1,a_2,\ldots,a_h) : 0 \leq a_i \leq n, 1 \leq i \leq h, a_1+\ldots+a_h = n-2t, 0 \leq t \leq \frac{n}{2}\},$$
$$\Lambda_n^m = \{(a_1,a_2,\ldots,a_h) : 0 \leq a_i \leq n, 1 \leq i \leq h, a_1+\ldots+a_h = m\}.$$

The following Axiom is equivalent to Axiom $(A2)$ by [16, Statement 7] or [51, Definition 3.3.1 and the remarks following].

Axiom 2.1.10. *(A2') For each $n \in \mathbb{N}$ the algebra A_n is quasi-hereditary, with heredity chain of the form*

$$0 \subset \cdots \subset A_n e_{n,i} A_n \subset \cdots \subset A_n e_{n,0} A_n = A_n.$$

As A_n is quasi-hereditary, there is for each $\lambda \in \Lambda_n$ a standard module $\Delta_n(\lambda)$, with simple head $L_n(\lambda)$. If we set $\lambda \leq \mu$ if either $\lambda = \mu$ or $\lambda \in \Lambda_n^r$ and $\mu \in \Lambda_n^s$ with $r > s$, then all other composition factors of $\Delta_n(\lambda)$ are labeled by weights μ with $\mu < \lambda$. Note that for

$\lambda \in \Lambda_n^n$, we have that $\Delta_n(\lambda) \cong L_n(\lambda)$, and that this is just the lift of a simple module for the quotient algebra $A_n/A_n e_n A_n$. We have [48, Proposition 3] that

$$G_n(\Delta_n(\lambda)) \cong \Delta_{n+2}(\lambda). \tag{2.1.2}$$

Similarly (see for example [18, A1]) we have

$$F_n(\Delta_n(\lambda)) \cong \begin{cases} \Delta_{n-2}(\lambda) & \text{if } \lambda \in \Lambda_{n-2}, \\ 0 & \text{if } \lambda \in \Lambda^n. \end{cases} \tag{2.1.3}$$

If $\lambda \in \Lambda^n$ then the cell module defined in the previous chapter coincides with the standard module $\Delta_n(\lambda)$. We can find the necessary explanation to this in [14, Corollary C.36., notes C.6:]. Now by repeated use of (2.1.2) it is easy to verify that we have for all $\lambda \in \Lambda$ that

$$\Delta_n(\lambda) \cong C^\lambda.$$

Lemma 2.1.11.

$$\Delta_n(\lambda)e_n \cong \Delta_{n-2}(\lambda), \tag{2.1.4}$$

where $\lambda = (a_1, a_2, \ldots, a_h)$ *and* $\Delta_n(\lambda)$ *is a* $TL_n^h -$ *module and* $a_1 + a_2 + \ldots + a_h \leq n - 2$*.*

Proof. Let D be a diagram in $\Delta_n(\lambda)$. We define the linear map ϕ as in Figure 2.20. The left-hand side of the diagram in $\Delta_n(\lambda)e_n$ maps to the right-hand side in $\Delta_{n-2}(\lambda)$ by removing the arc at the southern edge of e_n in the left diagram.

Map ϕ is a module homomorphism

If we want to show ϕ is a module homomorphism, we need to show

$$\phi(De_ne_nae_n) = \phi(De_n)\theta(e_nae_n).$$

However, this proof is very similar to the proof we have just done to show that θ a homomorphism. Therefore, we ignore the proof here.

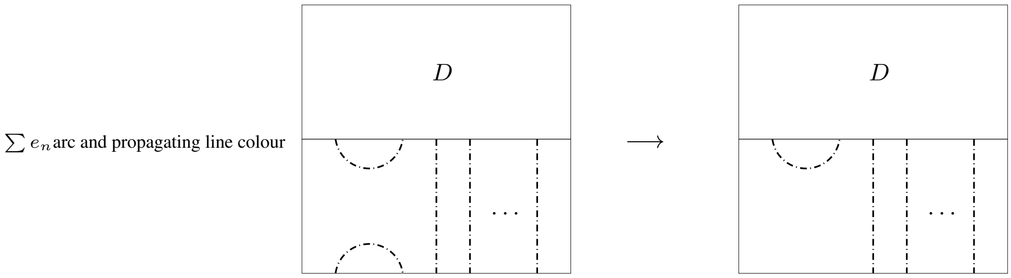


Figure 2.20:

ϕ is injective

Now we will show ϕ is injective. Suppose that

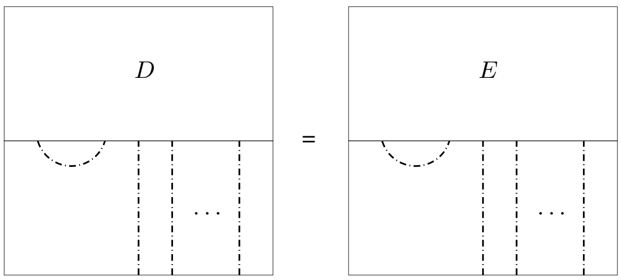


Figure 2.21:

$$\phi(Den) = \phi(Ee_n)$$

is given by Figure 2.21. For ϕ to be injective we need

$$De_n = Ee_n$$

as in Figure 2.22. This is obvious because if the diagram in Figure 2.22 are different we cannot get Figure 2.21.

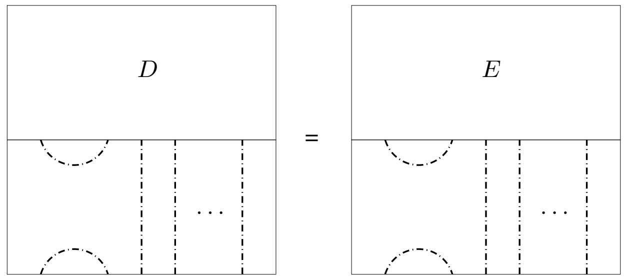


Figure 2.22:

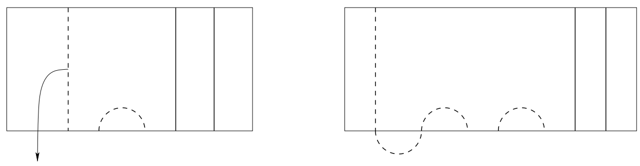


Figure 2.23:

ϕ is surjective

Now we will show that ϕ is surjective. Suppose that $X \in \Delta_{n-2}(\lambda)$; we want an element Ye_n of $\Delta_n(\lambda)e_n$ such that

$$\phi(Ye_n) = X$$

We will see how to make Y from $X \in TL_5^2$ in the following cases. If X is as in Fig-

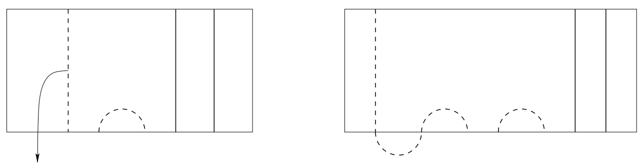


Figure 2.24:

ure 2.24, pull the green propagating line as in Figure to get the diagram on the right. The diagram inside the box is Y. If X as in Figure 2.25 pull the red arcs as in Figure to get the diagram on the right. In this case the diagram inside the box is Y. If we take our X as in

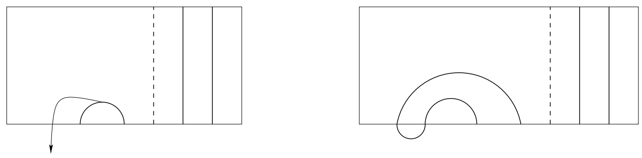


Figure 2.25:

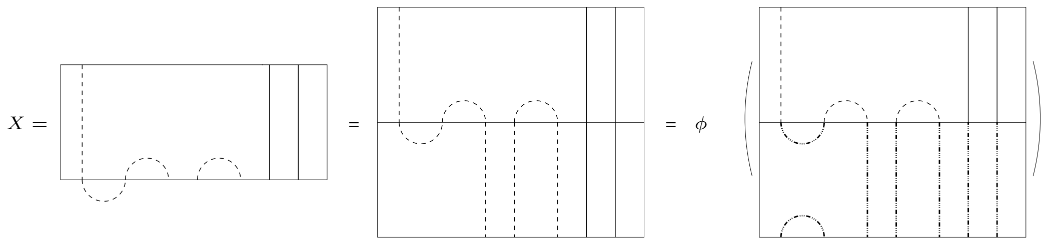


Figure 2.26:

Figure 2.24, then Figure 2.26 shows the way to write X as the multiplication of a diagram and a half-diagram. We call Y the top middle diagram. Let us choose same half diagram formed earlier for northern and southern edge which given an algebra element in e_n. We can replace for the algebra element by e_n as in far right bottom diagram in Figure 2.27 because except one diagram in e_n all the others give 0 when we multiply with Y. Therefore, we can say

$$\phi(Ye_n) = X.$$

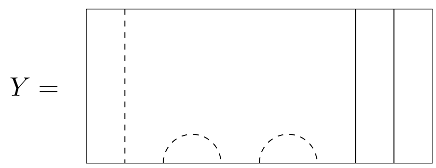


Figure 2.27:

By using this technique for any $X \in \Delta_n(\lambda)$ there is a $Ye_n \in \Delta_n(\lambda)e_n$ such that

$$\phi(Ye_n) = X.$$

Therefore, ϕ is surjective. We have shown that ϕ is injective and surjective and so

$$\Delta_n(\lambda)e_n \cong \Delta_{n-2}(\lambda).$$

□

Axiom 2.1.12. *(A3) For each $n \in \mathbb{N}$, the algebra A_n can be identified with a subalgebra of A_{n+1}.*

Algebra satisfies the third Axiom (A3)

Consider a diagram $D \in TL_n^h$. We add a propagating line to the front of D as in Figure 2.28. This notation represents the replacement of D by h copies of D each with a different coloured propagating line at the front. Now this is an element in TL_{n+1}^h. If we multiply two such elements we will get a third such element in TL_{n+1}^h. From this we can see that A_n can be identified with a subalgebra of A_{n+1}.

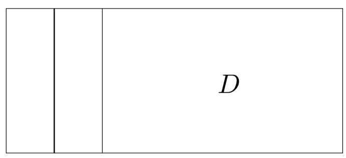


Figure 2.28:

Restriction of a module and induced module

Definition 2.1.13. Let B be a subalgebra of an algebra A. If V is an A-module, then V is also a B-module. This can be denoted by

$$\mathrm{res}_B V \ \text{ or } \ V \downarrow B$$

and is called the restriction of V to B.

Definition 2.1.14. Let B be a subalgebra of an algebra A. If V is a B-module then we denote by

$$\mathrm{ind}_A V \ \text{ or } \ V \uparrow A$$

the induced A-module.

These two operations are functorial. We have the restriction functor

$$\mathrm{res}_n : A_n\text{-mod} \to A_{n-1}\text{-mod}$$

and the induction functor

$$\mathrm{ind}_n : A_n\text{-mod} \to A_{n+1}\text{-mod}.$$

$$\mathrm{ind}_n(M) = M \otimes_{A_n} A_{n+1}.$$

Theorem 2.1.15. ***(Frobenius Reciprocity)*** *Assume that B is a subalgebra of an algebra A. Let U be a B-module and V be an A-module. Then we have*

$$\mathrm{Hom}_A(\mathrm{ind}_A U, V) \ \cong \ \mathrm{Hom}_B(U, \mathrm{res}_B V) \tag{2.1.5}$$

Axiom 2.1.16. *(A4) For all $n \geq 1$ we have that*

$$A_{n-1} \cong e_n A_n$$

as a left A_{n-2}-, right A_{n-1}-bimodule.

Note that the left action A_{n-2} on e_nA_n used here is given via the isomorphism in 2.1.3. We can immediately deduce from Axiom 2.1.16 that for each $\lambda \in \Lambda_n$ we have that

$$\mathrm{res}_{n+2}(G_n(\Delta_n(\lambda))) \cong \mathrm{ind}_n\Delta_n(\lambda). \tag{2.1.6}$$

Algebra satisfies the fourth Axiom

We next show that $A_n = TL_n^h$ satisfies Axiom 2.1.16. The left action of A_{n-2} on e_nA_n comes from the fact that $A_{n-2} \cong e_nA_ne_n$ via Axiom 2.1.3 and the right action of A_{n-1} on e_nA_n comes from restriction to A_{n-1} as a subalgebra of A_n. Similarly, the left action of A_{n-2} on A_{n-1} comes via restriction.

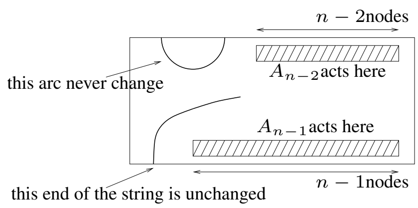


Figure 2.29:

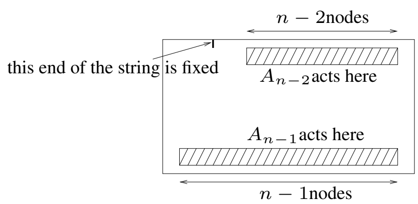


Figure 2.30:

In Figure 2.29, we illustrate a general diagram in e_nA_n. As we have shown, A_{n-2} acts at the northern edge of this diagram. Therefore, the arc on the northern left side of the edge never changes. Similarly, A_{n-1} acts at the southern edge of this diagram as shown. Therefore, the southern leftmost string will not be seen by A_{n-1} action.

If we look at Figure 2.30 that gives a diagram in A_{n-1}. As we have shown on the diagram, A_{n-2} acts at the northern edge. For this reason, the node at the northern edge left corner string will not be seen by A_{n-2} action. Southern edge of the diagram's nodes will be involved on the action of A_{n-1}.

Now we will define the linear map θ from e_nA_n to A_{n-1} as in Figure 2.31. The left-hand

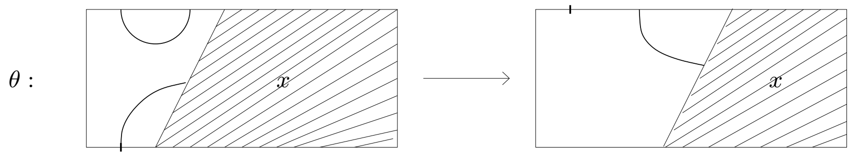


Figure 2.31:

side is a diagram in e_nA_n. We obtain the second diagram by deleting the arc at the northern edge of the first diagram and sliding the node at the southern edge of the first diagram to the northern edge of the diagram. Now the diagram obtained is a diagram in A_{n-1}.

We will show that θ is bijective. It is injective by the way it is defined. Let us prove θ is surjective. Let us take an arbitrary element $y \in A_{n-1}$. We will find an x in e_nA_n such that

$$\theta(x) = y.$$

We will obtain a diagram from y by sliding the top left-hand corner string to the left-hand corner of the southern edge. Now we have n nodes at the southern edge. At this point, the northern edge has $(n-2)$ nodes. We add a same colour arc to the northern edge as the sided string. Therefore at the northern edge we now have n nodes. This diagram is a diagram in

e_nA_n. If we take this as x then we will get

$$\theta(x) = y.$$

From this, we can say θ is surjective. We have shown θ is injective and surjective. Therefore θ is bijective.

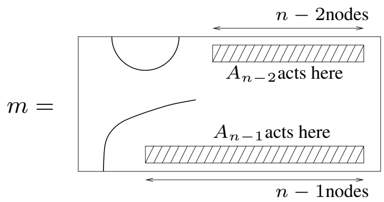


Figure 2.32:

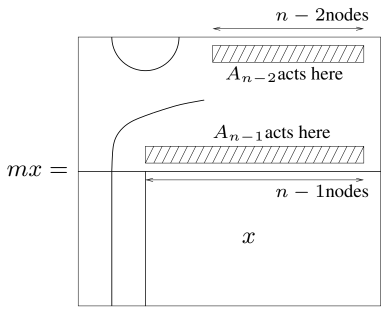


Figure 2.33:

We will show θ is a homomorphism from e_nA_n to A_{n-1} when the algebra A_{n-1} acts from the right. Let $m \in e_nA_n$ as in Figure 2.32. Let x be a diagram in A_{n-1}. If we find mx, the diagram will look like the Figure 2.33. If you see the bottom diagram, we have added an extra line to make a diagram in A_n. We did this to make the multiplication mx meaningful.

$\theta(mx) =$

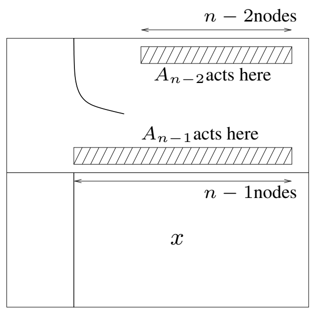


Figure 2.34:

$\theta(m)x =$

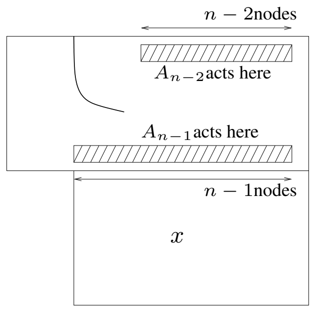


Figure 2.35:

If we calculate $\theta(mx)$, we will get Figure 2.34 because in mx the northern edge arc at m has been removed and the string at the southern edge of x has been slid up to the northern edge of m by the map θ. If we find $\theta(m)x$ we will get the Figure 2.35. Let us look at the reason. In $\theta(m)$, the arc at the northern edge of m has been removed and the string at the southern edge of m has been moved to the northern edge by the linear map θ. After that, when we apply x to that, we will get $\theta(m)x$. By looking at Figure 2.34 and Figure 2.35 we see that

$$\theta(mx) = \theta(m)x$$

as required. Now we will show θ is a homomorphism when the algebra elements act on the

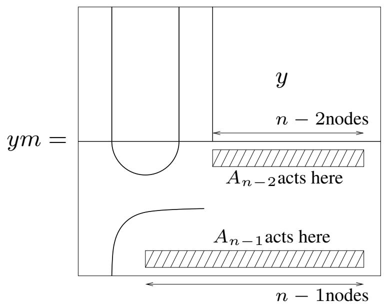


Figure 2.36:

left. Let y be a diagram in A_{n-2}. If we find ym we will get the diagram in Figure 2.36. In front of the northern and southern edge of diagram y we need to add two propagating lines to match the arc colour of the bottom diagram. If we find $\theta(ym)$ we will get Figure 2.37.

Now we will find $y\theta(m)$. This is given by Figure 2.38. We have added an extra propagating line in front of y to match the bottom string. By looking at Figure 2.37 and Figure 2.38 we see that

$$\theta(ym) = y\theta(m)$$

$\theta(ym) =$

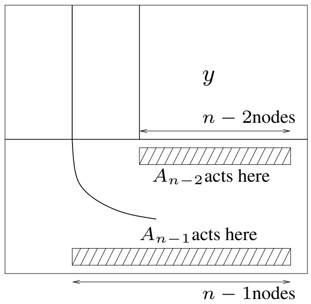


Figure 2.37:

$y\theta(m) =$

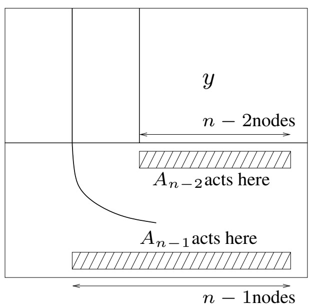


Figure 2.38:

as required.

Linear map θ is a homomorphism, when the algebra acts on the right or left, and bijective. From this we can say θ is an isomorphism. From this we have shown our algebra satisfies the Axiom $(A4)$.

Definition 2.1.17. A filtration of a module with successive quotients isomorphic to some $\Delta(\lambda)$s is called a Δ-filtration.

Definition 2.1.18. Let M in A_n have a Δ_n-filtration. The support of M is the set of labels λ for which $\Delta(\lambda)$ occurs in this filtration. It is denoted by $\mathrm{supp}_n(M)$.

Exact sequences and short exact sequences

We recall the definition of an exact sequence and short exact sequence as in [1].

Definition 2.1.19. Suppose A is an abelian category. Take an index set of consecutive integers. For each i in the index set, let A_i be an object in the category and let f_i be a morphism from A_i to A_{i+1}. This defines a sequence of objects and morphisms. The sequence is exact at A_i if the image of f_{i-1} is equal to the kernel of f_i. The sequence itself is exact if it is exact at each object except at the very first and very last object.

Definition 2.1.20.

A short exact sequence is an exact sequence of the form

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0 \tag{2.1.7}$$

By the above, we know that for any such short exact sequence, f is a monomorphism and g is an epimorphism. Furthermore, the image of f is equal to the kernel of g. It is helpful to think of A as a subobject of B with f being the embedding of A into B, and C

as the corresponding factor object B/A,with the map g being the natural projection from B to B/A (whose kernel is exactly A).

Axiom 2.1.21. *(A5) For each $\lambda \in \Lambda_n^m$ we have that res$(\Delta_n(\lambda))$ has a Δ-filtration and*

$$\mathrm{supp}(\mathrm{res}(\Delta_n(\lambda))) \subseteq \Lambda_{n-1}^{m-1} \sqcup \Lambda_{n-1}^{m+1}$$

Lets use the same argument as in [13] after the Axiom(A5). Equation (2.1.6) now implies the analogue of 2.1.21 for induction. Using (2.1.2) we deduce from Axiom 2.1.21 and 2.1.6 that for each $\lambda \in \Lambda_n^m$ the module ind$(\Delta_n(\lambda))$ has a Δ-filtration, and

$$\mathrm{supp}(\mathrm{ind}(\Delta_n(\lambda))) \subseteq \Lambda_{n+1}^{m-1} \sqcup \Lambda_{n+1}^{m+1} \tag{2.1.8}$$

Algebra satisfies the fifth Axiom (A5)

Let $\lambda = (t_1, t_2, \ldots, t_h) \in \Lambda_n^m$. From Claim 2.1.22 and Claim 2.1.23, that we are going to discuss soon, we will deduce Lemma 2.1.24 (which we are going to discuss after the two Claims). This Lemma says that we have a short exact sequence

$$\begin{aligned} 0 &\longrightarrow \oplus_{i=1}^h \Delta_{n-1}(t_1, t_2, \ldots, t_i - 1, \ldots, t_h) \\ &\longrightarrow \mathrm{res}_{n-1}^n \Delta_n(t_1, t_2, \ldots, t_i, \ldots, t_h) \\ &\longrightarrow \oplus_{i=1}^h \Delta_{n-1}(t_1, t_2, \ldots, t_i + 1, \ldots, t_h) \\ &\longrightarrow 0 \end{aligned}$$

where the modules in the sum $\oplus_{i=1}^h \Delta_{n-1}(t_1, t_2, \ldots, t_i - 1, \ldots, t_h)$ are taken to be zero when they are not defined. From this, we can say the support of res$(\Delta_n(\lambda))$ is going to be inside Λ_{n-1}^{m-1} and Λ_{n-1}^{m+1}. Therefore

$$\mathrm{supp}(\mathrm{res}(\Delta_n(\lambda))) \subseteq \Lambda_{n-1}^{m-1} \sqcup \Lambda_{n-1}^{m+1}.$$

Hence we have shown that our algebra satisfies the fifth Axiom.

Let us prove the Claims and the Lemma mentioned above. Let $\Delta_n(t_1, t_2, \ldots, t_h)$ be a TL_n^h-module. We restrict to TL_{n-1}^h using the embedding

$$TL_{n-1}^h \subset TL_n^h$$

via the map as in Figure 2.39. We can add colour C_1 or colour C_2 or $\ldots$ or colour C_h

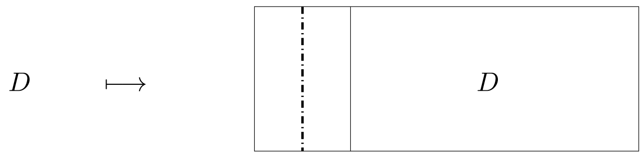


Figure 2.39:

propagating line at the front. Basis elements of $\Delta_n(t_1, t_2, \ldots, t_h)$ are of the form of a rectangle where there are n nodes at the southern edge and t_i propagating lines of colour C_i from north to south edge. Let $X \subseteq \Delta_n(t_1, t_2, \ldots, t_h)$ be the subspace spanned by

Figure 2.40:

diagrams with leftmost southern node propagating as in Figure 2.40. This line has some colour C_i, where i can take the values from 1 to h.

Claim 2.1.22.

X is a submodule of $\mathrm{res}_{n-1}^n \Delta_n(t_1, t_2, \ldots, t_h)$ and

$$X \cong \oplus_{i=1}^h \Delta_{n-1}(t_1, t_2, \ldots, t_i - 1, \ldots, t_h)$$

Proof. For the first part, it is enough to show that if $x \in X$ and d is a diagram in $TL_{n-1}^h \subset TL_n^h$ then $xd \in X$. We take x and d as in Figure 2.41. Here x has leftmost line colour C_i propagating and diagram d in TL_{n-1}^h a sum of diagrams in TL_n^h by adding the extra strings to the front as shown. In this particular situation the ith diagram of d will match with the front propagating line colour C_i of x. Therefore xd will be given by Figure 2.42. This has front line propagating, therefore $xd \in X$. Thus X is a submodule of $\text{res}_{n-1}^n \Delta_n(t_1, t_2, \ldots, t_h)$.

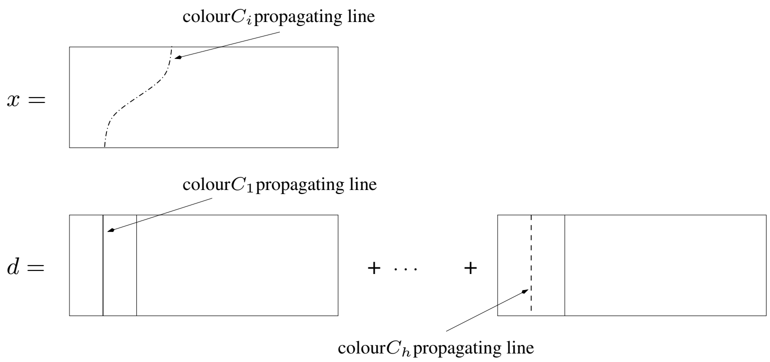


Figure 2.41:

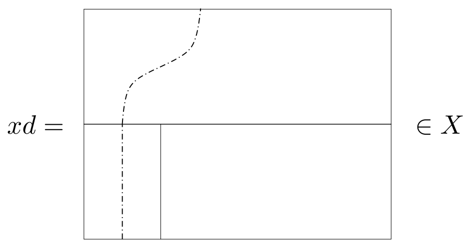


Figure 2.42:

Now TL_{n-1}^h acts on X as shown in Figure 2.43. The front propagating line cannot be seen by TL_{n-1}^h. Now we will show that

$$X \cong \oplus_{i=1}^h \Delta_{n-1}(t_1, t_2, \ldots, t_i - 1, \ldots, t_h).$$

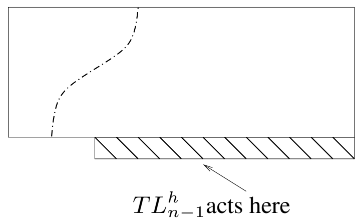


Figure 2.43:

Let the module element $x \in \oplus_{i=1}^h \Delta_{n-1}(t_1, t_2, \ldots, t_i - 1, \ldots, t_h)$ and the algebra element $d \in TL_{n-1}^h$. We want an isomorphism

$$\phi : \oplus_{i=1}^h \Delta_{n-1}(t_1, t_2, \ldots, t_i - 1, \ldots, t_h) \to X$$

as in Figure 2.44. Here we add an extra line to the front of x. If x has $t_i - 1$ colour C_i

Figure 2.44:

propagating lines then add one colour C_i propagating line at the front. Therefore, it is clear to see that ϕ is an isomorphism of vector spaces.

We will show that ϕ is a homomorphism of TL_{n-1}^h-modules. Element x has $n-1$ nodes at the southern edge and $t_1, \ldots, t_i - 1, \ldots, t_h$ number of colour C_1, ..., colour C_i, ..., colour C_h propagating lines from northern edge to southern edge and d has $n-1$ nodes at the northern and southern edge. If the southern edge of x and the northern edge of d do not match then $xd = 0$. This implies that $\phi(xd) = 0$. In this situation $\phi(x)d$ is also zero because of the colour sequence does not match. Therefore, we can say,

$$\phi(xd) = \phi(x)d.$$

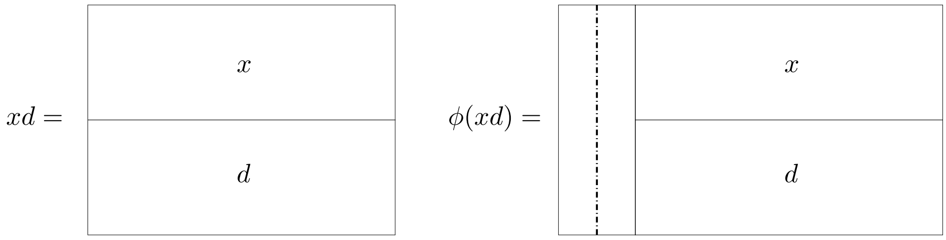


Figure 2.45:

If the colour sequence of the nodes match up then Figure 2.45 gives us xd, $\phi(xd)$. We add a colour C_i line if x has t_1 colour $C_1 \ldots t_i - 1$ colour $C_i \ldots t_h$ colour C_h propagating lines, where i can take the values from $1, \ldots, h$. Figure 2.46 gives us $\phi(x)d$. We draw the front

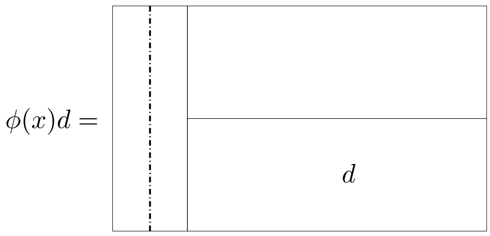


Figure 2.46:

colour C_i propagating line to d to match with the bottom diagram $\phi(x)$ front propagating line. Therefore we can say

$$\phi(xd) = \phi(x)d.$$

This implies that ϕ is a homomorphism of modules. Thus $\Delta_n(t_1, t_2, \ldots, t_h)$ has a TL^h_{n-1}-submodule.

$$X \cong \oplus_{i=1}^{h} \Delta_{n-1}(t_1, t_2, \ldots, t_i - 1, \ldots, t_h) \tag{2.1.9}$$

□

Claim 2.1.23. If we set

$$Y = \mathrm{res}^n_{n-1}\Delta_n(t_1, t_2, \ldots, t_h)/X \qquad (2.1.10)$$

then

$$Y \cong \oplus^h_{i=1}\Delta_{n-1}(t_1, t_2, \ldots, t_i + 1, \ldots, t_h) \qquad (2.1.11)$$

Proof. The subspace X is spanned by the set of diagrams whose leftmost southern node is propagating. If we find the quotient as in (2.1.10) then the leftmost southern node of Y will be on an arc as shown on Figure 2.47. Therefore, TL^h_{n-1} cannot see the front node. Let us try a map

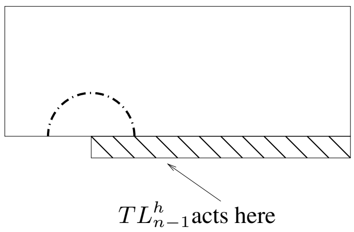


Figure 2.47:

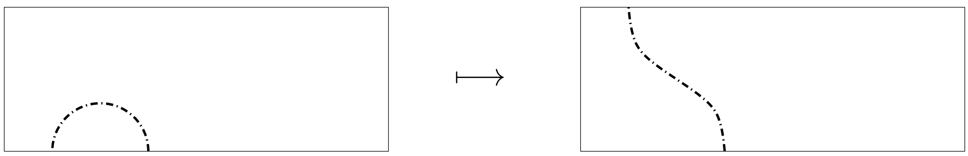


Figure 2.48:

$$\phi : Y \longrightarrow \oplus^h_{i=1}\Delta_{n-1}(t_1, t_2, \ldots, t_i + 1, \ldots, t_h)$$

as in Figure 2.48. We slide the node round the frame to the northern edge. First diagram has n nodes and second diagram has $n - 1$ nodes at the southern edge. This map ϕ is a vector space isomorphism.

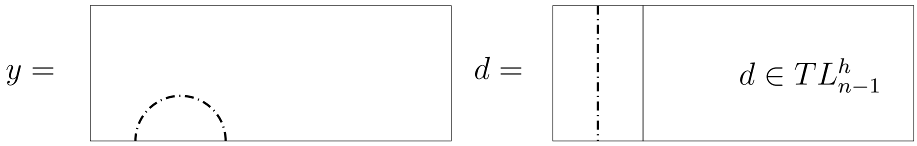


Figure 2.49:

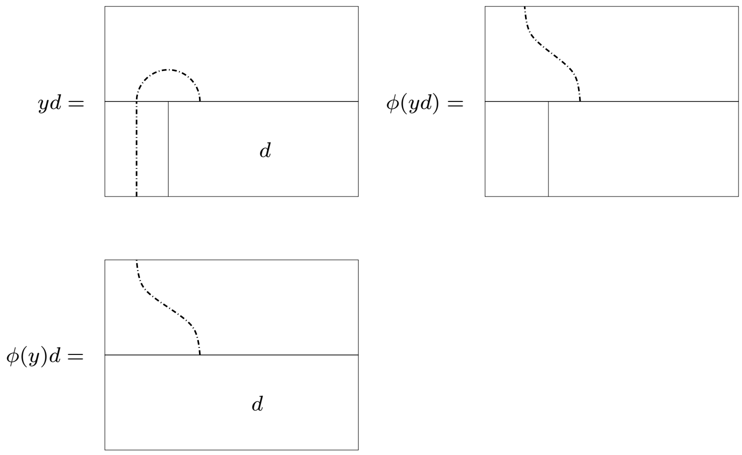


Figure 2.50:

We will show ϕ is a homomorphism. The cases we going to be discussed are based on what happens to the propagating arc which begins at the front part of the southern boundary. Let $y \in Y$ and $d \in TL_{n-1}^{h}$ as in Figure 2.49. Propagating line of d at the northern edge front to southern edge front is chosen to match with the southern edge arc colour at the front of y. Therefore, yd and $\phi(y)d$ are given by the Figure 2.50.

If the colour of y's southern edge and colour of d 's northern edge do not match then yd become 0. Therefore $\phi(yd)$ also become 0. In this situation colour of $\phi(y)$ and d also do not match. Therefore $\phi(y)d$ is 0. From these we can say $\phi(yd) = \phi(y)d$.

There are two cases to consider depending on whether the arc that begins at the front of the southern boundary of yd is propagating or not. First suppose $yd \notin X$, that is $yd \in Y$. Then yd, $\phi(y)d$ and now $\phi(yd)$ are given by the Figure 2.50, and it is clear that $\phi(yd) = \phi(y)d$.

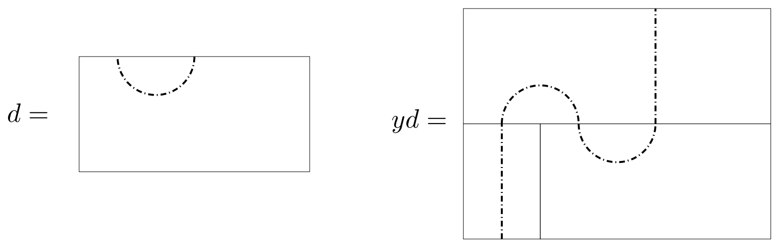


Figure 2.51:

Now suppose that $yd \in X$. An example of this case is shown in Figure 2.51. Here, we are considering the arc with an end in the south west corner on the right hand side of Figure 2.51, and figure assumes that the other end of the arc is on the northern edge of the diagram. The diagram yd will have t_1 number of colour C_1 propagating, ..., t_i number of colour C_i propagating, ... t_h number of colour C_h propagating. Therefore, we can say,

$$yd \in \Delta_n(t_1, t_2, \dots, t_h).$$

Since $yd \in X$, we have implies

$$yd = 0 \text{ in } \operatorname{res}^n_{n-1}\Delta_n(t_1, t_2, \ldots, t_h)/X = Y.$$

From this we can say

$$\phi(yd) = 0.$$

But $\phi(y)d$ is given by the Figure 2.52. This has t_1 colour C_1, ..., $t_i - 1$ colour C_i, ... and

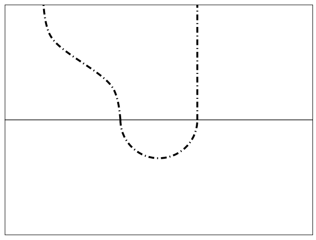
$\phi(y)d =$ 

Figure 2.52:

t_h colour C_h propagating lines. It does not have t_1 colour C_1, ..., $t_i + 1$ colour C_i, ... and t_h colour C_h propagating lines, where i take the values from $1, \ldots, h$. Therefore, $\phi(y)d$ is 0. From these we can say $\phi(yd) = \phi(y)d$.

From above cases we can say ϕ is a homomorphism. It is easy to see that ϕ is an isomorphism of vector spaces, and so gives a module isomorphism. □

Lemma 2.1.24. *The module* $\operatorname{res}^n_{n-1}\Delta_n(t_1, t_2, \ldots, t_i, \ldots, t_h)$ *has a submodule isomorphic to*

$$\oplus_{i=1}^h \Delta_{n-1}(t_1, t_2, \ldots, t_i - 1, \ldots, t_h)$$

and the quotient by this is isomorphic to

$$\oplus_{i=1}^h \Delta_{n-1}(t_1, t_2, \ldots, t_i + 1, \ldots, t_h).$$

Proof. Claims 2.1.22 and 2.1.23 proves this Lemma. □

Axiom 2.1.25. *(A6) For each $\Lambda \in \Lambda_n^n$ there exist $\mu \in \lambda_{n-1}^{n-1}$ such that*

$$\lambda \in \mathrm{supp}(\mathrm{ind}\Delta_{n-1}(\mu)).$$

Algebra satisfies the sixth Axiom (A6)

Let $\lambda = (a_1, a_2, \ldots, a_h) \in \Lambda_n^n$. Therefore, $\sum_{i=1}^{h} a_i$ will be n. We need to find a $\mu \in \Lambda_{n-1}^{n-1}$ such that $\lambda \in \mathrm{supp}(\mathrm{ind}\Delta(\mu))$.

For the value of $\mu = (x_1, x_2, \ldots, x_h) \in \Lambda_{n-1}^{n-1}$, so that $\sum_{i=1}^{h} x_i = n-1$, the module $\mathrm{ind}(\Delta_{n-1}(\mu))$ has the Δ-filtration

$$\begin{aligned} 0 &\longrightarrow \oplus_{i=1}^{h} \Delta_n(x_1, x_2, \ldots, x_i - 1, \ldots, x_h) \\ &\longrightarrow \mathrm{ind}_{n-1}^{n} \Delta_n(x_1, x_2, \ldots, x_i, \ldots, x_h) \\ &\longrightarrow \oplus_{i=1}^{h} \Delta_n(x_1, x_2, \ldots, x_i + 1, \ldots, x_h) \\ &\longrightarrow 0. \end{aligned}$$

Therefore, $\mathrm{supp}(\mathrm{ind}\Delta_{n-1}(\mu))$ is in $\Lambda_n^{n-2} \sqcup \Lambda_n^n$. However, $\lambda \in \Lambda_n^n$. Therefore, λ does not belong to Λ_n^{n-2}. From this, we can say

$$\lambda = (x_1, x_2, \ldots, x_i + 1, \ldots, x_h)$$

for some $i \in \{1, 2, \ldots, h\}$. If $a_i \neq 0$ take $x_i = a_i - 1$ and $x_k = a_k$ for k take the values from 1 to h except i. For these values of $x_1, x_2, \ldots, x_h$ there is a $\mu \in \Lambda_{n-1}^{n-1}$ such that $\lambda \in \mathrm{supp}(\mathrm{ind}\Delta(\mu))$.

The following Axiom is equivalent to the Axiom $(A6)$.

Axiom 2.1.26. *(A6') For each $\Lambda \in \Lambda_n^n$ there exist $\mu \in \lambda_{n+1}^{n-1}$ such that*

$$\lambda \in \mathrm{supp}(\mathrm{res}\Delta_{n+1}(\mu)).$$

Lets use the same argument as in [13] after the Axiom(A6') For a quasi-hereditary algebra we have that $\mathrm{Ext}(\Delta(\lambda), \Delta(\mu))) \neq 0$ implies that $\lambda < \mu$. Therefore, 2.1.25 is equivalent to the requirement that for each $\lambda \in \Lambda_n$ there exists $\mu \in \Lambda_{n-1}$ such that there is a surjection.

$$\mathrm{ind}\Delta_{n-1}(\mu) \to \Delta_n(\lambda) \to 0 \tag{2.1.12}$$

The axiomatic framework introduced so far in [13] is sufficient to reduce the study of various general homological problems. This broadly combines ideas from [9, 19].

The following Theorem is very helpful to understand the homological problem by reducing the size of the modules.

Theorem 2.1.27. *[13]*

(i) For all pairs of weights $\lambda \in \Lambda_n^m$ and $\mu \in \Lambda_n^l$ we have

$$\mathrm{Hom}(\Delta_n(\lambda), \Delta_n(\mu)) \cong \begin{cases} \mathrm{Hom}(\Delta_m(\lambda), \Delta_m(\mu)) & \textit{if } l \leq m, \\ 0 & \textit{otherwise.} \end{cases}$$

(ii)Suppose that for all $n \geq 0$ and pairs of weights $\lambda \in \Lambda_n^n$ and $\mu \in \Lambda_n^{n-2}$ we have

$$\mathrm{Hom}(\Delta_n(\lambda), \Delta_n(\mu)) = 0.$$

Then each of the algebras A_n is semisimple.

We will see how to find the homomorphism from a first module with no arcs to the second module with some arcs, if it exists, in Chapter 4 and 6. By using Theorem 2.1.27 we can find the homomorphism, if it exists, from the give first module with arcs in it to the given second module by reducing the size of the first module to all lines propagating.

2.2 Other algebras satisfy the axiomatic frame work

We list some of the algebras which satisfy the axiomatic framework to illustrate the utility of the tower of recollement theory. We have obtained the following from [13, Example 1.2].

(i) Temperley-Lieb algebra $TL_n(\delta)$: We can get more details about this algebra from [12, 45]. Temperley-Lieb algebra is the bubble algebra with one colour. In this case, indexing set $\Lambda_n = \{n, n-2, n-4, \ldots, 0 \text{ or } 1\}$ and $\Lambda^n = \{n\}$. We also have

$$0 \to \Delta_{n-1}(i-1) \to \mathrm{res}\Delta_n(i) \to \Delta_{n-1}(i+1) \to 0$$

for $0 \leq i < n$ and $\mathrm{res}\Delta_n(n) \cong \Delta_{n-1}(n-1)$, and similar sequences for $\mathrm{ind}(\Delta_n(i))$.

(ii) Blob algebra $b_n(\delta, \delta')$: This was introduced in [49]. In this case $\Lambda_n = \{n, n-2, n-4, \ldots, 2-n, -n\}$ with $\Lambda^n = \pm n$. We have a short exact sequence

$$0 \to \Delta_{n-1}(i \mp 1) \to \mathrm{res}\Delta_n(i) \to \Delta_{n-1}(i \pm 1) \to 0.$$

for $0 \leq i < n$ respectively $-n < i < 0$, and $\mathrm{res}\Delta_n(\pm n) \cong \Delta_{n-1}(\pm n \mp 1)$. There are similar sequences for $\mathrm{ind}\Delta_n(i)$.

(iii) Partition algebra: This was introduced in [46]. In this case application of the theory is a little more involved as the tower of algebras interleaves partition algebras with auxiliary intermediates. Details can be found in [47].

(iv) The Brauer and walled Brauer algebras and in characteristic zero with $\delta \neq 0$: These satisfies Axioms (A1)-(A6) according to [11] and [10]. They calculate when the Hom-spaces considered in Theorem 2.1.27(ii) are non-zero, and hence we can say when these algebras are semisimple [57].

If characteristic $p > 0$ the Brauer algebra is not quasi-hereditary. As the quotient algebras in (A2) are not semisimple. Apart from this, all of the other axioms (A1)-(A6) can be verified.

(v) Contour algebras: These were introduced in [13, Section 2]. It has been shown in [13, Section 2,3] that contour algebras satisfy the axiomatic framework of towers of recollement.

Chapter 3

Tensor products and Gram matrices

In this Chapter we will discuss certain special idempotent subalgebras of the bubble algebra, and show that they are isomorphic to products of Temperley-Lieb algebras. This also extends to the structure of the cell modules for these algebras.

We will then consider Gram matrices for cell modules, and show how these can be written as tensor products of Gram matrices for the Temperley-Lieb algebra. More details about the Gram matrix can be obtained from [49, 45].

Our main research goal is to find exactly when there are homomorphisms between two given cell modules. We will discuss this more in Chapters 4 and 6. If our first module has no arcs and the second module has one arc then the matrix responsible for the homomorphism (we will introduce this for the first time in Chapter 4) is some part of the whole Gram matrix. We will get non-zero homomorphisms for some special values of δ that are some of the roots of the Gram matrix determinant. These values of δ that give non-zero homomorphism are solutions of the part of the matrix we mentioned earlier.

3.1 An idempotent subalgebra of the bubble algebra

In this section we will discuss certain special idempotent elements in the bubble algebra. Idempotent elements plays a key role in our research. We also obtain certain results which will make our calculations easier in the Gram matrix.

3.1.1 Standard results associated with idempotent elements

Lemma 3.1.1. *If A is an algebra, M is a right A-module and $e \in A$ such that $e^2 = e$ then*

(a) eAe is an algebra

(b) Me is a right eAe-module.

It is easy to check the following Lemma.

Lemma 3.1.2. $\Phi : M \longrightarrow N$ *is an A-module homomorphism then* $\Phi : Me \longrightarrow Ne$ *is an eAe-module homomorphism.*

3.1.2 Idempotent element with all lines propagating

In this section we will consider a special idempotent element e. We take e to be the diagram with all strings propagating (and no crossings) where the first n_{C_1} nodes are colour C_1, the next n_{C_2} nodes are colour C_2 and so on up to the last n_{C_h} nodes which are colour C_h. We will obtain some interesting results relating $eTL_n^h e$ and $\Delta_n(\lambda)e$ to classical Temperley-Lieb theory.

Definition 3.1.3. Let A and B be algebras. Then $A \otimes B$ is also an algebra where multiplication is componentwise. If M is an A-module and N is a B-module then $M \otimes N$ is an $A \otimes B$-module via

$$(m \otimes n)(a \otimes b) = ma \otimes nb,$$

where $m \in M, n \in N, a \in A, b \in B$.

The main result of this section is the following.

Theorem 3.1.4. *With e defined as above we have that*

$$eTL_n^h(\delta_{C_1}, \dots, \delta_{C_h})e \cong \bigotimes_{k=1}^{h} TL_{n_{C_k}}(\delta_{C_k})$$

Proof. Let us denote the algebra $TL_n^h(\delta_{C_1}, \dots, \delta_{C_h})$ by A. Any diagram from eAe has first n_{C_1} nodes colour C_1, next n_{C_2} nodes colour C_2 and so on up to last n_{C_h} nodes colour C_h because all of the other elements of A will be killed by e. Now we define a linear map ψ from eAe to $\bigotimes_{k=1}^{h} TL_{n_{C_k}}(\delta_{C_k})$ such that

$$\psi : eAe \longrightarrow \bigotimes_{k=1}^{h} TL_{n_{C_k}}(\delta_{C_k})$$

as in Figure 3.1.

Let $a_1 \in A$ be as in Figure3.2, where $b_k \in TL_{n_{C_k}}$ for $1 \leq k \leq h$. Therefore

$$\psi(ea_1e) = b_1 \otimes b_2 \otimes \ldots \otimes b_h.$$

We will show ψ is an algebra homomorphism.

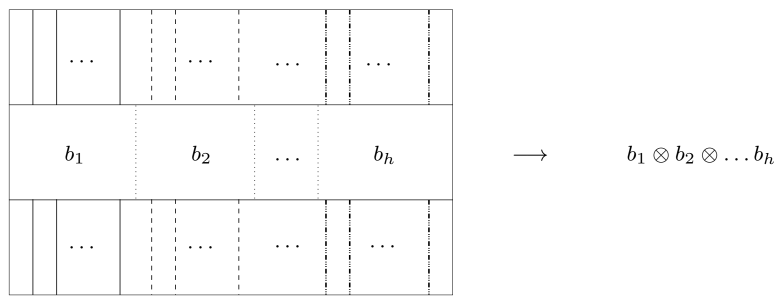


Figure 3.1:

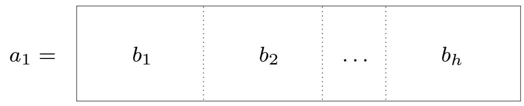


Figure 3.2:

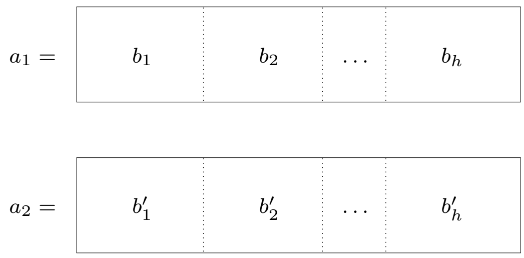


Figure 3.3:

Let a_1 and a_2 be as in Figure 3.3. Here $b_k \in TL_{n_{C_k}}$ and $b'_k \in TL_{n_{C_k}}$ for each $1 \leq k \leq h$. We need to show that

$$\psi((ea_1e)(ea_2e)) = \psi(ea_1e)\psi(ea_2e).$$

We have $(ea_1e)(ea_2e) = ea_1ea_2e$ because $e^2 = e$. If a_1, a_2 has the same northern and southern edge colour sequence as e then ea_1ea_2e can be given by a_1a_2, otherwise 0. If $a_1a_2 \neq 0$ then a_1a_2 can be given by Figure 3.4 which can be simplified into Figure 3.5. This can be written as

$$\psi((ea_1e)(ea_2e)) = \begin{cases} b_1b'_1 \otimes \ldots \otimes b_hb'_h, & \text{if } a_1a_2 \neq 0; \\ 0, & \text{otherwise.} \end{cases} \tag{3.1.1}$$

If we find $\psi(ea_1e)\psi(ea_2e)$ we will get

$$\begin{aligned} \psi(ea_1e)\psi(ea_2e) &= (b_1 \otimes \ldots \otimes b_h)(b'_1 \otimes \ldots \otimes b'_h) \\ &= b_1b'_1 \otimes \ldots \otimes b_hb'_h \end{aligned}$$

or $\psi(ea_1e)\psi(ea_2e)$ is 0. This can be written as

$$\psi(ea_1e)\psi(ea_2e) = \begin{cases} b_1b_1' \otimes \ldots \otimes b_hb_h', & \text{if } a_1, a_2 \text{ same colour sequence as } e; \\ 0, & \text{otherwise.} \end{cases} \tag{3.1.2}$$

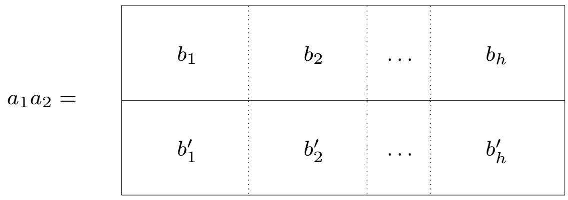


Figure 3.4:

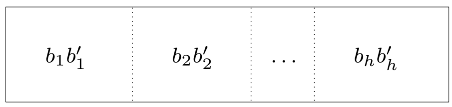


Figure 3.5:

From (3.1.1) and (3.1.2) we can say

$$\psi((ea_1e)(ea_2e)) = \psi(ea_1e)\psi(ea_2e).$$

This implies that ψ is an algebra homomorphism. At the same time, ψ is injective by the way it is defined.

We will show ψ is surjective. Let $y \in \bigotimes_{k=1}^{h} TL_{n_{C_k}}(\delta_{C_k})$ be a diagram. Therefore, $y = b_1 \otimes b_2 \ldots \otimes b_h$ for some $b_k \in TL_{n_{C_k}}$. However, ψ of Figure 3.6 gives us y. Choose $a \in A$ as in Figure 3.7. Therefore, eae given by the diagram in Figure 3.6, belongs to eAe. Now $\psi(eae) = y$ and so ψ is surjective.

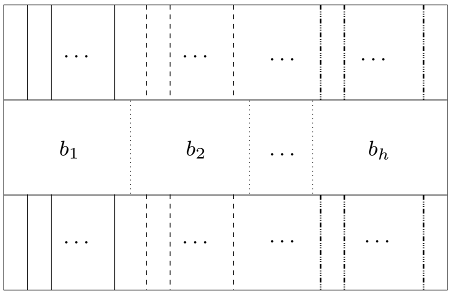


Figure 3.6:

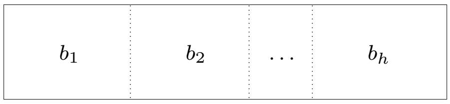


Figure 3.7:

We have shown ψ is homomorphism injective and surjective. Therefore, ψ is an algebra isomorphism. This implies that

$$eTL_n^h(\delta_{C_1}, \delta_{C_2} \ldots, \delta_{C_h})e \cong \bigotimes_{k=1}^{h} TL_{n_{C_k}}(\delta_{C_k}).$$

□

Definition 3.1.5. Given A and B are algebras. Suppose that $A \cong B$ via $\phi : A \longrightarrow B$ and that M is an A-module and N a B-module. If there is a vector space isomorphism $\theta : M \longrightarrow N$ and

$$\theta(ma) = \theta(m)\phi(a)$$

then we can say $M \cong N$, where $m \in M$ and $a \in A$.

Lemma 3.1.6.

$$\Delta_n(a_1, a_2, \ldots, a_h)e \cong \bigotimes_{k=1}^{h} \Delta_{n_{C_k}}(a_k),$$

where e is the idempotent defined above.

Proof. We are only going to prove this for two colours red and green. However, it is quite easily generalise into h colours. The diagrams become more complicated for h colours which is the reason we are doing it for 2 colours. Let us prove the following

$$\Delta_n(r,g)e \cong \Delta_{n-i,R}(r) \otimes \Delta_{i,G}(g),$$

where e has $n-i$ red and i green propagating lines and $r \leq n-i$ and $g \leq i$. We have proved in Lemma 3.1.4 that

$$eTL_n^2(\delta_R,\delta_G)e \cong TL_{n-i}(\delta_R) \otimes TL_i(\delta_G).$$

By using the function ψ as in Figure 3.1.

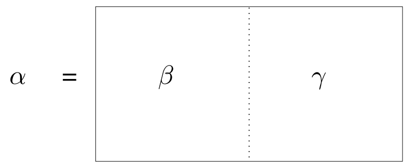


Figure 3.8:

Let αe be an element in $\Delta_n(r,g)e$. Here α is given by Figure 3.8. Let θ be a linear map such that

$$\theta : \Delta_n(r,g)e \longrightarrow \Delta_{n-i,R}(r) \otimes \Delta_{i,G}(g)$$

as in Figure 3.9. Therefore we can say

$$\theta(\alpha e) = \beta \otimes \gamma.$$

Clearly θ is a vector space isomorphism. Let $\alpha e \in \Delta_n(r,g)e$ and $eae \in eTL_n^2(\delta_R,\delta_G)e$. Now we will check the condition

$$\theta((\alpha e)(eae)) = \theta(\alpha e)\psi(eae)$$

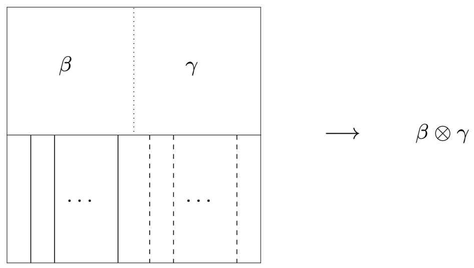


Figure 3.9:

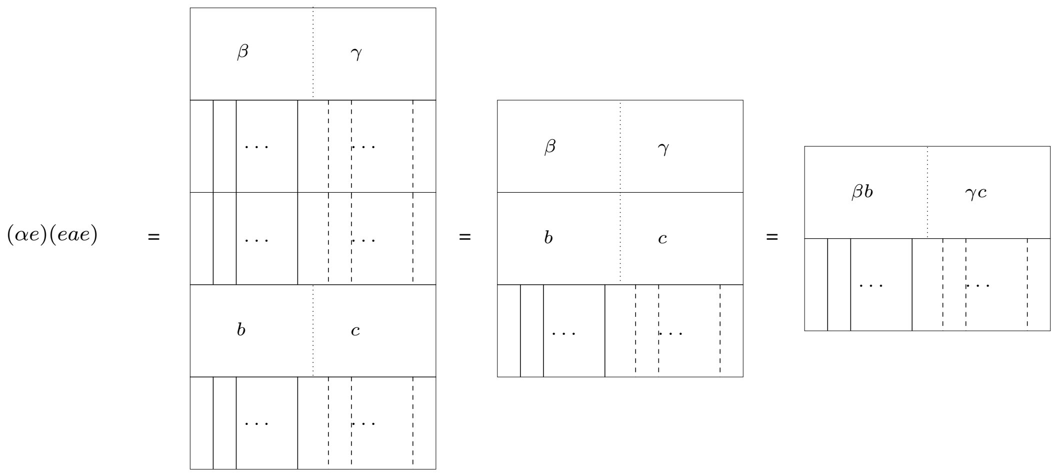


Figure 3.10:

Consider $(\alpha e)(eae)$, which is illustrated in Figure 3.10. From this Figure we can say

$$\theta((\alpha e)(eae)) = \beta b \otimes \gamma c$$

if the southern edge colour sequence of α and the northern edge colour sequence of a match with e, or 0 otherwise. We can write this as

$$\theta((\alpha e)(eae)) = \begin{cases} \beta b \otimes \gamma c, & \text{colour sequence of } \alpha,\ a \text{ and } e \text{ same;} \\ 0, & \text{otherwise.} \end{cases} \tag{3.1.3}$$

On the other hand

$$\begin{aligned} \theta(\alpha e)\phi(eae) &= (\beta \otimes \gamma)(b \otimes c) \\ &= \beta b \otimes \gamma c \end{aligned}$$

if the southern edge colour sequence of α and the northern edge colour sequence of a match with e, or 0 otherwise. This can be written as

$$\theta(\alpha e)\phi(eae) = \begin{cases} \beta b \otimes \gamma c, & \text{colour sequence of } \alpha,\ a \text{ and } e \text{ same;} \\ 0, & \text{otherwise.} \end{cases} \tag{3.1.4}$$

From (3.1.3) and (3.1.4) we can say

$$\theta((\alpha e)(eae)) = \theta(\alpha e)\phi(eae)$$

and hence

$$\Delta_n(r, g)e \cong \Delta_{n-i,R}(r) \otimes \Delta_{i,G}(g).$$

□

3.2 Gram matrix and its application

In this section we are going to discuss the Gram matrix and an application for finding the special values of δ for which a given cell module is not simple. The Gram matrix is named after Jorgen Pedersen Gram [21].

Definition 3.2.1. The Gram matrix of the module $\Delta_n(\lambda)$ is given by

$$M(\Delta_n(\lambda)) = (M_{ij}), \tag{3.2.1}$$

where

$$M_{ij} = \langle C_i^\lambda, C_j^\lambda \rangle \tag{3.2.2}$$

and C_i^λ, C_j^λ are cellular basis elements of the module $\Delta_n(\lambda)$. The inner product was defined earlier in 1.1.3 and given by (1.1.12).

When we calculate $\langle C_i^\lambda, C_j^\lambda \rangle$ only the southern half diagrams of C_i^λ and C_j^λ are involved. Therefore, it is enough to consider the southern half diagram basis elements of $\Delta_n(\lambda)$ to find the Gram matrix $M(\Delta_n(\lambda))$.

Lemma 3.2.2.

(a) M_{ij} *is* 0, 1 *or a monomial in* δ_{C_k}*'s.*

(b) The degree of M_{ii} *equals the number of arcs in* C_i^λ.

(c) If $i \neq j$ *then the degree of* M_{ij} *is strictly less than the number of arcs in* C_i^λ.

Proof. First of all we consider M_{ij}, where $i \neq j$. It is given by the inner product of C_i^λ and C_j^λ. According to (1.1.12) it can be calculated by multiplying C_i^λ followed by the multiplication of the upside down version of C_j^λ. From this we can say M_{ij} is 0 if the colours do not match or there are the wrong number of propagating lines, 1 if all the lines are propagating lines, and a monomial in δ's otherwise. This proves Lemma 3.2.2(a). The number of closed loops is less than number of arcs in C_i^λ and C_j^λ when $i \neq j$, because not all the arcs of C_i^λ will match with the arcs of C_j^λ exactly and some may end up with propagating lines or forming a single loop out of more than two arcs. Therefore, M_{ij} is a monomial of degree less than the number of arcs. This proves Lemma 3.2.2(c).

Now we consider M_{ii}. If we draw C_i^λ followed by the up side down of C_i^λ, then the number of closed loops obtained is equal to the number of arcs in C_i^λ. Therefore, M_{ii} is a monomial of degree equal to the number of arcs. This proves Lemma 3.2.2(b). □

3.2.1 Finding the Gram matrix of a given module

Consider $TL_n^h(\delta_{C_1}, \ldots, \delta_{C_h})$ and the module $\Delta_n(a_1, a_2, \ldots, a_h)$. We can find the special values of δ_{C_i} for which module become not simple. We can find this by solving

$$\det M(\Delta_n(a_1, a_2, \ldots, a_h)) = 0, \tag{3.2.3}$$

where i in δ_{C_i} can take the values 1 to h. These special values are very important when we find the homomorphism between given two modules. We will discuss this in a later Chapter.

Example 3.2.3.

Let us find the Gram matrix M for the module $\Delta_3(1, 0)$. This is a $TL_3^2(\delta_R, \delta_G)$-module with red and green colour nodes. Here the module has 3 nodes and 1 red propagating line and no green propagating line. Half diagrams of the basis elements of the module $\Delta_3(1, 0)$ are in Figure 3.11.

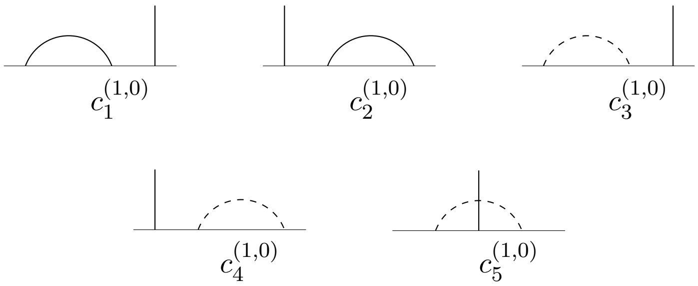
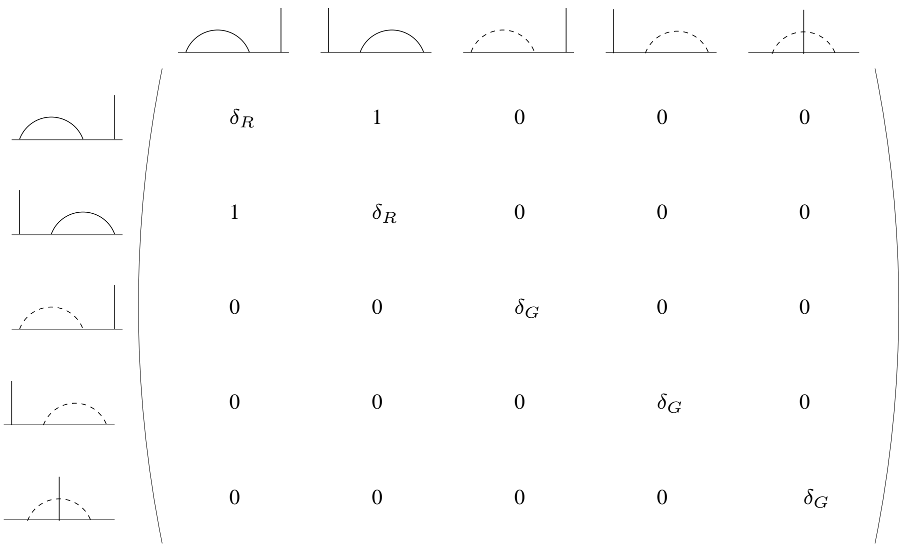


Figure 3.11: Half diagrams of the basis elements of the module $\Delta_3(1, 0)$



$$\begin{pmatrix} \delta_R & 1 & 0 & 0 & 0 \\ 1 & \delta_R & 0 & 0 & 0 \\ 0 & 0 & \delta_G & 0 & 0 \\ 0 & 0 & 0 & \delta_G & 0 \\ 0 & 0 & 0 & 0 & \delta_G \end{pmatrix}$$

Figure 3.12: Gram matrix of the module $\Delta_3(1, 0)$

From Figure 3.12 the Gram matrix M of the module $\Delta_3(1,0)$ can be calculated as

$$M(\Delta_3(1,0)) = \begin{pmatrix} \delta_R & 1 & 0 & 0 & 0 \\ 1 & \delta_R & 0 & 0 & 0 \\ 0 & 0 & \delta_G & 0 & 0 \\ 0 & 0 & 0 & \delta_G & 0 \\ 0 & 0 & 0 & 0 & \delta_G \end{pmatrix}.$$

If we find the determinant of the Gram matrix we obtain

$$\text{det}M(\Delta_3(1,0)) = \begin{vmatrix} \delta_R & 1 & 0 & 0 & 0 \\ 1 & \delta_R & 0 & 0 & 0 \\ 0 & 0 & \delta_G & 0 & 0 \\ 0 & 0 & 0 & \delta_G & 0 \\ 0 & 0 & 0 & 0 & \delta_G \end{vmatrix}$$
$$= (\delta_R^2 - 1)\delta_G^3.$$

We can find the special values of δ_R and δ_G by solving

$$\det M(\Delta_3(1,0)) = 0.$$

Therefore the special values of δ_R are ± 1 and of δ_G is 0.

Gram matrix with organised half diagram of the module according to colour sequence

We have seen how to find the Gram matrix and the special value of δ for a given module $\Delta_n(a_1, a_2, \ldots, a_h)$. However, if we increase the value of n then our Gram matrix will become very large very quickly. Therefore, it will be very hard to find the determinant. To avoid this problem, we organise the diagrams into collections by looking at half diagram organised by the colour sequences of their nodes. We define this as follows.

Definition 3.2.4. The list of colours of the nodes in a half diagram written as a sequence from left to right is called the colour sequence of the nodes. We denote this by

$$X_{t_1 \ldots t_n}$$

where t_i denotes the colour of node i.

Usually we will use the first letter of the colours in such a sequence, as in the following example.

Example 3.2.5. Let $X_{rrgrggg}$ be a set of half diagrams with nodes colour sequence red, red, green, red, green, green and green. If we look at Figure 3.13 the first half diagram belongs to $X_{rrgrggg}$. However, the second half diagram does not belong to $X_{rrgrggg}$.

Figure 3.13:

Let $X_1, X_2, \ldots X_p$ be the southern half diagram colour sequences of the module $\Delta_n(a_1, a_2, \ldots, a_h)$. If we organise the half diagrams according to the colour sequence our Gram matrix M of the module $\Delta_n(a_1, a_2, \ldots, a_h)$ may look as in Figure 3.14. We get non-zero matrices on the diagonals as shown and zero matrices in the other places, as we get non-zero entries in the Gram matrix if the colour sequences match, otherwise zero. If we find the determinant of this matrix, that is actually the product of the determinant of the diagonal matrices. This is summarised in the following Lemma.

Lemma 3.2.6. *The determinant of the Gram matrix of $\Delta_n(\lambda)$ is given by*

$$\det(M(\Delta_n(\lambda)) = \prod_i \det a(X_i), \tag{3.2.4}$$

$$
M = \begin{array}{c} \\ X_1 \\ X_2 \\ X_3 \\ \vdots \\ X_p \end{array}
\begin{array}{c}
\begin{array}{ccccc} X_1 & X_2 & X_3 & \dots & X_p \end{array} \\
\left(\begin{array}{c|c|c|c|c}
a(X_1) & 0 & 0 & \dots & 0 \\ \hline
0 & a(X_2) & 0 & \dots & 0 \\ \hline
0 & 0 & a(X_3) & \dots & 0 \\ \hline
\vdots & \vdots & \vdots & \ddots & \vdots \\ \hline
0 & 0 & 0 & 0 & a(X_p)
\end{array}\right)
\end{array}
$$

Figure 3.14: Appearance of the Gram matrix according to colour sequence

where $a(X_i)$ *is the sub-matrix comes from the collection of half diagrams of the module* $\Delta_n(\lambda)$ *whose southern edge colour sequence is* X_i*.*

3.2.2 Tensor Product of Matrices

We are going to discuss tensor products of matrices, known as the Kronecker product. This product is very useful in our Gram matrix calculation. We can write the Gram matrix as the tensor product of more than one matrix. By using the properties of tensor products we can then find the determinant of the Gram matrix quite easily.

Definition 3.2.7. Let $A = (a_{ij})_{1\leq i,j\leq m,n}$ and $B = (b_{kl})_{1\leq k,l\leq p,q}$ be two matrices. The tensor product of matrices A and B is given by

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{pmatrix}_{mn\times np}.$$

Some basic properties of tensor products of matrices

We recall some well-known properties of the tensor product of matrices.

(i) Suppose that A, A', B, B' are matrices where the usual matrix products $A.A'$ and $B.B'$ make sense. Then

$$(A \otimes B).(A' \otimes B') = (A.A') \otimes (B.B') \tag{3.2.5}$$

(ii) For all A and B we have

$$(A \otimes B)^T = A^T \otimes B^T \tag{3.2.6}$$

(iii) If A and B are invertible then

$$(A \otimes B)^{-1} = A^{-1} \otimes B^{-1} \tag{3.2.7}$$

(iv) For any scalar d and matrices A and B we have

$$d(A \otimes B) = dA \otimes B = A \otimes dB \tag{3.2.8}$$

(v) Let A, B, C and D be matrices and c, d, e and g be scalars. Then

$$(cA + dB) \otimes (eC + gD) = ceA \otimes C + cgA \otimes D + deB \otimes C + dgB \otimes D \tag{3.2.9}$$

(vi) If B and C are square matrices then

$$\begin{aligned} \det(B \otimes C) &= \det(B)^{\dim C} \times \det(C)^{\dim B} \\ &= \left(\det(B)^{\frac{1}{\dim B}}\ \det(C)^{\frac{1}{\dim C}}\right)^{\dim B \times \dim C} \end{aligned} \tag{3.2.10}$$

We will use this final property quite frequently.

Lemma 3.2.8. *Let A_1, A_2, $\ldots A_h$ be square matrices. Then the determinant of the tensor product of these matrices is given by*

$$\det A_1 \otimes A_2 \otimes \ldots \otimes A_h = \left(\prod_{i=1}^{h} \det A_i^{\frac{1}{\dim A_i}} \right)^{\prod_{i=1}^{h} \dim A_i}. \tag{3.2.11}$$

Example 3.2.9. Let us consider the module $\Delta_6(1,1)$ of the algebra $TL_6^2(\delta_R, \delta_G)$. This has one red and one green propagating line and two arcs of either colour. There are many possible colour sequence for our module.

We can divide the half diagrams of the module into the following three cases.

Case (i): One red propagating line, one green propagating and two red arcs: half diagrams in this case have five red nodes and one green node. Therefore, $\frac{6!}{5!} = 6$ number of possible colour sequences in this case.

Case (ii): One red propagating line, one green propagating line, one red arc and one green arc: half diagram in this case have three red nodes and three green nodes. Therefore, there will be $\frac{6!}{3!3!} = 20$ number of possible colour sequences.

Case (iii): one red propagating line, one green propagating line and two green arcs: half diagram in this case has five green nodes and one red node. Therefore, there will be $\frac{6!}{5!} = 6$ number of possible colour sequences in this case.

From these three cases, we can see that there will be 32 different possible colour sequences for the module $\Delta_6(1,1)$.

If we work out the matrix $a(X_{grrrgg})$ we will get as in Figure 3.15. We call this matrix

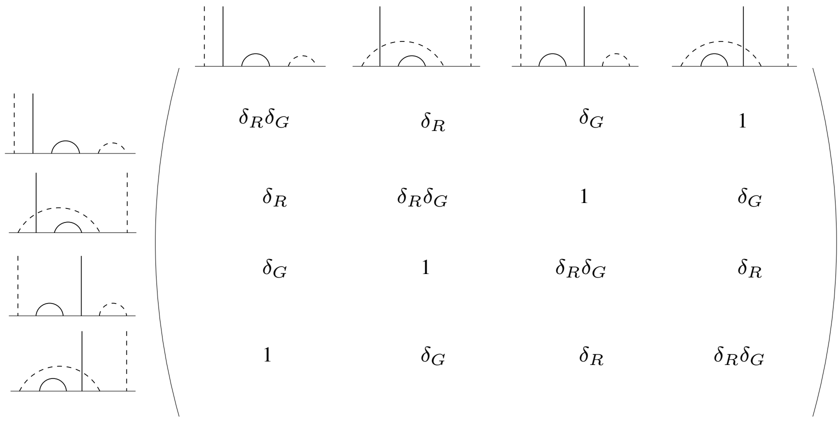


Figure 3.15: Matrix by considering the half diagrams with colour sequence X_{grrrgg}

A.

$$A = \begin{pmatrix} \delta_R.\delta_G & \delta_R.1 & 1.\delta_G & 1.1 \\ \delta_R.1 & \delta_R.\delta_G & 1.1 & 1.\delta_G \\ 1.\delta_G & 1.1 & \delta_R.\delta_G & \delta_R.1 \\ 1.1 & 1.\delta_G & \delta_R.1 & \delta_R.\delta_G \end{pmatrix} \tag{3.2.12}$$

If we look at this carefully we can observe the following.

$$A = \begin{pmatrix} \delta_R \begin{pmatrix} \delta_G & 1 \\ 1 & \delta_G \end{pmatrix} & 1 \begin{pmatrix} \delta_G & 1 \\ 1 & \delta_G \end{pmatrix} \\ 1 \begin{pmatrix} \delta_G & 1 \\ 1 & \delta_G \end{pmatrix} & \delta_R \begin{pmatrix} \delta_G & 1 \\ 1 & \delta_G \end{pmatrix} \end{pmatrix} \tag{3.2.13}$$

This matrix A can be written

$$\begin{aligned} A = & \begin{pmatrix} \delta_R & 1 \\ 1 & \delta_R \end{pmatrix} \otimes \begin{pmatrix} \delta_G & 1 \\ 1 & \delta_G \end{pmatrix} \\ = & B \otimes C, \end{aligned}$$

where B and C are the first and second matrix respectively. By using the property of the tensor product determinant in (3.2.10) we can say

$$\det(A) = \det(B)^{\dim C} \times \det(C)^{\dim B}.$$

However, we know

$$\det(B) = \delta_R^2 - 1 \quad \dim B = 2$$
$$\det(C) = \delta_G^2 - 1 \quad \dim C = 2$$

Therefore, we can say

$$\det(A) = (\delta_R^2 - 1)^2(\delta_G^2 - 1)^2.$$

However, $a(X_{grrrgg})$ is the matrix A. Therefore we can say

$$\det(a(X_{grrrgg})) = (\delta_R^2 - 1)^2(\delta_G^2 - 1)^2.$$

Application of tensor products in the Gram matrix calculation

We are going to find the determinant of the Gram matrix $M(\Delta_n(a_1, \ldots, a_h))$ by using one colour facts, which means by considering certain Temperley-Lieb modules. As in (3.2.4) we know $\det M(\Delta_n(a_1, \ldots, a_h))$ can be given as the product of $\det a(X_i)$. However, we are going to see very soon matrix $a(X_i)$ can be written as the tensor product of matrices as in Lemma 3.2.10. Each matrix in 3.2.14 correspond to a one colour module.

Lemma 3.2.10. *For $\Delta_n(a_1, \ldots, a_h)$ the matrix $a(X_i)$ in*

$$\det M(\Delta_n(a_1, \ldots, a_h)) = \prod_i \det a(X_i)$$

can be given by

$$a(X_i) = \bigotimes_{k=1}^{h} M(\Delta_{n_{C_k i}}(a_k)), \tag{3.2.14}$$

where $n_{C_k i}$ is the number of colour C_k nodes in X_i.

Proof. By using Lemma 3.1.6 we have

$$\Delta_n(a_1, \dots, a_h)e_i \cong \bigotimes_{k=1}^{h} \Delta_{n_{C_k i}}(a_k),$$

where e_i is the idempotent element with all propagating lines and colour sequence i. Therefore we have

$$M(\Delta_n(a_1, \dots, a_h)e_i) = \bigotimes_{k=1}^{h} M(\Delta_{n_{C_k i}}(a_k)).$$

Now $M(\Delta_n(a_1, \dots, a_h)e_i)$ is precisely the matrix $a(X_i)$ and hence we have proved the Lemma. □

Application of the above results in Gram-matrix

We can construct the Gram matrix of $\Delta_n(a_1, a_2, \dots, a_h)$ by grouping the half diagrams with the same colour sequence. Each colour sequence i corresponds to an idempotent element e_i. If we say there are s colour sequences then

$$\begin{aligned} \det M(\Delta_n(a_1, a_2, \dots, a_h)) &= \prod_{i=1}^{s} \det(a(X_i)) \\ &= \prod_{i=1}^{s} \det(M(\Delta_n(a_1, a_2, \dots, a_h)e_i)) \end{aligned}$$

Example 3.2.11. Let us work out the determinant of the Gram matrix of the module $\Delta_6(1,1)$. If we use the above Lemma we will get

$$\det M(\Delta_6(1,1)) = \prod_{i=1}^{32} \det a(X_i).$$

Here $X_1, X_2, \dots, X_{32}$ are the possible southern edge colour sequence of the module $\Delta_6(1,1)$. Among these, six of them have the determinant $\delta_R^2(\delta_R^2 - 1)^4$, twenty of them have the determinant $(\delta_R^2 - 1)^2(\delta_G^2 - 1)^2$ and another six of them have the determinant $\delta_G^2(\delta_G^2 - 1)^4$. Therefore

$$\det M(\Delta_6(1,1)) = \left(\delta_R^2(\delta_R^2 - 1)^4\right)^6 \left((\delta_R^2 - 1)^2(\delta_G^2 - 1)^2\right)^{20} \left(\delta_G^2(\delta_G^2 - 1)^4\right)^6$$

Lemma 3.2.12. *The determinant of the Gram matrix of* $\Delta_n(a_1, a_2, \dots a_h)$ *is given by*

$$\det M(\Delta_n(a_1, a_2, \dots, a_h)) = \prod_{i=1}^{s}[\prod_{k=1}^{h} \det M(\Delta_{n_{C_k i}}(a_k))^{\frac{1}{d_{C_k i}}}]^{\prod_{k=1}^{h} d_{C_k i}}, \qquad (3.2.15)$$

where $n_{C_k i}$ *denotes the number of* C_k *colour nodes in the colour sequence* X_i*,* $d_{C_k i}$ *denote the dimension of the matrix* $M(\Delta_{n_{C_k i}}(a_k))$ *and* s *denotes the number of colour sequences.*

Proof. We prove this by induction on the number of colours h.

When we consider $h = 1$, that means one colour, result is obvious because there will be only one colour sequence.

When we consider $h = 2$, that means two colours C_1 and C_2. From (3.2.14) and (3.2.4) we get

$$\begin{aligned}
\det M(\Delta_n(a_1, a_2)) &= \prod_{i=1}^{s} \det M(\Delta_{n_{C_1 i}}(a_1))^{d_{C_2 i}} \ \det M(\Delta_{n_{C_2 i}}(a_2))^{d_{C_1 i}} \\
&= \prod_{i=1}^{s} \left(\det M(\Delta_{n_{C_1 i}}(a_1))^{\frac{1}{d_{C_1 i}}} \ \det M(\Delta_{n_{C_2 i}}(a_2))^{\frac{1}{d_{C_2 i}}} \right)^{d_{C_1 i} d_{C_2 i}} \\
&= \prod_{i=1}^{s} \left(\det M(\Delta_{n_{C_1 i}}(a_1))^{\frac{1}{d_{C_1 i}}} \ \det M(\Delta_{n_{C_2 i}}(a_2))^{\frac{1}{d_{C_2 i}}} \right)^{d_{C_1 i} d_{C_2 i}}
\end{aligned}$$

Suppose this is true for $h = p$ colours. Therefore,

$$\det M(\Delta_{n_{C_1 i}+n_{C_2 i}+\dots+n_{C_p i}}(a_1, a_2, \dots, a_p)) = \prod_{i=1}^{s}[\prod_{k=1}^{p} \det M(\Delta_{n_{C_k i}}(a_k))^{\frac{1}{d_{C_k i}}}]^{\prod_{k=1}^{p} d_{C_k i}}. \qquad (3.2.16)$$

Let us say e_i is the idempotent element with the colour sequence X_i at this stage.

Now we will prove this for $h = p + 1$ colours. Consider the idempotent element e'_i with the colour sequence X'_i. This can be obtained by adding $n_{C_{p+1} i}$ colour nodes to the end of the idempotent in the case with p colours. Therefore, we can say

$$\begin{aligned}
\Delta_n(a_1, a_2, \dots, a_{p+1})e'_i \cong & \Delta_{n_{C_1 i}+n_{C_2 i}+\dots+n_{C_p i}}(a_1, a_2, \dots, a_p)) \\
& \otimes \Delta_{n_{C_{p+1} i}}(a_{p+1}))
\end{aligned}$$

$$M(\Delta_n(a_1, a_2, \ldots, a_{p+1})e_i') = M(\Delta_{n_{C_1 i}+n_{C_2 i}+\ldots+n_{C_p i}}(a_1, a_2, \ldots, a_p)) \otimes M(\Delta_{n_{C_{p+1} i}}(a_{p+1}))$$

From this we can say

$$\det M(\Delta_n(a_1, a_2, \ldots, a_{p+1})e_i') = [\det M(\Delta_{n_{C_1 i}+n_{C_2 i}+\ldots+n_{C_p i}}(a_1, a_2, \ldots, a_p))]^{d_{C_{p+1} i}} \det M(\Delta_{n_{C_{p+1} i}}(a_{p+1}))^{\prod_{k=1}^{p} d_{C_k i}} \tag{3.2.17}$$

If we substitute (3.2.16) in (3.2.17) this implies

$$\det M(\Delta_n(a_1, a_2, \ldots, a_{p+1})) = \prod_{i=1}^{s} \left([\prod_{k=1}^{p} \det M(\Delta_{n_{C_k i}}(a_k))^{\frac{1}{d_{C_k i}}}]^{\prod_{k=1}^{p} d_{C_k i}} \right)^{d_{C_{p+1} i}} \det M(\Delta_{n_{C_{p+1} i}}(a_{p+1}))^{\prod_{k=1}^{p} d_{C_k i}}$$

We can simplify this as

$$M(\Delta_n(a_1, a_2, \ldots, a_{p+1})) = \prod_{i=1}^{s} [\prod_{k=1}^{p+1} \det M(\Delta_{n_{C_k i}}(a_k))^{\frac{1}{d_{C_k i}}}]^{\prod_{k=1}^{p+1} d_{C_k i}}$$

and so the Lemma holds. □

Chapter 4

Homomorphism between modules: the one arc case

In this chapter, we will show how to find the non-zero homomorphisms between two given modules where the number of arcs differs by one. By Theorem 2.1.27 we can assume that the first module has all line propagating and the second module has only one arc. We will generalise this to h colours and allow the second module to have more than one arc in Chapter 6. Figure 4.1 denotes the labeling for the red, green and black propagating lines and arc colour of the figures in the remaining Chapters.

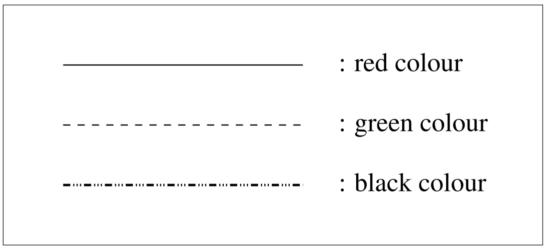


Figure 4.1: Colour labeling

4.1 Finding homomorphisms

Let $\Delta_n(a,b)$ and $\Delta_n(c,d)$ be the two given modules of the algebra $TL_n^2(\delta_R,\delta_G)$. We would like to find the non-zero homomorphisms, $\theta : \Delta_n(a,b) \longrightarrow \Delta_n(c,d)$, so as to gain a better understanding of the relationship between the modules $\Delta_n(a,b)$ and $\Delta_n(c,d)$. It should satisfy the homomorphism condition

$$\theta(mx) = \theta(m)x, \tag{4.1.1}$$

for $m \in \Delta_n(a,b)$ and $x \in TL_n^2(\delta_R,\delta_G)$.If there is a non-zero homomorphism, then

$$a \geq c,\ b \geq d \text{ and both } a-c \text{ and } b-d \text{ are divisible by two.} \tag{4.1.2}$$

However, (4.1.2) alone does not imply that there is a non-zero homomorphism because this will also depends on the values of δ_R and δ_G. Therefore, we need to find these special values of δ_R and δ_G.

4.1.1 Introduction to the notation for modules and algebra elements

Our modules and algebra have two colour nodes. Therefore, more than one colour sequence is possible for the nodes. We denote the ith colour sequence by X_i. Module $\Delta_n(a,b)$ has all nodes propagating, $a+b=n$. Therefore, there will be only one basis element possible for each colour sequence. We call that basis element m_i. If we consider the module $\Delta_n(c,d)$ there will be more than one basis element for each colour sequence. Therefore, we label the basis elements by n_{ij} where i represents the colour sequence X_i and j represents the label for the southern edge half diagram of the basis element. We label the colour sequence as X_1 if it has first a nodes red colour and the next b nodes green colour. Please see the beginning of Example 4.1.2 for the labeling of the basis elements of the modules.

Lemma 4.1.1. *Let θ be a non-zero homomorphism from $\Delta_n(a,b)$ to $\Delta_n(c,d)$. Then we can write*

$$\theta(m_i) = \sum_j s_j n_{ij},$$

where n_{ij} is a basis element of the second module with the southern edge colour sequence X_i and s_j is the coefficient of n_{ij} which is independent of i.

Proof. First consider the case $i = 1$. Let x be an element of the algebra. We know θ should satisfy the homomorphism condition

$$\theta(m_1 x) = \theta(m_1)x.$$

However, $m_1 x$ becomes 0 if the southern edge colour sequence of m_1 and northern edge colour sequence of x are different. We may get non-zero value for $m_1 x$, unless multiplication gives the wrong number of propagating lines as m_1, if the southern edge of m_1 and northern edge of x are same. Therefore, the colour sequence of $\theta(m_1)$ should have the same colour sequence as the northern edge of x. From this we see that the southern edge colour sequences of m_1 and $\theta(m_1)$ are the same. Therefore, we can say $\theta(m_1)$ is a linear combination of the second module basis elements with the same southern edge colour sequence as m_1. Let n_{1j} be the second module basis element with the southern edge colour sequence as m_1. Therefore, we can write

$$\theta(m_1) = \sum_j s_j n_{1j}.$$

Now consider the case $i \neq 1$. By using diagrams with one crossing and all lines propagating, the position of the colour nodes can be moved. (These diagrams are some of the generators of the algebra. We will see more of this kind of diagrams in Chapter 5.) First module basis elements m_i and m_1 have the same number of each colour propagating line.

However, the colour sequence of the nodes is different. Therefore, we can write m_i as the multiplication of m_1 with a sequence of diagrams of the algebra TL_n^2 with one crossing. Let us say

$$m_i = m_1 g_{i_1} \dots g_{i_k}.$$

We have

$$\theta(m_i) = \theta(m_1 g_{i_1} \dots g_{i_k}).$$

and hence as θ is a homomorphism we have

$$\theta(m_i) = \theta(m_1) g_{i_1} \dots g_{i_k}.$$

If we substitute for $\theta(m_1)$ we will get

$$\theta(m_i) = (\sum_j s_j n_{1j}) g_{i_1} \dots g_{i_k} = \sum_j s_j n_{1j} g_{i_1} \dots g_{i_k}.$$

Multiplication of n_{1j} by the sequence of diagrams $g_{i_1} \dots g_{i_k}$ cannot change the shape of each colour because crossing of same colour is not allowed in our algebra even though, it changes the colour sequence of the nodes. However, the southern edge colour sequence of m_i and $\theta(m_i)$ should be the same. Therefore, $n_{1j} g_{i_1} \dots g_{i_k}$ colour sequence should be same as m_i. For this reason, we can write $n_{1j} g_{i_1} \dots g_{i_k}$ as n_{ij}. From this we can say

$$\theta(m_i) = \sum_j s_j n_{ij}.$$

□

Example 4.1.2. Let us find a non-zero homomorphisms between $\Delta_4(3, 1)$ to $\Delta_4(1, 1)$. We list the possible color sequences of the half diagrams of the modules, which are

$$X_1 = X_{rrrg} \qquad X_5 = X_{rggg}$$

$$X_2 = X_{rrgr} \qquad X_6 = X_{grgg}$$

$$X_3 = X_{rgrr} \qquad X_7 = X_{ggrg}$$

$$X_4 = X_{grrr} \qquad X_8 = X_{gggr}.$$

Module $\Delta_4(3,1)$ has three red and one green propagating lines. Therefore, its basis elements colour sequences can be X_1, X_2, X_3 and X_4. However, module $\Delta_4(1,1)$ basis elements can have the colour sequence $X_1, X_2, \ldots, X_8$. Basis elements of $\Delta_4(3,1)$ are displayed in Figure 4.2 and basis elements of $\Delta_4(1,1)$ are displayed in Figure 4.3. According to our labeling system, we call the first module $\Delta_4(3,1)$ basis elements as m_1, m_2, m_3 and m_4. There is only one basis element possible for each colour sequence. However, if we consider the second module $\Delta_4(1,1)$, it has two basis elements possible for each colour sequence. Therefore, basis elements of $\Delta_4(1,1)$ can be labeled as $n_{11}, n_{12}, \ldots, n_{82}$.

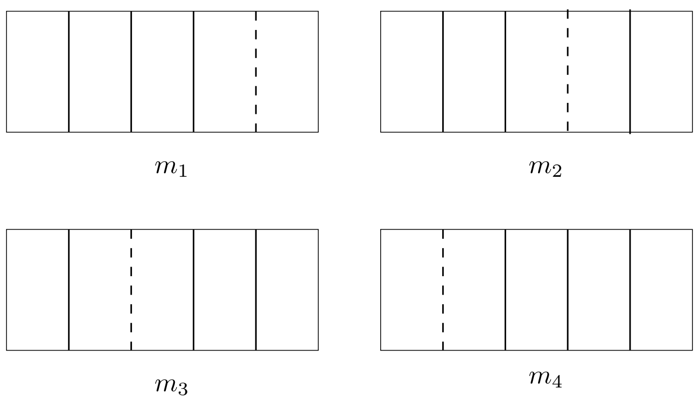


Figure 4.2: Basis elements of $\Delta_4(3,1)$

By Lemma 4.1.1, each basis element in $\Delta_4(3,1)$ maps to a linear combination of two

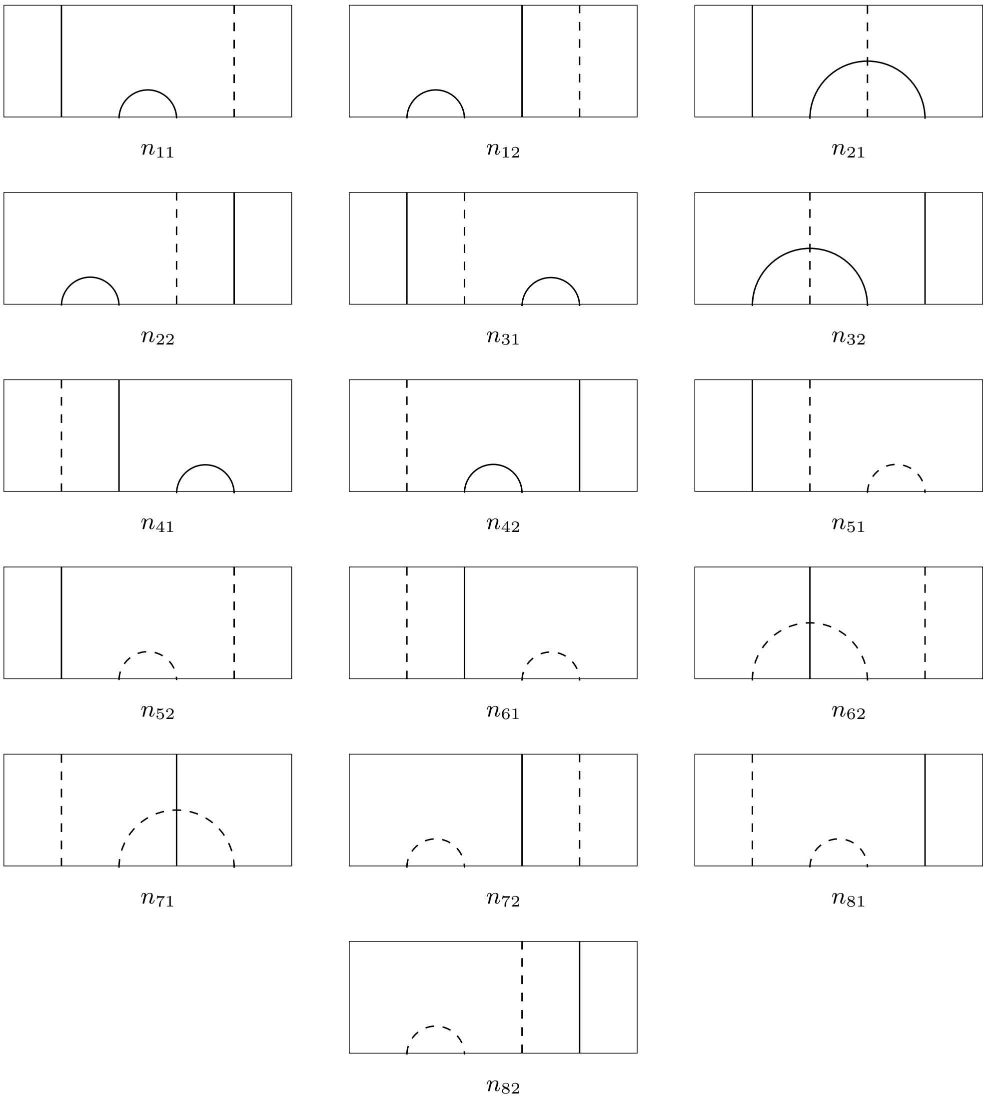


Figure 4.3: Basis elements of $\Delta_4(1,1)$

basis elements in the second module $\Delta_4(1,1)$, because there are exactly two basis elements in the second module $\Delta_4(1,1)$ with the same colour sequence. Therefore, by Lemma 4.1.1 we can say

$$\theta(m_1) = s_1 n_{11} + s_2 n_{12} \tag{4.1.3}$$

$$\theta(m_2) = s_1 n_{21} + s_2 n_{22} \tag{4.1.4}$$

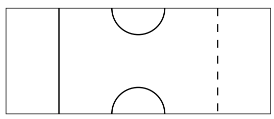
$$\theta(m_3) = s_1 n_{31} + s_2 n_{32} \tag{4.1.5}$$

$$\theta(m_4) = s_1 n_{41} + s_2 n_{42} \tag{4.1.6}$$

for some $s_1, s_2 \in \mathbb{C}$.

Finding a necessary condition

In order to obtain a necessary condition for a non-zero homomorphism to exist, we choose algebra elements with the same northern edge half diagram as the southern edge half diagram of the second module basis element. After that by taking a basis element of the first module and applying the homomorphism condition (4.1.1) we get a system of equations. By solving these we can find the solution for the unknown constants s_1 and s_2.



a

Figure 4.4:

We choose an algebra element a with the same northern and southern edge half diagram as the southern edge half diagram of n_{11} as in Figure 4.4. If we apply the homomorphism condition (4.1.1) for m_1 and the algebra element a we need

$$\theta(m_1 a) = \theta(m_1)a.$$

If we multiply m_1 by a we will get Figure 4.5. From this we can say $m_1 a$ is 0 which in turn gives us $\theta(m_1 a)$ is 0. We have already obtained an expression for $\theta(m_1)$ and by

$m_1 a =$

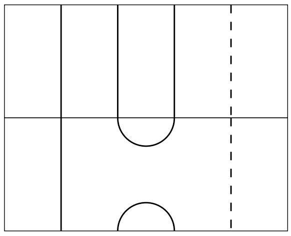
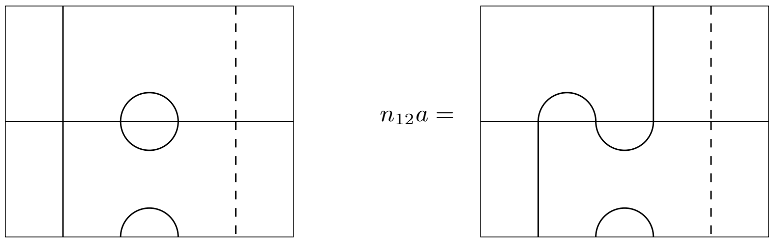


Figure 4.5:

$n_{11} a =$ $\qquad$ $n_{12} a =$



$= \delta_R n_{11}$ $\qquad$ $= n_{11}$

Figure 4.6:

substitution we obtain

$$\begin{aligned} \theta(m_1)a =& 0 \\ (s_1 n_{11} + s_2 n_{12})a =& 0 \\ s_1 n_{11} a + s_2 n_{12} a =& 0. \end{aligned}$$

By using Figure 4.6 we can get $n_{11}a = \delta_R n_{11}$ and $n_{12}a = n_{11}$. This gives us

$$\begin{aligned} s_1\delta_R n_{11} + s_2 n_{11} =& 0 \\ (s_1\delta_R + s_2)n_{11} =& 0. \end{aligned}$$

However, we know $n_{11} \neq 0$. Therefore, we can say

$$s_1\delta_R + s_2 = 0. \tag{4.1.7}$$

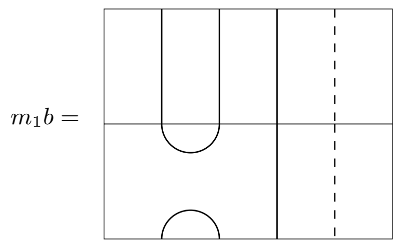


Figure 4.7:

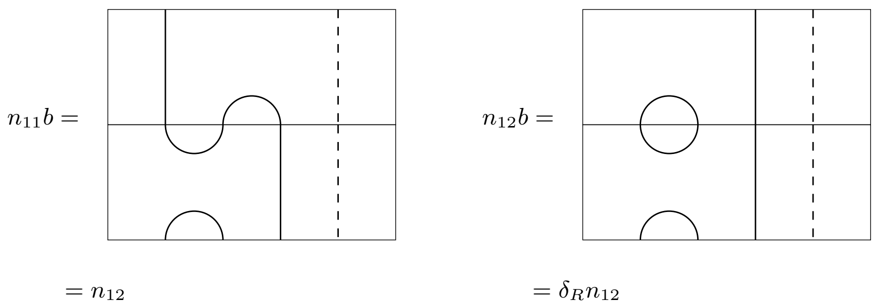


Figure 4.8:

We choose algebra element b with the northern and southern edge half diagram as the southern edge half diagram of n_{12}. If we apply the homomorphism condition (4.1.1) by choosing m as m_1 and x as b we will get

$$\theta(m_1 b) = \theta(m_1)b.$$

If we multiply m_1 by b we will get Figure 4.18. From this we can say $m_1 b$ is 0 which in turns gives us $\theta(m_1 b)$ is 0. We have already obtained an expression for $\theta(m_1)$ and by substitution we obtain

$$\begin{aligned} \theta(m_1)b =& 0 \\ (s_1 n_{11} + s_2 n_{12})b =& 0 \\ s_1 n_{11} b + s_2 n_{12} b =& 0. \end{aligned}$$

By using Figure 4.8 we can get $n_{11}b = n_{12}$ and $n_{12}b = \delta_R n_{12}$. This gives us

$$\begin{aligned} s_1 \delta_R n_{12} + s_2 n_{12} =& 0 \\ (s_1 \delta_R + s_2) n_{12} =& 0. \end{aligned}$$

However, we know $n_{12} \neq 0$. Therefore, we can say

$$s_1 + s_2 \delta_R = 0. \tag{4.1.8}$$

Forming matrix equation

We can write (4.1.7) and (4.1.8) as

$$\begin{pmatrix} \delta_R & 1 \\ 1 & \delta_R \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{4.1.9}$$

which we write as

$$\begin{pmatrix} \delta_R & 1 \\ 1 & \delta_R \end{pmatrix} \underline{S} = \underline{0}.$$

Solving matrix equation

We get non-zero solutions to $\underline{S}$ only if the determinant is zero. This gives us $\delta_R^2 - 1 = 0$, so $\delta_R = \pm 1$. If we substitute this into (4.1.9) we will get

$$s_2 = \mp s_1.$$

Finding the homomorphism

From the above equation and (4.1.3) we can say

$$\theta(m_1) = s_1 n_{11} \mp s_1 n_{12}.$$

Similarly, we can find $\theta(m_2)$, $\theta(m_3)$ and $\theta(m_4)$. These can be written as

$$\theta(m_i) = s_1(n_{i1} \mp n_{i2}).$$

Let m be an arbitrary element of the first module. Therefore, we can write

$$m = \sum_{i=1}^{4} c_i m_i$$

where $c_i \in \mathbb{C}$. If we apply θ to both sides and simplify, we get

$$\theta(m) = \sum_{i=1}^{4} c_i \theta(m_i).$$

Therefore, we can write the above equation as

$$\theta(m) = s_1 \sum_{i=1}^{4} c_i(n_{i1} \mp n_{i2}).$$

This is our non-zero homomorphism for $\delta_R = \pm 1$.

Special value of δ_R and δ_G and involvement with the Gram matrix

In order to investigate the link between the determinant of the Gram matrix and the conditions $\delta_R^2 - 1 = 0$, which is derived from the matrix equation (4.1.9), we look at the Gram matrix of the module $\Delta_4(1,1)$. This is a 16×16 matrix and to find its determinant, $\det M(\Delta_4(1,1))$, we will use the knowledge we got earlier from the Gram matrix determinant. As a first step we find the matrix associated with each colour sequence. The matrix $a(X_1)$ is constructed by the half diagrams with 3 red nodes and 1 green node, such that among these there should be 1 red propagating line, 1 green propagating line and 1 red arc. By (3.2.14) we have

$$a(X_1) = M(\Delta_{3R}(1)) \otimes M(\Delta_{1G}(1)).$$

The matrix $M(\Delta_{3R}(1))$ is given by Figure 4.9. Matrix $M(\Delta_{1G}(1))$ is constructed by the

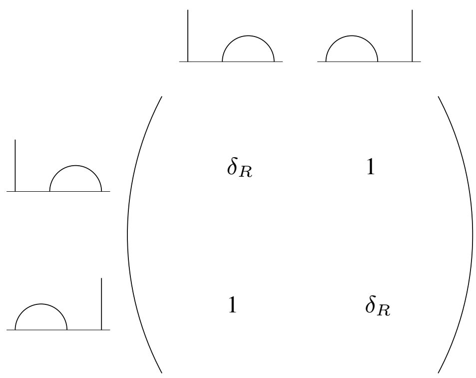


Figure 4.9:

half diagram with one green propagating line. This diagram has only one green propagating line, and hence $M(\Delta_{1G}(1))$ is the identity matrix. It follows that the determinant of $a(X_1)$ is given by

$$\det a(X_1) = (\delta_R^2 - 1)^1 \times 1^2 = \delta_R^2 - 1.$$

Similarly, we can derive the determinant of $a(X_2)$, $a(X_3)$ and $a(X_4)$ as shown below

$$\det a(X_2) = (\delta_R^2 - 1),$$
$$\det a(X_3) = (\delta_R^2 - 1),$$
$$\det a(X_4) = (\delta_R^2 - 1).$$

We use the same procedure to derive the determinant of $a(X_5)$, $a(X_6)$, $a(X_7)$ and $a(X_8)$, thus obtaining

$$\det a(X_5) = (\delta_G^2 - 1)^1 \times 1^2 = \delta_G^2 - 1,$$
$$\det a(X_6) = (\delta_G^2 - 1),$$
$$\det a(X_7) = (\delta_G^2 - 1),$$
$$\det a(X_8) = (\delta_G^2 - 1).$$

Therefore, by using (3.2.4), the determinant of $M(\Delta_4(1,1))$ can be written as

$$\begin{aligned}\det M(\Delta_4(1,1)) &= \prod_{i=1}^{8} \det a(X_i) \\ &= (\delta_R^2 - 1)^4 (\delta_G^2 - 1)^4.\end{aligned}$$

Example 4.1.3. Let us find the possible first module when the second module is $\Delta_6(2,2)$. First we obtain the Gram matrix of the module $\Delta_6(2,2)$ and find the determinant of the Gram matrix. Upon solving

$$\det M(\Delta_6(2,2)) = 0$$

we are able to derive the conditions for the possible homomorphisms from first module in terms of δ_R and δ_G. The possible first modules are $\Delta_6(4,2)$ and $\Delta_6(2,4)$ because the partial ordering in Λ of the algebra $TL_6^2(\delta_R, \delta_G)$ is

$$(2,2) \geq (4,2) \text{ and } (2,2) \geq (2,4)$$

and there are no more branches below $(4, 2)$ and $(2, 4)$. To derive the Gram matrix determinant of $\Delta_6(2, 2)$ we group the half diagrams according to the colour sequence as follows.

$$
\begin{array}{lllll}
X_1 = X_{rrrrgg} & X_2 = X_{rrrggr} & X_3 = X_{rrggrr} & X_4 = X_{rggrrr} & X_5 = X_{ggrrrr} \\
X_6 = X_{rrrgrg} & X_7 = X_{rrgrrg} & X_8 = X_{rgrrrg} & X_9 = X_{grrrrg} & X_{10} = X_{rrgrgr} \\
X_{11} = X_{rgrrgr} & X_{12} = X_{grrrgr} & X_{13} = X_{rgrgrr} & X_{14} = X_{grrgrr} & X_{15} = X_{grgrrr} \\
X_{16} = X_{rrgggg} & X_{17} = X_{grrggg} & X_{18} = X_{ggrrgg} & X_{19} = X_{gggrrg} & X_{20} = X_{ggggrr} \\
X_{21} = X_{rgrggg} & X_{22} = X_{rggrgg} & X_{23} = X_{rgggrg} & X_{24} = X_{rggggr} & X_{25} = X_{grgrgg} \\
X_{26} = X_{grggrg} & X_{27} = X_{grgggr} & X_{28} = X_{ggrgrg} & X_{29} = X_{ggrggr} & X_{30} = X_{gggrgr}.
\end{array}
$$

We obtain the matrix $a(X_1)$ by considering the half diagrams of the module $\Delta_6(2, 2)$ shown in Figure 4.10. We can simplify $a(X_1)$ as

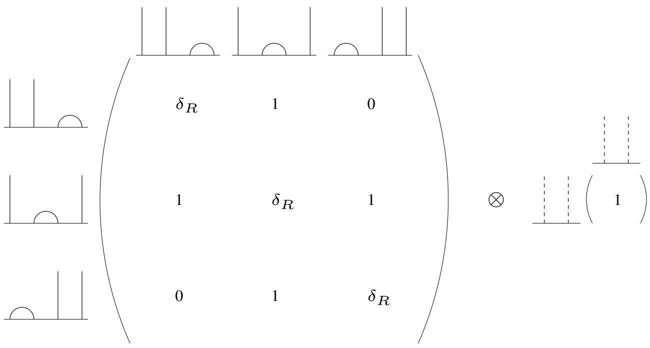


Figure 4.10:

$$
a(X_1) = \begin{pmatrix} \delta_R & 1 & 0 \\ 1 & \delta_R & 1 \\ 0 & 1 & \delta_R \end{pmatrix}
$$

and then

$$\det a(X_1) = \begin{vmatrix} \delta_R & 1 & 0 \\ 1 & \delta_R & 1 \\ 0 & 1 & \delta_R \end{vmatrix} = [\delta_R(\delta_R^2 - 1)] - \delta_R = [\delta_R(\delta_R^2 - 2)]$$

We find that each matrix $a(X_i)$ for $1 \leq i \leq 15$ is the same since each half diagrams to construct $a(X_i)$ has colour sequence X_i for $1 \leq i \leq 15$, has two red and two green propagating lines and one red arc. Hence, we have $\det a(X_i) = \delta_R(\delta_R^2 - 2)$ for $1 \leq i \leq 15$. Similarly, we find matrix $a(X_{16})$ as we did for $a(X_1)$. Figure 4.11 illustrates how we found the matrix $a(X_{16})$ which can be simplified as

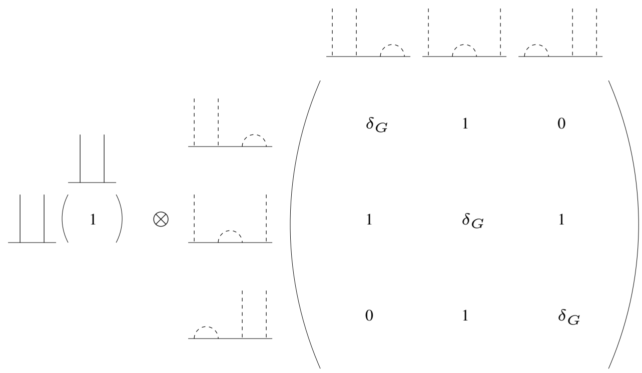


Figure 4.11:

$$a(X_{16}) = \begin{pmatrix} \delta_G & 1 & 0 \\ 1 & \delta_G & 1 \\ 0 & 1 & \delta_G \end{pmatrix}.$$

If we find the determinant of $a(X_{16})$ we get

$$\det(a(X_{16})) = [\delta_G(\delta_G^2 - 2)].$$

We find that each matrix $a(X_i)$ for $16 \leq i \leq 30$ is the same since each colour sequence has two red and two green propagating line and one green arc. Therefore, we have $\det a(X_i) = \delta_R(\delta_R^2 - 2)$ for $16 \leq i \leq 30$.

We can write the determinant of our Gram matrix $M(\Delta_6(2,2))$ by using (3.2.4) as

$$\begin{aligned}\det M(\Delta_6(2,2)) &= \prod_{i=1}^{30} \det a(X_i) \\ &= [\delta_R(\delta_R^2 - 2)]^{15}[\delta_G(\delta_G^2 - 2)]^{15}.\end{aligned}$$

Therefore, we are able to obtain the solution to $\det M(\Delta_6(2,2)) = 0$ from which we derive the condition $\delta_R(\delta_R^2 - 2) = 0$ which will give the non-zero homomorphism from $\Delta_6(4,2)$ to $\Delta_6(2,2)$ and the condition $\delta_G(\delta_G^2 - 2) = 0$ which will give the non-zero homomorphism from $\Delta_6(2,4)$ to $\Delta_6(2,2)$.

Definition 4.1.4. Let R_n be the $n \times n$ matrix

$$\begin{pmatrix} \delta_R & 1 & 0 & 0 & \cdots & 0 \\ 1 & \delta_R & 1 & 0 & \cdots & 0 \\ 0 & 1 & \delta_R & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & & \vdots \\ & & & & \ddots & \\ 0 & 0 & 0 & \cdots & \cdots & \delta_R \end{pmatrix}, \tag{4.1.10}$$

where δ_R is a scalar.

We can find $\det R_n$ by using the following difference equation.

Lemma 4.1.5. *The determinant of R_n can be obtained by the difference equation*

$$|R_{n+2}| = \delta_R |R_{n+1}| - |R_n|, \tag{4.1.11}$$

where $|R_1| = \delta_R$ and $|R_2| = \delta_R^2 - 1$.

Proof. Let us consider the determinant of R_{n+2}.

We can find the determinant of any matrix by expanding any of the row or column. If we expand $|R_{n+2}|$ by using the first row, then we will get

$$|R_{n+2}| = \delta_R \begin{vmatrix} \delta_R & 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & \delta_R & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \delta_R & 1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \delta_R & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & \delta_R \end{vmatrix}_{(n+1)\times(n+1)}$$
$$- \begin{vmatrix} 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \delta_R & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \delta_R & 1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \delta_R & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & \delta_R \end{vmatrix}_{(n+1)\times(n+1)}.$$

According to our notation in (4.1.10), the first determinant in the above equation can be written as $|R_{n+1}|$. Let us label the second determinant $|M|$. Therefore, we can say

$$|R_{n+2}| = \delta_R |R_{n+1}| - |M|. \tag{4.1.12}$$

We can expand $|M|$ as follows

$$|M| = \begin{vmatrix} \delta_R & 1 & 0 & \cdots & 0 & 0 \\ 1 & \delta_R & 1 & \cdots & 0 & 0 \\ 0 & 1 & \delta_R & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & \delta_R \end{vmatrix}_{n\times n} - \begin{vmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & \delta_R & 1 & \cdots & 0 & 0 \\ 0 & 1 & \delta_R & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & \delta_R \end{vmatrix}_{n\times n} .$$

In the above equation, the first determinant is $|R_n|$ and the second is 0 because the first column of this matrix has all entries 0. Thus

$$|R_{n+2}| = \delta_R|R_{n+1}| - |R_n|$$

as required. □

Lemma 4.1.5 will have a variety of applications later on.

4.1.2 Introducing the matrix corresponding to the homomorphism

Example 4.1.6. We find a homomorphism from $\Delta_6(4,2)$ to $\Delta_6(2,2)$, where m_1 in Figure 4.12 is a basis element of $\Delta_6(4,2)$. By using the permutation of 4 red and 2 green

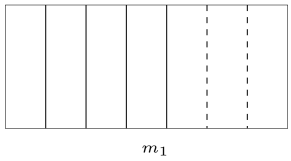


Figure 4.12:

propagating lines we can say there are 15 basis elements for $\Delta_6(4,2)$. Basis elements of $\Delta_6(2,2)$, which have the same colour sequence as southern edge of m_1, is displayed in Figure 4.13. Therefore, we write the map $\theta(m_1)$ as the linear combination of the basis element of n_{11}, n_{12} and n_3 thus obtaining

$$\theta(m_1) = s_1 n_{11} + s_{12} n_{12} + s_3 n_{13},$$

where s_1, s_2 and s_3 are unknown constants needed to find. Suppose we choose the algebra

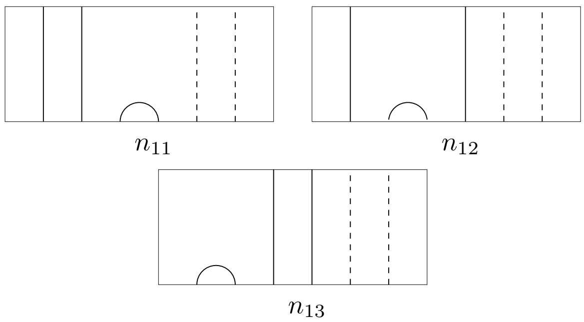


Figure 4.13:

elements a, b and c as in Figure 4.14 to find the conditions for non-zero homomorphism. We use the condition that

$$\theta(m_1 a) = 0$$

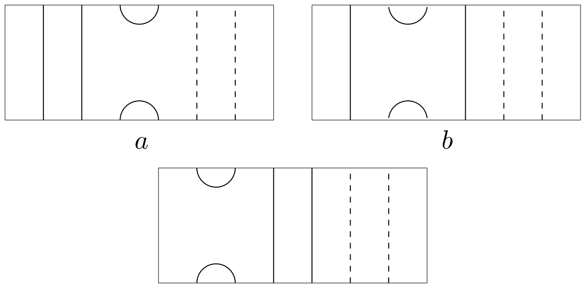


Figure 4.14:

along with $\theta(m_1)a$ which is given by

$$\begin{aligned}\theta(m_1)a &= (s_1 n_{11} + s_2 n_{12} + s_3 n_{13})a \\ &= s_1 n_{11} a + s_2 n_{12} a + s_3 n_{13} a.\end{aligned}$$

From Figure 4.15 we can say $n_{11}a = \delta_R n_{11}$, $n_{12}a = n_{11}$ and $n_{13}a = 0$. From this we can say

$$\begin{aligned}\theta(m_1)a &= s_1 \delta_R n_{11} + s_2 n_{11} \\ &= (s_1 \delta_R + s_2) n_{11}.\end{aligned}$$

However, with $\theta(m_1 a) = \theta(m_1)a$ and $n_{11} \neq 0$ we can explicitly derive

$$s_1 \delta_R + s_2 = 0. \tag{4.1.13}$$

By considering $m_1 b$ and $m_1 c$ we see in a similar way that

$$s_1 + s_2 \delta_R + s_3 = 0. \tag{4.1.14}$$

and

$$s_2 + s_3 \delta_R = 0. \tag{4.1.15}$$

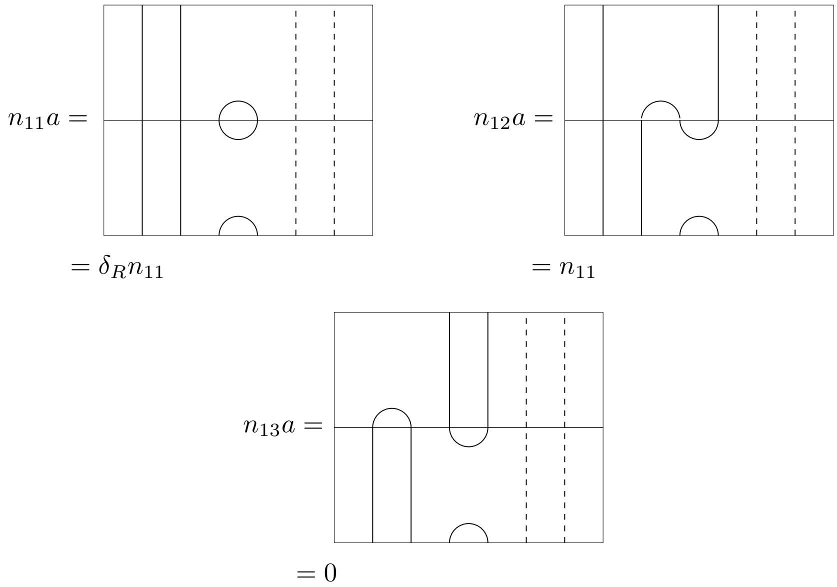


Figure 4.15:

If we solve (4.1.14) and (4.1.15) we obtain

$$s_1\delta_R + s_2\delta_R^2 - s_2 = 0 \tag{4.1.16}$$

which can be simplified to

$$s_1\delta_R + s_2(\delta_R^2 - 1) = 0. \tag{4.1.17}$$

By solving (4.1.13) and (4.1.17) we obtain

$$-s_2 + s_2(\delta_R^2 - 1) = 0$$

which can be simplified to

$$s_2(\delta_R^2 - 2) = 0. \tag{4.1.18}$$

If $\delta_R^2 - 2 \neq 0$ then (4.1.18) implies $s_2 = 0$ and if $\delta_R \neq 0$ then (4.1.17) implies $s_1 = 0$ and (4.1.14) implies $s_3 = 0$. Therefore, $\theta(m_1) = 0$ and we can show that $\theta(m_i) = 0$ for all $i = 1, 2, ..., 15$. Since all the basis elements are mapped to 0, we can say there is no non-zero homomorphism from $\Delta_6(4,2)$ to $\Delta_6(2,2)$. If $\delta_R^2 - 2 \neq 0$ and $\delta_R = 0$ then $s_2 = 0$ and (4.1.14) implies that $s_1 + s_3 = 0$. Therefore, $s_3 = -s_1$ and thus

$$\begin{aligned}\theta(m_1) &= s_1 n_{11} + s_2 n_{12} + s_3 n_{13} \\ &= s_1 n_{11} - s_1 n_{13} \\ &= s_1(n_{11} - n_{13}).\end{aligned}$$

Similarly, we can show

$$\theta(m_i) = s_1(n_{i1} - n_{i3})$$

for all i taking the values 1, 2,...,15. Where n_{i1} and n_{i3} are the basis element of $\Delta_6(2,2)$ with the same colour sequence as the southern edge of m_i.

Let m be an arbitrary element of the first module

$$m = \sum_{i=1}^{15} c_i m_i.$$

Then when $\delta_R = 0$ the homomorphism can be written as

$$\theta(m) = s_1 \sum_{i=1}^{15} c_i(n_{i1} - n_{i3}).$$

If $\delta_R = \pm\sqrt{2}$ then

$$\begin{aligned}s_2 &= \mp\sqrt{2}s_1 \\ s_3 &= -\frac{1}{\delta_R}s_2 \\ s_3 &= s_1\end{aligned}$$

and we can deduce that

$$\theta(m) = s_1 \sum_{i=1}^{15} c_i(n_{i1} \mp \sqrt{2}n_{i2} + n_{i3}). \tag{4.1.19}$$

Formation of matrix equation

Equations (4.1.13), (4.1.14) and (4.1.15) can be written as

$$\begin{pmatrix} \delta_R & 1 & 0 \\ 1 & \delta_R & 1 \\ 0 & 1 & \delta_R \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{4.1.20}$$

Therefore, the matrix equation (4.1.20) can be written as

$$\mathbf{R_3}\underline{\mathbf{S}} = \underline{\mathbf{0}}, \tag{4.1.21}$$

where $\mathbf{R_3}$ is the matrix in (4.1.10). We have derived this matrix when calculating the determinant of the Gram matrix of $\Delta_6(2,2)$. This is actually the matrix $a(X_i)$, for $1 \leq i \leq 15$. There is a connection between the basis elements of $\Delta_6(4,2)$ and the half diagrams used to construct $a(X_i)$. Those half diagrams with four red nodes and two green nodes are the basis elements of $\Delta_6(4,2)$ and the X_i's colour sequence is fixed. For example, consider the matrix constructed by choosing a basis element m_1 in Figure 4.12 of $\Delta_6(4,2)$ and multiply by the selected algebra elements. We get a set of equations which can be written as in (4.1.21) and the matrix form at the front R_3 is same as choosing one of the colour sequence with same number of red and green nodes as m_1, for example X_1, and finding the matrix of it. We call the matrix $a(X_1)$ the matrix corresponding to the homomorphism.

We can use the Gram matrix to work out the homomorphism between two modules. The following result will explain how we can find the homomorphism between two modules.

Definition 4.1.7. Let $\Delta_n(\lambda)$ and $\Delta_n(\mu)$ be two modules, where λ and μ are h tuples. If X is a common colour sequence for both modules then let

$$a(X)\underline{S} = \underline{0} \tag{4.1.22}$$

be the matrix equation needed to be solved to find the non-zero homomorphism, where $\underline{S}$ is the column vector which contain all the coefficients of the basis element map. Matrix $a(X)$ being the part of the Gram matrix of the module $\Delta_n(\mu)$ for the colour sequence X. We call this the **matrix corresponding to the homomorphism**. In this chapter our modules can have two colour arcs or strings. Therefore, h takes the value two.

The matrix corresponding to the homomorphism

$$\theta : \Delta_{n+1}(n+1,0) \longrightarrow \Delta_{n+1}(n-1,0) \tag{4.1.23}$$

can be written as $a(X) = R_n$, where X is the colour sequence and R_n is given by (4.1.10). In this example all the nodes are red.

Finding the special coefficient by using the Gram matrix

Let the homomorphism be

$$\theta : \Delta_n(a,b) \longrightarrow \Delta_n(a-2,b)$$

where $a+b=n$. First we find the matrix corresponding to this homomorphism. Therefore, look at the colour sequence $X_1 = X_{r\ldots rg\ldots g}$ in $\Delta_n(a-2,b)$, which is a collection of half diagrams with first a red nodes and last b green nodes. These half diagrams have $a-2$ red propagating lines, b green propagating lines and one red arc in it. Therefore, matrix $a(X_1)$ can be given by

$$a(X_1) = R_{a-1} \otimes (1). \tag{4.1.24}$$

This can be simplified into

$$a(X_1) = R_{a-1}.$$

When we construct the matrix $a(X_1)$ by using the half diagrams, we can see only the red bit matches together and the green bit matches together. Therefore, we can separate the red part and green part which leads to writing the matrix as tensor product in (4.1.24). R_{a-1} is normally the Gram matrix of the module $\Delta_a(a-2)$ where all the basis elements have red nodes.

Determinant $a(X_1) = 0$ gives the value of δ_R for which we get non-zero homomorphism. When δ_R take a value other than this value we cannot find the non zero homomorphism.

Replace the δ_R value by those special value in R_{a-1} and solve

$$\mathbf{R_{a-1}}\underline{\mathbf{S}} = \underline{\mathbf{0}}. \tag{4.1.25}$$

Here, $\underline{S}$ is the column vector of the special coefficients. By row reducing R_{a-1} we can calculate $\underline{S}$ quite easily.

Example 4.1.8. Let us find a non-zero homomorphism $\theta : \Delta_6(4,2) \longrightarrow \Delta_6(2,2)$ again by using the matrix corresponding to the homomorphism idea. First we find the matrix correspond to the above homomorphism. Actually, it is one of the matrices which comes in the calculation of the determinant of the Gram matrix $\Delta_6(2,2)$. We need a colour sequence with four red and two green nodes because it should be common for both modules. Therefore, we look at the colour sequence $X_1 = X_{rrrrgg}$. Matrix associated with this colour sequence, $a(X_1)$, given by the tensor product of two matrices as below. Figure 4.10 illustrate this formation.

$$\begin{aligned} a(X_1) &= \begin{pmatrix} \delta_R & 1 & 0 \\ 1 & \delta_R & 1 \\ 0 & 1 & \delta_R \end{pmatrix} \otimes (1) \\ &= R_3 \otimes (1) \\ &= R_3. \end{aligned}$$

We know that R_3 is a function of δ_R. Suppose determinant of $a(X_1) = 0$, this will give the possible values of δ_R for which we get non-zero homomorphism, that is determinant of $R_3 = 0$ which gives us

$$\delta_R(\delta_R^2 - 1) - 1.\delta_R = 0$$

This can be simplified into

$$\delta_R(\delta_R^2 - 2) = 0.$$

Therefore,

$$\delta_R = 0 \text{ or } \delta_R = \pm\sqrt{2}$$

When $\delta_R = 0$ matrix R_3 becomes as

$$R_3(0) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$R_3(0)S_3 = \underline{0}.$$

According to (4.1.22), matrix equation can be given by

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} = \underline{0}.$$

If we simplify this we will get

$$s_2 = 0, \quad s_1 + s_3 = 0.$$

This implies that

$$s_2 = 0, \quad s_3 = -s_1.$$

Therefore, by using Lemma 4.1.1, we can say,

$$\theta(m_i) = s_1(n_{i1} - n_{i2}),$$

for all $i = 1, 2, \ldots, 15$. Let m be an arbitrary element of the first module

$$m = \sum_{i=1}^{15} c_i m_i,$$

where $c_i \in \mathbb{C}$. Then

$$\theta(m) = s_1 \sum_{i=1}^{15} c_i(n_{i1} - n_{i2}).$$

When $\delta_R = \pm\sqrt{2}$ matrix R_3 becomes

$$R_3(\pm\sqrt{2}) = \begin{pmatrix} \pm\sqrt{2} & 1 & 0 \\ 1 & \pm\sqrt{2} & 1 \\ 0 & 1 & \pm\sqrt{2} \end{pmatrix}.$$

If we row reduce this, we will get

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & \pm\sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}.$$

Therefore, our matrix equation becomes

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & \pm\sqrt{2} \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} = \underline{0},$$

which gives us

$$s_2 \pm \sqrt{2} s_2 = 0, \quad s_1 - s_2 = 0.$$

This implies that

$$s_2 = \mp\sqrt{2} s_1, \quad s_3 = s_1.$$

From this, we deduce that the homomorphism can be written as

$$\theta(m) = s_1 \sum_{i=1}^{15} c_i(n_{i1} \mp \sqrt{2} n_{i2} + n_{i3}).$$

4.1.3 Verifying that the matrix corresponding to the homomorphism and the matrix coming from the matrix equation are the same

Example 4.1.9. Equations we get by choosing the algebra element multiplied with the basis element could be written in the matrix equation format and the matrix corresponding to the homomorphism which appears in the Gram matrix calculations are the same. We investigate this statement through the following homomorphism

$$\theta : \Delta_4(3,1) \longrightarrow \Delta_4(1,1).$$

We found this non zero homomorphism in Example 4.1.2. In this Example we obtained

$$s_1\delta_R + s_2 = 0 \tag{4.1.26}$$

$$s_1 + s_2\delta_R = 0 \tag{4.1.27}$$

We can write (4.1.26) and (4.1.27) as

$$\begin{pmatrix} \delta_R & 1 \\ 1 & \delta_R \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0. \tag{4.1.28}$$

This can be written as

$$\mathbf{R_2}\underline{\mathbf{S}} = \underline{\mathbf{0}}$$

Now we look at the module $\Delta_4(1,1)$. When we work out the Gram matrix of $\Delta_4(1,1)$, we look at the following major groups of half diagrams.

$$X_1 = X_{rrrg}$$

$$X_2 = X_{rggg}$$

First module $\Delta_4(3,1)$ has basis element with three red and one green nodes. Therefore, the colour sequence corresponds to the homomorphism is X_1. Let us find the matrix $a(X_1)$ which is displayed in Figure 4.16. We obtain

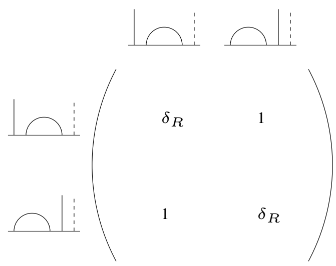


Figure 4.16:

$$a(X_1) = \begin{pmatrix} \delta_R & 1 \\ 1 & \delta_R \end{pmatrix}$$

This is the matrix in (4.1.28). Therefore,

$$a(X_1)\underline{\mathbf{S}} = \underline{\mathbf{0}}.$$

Let us see why we end up with the same equations in both ways

Half diagrams correspond to the basis element n_{11} and n_{12} are in Figure 4.17. Similarly, the same half diagrams correspond to the top edge of the algebra elements a and b respectively. First module basis element m_1 mapped to $s_1 n_{11} + s_2 n_{12}$. Here s_1 and s_2 are unknown constants in $\mathbb{C}$. Multiplication of m_1 and a become 0 because of the wrong number of

Figure 4.17: Half diagrams of the basis elements n_{11} and n_{12}

propagating lines. Therefore, $\theta(m_1 a)$ also 0. This gives $\theta(m_1)a = 0$. However,

$$\theta(m_1)a = s_1 n_{11} a + s_2 n_{12} a$$

From Figure 4.18 we know $n_{11}a = \delta_R n_{11}$ and $n_{12}a = n_{11}$. From this we can get

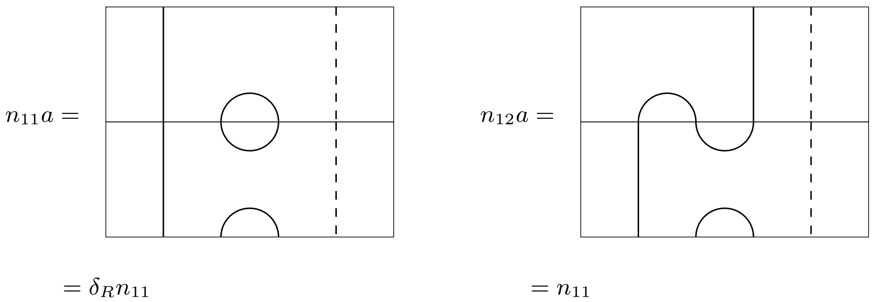


Figure 4.18:

$$\theta(m_1)a = (s_1\delta_R + s_2.1)n_{11}.$$

From this and $\theta(m_1)a = 0$ we can say

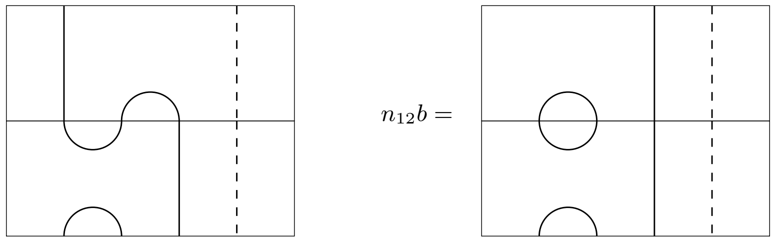
$$s_1\delta_R + s_2.1 = 0 \tag{4.1.29}$$

If you look at the middle line of $n_{11}a$ and $n_{12}a$ in Figure 4.18, it corresponds to the first row of our matrix $a(X_1)$ in Figure 4.16.

Similarly,

$$\theta(m_1)b = s_1 n_{11} b + s_2 n_{12} b$$

$n_{11}b =$ $\qquad$ $n_{12}b =$



$= n_{12}$ $\qquad$ $= \delta_R n_{12}$

Figure 4.19:

By using Figure 4.19 we can get $n_{11}b = n_{12}$ and $n_{12}b = \delta_R n_{12}$. If you look at the middle line of $n_{11}b$ and $n_{12}b$, it corresponds to the second row of our matrix $a(X_1)$ in Figure 4.16. Multiplication of m_1 and b become 0 because of the wrong number of propagating lines. Therefore, $\theta(m_1 b)$ also 0. This implies that $\theta(m_1)b = 0$. However,

$$\theta(m_1)b = (s_1.1 + s_2\delta_R)n_{12}$$

Therefore, we can say

$$s_1.1 + s_2\delta_R = 0. \tag{4.1.30}$$

Equations (4.1.29) and (4.1.30) gives us

$$\begin{pmatrix} \delta_R & 1 \\ 1 & \delta_R \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Both methods lead to the same equation.

Finding the homomorphism in the general case

How can we find a non-zero homomorphism θ from the module $\Delta_n(a, b)$ with no arcs to the module $\Delta_n(c, d)$ with one arc as listed in (4.1.31)?

$$\theta : \Delta_n(a, b) \longrightarrow \Delta_n(c, d), \tag{4.1.31}$$

where $a+b=n$ and either $a-c=2$ or $b-d=2$.

First we list all possible colour sequences which help to find the determinant of the Gram matrix $\Delta_n(c,d)$. After that, we choose the colour sequence from this list with first a nodes red and last b nodes green and call it X_1. The colour sequence X_1 has c number of red and d number of green propagating lines. We construct the matrix $a\,(X_1)$ by considering the half diagrams in order, which have the colour sequence X_1, and drawing one half diagram up and the other half diagram down. This is exactly the same way that we calculate the Gram matrix. The matrix $a\,(X_1)$ can be seen inside the Gram matrix of $\Delta_n(c,d)$.

We find the determinant of $a\,(X_1)$ and solve $\det a\,(X_1)=0$, which gives the special values of either δ_R or δ_G for which $\Delta_n(c,d)$ is reducible. For each of these special values of δ solve the equation

$$a\,(X_1)\,\underline{\mathbf{S}}=\underline{\mathbf{0}}. \tag{4.1.32}$$

Let m_1 be a basis element of $\Delta_n(a,b)$ with colour sequence X_1, and n_{1j} be the basis elements of $\Delta_n(c,d)$ with the same colour sequence. Therefore, we can write

$$\theta(m_1)=\sum_{j=1}^{t}s_j n_{1j}.$$

Here s_j for all $j=1,\ldots,t$ are solution of (4.1.32) for each special value of either δ_R or δ_G. From Lemma 4.1.1, we can say all the basis elements of $\Delta_n(a,b)$ also map in the same way as m_1 to the basis element of the module $\Delta_n(c,d)$. Therefore, if m is an arbitrary element of the module $\Delta_n(a,b)$:

$$m=\sum_i c_i m_i$$

then we claim that

$$\theta(m)=\sum_i c_i\sum_{j=1}^{t}s_j n_{ij}.$$

is a non-zero homomorphism from $\Delta_n(a,b)$ to $\Delta_n(c,d)$.

We will prove this claim in the next section.

4.1.4 Proving the constructed map is a homomorphism

We would like to use the following notation for modules and algebra basis elements to prove the Theorem 4.1.10. At this instance, we need to thank Prof.Robert Marsh, who helped us to improve the notation of modules and algebra basis elements and the proof of the Theorem. **Introduction to the notation for modules and algebra elements**

Our modules and algebra have two colour nodes. Therefore, more than one colour sequence is possible for the nodes. We denote the ith colour sequence by X_i. Module $\Delta_n(a, b)$ has all nodes propagating. Therefore, there will be only one basis element possible for each colour sequence. We call that basis element m_i. If we consider the module $\Delta_n(c, d)$ there will be more than one basis element for each colour sequence. Therefore, we label the basis elements by n_{ij} where i represents the colour sequence X_i and touple j represent a collection of uncoloured half-diagrams, one for each colour (with the number of nodes in the half diagram associated to colour C_r equal to the number of nodes coloured C_r in the colour sequence i). We label the colour sequence as X_1 if it has first a nodes red colour and the next b nodes green colour.

A basis diagram of the bubble algebra can be regarded as a pair of half diagrams. (by cutting in two along a horizontal line). Each half-diagram has its own colour sequence along the boundary. Such a half-diagram can in turn be regarded as a pair, consisting of its colour sequence together with a tuple of uncoloured half-diagrams, one for each colour- obtained by restricting to the lines of that fixed colour.

Thus an algebra basis diagram can be written in the form $x_{u,u'}$ where u, u' are half-diagrams. In turn, u and u' can be written $u = (\alpha, k)$ where X_α is the colour sequence of u and k is its tuple of uncoloured half diagrams (i.e. one for each colour). Similarly $u' = (\alpha', k')$. Thus we have $x_{u,u'} = x_{\alpha,k,\alpha',k'}$.

Theorem 4.1.10. *Suppose that the s_k satisfy the matrix equation*

$$a\,(X_1)\,\underline{\mathbf{S}} = \underline{\mathbf{0}} \tag{4.1.33}$$

where $\underline{S} = (s_k)$ and $\underline{S} \neq \underline{0}$, and $a\,(X_1)$ is the "matrix corresponding to the homomorphism" between the two given modules. Then, the linear map

$$\theta(\sum_i c_i m_i) = \sum_i c_i \sum_k s_k n_{i,k} \tag{4.1.34}$$

is a non-zero homomorphism from $\Delta_n(a,b)$ to $\Delta_n(c,d)$.

Proof. Let us show that the constructed map is a non-zero homomorphism. Let m be an arbitrary element of the module $\Delta_n(a,b)$ and x be an arbitrary element of the algebra $TL_n^2(\delta_R, \delta_G)$. We can write x as the linear combination of the basis element of the algebra as follows

$$x = \sum_{\alpha,k,\alpha',k'} d_{\alpha,k,\alpha',k'} x_{\alpha,k,\alpha',k'} + \sum_{\alpha,\alpha'} e_{\alpha,\alpha'} y_{\alpha,\alpha'} + \sum_{u,v} f_{u,v} z_{u,v}. \tag{4.1.35}$$

Here $x_{\alpha,k,\alpha',k'}$ represents the bubble algebra basis element with colour sequence X_α on the northern edge and tuple k of uncloured half-diagrams and colour sequence $X_{\alpha'}$ on the southern edge and tuple k' of uncoloured half-diagrams. The first sum runs over basis elements $x_{\alpha,k,\alpha',k'}$ for which X_α is the colour sequence of some basis element m_α in $\Delta_n(a,b)$ and (α, k), representing the (coloured) northern half-diagram, is a basis element of $\Delta_n(c,d)$.

The second sum runs over basis elements $y_{\alpha,\alpha'} = x_{\alpha,k_0,\alpha',k_0}$ where k_0 is the tuple of uncoloured half-diagrams which have all lines propagating and X_α is again a colour sequence of some basis element of $\Delta_n(a,b)$. Thus $y_{\alpha,\alpha'}$ is an algebra basis element with all lines propagating and colour sequence X_α at the northern edge and colour sequence $X_{\alpha'}$ at the southern edge.

Finally, $z_{u,v}$ is the bubble algebra basis element with northern (coloured) half-diagram u and southern (coloured) half-diagram v. The third sum is over such elements with the property that either the northern edge colour sequence does not occur in $\Delta_n(a,b)$ or, if it does, then either the northern half diagram (u) is not a basis element of the module $\Delta_n(c,d)$ or it is not the case that all lines in $z_{u,v}$ are propagating.

We know m_i is a basis element of $\Delta_n(a,b)$, which has the colour sequence X_i. Therefore, if we find $m_i x$ we will get

$$\begin{aligned} m_i x &= m_i \left(\sum_{\alpha,k,\alpha',k'} d_{\alpha,k,\alpha',k'} x_{\alpha,k,\alpha',k'} + \sum_{\alpha,\alpha'} e_{\alpha,\alpha'} y_{\alpha,\alpha'} + \sum_{u,v} f_{u,v} z_{u,v} \right), \\ &= \sum_{\alpha,k,\alpha',k'} d_{\alpha,k,\alpha',k'} m_i x_{\alpha,k,\alpha',k'} + \sum_{\alpha,\alpha'} e_{\alpha,\alpha'} m_i y_{\alpha,\alpha'} + \sum_{u,v} f_{u,v} m_i z_{u,v}. \end{aligned}$$

Consider the sum $\sum_{\alpha,k,\alpha',k'} d_{\alpha,k,\alpha',k'} m_i x_{\alpha,k,\alpha',k'}$. When $\alpha = i$, $m_i x_{\alpha,k,\alpha',k'} = 0$ since it gives the wrong number of propagating lines. When $\alpha \neq i$, the colour sequences do not match, so $m_i x_{\alpha,k,\alpha',k'} = 0$. Hence

$$\sum_{\alpha,k,\alpha',k'} d_{\alpha,k,\alpha',k'} m_i x_{\alpha,k,\alpha',k'} = 0.$$

Next consider the sum $\sum_{\alpha,\alpha'} e_{\alpha,\alpha'} m_i y_{\alpha,\alpha'}$. Since, if $\alpha \neq i$, $m_i y_{\alpha,\alpha'} = 0$ (as the colour sequences do not match). Hence

$$\sum_{\alpha,\alpha'} e_{\alpha,\alpha'} m_i y_{\alpha,\alpha'} = \sum_{\alpha'} e_{i,\alpha'} m_i y_{i,\alpha'}.$$

The last sum, $\sum_{u,v} f_{u,v} m_i z_{u,v}$ is zero, since either m_i and $z_{u,v}$ do not match colour sequences or the concatenation has the wrong number of propagating lines.(Since, if the colour sequences match, $z_{u,v}$ does not have all propagating lines.)

From these we can say

$$m_i x = \sum_{\alpha'} e_{i,\alpha'} m_i y_{i,\alpha'}.$$

Since, $y_{i,\alpha'}$ has all lines propagating (and so does m_i), we see that $m_i y_{i,\alpha'} = m_{\alpha'}$. Hence

$$m_i x = \sum_{\alpha'} e_{i,\alpha'} m_{\alpha'} = \sum_{\alpha} e_{i,\alpha} m_\alpha.$$

If $m = \sum_i c_i m_i$ is an arbitrary element of $\Delta_n(a, b)$, we have

$$\begin{aligned} mx &= \sum_i c_i m_i x \\ &= \sum_i c_i \sum_\alpha e_{i,\alpha} m_\alpha \\ &= \sum_{\alpha,i} c_i e_{i,\alpha} m_\alpha. \end{aligned}$$

We have

$$\begin{aligned} \theta(mx) &= \sum_{\alpha,i} c_i e_{i,\alpha} \theta(m_\alpha) \\ &= \sum_{\alpha,i} c_i e_{i,\alpha} \sum_k s_k n_{\alpha,k} \\ &= \sum_{i,k,\alpha} c_i e_{i,\alpha} s_k n_{\alpha,k}. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \theta(m)x &= \theta(\sum_i c_i m_i)x \\ &= \sum_i c_i \theta(m_i) x \\ &= \sum_i c_i \sum_k s_k n_{i,k} x \\ &= \sum_{i,l} c_i s_l n_{i,l} x \end{aligned}$$

(changing name of variable k)

Substituting for x, we have:

$$\theta(m)x = \theta(m)\sum_{\alpha,k,\alpha',k'} d_{\alpha,k,\alpha',k'}x_{\alpha,k,\alpha',k'}$$
$$+ \theta(m)\sum_{\alpha,\alpha'} e_{\alpha,\alpha'}y_{\alpha,\alpha'}$$
$$+ \theta(m)\sum_{u,v} f_{u,v}z_{u,v}.$$

The first term is

$$\theta(m)\sum_{\alpha,k,\alpha',k'} d_{\alpha,k,\alpha',k'}x_{\alpha,k,\alpha',k'} = \sum_{i,l} c_i s_l n_{i,l} \sum_{\alpha,k,\alpha',k'} d_{\alpha,k,\alpha',k'}x_{\alpha,k,\alpha',k'}$$
$$= \sum_{i,l,\alpha,k,\alpha',k'} c_i s_l d_{\alpha,k,\alpha',k'} n_{i,l} x_{\alpha,k,\alpha',k'}$$
$$= \sum_{i,l,k,\alpha',k'} c_i s_l d_{i,k,\alpha',k'} n_{i,l} x_{i,k,\alpha',k'}.$$

(Since the colours must match). From Proposition 1.1.4(iii) we can write the above sum as

$$\sum_{i,l,k,\alpha',k'} c_i s_l d_{i,k,\alpha',k'} n_{i,l} x_{i,k,\alpha',k'} = \sum_{i,l,k,\alpha',k'} c_i s_l d_{i,k,\alpha',k'} \langle n_{i,l}, n_{i,k}\rangle n_{\alpha',k'}.$$

Note that X_1 and X_i are both colour sequences for $\Delta_n(a,b)$. So, there is an algebra element u_{1i} with all lines propagating such that $m_1 u_{1i} = m_i$. We then have $n_{1l}u_{1i} = n_{i,l}$ and $n_{1,k}u_{1,i} = n_{i,k}$. Hence

$$\langle n_{i,l}, n_{i,k}\rangle = \langle n_{1l}u_{1i}, n_{1,k}u_{1,i}\rangle.$$

From Proposition 1.1.4(ii)

$$\langle n_{1l}u_{1i}, n_{1,k}u_{1,i}\rangle = \langle n_{1l}, n_{1,k}u_{1,i}u_{1i}^*\rangle.$$

But $u_{1,i}u_{1i}^*$ has all lines propagating and no crossings, with northern and southern colour sequence X_1. Hence $n_{1,k}u_{1,i}u^*_{1i} = n_{1k}$. Therefore

$$\begin{aligned}\langle n_{i,l}, n_{i,k}\rangle &= \langle n_{1,l}, n_{1,k}\rangle \\ &= a(X_1)_{lk} \\ &= a(X_1)_{kl}.\end{aligned}$$

Note $\langle , \rangle$ is symmetric by Proposition 1.1.4(i). Thus we have

$$\sum_{i,l,k,\alpha',k'} c_i s_l d_{i,k,\alpha',k'} \langle n_{i,l}, n_{i,k}\rangle n_{\alpha',k'} = \sum_{i,l,k,\alpha',k'} c_i d_{i,k,\alpha',k'} \left(\sum_l s_l a(X_1)_{kl}\right) n_{\alpha',k'}.$$

Since $a(X_1)\underline{S} = \underline{0}$,

$$\sum_l s_l a(X_1)_{kl} = 0.$$

So, this reduce to zero. That is

$$\theta(m) \sum_{\alpha,k,\alpha',k'} d_{\alpha,k,\alpha',k'} x_{\alpha,k,\alpha',k'} = 0.$$

The second term is

$$\begin{aligned}\theta(m)\sum_{\alpha,\alpha'} e_{\alpha,\alpha'} y_{\alpha,\alpha'} &= \left(\sum_i c_i \sum_l s_l n_{i,l}\right)\left(\sum_{\alpha,\alpha'} e_{\alpha,\alpha'} y_{\alpha,\alpha'}\right) \\ &= \sum_{i,l,\alpha,\alpha'} c_i s_l e_{\alpha,\alpha'} n_{i,l} y_{\alpha,\alpha'}.\end{aligned}$$

Since the colours must match

$$\sum_{i,l,\alpha,\alpha'} c_i s_l e_{\alpha,\alpha'} n_{i,l} y_{\alpha,\alpha'} = \sum_{i,l,\alpha'} c_i s_l e_{i,\alpha'} n_{i,l} y_{i,\alpha'}.$$

As $y_{i,\alpha'}$ has all lines propagating

$$\sum_{i,l,\alpha'} c_i s_l e_{i,\alpha'} n_{i,l} y_{i,\alpha'} = \sum_{i,l,\alpha'} c_i s_l e_{i,\alpha'} n_{\alpha',l}.$$

Changing the names of the summation variables

$$\sum_{i,l,\alpha'} c_i s_l e_{i,\alpha'} n_{\alpha',l} = \sum_{i,k,\alpha} c_i s_k e_{i,\alpha} n_{\alpha,k}.$$

That is

$$\theta(m) \sum_{\alpha,\alpha'} e_{\alpha,\alpha'} y_{\alpha,\alpha'} = \sum_{i,k,\alpha} c_i s_k e_{i,\alpha} n_{\alpha,k}.$$

The third term is

$$\begin{aligned}\theta(m) \sum_{u,v} f_{u,v} z_{u,v} &= \sum_{i} c_i \sum_{l} s_l n_{i,l} \sum_{u,v} f_{u,v} z_{u,v} \\ &= \sum_{i,l,u,v} c_i s_l f_{u,v} n_{i,l} z_{u,v}.\end{aligned}$$

But $n_{i,l} z_{u,v}$ is always zero since either the colour sequences do not match or if they do the concatenation does not give a basis element of $\Delta_n(c, d)$. Let us discuss the last statement. If it did, every propagating lines in $n_{i,l}$ would have to match with a propagating line in $z_{u,v}$, so $z_{u,v}$ would have c red propagating lines and d green propagating lines, with $c + d = n - 2$, leaving two nodes at the top and bottom. If the two nodes at the top were joined in an arc, the northern half-diagram would be a basis diagram of $\Delta_n(c, d)$. This contradict since such elements are excluded in the sum. Hence every line in $z_{u,v}$ is propagating. This also contradict. From this we can say $n_{i,l} z_{u,v}$ does not give a basis element of $\Delta_n(c, d)$. From this we can say third term is zero. That is

$$\theta(m) \sum_{u,v} f_{u,v} z_{u,v} = 0.$$

Hence,

$$\begin{aligned}\theta(m)x &= \sum_{i,k,\alpha} c_i s_k e_{i,\alpha} n_{\alpha,k} \\ &= \theta(mx).\end{aligned}$$

It follows that θ is a homomorphism. Since $\underline{S} \neq 0$ by assumption we can say $\theta \neq 0$. $\square$

4.2 Finding non-zero homomorphism for fixed δ

In this section we will find families of modules which give non-zero homomorphisms. We know the matrix corresponding to the homomorphism

$$\theta : \Delta_{n+1}(n+1,0) \to \Delta_{n+1}(n-1,0)$$

can be written as $a(X_1) = R_n$. From Lemma 4.1.5 we can find $\det a(X_1)$ by using the difference equation (4.1.11). If we find $|R_n|$, then by solving $|R_n| = 0$ we can find all the possible δ_R values for which the homomorphism become non-zero.

4.2.1 Solving the difference equation

The difference equation (4.1.11) can be written as in (4.2.1). We will use this to find the special value of δ_R to have a non-zero homomorphism from $\Delta_{n+1}(n+1,0)$ to $\Delta_{n+1}(n-1,0)$.

Proposition 4.2.1. *Suppose $\delta_R \neq \pm 2$. The solution to the difference equation*

$$|R_{n+2}| - \delta_R |R_{n+1}| + |R_n| = 0 \tag{4.2.1}$$

can be given by

$$|R_n| = \left(\frac{\alpha^2}{\alpha^2-1}\right)\alpha^n + \left(\frac{\beta^2}{\beta^2-1}\right)\beta^n, \tag{4.2.2}$$

where

$$\alpha = \frac{\delta_R + \sqrt{\delta_R^2 - 4}}{2}, \tag{4.2.3}$$

$$\beta = \frac{\delta_R - \sqrt{\delta_R^2 - 4}}{2}.. \tag{4.2.4}$$

Proof. Let us solve the difference equation

$$|R_{n+2}| - \delta_R |R_{n+1}| + |R_n| = 0.$$

The characteristic equation of the above difference equation can be given by

$$m^2 - \delta_R m + 1 = 0.$$

If we solve this equation we will get the solutions α and β as follows

$$\alpha = \frac{\delta_R + \sqrt{\delta_R^2 - 4}}{2},$$
$$\beta = \frac{\delta_R - \sqrt{\delta_R^2 - 4}}{2}.$$

Therefore, the general solution of the difference equation can be given by

$$|R_n| = A\alpha^n + B\beta^n. \tag{4.2.5}$$

We know matrix R_1 and R_2 are given by

$$R_1 = (\delta_R) \text{ and } R_2 = \begin{pmatrix} \delta_R & 1 \\ 1 & \delta_R \end{pmatrix}.$$

From this we can get $|R_1| = \delta_R$ and $|R_2| = \delta_R^2 - 1$. These imply that

$$A\alpha + B\beta = \delta_R, \tag{4.2.6}$$

$$A\alpha^2 + B\beta^2 = \delta_R^2 - 1. \tag{4.2.7}$$

By solving (4.2.6) and (4.2.7) we can obtain

$$A\alpha^2 + (\delta_R - A\alpha)\beta = \delta_R^2 - 1,$$
$$A(\alpha^2 - \alpha\beta) = (\delta_R^2 - 1) - \delta_R\beta.$$

From this, if we make A the subject and simplify, we will obtain

$$\begin{aligned} A &= \frac{(\delta_R^2 - 1) - \delta_R \beta}{\alpha^2 - \alpha\beta}, \\ &= \frac{(\delta_R^2 - 1) - \delta_R \left(\frac{\delta_R - \sqrt{\delta_R^2 - 4}}{2} \right)}{\left(\frac{\delta_R + \sqrt{\delta_R^2 - 4}}{2} \right)^2 - 1}, \\ &= \frac{\left(\delta_R + \sqrt{\delta_R^2 - 4} \right)^2}{\left(\delta_R + \sqrt{\delta_R^2 - 4} \right)^2 - 4}, \\ &= \frac{\alpha^2}{\alpha^2 - 1}. \end{aligned}$$

If we find (4.2.7)- (4.2.6)$\times\alpha$, we will obtain

$$\begin{aligned} B\left(\beta^2 - \alpha\beta \right) &= (\delta_R^2 - 1) - \delta_R \alpha, \\ B &= \frac{(\delta_R^2 - 1) - \delta_R \alpha}{\beta^2 - \alpha\beta}. \end{aligned}$$

By looking at the value we found for A above, we can deduce B as being

$$B = \frac{\beta^2}{\beta^2 - 1}.$$

By substituting for A and B in (4.2.2), we have proved the result. □

For the value of α and β as in (4.2.3) and (4.2.4) we can get the following result.

Proposition 4.2.2. *We have*

$$\alpha + \beta = \delta_R, \tag{4.2.8}$$

$$\alpha\beta = 1. \tag{4.2.9}$$

Proof. By adding and multiplying (4.2.3) and (4.2.4) we can easily prove this proposition.

□

This proposition has been discussed in [30, Subsection 2.2]. They denote α and β by q and q^{-1} respectively, where q is an invertible indeterminate.

Example 4.2.3. Let us verify the Proposition 4.2.1 for $n = 3$. If we substitute $n = 3$ in (4.2.2), we will obtain

$$|R_3| = \left(\frac{\alpha^2}{\alpha^2 - 1}\right)\alpha^3 + \left(\frac{\beta^2}{\beta^2 - 1}\right)\beta^3,$$
$$= \frac{\alpha^2\beta^2(\alpha^3 + \beta^3) - (\alpha^5 + \beta^5)}{(\alpha^2 - 1)(\beta^2 - 1)}.$$

Let us find the sum $\alpha^5 + \beta^5$ by writing it in a form in which we can easily substitute the results in (4.2.8) and (4.2.9). This will give us

$$\alpha^5 + \beta^5 = (\alpha + \beta)[(\alpha + \beta)^4 - 5\alpha\beta\left((\alpha + \beta)^2 - \alpha\beta\right)],$$
$$= \delta_R[\delta_R^4 - 5(\delta_R^2 - 1)],$$
$$= \delta_R\left(\delta_R^4 - 5\delta_R^2 + 5)\right).$$

Similarly, let us find the sum of $\alpha^3 + \beta^3$. This will give us

$$\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta),$$
$$= \delta_R^3 - 3\delta_R.$$

Now we find the product $(\alpha^2 - 1)(\beta^2 - 1)$. This will give us

$$(\alpha^2 - 1)(\beta^2 - 1) = \alpha^2\beta^2 - (\alpha^2 + \beta^2) + 1,$$
$$= (\alpha\beta)^2 - [(\alpha + \beta)^2 - 2\alpha\beta] + 1,$$
$$= 1 - (\delta_R^2 - 2) + 1,$$
$$= 4 - \delta_R^2.$$

$$
\begin{aligned}
|R_3| &= \frac{1(\delta_R^3 - 3\delta_R) - \delta_R\,(\delta_R^4 - 5\delta_R^2 + 5))}{4 - \delta_R^2}, \\
&= \frac{\delta_R^5 - 6\delta_R^3 + 8\delta R}{\delta_R^2 - 4}, \\
&= \delta_R(\delta_R^2 - 2).
\end{aligned}
$$

If we find $|R_3|$ by substituting $n = 1$ in (4.1.11) we can get

$$
\begin{aligned}
|R_3| &= \delta_R|R_2| - |R_1|, \\
&= \delta_R(\delta_R^2 - 1) - \delta_R, \\
&= \delta_R(\delta_R^2 - 2).
\end{aligned}
$$

By substituting $n = 3$ in (4.2.2) we obtained $|R_3|$. From the difference equation (4.1.11), we obtained $|R_3|$. Both values are the same. Therefore, this verifies our Proposition 4.2.1 when $n = 3$.

4.2.2 Finding the value of δ_R for which determinant of $R_n = 0$

Solution of $|R_n| = 0$ give the special value of δ_R for which we get the non zero homomorphism from $\Delta_{n+1}(n+1, 0)$ to $\Delta_{n+1}(n-1, 0)$. We know R_n is the matrix corresponding to the homomorphism, $a(X)$, between the modules. By solving $a(X)\underline{\mathbf{S}} = \underline{\mathbf{0}}$ for the special value of δ_R we can find the non-zero homomorphism.

Proposition 4.2.4. *We again assume that $\delta_R \neq \pm 2$. Determinant of R_n becomes zero when*

$$
\delta_R = 2\cos\left(\frac{k\pi}{n+1}\right), \tag{4.2.10}
$$

where $k \in \{1, 2, \cdots, n\} \cup \{n+2, \cdots, 2n+1\}$.

Therefore, non-zero homomorphism exists from $\Delta_{n+1}(n+1, 0)$ to $\Delta_{n+1}(n-1, 0)$ when δ_R takes the values as in (4.2.10) for each value of k.

Proof. Let us solve $|R_n| = 0$. From (4.2.2) we can get

$$\left(\frac{\alpha^2}{\alpha^2 - 1}\right)\alpha^n + \left(\frac{\beta^2}{\beta^2 - 1}\right)\beta^n = 0.$$

This implies

$$\begin{aligned}\left(\frac{\alpha}{\beta}\right)^n &= \frac{-\beta^2(\alpha^2 - 1)}{\alpha^2(\beta^2 - 1)} \\ &= \frac{-\beta^2\alpha^2 + \beta^2}{\alpha^2\beta^2 - \alpha^2}.\end{aligned}$$

However, we can simplify this by using $\alpha\beta = 1$ from (4.2.9) as follows

$$\left(\frac{\alpha}{\beta}\right)^n = \frac{\beta^2 - 1}{1 - \alpha^2}.$$

By multiplying the numerator and denominator of the right-hand side by α^2 we can obtain

$$\left(\frac{\alpha}{\beta}\right)^n = \frac{\alpha^2\beta^2 - \alpha^2}{\alpha^2(1 - \alpha^2)}.$$

By substituting $\alpha\beta = 1$, we can get

$$\left(\frac{\alpha}{\beta}\right)^n = \frac{1 - \alpha^2}{\alpha^2(1 - \alpha^2)}.$$

This can be simplified into

$$\left(\frac{\alpha}{\beta}\right)^n = \frac{1}{\alpha^2}.$$

By substituting $\alpha\beta = 1$ followed by the simplification we can get

$$\begin{aligned}\left(\frac{\alpha^2}{\alpha\beta}\right)^n &= \frac{1}{\alpha^2}, \\ \alpha^{2n} &= \frac{1}{\alpha^2}.\end{aligned}$$

From this we can get the equation

$$\alpha^{2n+2} = 1.$$

We can find the solutions to the above equation by using the roots of unity. This gives us

$$\alpha = \cos\left(\frac{\pi k}{n+1}\right) + i\sin\left(\frac{\pi k}{n+1}\right), \tag{4.2.11}$$

where $k = 0, 1, 2, \ldots, 2n+1$. We know from (4.2.9) that $\alpha\beta = 1$. Therefore,

$$\beta = \cos\left(\frac{\pi k}{n+1}\right) - i\sin\left(\frac{\pi k}{n+1}\right), \tag{4.2.12}$$

where $k = 0, 1, 2, \ldots, 2n+1$. We know from (4.2.8) $\alpha + \beta = \delta_R$ therefore,

$$2\cos\left(\frac{\pi k}{n+1}\right) = \delta_R.$$

That is,

$$\delta_R = 2\cos\left(\frac{k\pi}{n+1}\right),$$

where $k = 0, 1, 2, \ldots, 2n+1$. From Proposition 4.2.1 we know that $\alpha^2 \neq 1$. This implies that $\alpha \neq \pm 1$. However, we know from (4.2.3)

$$\alpha = \frac{\delta_R + \sqrt{\delta_R^2 - 4}}{2}.$$

By substituting this into $\alpha \neq \pm 1$ and simplifying we can get

$$\begin{aligned} \frac{\delta_R + \sqrt{\delta_R^2 - 4}}{2} &\neq \pm 1, \\ \sqrt{\delta_R^2 - 4} &\neq \pm 2 - \delta_R, \\ \delta_R^2 - 4 &\neq 4 \mp 4\delta_R + \delta_R^2, \\ \pm 4\delta_R &\neq 8, \\ \delta_R &\neq \pm 2. \end{aligned}$$

Therefore, we can say

$$\delta_R = 2\cos\left(\frac{k\pi}{n+1}\right),$$

where $k \neq 0$, $k \neq n+1$. □

Example 4.2.5. If we choose the value of $n = 3$, from Proposition 4.2.4 we will get

$$\delta_R = 2\cos\left(\frac{k\pi}{4}\right)$$

where $k \in \{1, 2, 3\} \cup \{5, 6, 7\}$. This gives us

$$\delta_R = \sqrt{2}, 0 \text{ and } -\sqrt{2}.$$

From this we can say, we have non-zero homomorphism from $\Delta_4(4,0)$ to $\Delta_4(2,0)$ from (4.1.23) for the value of $\delta_R = \sqrt{2}, 0$ and $-\sqrt{2}$. Similarly, if we choose $n = 5$ we will get

$$\delta_R = 2\cos\left(\frac{k\pi}{6}\right)$$

where $k \in \{1, 2, 3, 4, 5\} \cup \{7, 8, 9, 10, 11\}$. This gives us $\delta_R = \sqrt{3}, 1, 0, -1$ and $-\sqrt{3}$. From this we can say, we have non-zero homomorphism from $\Delta_6(6,0)$ to $\Delta_6(4,0)$ for the values of $\delta_R = \sqrt{3}, 1, 0, -1$ and $-\sqrt{3}$.

4.2.3 Families of modules giving non-zero homomorphisms

For a given value of δ_R we find a family of non-zero homomorphism of the form

$$\theta : \Delta_{n+1}(n+1, 0) \longrightarrow \Delta_{n+1}(n-1, 0). \tag{4.2.13}$$

By adding the same number of green propagating lines to both modules we can get more non-zero homomorphism for the same value of δ_R.

Example 4.2.6. Let us find the values of n for which (4.2.13) will give non-zero homomorphism when $\delta_R = \pm\sqrt{2}$. From (4.2.10) we can say

$$2\cos\left(\frac{k\pi}{n+1}\right) = \pm\sqrt{2},$$

where $k \in \{1, 2, \ldots, n\} \cup \{n+2, \ldots, 2n+1\}$. Therefore, we can say

$$\cos\left(\frac{k\pi}{n+1}\right) = \pm\frac{\sqrt{2}}{2},$$

$$0 < \frac{k}{n+1} < 1 \text{ or } 1 < \frac{k}{n+1} < 2. \tag{4.2.14}$$

When $\cos\left(\frac{k\pi}{n+1}\right) = +\frac{\sqrt{2}}{2}$ we get $\frac{k\pi}{n+1} = \frac{\pi}{4}$ or $\frac{7\pi}{4}$ because of (4.2.14). Consider the case $\frac{k\pi}{n+1} = \frac{\pi}{4}$. This implies that $n = 4k - 1$. In this situation we can obtain the table as below.

k	1	2	3	4	5	$\cdots$
n	3	7	11	15	19	$\cdots$

If we consider $\frac{k\pi}{n+1} = \frac{7\pi}{4}$ we get $n = \frac{4k}{7} - 1$. Therefore, the table becomes as follows.

k	7	14	21	28	35	$\cdots$
n	3	7	11	15	19	$\cdots$

When $\cos\left(\frac{k\pi}{n+1}\right) = -\frac{\sqrt{2}}{2}$ we get $\frac{k\pi}{n+1} = \frac{3\pi}{4}$ or $\frac{5\pi}{4}$ because of (4.2.14). Consider the case $\frac{k\pi}{n+1} = \frac{3\pi}{4}$ Therefore, $n = \frac{4k}{3} - 1$. In this situation the table becomes as follows.

k	3	6	9	12	15	$\cdots$
n	3	7	11	15	19	$\cdots$

If we look at the case $\frac{k\pi}{n+1} = \frac{5\pi}{4}$ we will get $n = \frac{4k}{5} - 1$. Therefore,

k	5	10	15	20	25	$\cdots$
n	3	7	11	15	19	$\cdots$

From these cases we can say for the value of $\delta_R = \pm\sqrt{2}$ we can get non-zero homomorphism of the form in (4.2.13) for $n = 3, 7, 11, 15, 19$ and etc. This implies the following result.

Proposition 4.2.7.

$$\theta : \Delta_n(r+1, g) \longrightarrow \Delta_n(r-1, g),$$

where $n = r + g + 1$*, will give a non-zero homomorphism when* $\delta_R = \pm\sqrt{2}$ *and* $r = 3, 7, 11, 15, 19$ *etc.*

We can verify the above result by choosing $g = 2$, $r = 3$ and $n = 6$. Therefore, we are looking at the homomorphism

$$\Delta_6(4, 2) \longrightarrow \Delta_6(2, 2). \tag{4.2.15}$$

Matrix corresponding to the homomorphism can be obtained as

$$\begin{aligned} a\left(X_1\right) &= \begin{pmatrix} \delta_R & 1 & 0 \\ 1 & \delta_R & 1 \\ 0 & 1 & \delta_R \end{pmatrix} \\ &= R_3 \end{aligned}$$

Therefore, determinant of $a(X_1)$ becomes

$$\begin{aligned} \det a(X_1) &= |R_3| \\ &= \delta_R(\delta_R^2 - 1) - \delta_R \\ &= \delta_R(\delta_R^2 - 2) \end{aligned}$$

From this, we can say that $\delta_R = \pm\sqrt{2}$ will give us the non-zero homomorphism as in (4.2.15).

If we ask the question is that the Proposition 4.2.7 gives us the only possible homomorphism of that form for the value of $\delta_R = \pm\sqrt{2}$? At this stage we are not quite sure.

Example 4.2.8. Let us find the n values for which (4.2.13) will give non-zero homomorphism when $\delta_R = \pm\sqrt{3}$.

$$2\cos\left(\frac{k\pi}{n+1}\right) = \pm\sqrt{3}$$

Where $k \in \{1, 2, \cdots, n\} \cup \{n+2, \cdots, 2n+1\}$. Therefore,

$$0 < \frac{k}{n+1} < 1 \text{ or } 1 < \frac{k}{n+1} < 2$$

When $\cos\left(\frac{k\pi}{n+1}\right) = +\frac{\sqrt{3}}{2}$ we get $\frac{k\pi}{n+1} = \frac{\pi}{6}$ or $\frac{11\pi}{6}$ because of (4.2.14). Consider the case $\frac{k\pi}{n+1} = \frac{\pi}{6}$. This implies that $n = 6k - 1$ so we can obtain the table as below.

k	1	2	3	4	5	$\cdots$
n	5	11	17	23	29	$\cdots$

If we consider $\frac{k\pi}{n+1} = \frac{11\pi}{6}$ we get $n = \frac{6k}{11} - 1$. Therefore,

k	11	22	33	44	55	$\cdots$
n	5	11	17	23	29	$\cdots$

When $\cos\left(\frac{k\pi}{n+1}\right) = -\frac{\sqrt{3}}{2}$ we get $\frac{k\pi}{n+1} = \frac{5\pi}{6}$ or $\frac{7\pi}{6}$ because of (4.2.14). Consider the case $\frac{k\pi}{n+1} = \frac{5\pi}{6}$ Therefore, $n = \frac{6k}{5} - 1$

k	5	10	15	20	25	$\cdots$
n	5	11	17	23	29	$\cdots$

If we look at the case $\frac{k\pi}{n+1} = \frac{7\pi}{6}$ we will get $n = \frac{6k}{7} - 1$. Therefore,

k	7	14	21	28	35	$\cdots$
n	5	11	17	23	29	$\cdots$

From these cases, we can say that $\delta_R = \pm\sqrt{3}$ will give a non-zero homomorphism of the form in (4.2.13) for $n = 5, 11, 17, 23, 29$ and etc. This implies the following result.

Proposition 4.2.9.

$$\theta : \Delta_n(r+1, g) \longrightarrow \Delta_n(r-1, g),$$

where $n = r + g + 1$*, will give a non-zero homomorphism when* $\delta_R = \pm\sqrt{3}$ *and* $r = 5, 11, 17, 23, \cdots$

We can verify the above proposition by choosing $g = 0$, $r = 5$ and $n = 6$. Therefore, we are looking at the homomorphism

$$\Delta_6(6, 0) \longrightarrow \Delta_6(4, 0). \tag{4.2.16}$$

Matrix corresponding to the homomorphism is $a(X_1) = R_5$ Therefore,

$$\begin{aligned} \det a(X_1) &= |R_5| \\ &= \delta_R|R_4| - |R_3| \\ &= \delta_R(\delta_R|R_3| - |R_2|) - |R_3| \\ &= \delta_R(\delta_R^2 - 1)(\delta_R^2 - 3) \end{aligned}$$

We have obtained this by substituting $|R_3| = \delta_R(\delta_R^2 - 2)$ and $|R_2| = \delta_R^2 - 1$. From the determinant, we can say $\delta_R = \pm\sqrt{3}$ will give a non-zero homomorphism (4.2.16).

Chapter 5

Generators

In this chapter we will introduce a simple set of generators for our algebra. These will be useful when we return to determining homomorphisms in the following chapters. We also determine the number of generators in our generating set. We illustrate our methods by considering the cases of 2 and 3 colours before moving to the general case.

5.1 Finding generators of $TL_n^2(\delta_R, \delta_G)$

The above algebra consists of two colours which are red and green. We are going to find a generating set for this algebra by taking the diagrams with (i) all (non-crossing) propagating lines, (ii) diagrams with exactly one crossing of red and green propagating lines but no arcs, and (iii) diagrams with one arc at the northern and southern edge and all other lines noncrossing and propagating .

Given any diagram in our algebra, it can be regarded as a superposition of two single-colour Temperley-Lieb diagrams. By considering a crossing generator and the same diagram with the crossing reversed, we can generate all diagrams with propagating lines. These generators and those of type (iii) are well-known (from the Temperley-Lieb case) to generate all ordinary Temperley-Lieb diagrams. The diagrams with one crossing can now be used to overlay one colour on top of the other in the required configuration. Thus these diagrams will generate, Theorem 5.3.2, the bubble algebra.

Based on the above, we attempt to derive at all possible generators as follows. We should have 2 generators, one with all red propagating lines and the other with all green propagating lines. The simple reason is that we cannot derive these algebra elements by any of the one cross diagrams or diagrams with one arc. Any algebra element with all lines propagating with both colours can be written as the multiplication of the generators with one cross as in Figure5.1.

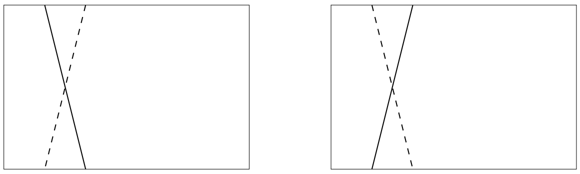


Figure 5.1: Generators with one cross

First, we find the generators with the cross as in Figure 5.1 and the remaining $n-2$ nodes with all green propagating lines. In this situation, we can have $\frac{(n-1)!2!}{(n-2)!}$ arrangements which results in $2(n-1)$. In addition to the illustration within Figure 5.1, if we have a red propagating line and the rest of the $(n-3)$ are green propagating lines then we can get $\frac{(n-1)!2!}{1!(n-3)!}$ generators which results in $2(n-1)(n-2)$. This can be generalised by letting a cross, r number of red propagating lines (excluding the red propagating lines in the cross), and $n-(r+2)$ green propagating lines (excluding the green in the cross) given by

$$\frac{(n-1)!2!}{r!(n-(r+2))!}. \tag{5.1.1}$$

By using the generators with the cross lines we can not make the algebra elements with arcs. Therefore, we need generators with arcs in them. Here we are going to find the

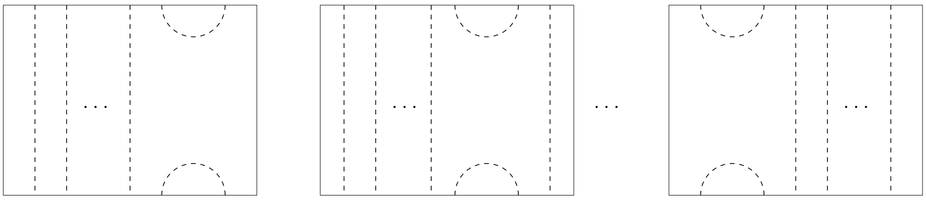


Figure 5.2:

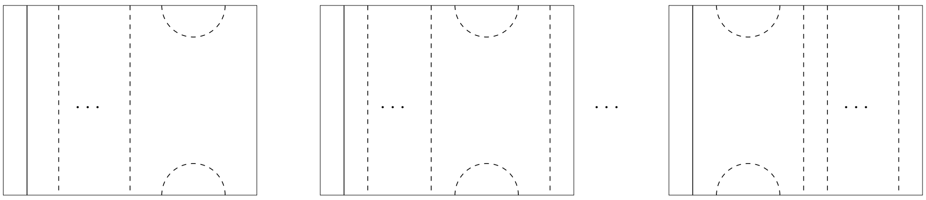


Figure 5.3: Generators with one red propagating line and one pair of green arcs

generators with on the left of the diagram all red nodes and the right of the diagram with all green nodes. Fixing one colour to a side with propagating lines and the other side with the other colour consisting an arc on that side. In addition, if there are any unused nodes left on the same side as the arc, this will result in additional propagating lines of the same colour as the arc appearing on the same side.

First we fix the number of red propagating lines and allow the green side to have an arc: If there are no red propagating lines we can have generators in the form as in Figure 5.2. Here there are $n - 2 + 1 = n - 1$ generators. If there is one red propagating line we can list the green as in Figure 5.3. There are $n - (1 + 2) + 1 = n - 2$ generators. This can be generalised by assuming that there are r number of red propagating lines and allowing the arc to be on the green side as in Figure 5.4. This will result with

$$n - (r + 2) + 1 = n - r - 1 \tag{5.1.2}$$

generators.

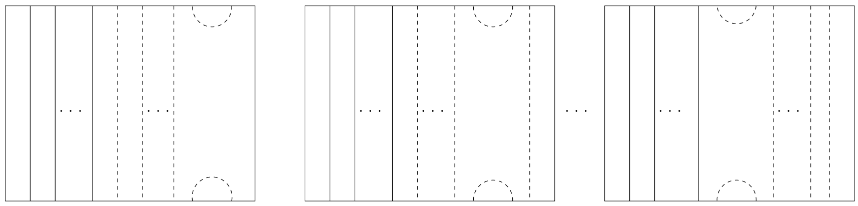


Figure 5.4:

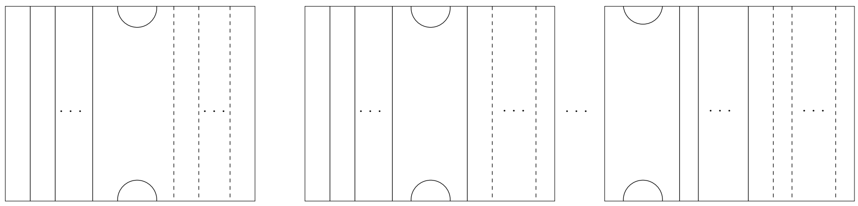


Figure 5.5:

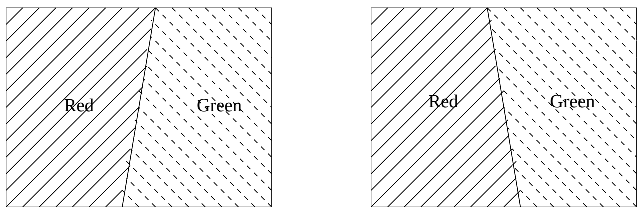


Figure 5.6:

Secondly, we fix the number of green propagating lines and allow red side to have an arc: Assuming there are g number of green propagating lines and allowing the arc to be on the red side as in Figure 5.5 will result with

$$n-(g+2)+1=n-g-1 \tag{5.1.3}$$

generators.

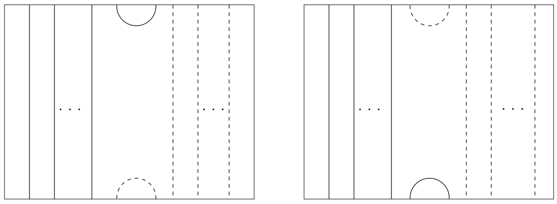


Figure 5.7:

There are algebra elements which have different numbers of red and green nodes at the northern edge and southern edge. Generators we have created so far have the same number of red and green nodes in both edges. Therefore, we need generators as in Figure 5.6. In this Figure the number of red and green nodes are not the same. For this illustration we need generators as in Figure 5.7. In this Figure, the number of red and green nodes at the northern edge and southern edge differ by 2 . If there are r number of red and g number of green propagating lines then we can get two generators where

$$r + g = n - 2. \tag{5.1.4}$$

Within the above illustrations we have identified all necessary generators of our algebra.

Proposition 5.1.1. *The algebra $TL_n^2(\delta_R, \delta_G)$ may be generated by*

$$2^{n-1}(n-1) + n(n+1) \tag{5.1.5}$$

elements.

Proof. Let us find this by analysing the possibilities of the generators. The number of generators with all red or all green propagating lines is 2.

From (5.1.1) we can say that the total number of the generators with one red and one green crossing is given by

$$\sum_{r=0}^{n-2} \frac{(n-1)!2!}{r!(n-(r+2))!}.$$

This can be simplified as

$$\begin{aligned}
\sum_{r=0}^{n-2} \frac{(n-1)!2!}{r!(n-(r+2))!} &= \sum_{r=0}^{n-2} \frac{(n-1)(n-2)!2!}{r!(n-(r+2))!} \\
&= (n-1)2! \sum_{r=0}^{n-2} \frac{(n-2)!}{r!(n-(r+2))!} \\
&= (n-1) \times 2 \sum_{r=0}^{n-2} \frac{(n-2)!}{r!(n-2-r)!} \\
&= (n-1) \times 2 \times (1+1)^{n-2} \\
&= (n-1) \times 2 \times 2^{n-2} \\
&= 2^{n-1}(n-1)
\end{aligned}$$

From (5.1.2) we can say that the total number of generators with 1 green arc and r red propagating lines is given by

$$\sum_{r=0}^{n-2} n-(r+2)+1,$$

which is

$$\begin{aligned}
\sum_{r=0}^{n-2} n-(r+2)+1 &= \sum_{r=0}^{n-2} n-r-1 \\
&= (n-1)+(n-2)+\ldots+1 \\
&= \frac{(n-1)((n-1)+1)}{2} \\
&= \frac{n(n-1)}{2}.
\end{aligned}$$

From (5.1.3) we can say that the total number of generators with 1 red arc and g green propagating lines is given by

$$\sum_{g=0}^{n-2} n-(g+2)+1,$$

which is

$$\begin{aligned}\sum_{g=0}^{n-2} n-(g+2)+1 &= \sum_{g=0}^{n-2} n-g-1\\ &= (n-1)+(n-2)+\ldots+1\\ &= \frac{(n-1)((n-1)+1)}{2}\\ &= \frac{n(n-1)}{2}.\end{aligned}$$

From (5.1.4) we can say that the total number of generators with 1 arc but northern edge and southern edge differing by 2 red and 2 green nodes are given by

$$\sum_{r+g=n-2} 2 = 2(n-1).$$

If we add all of these we will get

$$\begin{aligned}2+2^{n-1}(n-1)+&\frac{n(n-1)}{2}+\frac{n(n-1)}{2}+2(n-1)\\ &= 2+2^{n-1}(n-1)+n(n-1)+2(n-1)\\ &= 2^{n-1}(n-1)+n^2-n+2n-2+2\\ &= 2^{n-1}(n-1)+n^2+n\\ &= 2^{n-1}(n-1)+n(n+1)\end{aligned}$$

Hence, we have proved the proposition. □

Example 5.1.2. Let us find our generating set for the algebra $TL_4^2(\delta_R, \delta_G)$.

First we find the generators with all lines propagating by the same colour. There are 2 diagrams possible as in Figure 5.8.

Now we find the generators with one cross in them. In this case, first we find the generators with 0 red propagating and 2 green propagating(excluding the red, green propagating lines in the cross). There are $\frac{(4-1)!2!}{2!}$ arrangements possible. This gives us 6 generators.

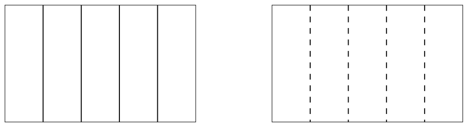


Figure 5.8: Generators for all same colour propagating

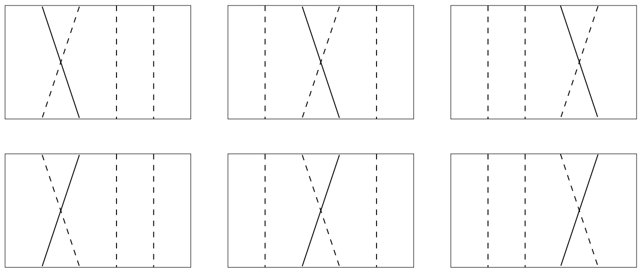


Figure 5.9: 0 red and 2 green propagating lines excluding the ones in the cross

These generators are given by the Figure 5.9. Let us look at the generators with 1 red propagating and 1 green propagating(excluding the red, green propagating lines in the cross). There are $\frac{(4-1)!2!}{1!1!}$ arrangements possible. This gives us 12 generators. These generators are given by Figure 5.10. Similarly if we look at the generators with 2 red propagating and 0 green propagating(excluding the red, green propagating lines in the cross). There are $\frac{(4-1)!2!}{2!1!}$ arrangements possible. This gives us 6 generators. These generators are given by Figure 5.11.

Let us find the generators with one same colour arc at the northern edge and southern edge. As discussed earlier, first we fix the red colour strings and allow the green colour strings and to move around on the edges . If we say there are 0 red propagating lines then we can have $4-(0+2)+1$ generators, this gives us 3 generators. These are in Figure 5.12. If there is 1 red propagating lines, then we can have $4-(1+2)+1$ generators. This gives us 2 generators. These are in Figure 5.13. Similarly, if we say there are 2 red propagating lines then we can have $4-(2+2)+1$ generators, this gives us only 1 generator, which is in Figure 5.14. Now we fix the green bit and generate the red bit as follows. If we say there

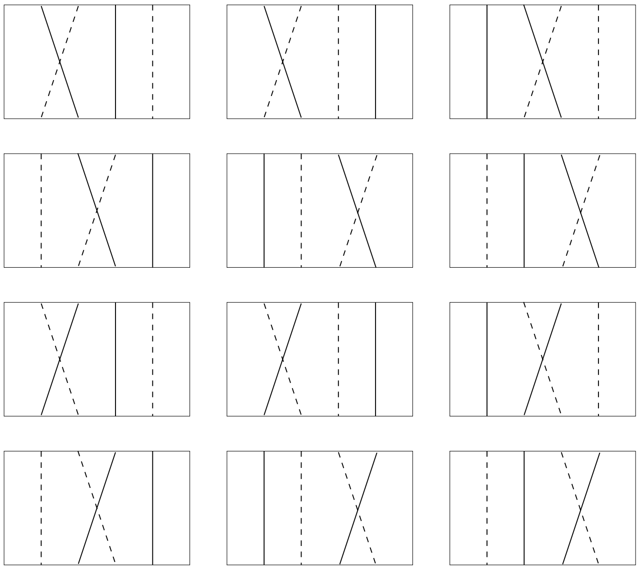


Figure 5.10: 1 red and 1 green propagating lines excluding the ones in the cross

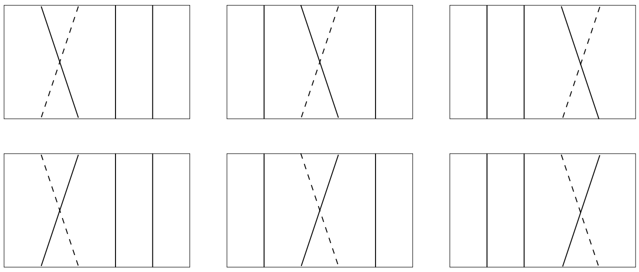


Figure 5.11: 2 red and 0 green propagating lines excluding the ones in the cross

are 0 green propagating lines then we can have $4-(0+2)+1$ generators, that is, there are 3 generators possible. These are in Figure 5.15. If we say there is 1 green propagating line, then we can have $4-(1+2)+1$ generators. This gives us 2 generators. This is in Figure 5.16.If we say there are 2 green propagating lines, then we can have $4-(2+2)+1$ generators. This gives us only 1 generator. This is in Figure 5.17.

Let us find the generators with different colour arcs at the northern edge and southern edges. If we say there are 0 red propagating lines, we could have 2 generators as in Figure 5.18. If there is 1 red propagating line, we could have 2 generators as in Figure 5.19.

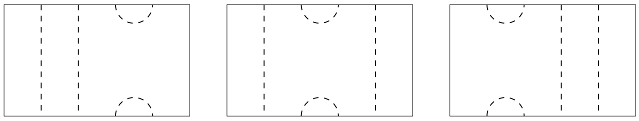


Figure 5.12: 0 red propagating lines

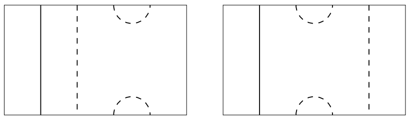


Figure 5.13: 1 red propagating lines

Similarly if there are 2 red propagating lines we could have 2 generators as in Figure 5.20.

From these we can say our generating set for the $TL_4^2(\delta_R, \delta_G)$ has 44 generators.

Let us verify the Proposition 5.1.1. We have found in Example 5.1.2 that algebra $TL_4^2(\delta_R, \delta_G)$ has 44 generators.

When $n = 4$, (5.1.5) implies that

$$\begin{aligned} 2^{n-1}(n-1) + n(n+1) &= 2^{4-1}(4-1) + 4(4+1) \\ &= 2^3 \times 3 + 4 \times 5 \\ &= 24 + 20 \\ &= 44 \end{aligned}$$

Therefore, $TL_4^2(\delta_R, \delta_G)$ has 44 generators, which is true. Thus our proposition has been verified in this case.

5.2 Finding the generators of $TL_n^3(\delta_R, \delta_G, \delta_B)$

The above algebra has red, green and black colour nodes. Same colour lines are not allowed to cross in this algebra. Therefore, we should have 3 generators with all red propagating

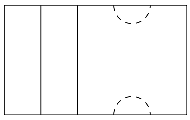


Figure 5.14: 2 red propagating lines

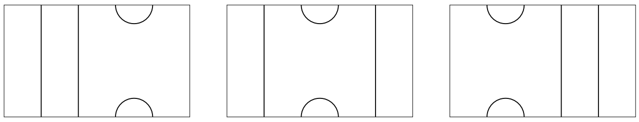


Figure 5.15: 0 green propagating lines

lines, all green propagating lines and all black propagating lines.

Any algebra element with all lines propagating can be written as the multiplication of the generators with one red and green crossing or green and black crossing or black and red crossing. These generators help to move the propagating lines where we wanted.

Let us find the number of generators with one red and green cross, r number of red propagating lines, g number of green propagating lines and b number of black propagating lines. Here the red and green propagating lines in the cross are not counted as the propagating lines. Therefore, the number of generators are $\frac{(n-1)!2!}{r!g!b!}$, where $r+g+b=n-2$. Similarly, we find the number of generators with one green and one black cross given by $\frac{(n-1)!2!}{r!g!b!}$ and one black and one red cross also given by $\frac{(n-1)!2!}{r!g!b!}$.

By using the generators with the cross lines we cannot make the algebra elements with arcs in them. Therefore, we need generators with arcs. Here, we find the generators with a pair of same colour arcs at the northern and southern edge. We fix the left side of the northern and southern edge with red nodes, right side of the northern and southern edge with black nodes and middle of the northern and southern edge by green nodes. By fixing any two sides by all propagating and third side generating we can find the generators of this type. Let us find the generators which generate(allow to have arc) a black bit with r number

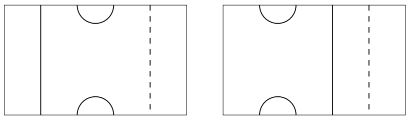


Figure 5.16: 1 green propagating lines

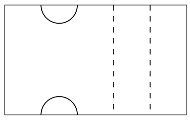


Figure 5.17: 2 green propagating lines

of red propagating lines, g number of green propagating lines. There are $n-(r+g+2)+1$ generators that is $n-1-(r+g)$. Where $r+g \leq n-2$. If we find the generators which generate(allow to have arc) the green bit with r number of red and b number of black propagating lines given by $n-(r+b+2)+1$ that is $n-1-(r+b)$, where $r+g \leq n-2$. Similarly if we find the generators which generate(allow to have arc) the red bit with g number of green and b number of black propagating lines are given by $n-(g+b)+1$ generators that is $n-1-(g+b)$, where $g+b \leq n-2$.

There are algebra elements which have different numbers of red, green and black nodes at the northern edge and southern edge. Generators we found so far have same number of red, green and black nodes at the northern edge and southern edge. Therefore, we find the generators in the form as in Figure 5.21. By using these generators and generators with one cross, we can get any diagram with different number of colour arcs at the northern and southern edge. Here, generators with one cross help to move the nodes to the desired positions. If we look at the top left and bottom left diagrams in Figure 5.21, they have red and green nodes at the northern edge and southern edge differ by 2. If there are r number of red, g number of green and b number of black propagating lines then we can get those two as the generators in this case. If the number of green and black nodes differ by 2

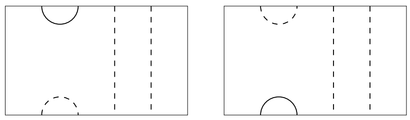


Figure 5.18: 0 red propagating line

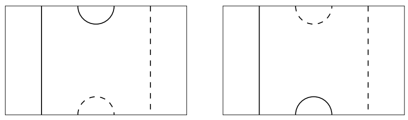


Figure 5.19: 1 red propagating line

at the northern and southern edges then middle diagrams in Figure 5.21 are the possible generators. Similarly if the red and black nodes differ by 2, then top right and the bottom right diagrams in Figure 5.21 are the possible generators. Here we put the black nodes at the front, red at the middle and green at the back.

Proposition 5.2.1. *The algebra $TL_n^3(\delta_R, \delta_G)$ may be generated by*

$$2(n-1)3^{n-1} + \frac{n(n+1)(n-1)}{2} + 3n(n-1) + 3 \tag{5.2.1}$$

elements.

Proof. Number of generators with all red or all green or all black propagating lines is 3.

According to the third paragraph in Section 5.2, we can say the total number of generators with 2 colours crossing is given by

$$\binom{3}{2} \times \sum_{r+g+b=n-2} \frac{(n-1)!2!}{r!g!b!}.$$

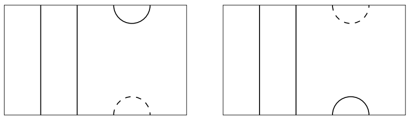


Figure 5.20: 2 red propagating lines

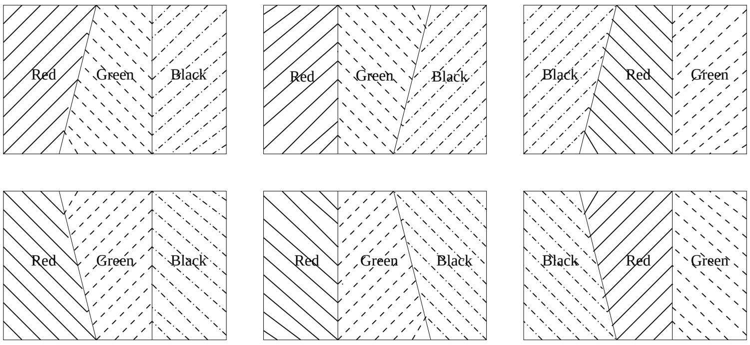


Figure 5.21:

This can be written as

$$\begin{pmatrix} 3 \\ 2 \end{pmatrix} \sum \frac{(n-1)!2!}{r!g!b!} \qquad \text{where } r+g+b=n-2$$

$$\begin{aligned}
&= 3\sum_{r=0}^{n-2}\sum_{g=0}^{n-2-r}\frac{(n-1)!2!}{r!g!(n-r-g-2)!} \\
&= 3\sum_{r=0}^{n-2}\sum_{g=0}^{n-2-r}\frac{(n-1)!2!}{r!g!(n-2-r-g)!} \\
&= 3\sum_{r=0}^{n-2}\sum_{g=0}^{n-2-r}\frac{(n-1)\dots(n-1-r)(n-2-r)!2!}{r!g!(n-2-r-g)!} \\
&= 3\sum_{r=0}^{n-2}\frac{(n-1)\dots(n-1-r)}{r!}\sum_{g=0}^{n-2-r}\frac{(n-2-r)!2!}{(n-2-r-g)!g!} \\
&= 3\sum_{r=0}^{n-2}\frac{(n-1)\dots(n-1-r)}{r!}\sum_{g=0}^{n-2-r}\binom{n-2-r}{g}\times 2.
\end{aligned}$$

This can be further simplified as follows by using the binomial expansion

$$\begin{aligned}
3\sum_{r=0}^{n-2}\frac{(n-1)\dots(n-1-r)}{r!}\sum_{g=0}^{n-2-r}\binom{n-2-r}{g}\times 2 \\
&= 3\sum_{r=0}^{n-2}\frac{(n-1)\dots(n-1-r)}{r!}(1+1)^{n-2-r}\times 2\\
&= 3\sum_{r=0}^{n-2}\frac{(n-1)\dots(n-1-r)}{r!}2^{n-2-r}\times 2\\
&= 3\times 2\sum_{r=0}^{n-2}\frac{(n-1)\dots(n-1-r)(n-2-r)!}{(n-2-r)!r!}2^{n-2-r}\\
&= 3\times 2\sum_{r=0}^{n-2}\frac{(n-1)!}{(n-2-r)!r!}2^{n-2-r}\\
&= 3\times 2(n-1)\sum_{r=0}^{n-2}\frac{(n-2)!}{(n-2-r)!r!}2^{n-2-r}\\
&= 3\times 2(n-1)\sum_{r=0}^{n-2}\binom{n-2}{r}2^{n-2-r}1^{r}\\
&= 2\times 3(n-1)\times 3^{n-2}\\
&= 2(n-1)3^{n-1}.
\end{aligned}$$

According to the fourth paragraph in Section 5.2, we can say the total number of generators with 1 black arc at the northern edge and southern edge, red (r) and green (g) propagating lines fixed given by

$$
\begin{aligned}
\sum_{r+g\leq n-2} n-(r+g+2)+1 &= \sum_{r+g\leq n-2} n-1-(r+g) \\
&= \sum_{r=0}^{n-2}\sum_{g=0}^{n-2-r} n-1-r-g \\
&= \sum_{r=0}^{n-2} (n-1-r)+(n-2-r)+\ldots+1 \\
&= \sum_{r=0}^{n-2} \frac{(n-1-r)(n-r)}{2} \\
&= \frac{1}{2}\sum_{r=0}^{n-2}(n-r-1)(n-r) \\
&= \frac{1}{2}\sum_{r=0}^{n-2}(n-r)^2-(n-r) \\
&= \frac{1}{2}[(n^2+(n-1)^2+\ldots+2^2)-(n+(n-1)+\ldots+2)] \\
&= \frac{1}{2}[(n^2+\ldots+2^2+1^2)-(n+\ldots+2+1)] \\
&= \frac{1}{2}[\frac{n(n+1)(2n+1)}{6}-\frac{n(n+1)}{2}] \\
&= \frac{n(n+1)}{12}[(2n+1)-3] \\
&= \frac{n(n+1)(2n-2)}{12} \\
&= \frac{n(n+1)(n-1)}{6}.
\end{aligned}
$$

Similarly, we can show that the total number of generators with 1 green arc at the northern and southern and red (r) and black (b) propagating lines fixed is given by

$$\sum n-(r+b+2)+1 = \frac{n(n+1)(n-1)}{6},$$

and the total number of generators with 1 red arc at the northern and southern and green (g) and black (b) propagating lines fixed is given by

$$\sum n-(g+b+2)+1 = \frac{n(n+1)(n-1)}{6}.$$

According to the fifth paragraph in Section 5.2, we can say that the total number of generators with 1 arc but northern edge and southern edge differ by 2 red and 2 green nodes is given by

$$\begin{aligned}\sum_{r+g+b=n-2} 2 &= \sum_{b=0}^{n-2} \sum_{r+g=n-2-b} 2 \\ &= \sum_{b=0}^{n-2} 2(n-2-b+1) \\ &= \sum_{b=0}^{n-2} 2(n-1-b) \\ &= 2\sum_{b=0}^{n-2} n-1-b \\ &= 2[(n-1)+(n-2)+\ldots+1] \\ &= 2\times\frac{(n-1)(n-1+1)}{2} \\ &= n(n-1).\end{aligned}$$

Similarly, the total number of generators with 1 arc but northern edge and southern edge differing by 2 green and 2 black nodes is given by

$$\sum_{r+g+b=n-2} 2 = n(n-1)$$

and that the total number of generators with 1 arc but northern edge and southern edge differing by 2 black and 2 red nodes is given by

$$\sum_{r+g+b=n-2} 2 = n(n-1).$$

If we find the total of all of these generators we will get

$$\begin{aligned}3+2(n-1)3^{n-1}+&\frac{n(n+1)(n-1)}{6}\times 3+n(n-1)\times 3 \\ &= 2(n-1)3^{n-1}+\frac{n(n+1)(n-1)}{2}+3n(n-1)+3.\end{aligned}$$

□

Let us verify this result for a very small algebra $TL_2^3(\delta_R, \delta_G, \delta_B)$.

According to (5.2.1), the total number of generators of the algebra $TL_2^3(\delta_R, \delta_G, \delta_B)$ is given by

$$2(2-1)3^{2-1} + \frac{2(2+1)(2-1)}{2} + 3 \times 2(2-1) + 3$$
$$= 2 \times 1 \times 3 + \frac{2 \times 3 \times 1}{2} + 3 \times 2 \times 1 + 3$$
$$= 18.$$

Algebra $TL_2^3(\delta_R, \delta_G, \delta_B)$ has diagrams with two nodes. Let us find the total number of generators by analysing the possibilities of both nodes.

(i) Number of generators with both nodes propagating with the same colour $\binom{3}{1} = 3$

(ii) Number of generators with both nodes involved in different colour propagating line crossing is given by $\binom{3}{1}2! = 6$

(iii) Number of generators with both nodes involved in same colour arc given by $\binom{3}{1} = 3$

(iv) Number of generators with both nodes involved in different colour arcs is given by $\binom{3}{1}2! = 6$

Total number of generators is $3 + 6 + 3 + 6 = 18$. From this we can say, Proposition 5.2.1 is true for $n = 2$.

5.3 Finding the generators of $TL_n^h(\delta_{C_1}, \delta_{C_2}, \ldots, \delta_{C_h})$

In the previous sections, we found formulas to find the total number of generators of the algebra which has 2 or 3 colours. Now we are going to generalise the result. Suppose our algebra has h colours, then following the Proposition gives the total number of generators in this case.

Proposition 5.3.1. *The algebra $TL_n^h(\delta_{C_1}, \ldots, \delta_{C_h})$ may be generated by*

$$(h-1)(n-1)h^{n-1} + h\binom{n+h-2}{h} + h(h-1)\binom{n+h-3}{h-1} + h \tag{5.3.1}$$

elements.

Proof. To prove this we need the Claims 5.3.3, 5.3.4 and 5.3.5 that we are going to discuss soon. We provide the interpretation of these Claims at the beginning of the proof of each Claim.

First we find the number of generators with all lines propagating with the same colour. Algebra has diagrams with h colour nodes. Therefore, there are h generators in this case.

Let us find the generators with one cross. A cross could be made by choosing 2 colours from h in $\binom{h}{2}$ ways. If we say there are c_1 colour 1 propagating lines and c_2 colour 2 propagating lines and so on up to c_h colour h propagating lines, then by using the permutation we can say there are

$$\binom{h}{2} \sum \frac{(n-1)!2!}{c_1!c_2!\ldots c_h!} \tag{5.3.2}$$

generators with one cross, where $c_1 + c_2 + \ldots + c_h = n-2$. If you see the above expression there is a number $2!$. This says a cross could be drawn in two ways. Let us simplify the above expression.

$$\begin{aligned}&\binom{h}{2} \sum \frac{(n-1)!2!}{c_1!c_2!\ldots c_h!} \\ &\quad = \binom{h}{2} \sum_{c_1+c_2+\ldots+c_h=n-2} \frac{(n-1)(n-2)!2!}{c_1!c_2!\ldots c_h!}.\end{aligned}$$

We can simplify the right-hand side as $2\binom{h}{2}(n-1)h^{n-2}$ by using the Claim 5.3.5. Therefore, we will get

$$\binom{h}{2}\sum\frac{(n-1)!2!}{c_1!c_2!\dots c_h!} \tag{5.3.3}$$

$$= 2\binom{h}{2}(n-1)h^{n-2} \tag{5.3.4}$$

$$= 2\times\frac{h!}{(h-2)!2!}(n-1)h^{n-2} \tag{5.3.5}$$

$$= 2\times\frac{h(h-1)}{2}(n-1)h^{n-2} \tag{5.3.6}$$

$$= (h-1)(n-1)h^{n-1}. \tag{5.3.7}$$

From this we can say there are $(h-1)(n-1)h^{n-1}$ generators with one cross.

Let us find the number of generators with one arc at the northern and edge and one at the southern edge. First we consider the case in which both arcs are the same in colour. If we say that both are coloured with colour c_h then we can have c_h+1 generators. Total number of generators with arc colour c_h is given by

$$\sum_{\sum_{i=1}^{h}c_i=n-2} c_h+1.$$

However we can choose the colours in h ways. Therefore, the total number of generators with one arc with same colour given by

$$h\sum_{\sum_{i=1}^{h}c_i=n-2} c_h+1. \tag{5.3.8}$$

We can write the above sum as follows.

$$h\left(\sum_{\sum_{i=1}^{h}c_i=n-2} c_h+\sum_{\sum_{i=1}^{h}c_i=n-2} 1\right) \tag{5.3.9}$$

By using the Claim 5.3.4 and Claim 5.3.3 respectively we can simplify it as

$$h\left(\binom{n-2+h-1}{h}+\binom{n-2+h-1}{h-1}\right).$$

This can be simplified into

$$h(\binom{n+h-3}{h} + \binom{n+h-3}{h-1}).$$

By using

$$\binom{n}{r} + \binom{n}{r-1} = \binom{n+1}{r}.$$

we can further simplified into

$$h\left(\binom{n+h-3}{h} + \binom{n+h-3}{h-1}\right) = h\binom{n+h-2}{h}. \tag{5.3.10}$$

Lastly, we look at the case with one arc at the northern edge and southern edge with different colour arcs. We can choose two colours in $\binom{h}{2}$ ways. In particular choice of colours and propagating lines we can have 2 generators. Therefore, as a total we can get

$$\binom{h}{2} \sum_{\sum_{i=1}^{h} c_i = n-2} 2. \tag{5.3.11}$$

We can simplify this by using Claim 5.3.3 as follows.

$$\begin{aligned} 2\binom{h}{2} \sum_{\sum_{i=1}^{h} c_i = n-2} 1 &= 2 \times \frac{h(h-1)}{2} \binom{n-2+h-1}{h-1} && (5.3.12) \\ &= h(h-1)\binom{n+h-3}{h-1}. && (5.3.13) \end{aligned}$$

Let us find the total number of generators from (5.3.7), (5.3.10) and (5.3.13). This is given by

$$\begin{aligned} &h + (h-1)(n-1)h^{n-1} + h\binom{n+h-2}{h} + h(h-1)\binom{n+h-3}{h-1} \\ &= (h-1)(n-1)h^{n-1} + h\binom{n+h-2}{h} + h(h-1)\binom{n+h-3}{h-1} + h. \end{aligned}$$

Hence we are done. □

Theorem 5.3.2. *Algebra TL_n^h elements can be generated by our generating set.*

Proof. Let x be an arbitrary element of the algebra TL_n^h. Therefore x can be written as the linear combination of the basis elements of the algebra. Therefore

$$x = \sum dD,$$

where d is a scalar in $\mathbb{C}$ and D is a basis element of the algebra. Basis elemnt D can have colours mixed. For our convenient, we seperate the colours. This could be done by using our generators with one cross. Therefore

$$D = kD'k',$$

where k and k' are product of sequence of bubble algebra generators with one cross and D' is the diagram with all colour seperated. Purpose of seperate colours is to avoid crossing. For example, if D as in left hand side of Figure 5.22 then D' is the diagram in right hand side of Figure 5.22. If we igonore the colours of D' we get a Temperley-Lieb dia-

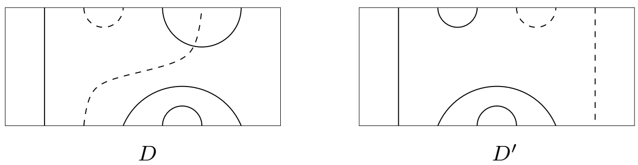


Figure 5.22:

gram. Therefore, we can write D' as a sequence of product of Temperley-Lieb generators. Therefore

$$D' = U_1 U_2 \dots U_p,$$

where U_i is a Temperley-Lieb generator. Now we through the colours back into $D', U_1, U_2, \dots, U_p$. For example, if we take right hand side of Figure 5.22 and ignore the colours then we get the left hand side of Figure 5.23. That diagram can be given by the product of Temperley-Lieb generators as in the right hand side of Figure 5.23 when we ignore the colour. That

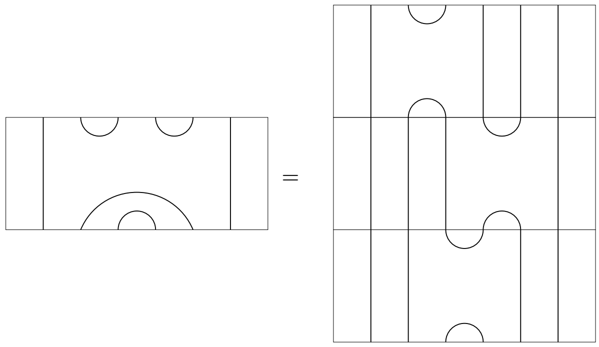


Figure 5.23:

diaram can be given by products of Temperley-Lieb generators as in the right hand side of Figure 5.23. If we through the clours back into Figure 5.23, then we get the Figure 5.24. By using the toplogical argument, stretch the strings in a way to slice easily, we can say

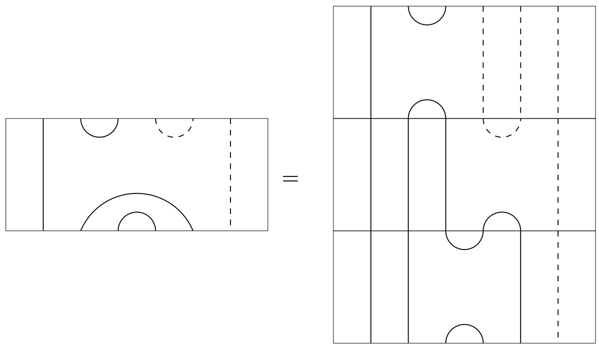


Figure 5.24:

$U_1, U_2, \ldots, U_p$ with colours generate D'. These generators with colours are actually elements of bubble algebra generators with one arc. If we do not want to use the topological argument, colours could be anywhere in each slice U_i, then we need to seperate the colours of each U_i as follows.

$$U_i = k_i U_i' k_i'.$$

Here U_i' has all colours seperated according to colour order, and k_i and k_i' are products of sequence of generators of the bubble algebra with a cross. For example, if U_i is a diagram

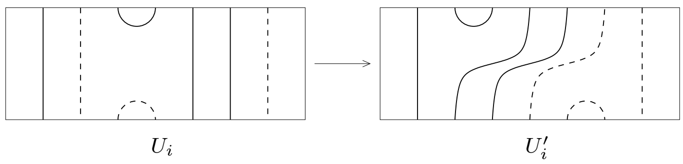


Figure 5.25:

as in the left hand side of Figure 5.25 then U_i' become as the right hand side of Figure 5.25. We can write each U_i' as

$$U_i' = TV_iB,$$

where V_i is a generator of the algebra with one pair of arc, T and B are product of sequence of permutation generators with a pair of same colour arc or different colour arc. We can understand this by taking U_i' as in Figure 5.26. In the right hand side of the Figure 5.26 top

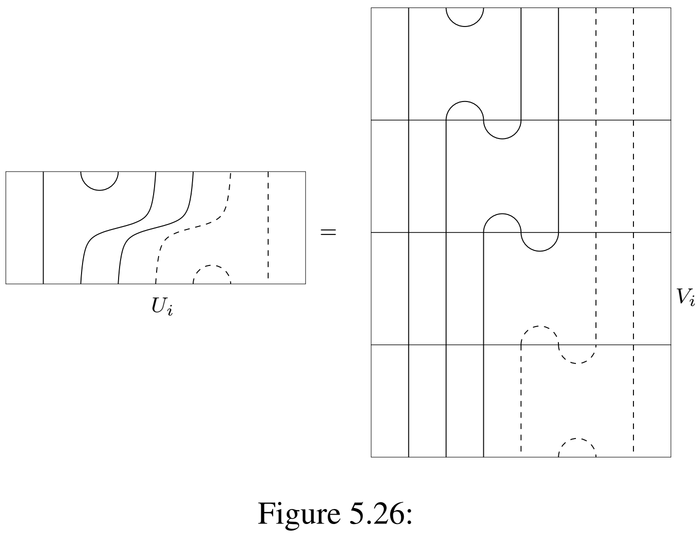


Figure 5.26:

two diagrams product is T and third diagram is V_i and fourth diagram is B. Each U_i' can be written as the product of generators in our generating set. Therefore U_i, D' and D follows this. From this we can say each and every basis elements can be written as the product of generators. This implies taht every algebra elment x can be generated by our generating set. □

Claim 5.3.3.

$$\sum_{\sum_{i=1}^{h} c_i = n} 1 = \binom{n+h-1}{h-1} \tag{5.3.14}$$

You can find this claim in [59, section 1.2, page15]

Proof. The sum in the lefthand side of the above Claim can be interpreted as counting the number of different combinations we could make with h colour strings in such a way that total number of strings in each diagram is n. Allow that there are c_i number of colour i strings, where i take the values from 1 to h and c_i can take the values from 0 to n. We make the diagrams as in Figure 5.27. First we draw c_1 strings of colour 1 then c_2 strings of colour 2 and so on up to c_h strings of colour h. Here permutation is not allowed and only combination is possible. Now we need to count the number of possible different first diagrams which could be made. Note that we could have diagrams without a particular colour string. This is quite hard to count. Therefore, we make the second diagram by adding one extra string of each colour. That is

$$d_i = c_i + 1.$$

Therefore, our second diagram will look like Figure 5.28. Therefore, all bands have positive length. That is, every colour string appears in the diagram. Number of first diagram equal to the number of second diagram because there is a bijection between them. Now we construct the third diagram by adding dots to the second diagram between each consecutive

Figure 5.27:

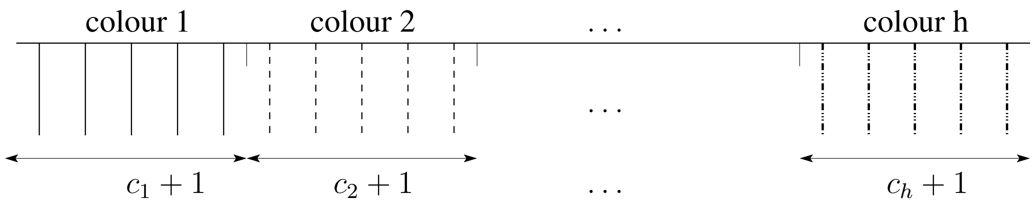


Figure 5.28:

pair of strings as in Figure 5.29. If we look at Figure 5.29 that shows the dots between the strings. By choosing the position of $h - 1$ of the dots we can separate the colour strings. Actually, the number of third diagram (Figure 5.29) is the same as the number of second diagrams (Figure 5.28) because there is a bijection between these diagrams. There are $d_1 + d_2 + \ldots + d_h$ number of strings. This can be simplified as

$$\begin{aligned} d_1 + d_2 + \ldots + d_h &= (c_1 + 1) + (c_2 + 1) + \ldots + (c_h + 1) \\ &= (c_1 + c_2 + \ldots + c_h) + h \\ &= n + h \end{aligned}$$

Therefore, the number of dots between these nodes is $n + h - 1$. The number of ways to choose the h colour strings is the same as choosing the $h - 1$ dots. Therefore, there are $\binom{n+h-1}{h-1}$ number of third diagrams we could make. The number of ways to make first diagrams (Figure 5.27) is the same as the ways to make the second diagram and the number of ways to make the second diagram is the same as the way to make the third diagram.

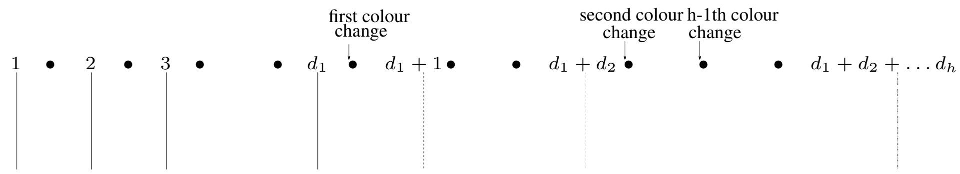


Figure 5.29:

Therefore, we can make $\binom{n+h-1}{h-1}$ first diagrams. Hence, we have proved the claim. □

Claim 5.3.4.

$$\sum_{\sum_{i=1}^{h} c_i = n} c_h = \binom{n+h-1}{h} \tag{5.3.15}$$

Proof. By modifying the previous proof we can prove this claim. The sum in the left hand side of the above claim could be interpreted as finding the total of colour h strings c_h in each different combination, we could make with h colour strings in such a way that total number of strings in each diagram is n.

Each first diagram (Figure 5.27) in Claim 5.3.3's c_h copy should be counted. This is the same as counting the second diagram in Figure 5.28's c_h copies. Similarly, the number of second diagram's c_h copy is the same as counting the third diagram in Figure 5.29's c_h copies. We can find the c_h copies of the third diagram by choosing one more dot from Figure 5.29. Let us see the reason. After the string $d_1 + d_2 + \ldots + d_{h-1}$ there are d_h number of colour h strings until the string $d_1 + d_2 + \ldots + d_h$. Actually, there are d_h number of gaps after the string $d_1 + d_2 + \ldots + d_{h-1}$. Therefore, d_h number of dots, that is $c_h + 1$ string, between the strings. If we need to count the third diagram in Figure 5.29 ones, we need to choose the dot between the string $d_1 + d_2 + \ldots + d_{h-1} + 1$ and $d_1 + d_2 + \ldots + d_{h-1} + 2$. Similarly, if we need to count it twice, we need to choose the dot between the string $d_1 + d_2 + \ldots + d_{h-1} + 2$ and $d_1 + d_2 + \ldots + d_{h-1} + 3$. Similarly, if we choose the node between the string $d_1 + d_2 + \ldots + d_h - 1$ and $d_1 + d_2 + \ldots + d_h$ as in Figure 5.30, we will count the

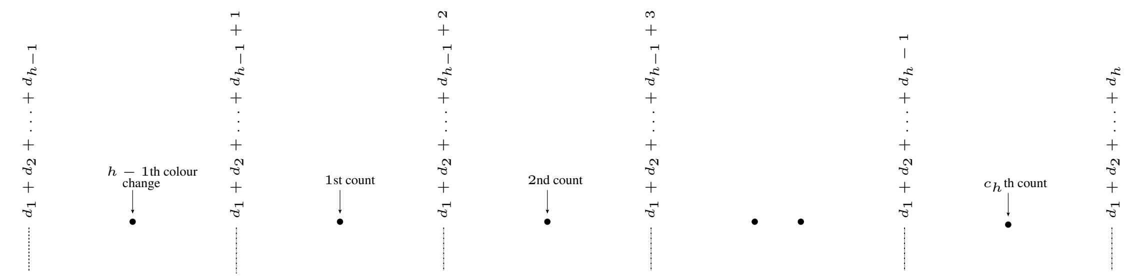


Figure 5.30:

third diagram in Figure 5.29 c_h times. We need to choose $h-1$ dots to separate the colour string and 1 more dot to count the number of c_h string. That is, we need to choose h dots from $n+h-1$ dots. Therefore, there are $\binom{n+h-1}{h}$ number of c_h copies of first diagram there. Hence, we have proved the claim. □

Claim 5.3.5.

$$\sum_{\sum_{i=1}^{h} c_i = n} \frac{n!}{c_1! c_2! \dots c_h!} = h^n \tag{5.3.16}$$

This claim follows from the Multinomial theorem [59, section 1.2, page16 and 17]

Proof. Sum in the left-hand side of the above claim could be interpreted as the number of different arrangements we could make with h colour strings in such a way that the total number of strings in each diagram is n. Allow that there are c_i number of colour i strings, where i takes the values from 1 to h. Therefore, the number of arrangement we could make with these propagating lines is (number of diagrams)

$$\frac{n!}{c_1! c_2! \dots c_h!}.$$

If we consider the colour i which can have $0, 1, \dots, n$ propagating lines, that is c_i can take the values $0, 1, \dots, n$, the total number of propagating lines is n. That is

$$\sum_{i=1}^{n} c_i = n.$$

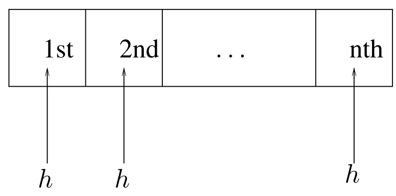


Figure 5.31:

Therefore, the total number of different diagrams we could get from colour 1, colour 2, colour h propagating lines is given by

$$\sum_{\sum_{i=1}^{h} c_i = n} \frac{n!}{c_1!c_2!\ldots c_h!}.$$

Let us find the total number of different diagrams we could get from the colour 1, colour 2, colour h propagating lines in a different way. First propagating line could be colour 1 propagating or colour 2 propagating or, $\ldots$ colour h propagating. Similarly, the second propagating line could be filled in h ways and so on up to nth propagating lines can be filled in h ways as in Figure 5.31. Therefore, total number of different diagrams is given by

$$h \times h \times \ldots \times h = h^n.$$

Hence, we have proved the Claim. $\square$

Let us verify this result. For the value of $h = 2$, that is two colour case, total number of generators is given by substituting $h = 2$ in the Proposition 5.3.1. This gives us

$$\begin{aligned}
&(2-1)(n-1)2^{n-1} + 2\binom{n}{2} + 2(2-1)\binom{n-1}{1} + 2 \\
&= (n-1)2^{n-1} + \frac{2n(n-1)}{2} + 2n - 2 + 2 \\
&= (n-1)2^{n-1} + n(n+1).
\end{aligned}$$

From Proposition 5.1.1 we can this is true.

For the value of $h = 3$, that is three colour case, total number of generators is given by substituting $h = 3$ in the Proposition 5.3.1. This gives us

$$\begin{aligned}
&(3-1)(n-1)3^{n-1} + 3\binom{n+1}{3} + 3 \times 2\binom{n}{2} + 3 \\
&= 2(n-1)3^{n-1} + \frac{3(n+1)!}{(n-2)!3!} + \frac{6n!}{(n-2)!2!} + 3 \\
&= 2(n-1)3^{n-1} + \frac{n(n+1)(n-1)}{2} + 3n(n-1) + 3
\end{aligned}$$

From Proposition 5.2.1, we can see this is true.

Chapter 6

Homomorphism between modules: the general case

In this chapter, we will show how to find the non-zero homomorphisms between cell modules with no arcs to a general cell module by using the hypercuboid. We will discuss this method in Section 6.2 in great detail. This is our first general way to find the homomorphism between two modules. We do this by finding the homomorphism between Temperley-Lieb algebra modules and gluing these together to find the homomorphism between any given two cell modules.

We will introduced the new convention of notation in Section 6.1 to label our module basis elements and generators of the algebra. This will help us to extend our investigation compared with Chapter 4 to more than two colours, and more than one arc in the second module.

Basic condition for existence of a non-zero homomorphism

Suppose that

$$\theta : \Delta_n(a_1, \ldots, a_h) \longrightarrow \Delta_n(b_1, \ldots, b_h)$$

is our desired homomorphism. Therefore, it should satisfy the homomorphism condition

$$\theta(mx) = \theta(m)x,$$

where m is an element of the first module and x is an element of the second module. As we saw earlier $\theta(m)$ must be a linear combination of basis elements of the second module with northern edge having the same colour sequence as x. Therefore, the second module should have basis elements with the same colour sequence of nodes as the first module basis element otherwise non-zero homomorphism is not possible to define. From this we can say that , if there is a non-zero homomorphism between the two modules, then the difference between the number of propagating lines of the first module and the second module of each colour will be twice the number of arcs of the same colour. That is

$$a_i = b_i + 2t_i, \qquad (6.0.1)$$

Our module and algebra have more than one colour. We mainly use red, green and black colour strings in this chapter. We follow the same labeling as in Chapter 4(Figure 4.1).

6.1 A new notation for module and algebra elements

Suppose that $\Delta_n(a_1, a_2, \ldots, a_h)$ and $\Delta_n(b_1, b_2, \ldots, b_h)$ are the given two modules. We give a convenient notation for basis elements of the module and algebra. We gave a notation in Chapter 4 as in subsection 4.1.1. However, this notation is quite messy to handle if we deal with more than one colour.

In the new notation, we write the basis elements of the modules and algebras as $h + 1$ tuples. Let us denote the first module basis element by m_u, second module basis element by n_v and the algebra element by g_w. Here, u, v and w are the $h + 1$ tuples. The first place of the tuple represents the colour sequence, the second place represents the colour C_1 half diagram, the third place represents the colour C_2 half diagram and so on up to $h+1$th place which represents the colour C_h half diagram.

Labelling the basis elements of the modules and the generators

The module $\Delta_n(a_1, a_2, \ldots, a_h)$ has all lines propagating by assumption. Therefore, there will be $\frac{n!}{a_1!a_2!\ldots a_h!}$ arrangements we can make of the propagating lines. Each arrangement gives a colour sequence. We label the colour sequence of the nodes by numbers. If a colour sequence has first n_{C_1} nodes colour C_1, next n_{C_2} nodes colour C_2 and so on up to last n_{C_h} nodes colour C_h then we call that colour sequence X_1. In this situation, the first place of the tuples take the value 1. Similarly, we label each colour shape half diagram by numbers. If a particular colour shape has all nodes propagating lines then we label it 0.

We know that the first module does not have any arcs. Therefore, n_{C_i} is equal to a_i for all i from 1 to h. We label all the possible shapes which could arrive from each colour C_i by the numbers 0 to p_i. 0 represents the shape which has all lines propagating. If we take a basis element of the module $\Delta_n(a_1, a_2, \ldots, a_h)$ then this can be written as $m_{i,\alpha_1,\alpha_2,\ldots,\alpha_h}$. Here i represents the colour sequence, α_1 represents the shape of the colour C_1, α_2 represents the shape of the colour C_2 and so on. Similarly, if we take a basis element of the module $\Delta_n(b_1, b_2, \ldots, b_h)$, that can be written as $n_{i,\beta_1,\beta_2,\ldots,\beta_h}$.

If we take a generator of the algebra, with the same colour arc at the northern and southern edge, that can be written as $g_{i,\gamma_1,\gamma_2,\ldots,\gamma_h}$. If the northern edge and southern edge of the generator have different colour sequences then we can write the generator as $g_{i,\gamma_1,\gamma_2,\ldots,\gamma_h}^{j,\gamma'_1,\gamma'_2,\ldots,\gamma'_h}$. Here $i, \gamma_1, \gamma_2, \ldots, \gamma_h$ represents the northern edge colour sequence and $j, \gamma'_1, \gamma'_2, \ldots, \gamma'_h$ represents the southern edge colour sequence. However, we do not need the generators with the different colour arc at the northern and southern edge to find the non-zero homomorphism between the modules according to Lemma 6.1.2. Therefore, the colour sequence of the generators at both places is same. For this reason, we do not really show the southern edge colour sequence.

With this new notation Lemma 4.1.1 becomes

Lemma 6.1.1. *Suppose that θ is a non-zero homomorphism from $\Delta_n(a_1, a_2, \ldots, a_h)$ to $\Delta_n(b_1, b_2, \ldots, b_h)$. If $m_{i,0,0,\ldots,0}$ is a basis element of the first module with the southern edge colour sequence as X_i then we can write*

$$\theta(m_{i,0,0,\ldots,0}) = \sum s_{\beta_1,\beta_2,\ldots,\beta_h} n_{i,\beta_1,\beta_2,\ldots,\beta_h}, \tag{6.1.1}$$

where $n_{i,\beta_1,\beta_2,\ldots,\beta_h}$ is a basis element of the second module with the southern edge colour sequence as X_i and $s_{\beta_1,\beta_2,\ldots,\beta_h}$ is the coefficient of $n_{i,\beta_1,\beta_2,\ldots,\beta_h}$ which is independent of i.

Lemma 6.1.2. *In the process of finding the non-zero homomorphism from one module to another it is enough to consider the generators with same colour arc at the northern edge and southern edge.*

Proof. Let θ be the non-zero homomorphism between the given two modules. So, it should satisfy the homomorphism condition

$$\theta(ma) = \theta(m)a. \tag{6.1.2}$$

It is enough to check the above condition by choosing m as a basis element of the module and a as a generator of the algebra. In Chapter 5 we have classified the generators of the algebra in to following five categories. They are generators with all same colour propagating, generators with a cross, generators with same colour arc and generators with different colour arcs.

If we choose the generators with all the same colour propagating we will get 0 on both sides of (6.1.2) if the colours do not match. If the southern edge colour sequence of m and the northern edge colour sequence of generator match the homomorphism the condition in (6.1.2) will be satisfied because generator does not do anything; it behaves as an identity.

If we use the generators with one cross, two different colour propagating lines crossing, we will not get any condition because this type of generator only moves the position of the nodes. Therefore, homomorphism condition in (6.1.2) is satisfied.

If we consider the generators with same colour arcs we will get 0 in the left-hand side of (6.1.2) because ma has wrong number of propagating lines. Therefore, the right-hand side of (6.1.2) should be 0. This gives us the necessary condition to find the non-zero homomorphism.

However, if we consider the generators with different colour arcs at the northern edge and southern edge we get conditions which already arose from considering generators with the same colour arcs. Further, as we have seen in cellular algebra, the southern edge of the module element and the northern edge of algebra element are alone responsible for the formation of the constant which arises in any condition. For this reason, we get the same equation as we obtain by considering the same colour arc at the northern and southern edge of a generator and the same colour arc at the northern edge and different colour arc at the southern edge of another generator. □

6.2 Using one colour fact to construct the hypercuboid

In this section we are going to find the non-zero homomorphism between the two given modules with more than one colour, just by finding the non-zero homomorphism between each colour of the modules separately. In the process of finding the non-zero homomorphism, we should know how the basis element of the first module maps to a linear combination of the second module basis elements. In this linear combination, the relationship between the coefficients of the second module basis element can be represented by a straight line segment if the modules have only one colour. On the other hand, if the modules have two colours, then we will get a rectangle, three colours then a cuboid and n-dimensional hypercuboid.

Definition 6.2.1. A hypercuboid is a set of the form

$$\mathbb{H} = [1, \ldots, n_1] \times [1, \ldots, n_2] \times \ldots [1, \ldots, n_h]$$

inside $\mathbb{R}^h$. Elements $x = (x_1, \ldots, x_h)$ and $y = (y_1, \ldots, y_h)$ in $\mathbb{H}$ are connected if there exist t with $1 \leq t \leq h$ such that

$$\begin{aligned} x_i &= y_i \quad \text{if } i \neq t \\ x_t &= y_t \pm 1. \end{aligned}$$

We would like to thank Prof.Joseph Chuang to improve the following theorem.

Theorem 6.2.2. *Suppose there exists non-homomorphisms from $\Delta_{n_{C_k}}(a_k)$ to $\Delta_{n_{C_k}}(b_k)$, as k varies from 1 to h, such that*

$$\theta_{C_k}(\overset{(k)}{m_0}) = \sum_{\beta_k} s_{\beta_k} n_{\beta_k} \tag{6.2.1}$$

where $a_k = n_{C_k}$, $\overset{(k)}{m_0}$ is the basis element of $\Delta_{n_{C_k}}(a_k)$, and n_{β_k} is a basis element of $\Delta_{n_{C_k}}(b_k)$ and s_{β_k} is the coefficient of n_{β_k}. Then there exists a non-zero homomorphism from $\Delta_n(a_1, a_2, \ldots, a_h)$ to $\Delta_n(b_1, b_2, \ldots, b_h)$ such that

$$\theta(\sum_i c_i m_{i,0,0,\ldots,0}) = \sum_i c_i \sum_{\beta_1,\beta_2,\ldots,\beta_h} s_{\beta_1,\beta_2,\ldots,\beta_h} n_{i,\beta_1,\beta_2,\ldots,\beta_h}, \tag{6.2.2}$$

where $n = \sum_{k=1}^{h} n_{C_k}$, and $s_{\beta_1,\beta_2,\ldots,\beta_h}$ is a coefficient of $n_{i,\beta_1,\beta_2,\ldots,\beta_h}$ and which is given by

$$s_{\beta_1,\beta_2,\ldots,\beta_h} = s_{\beta_1} \times s_{\beta_2} \times \ldots \times s_{\beta_h}. \quad (6.2.3)$$

Proof. We need to show θ should satisfy the homomorphism condition

$$\theta(mx) = \theta(m)x$$

for all m in $\Delta_n(a_1, a_2, \ldots, a_h)$ and x in $TL_n^h(\delta_{C_1}, \ldots, \delta_{c_h})$. If we can show above homomorphism condition hold for any m and each generators of the algebra then it will be true for any m and any element x of the algebra.

We have classified the generators of the algebra into four cases in Chapter 5. They are (i)generators with all same colour propagating lines, (ii)generator with one cross, (iii)generator with one pair of arcs of the same colour at the northern and southern edge and (iv)generator with one pair of arcs with different colour at the northern and southern edge.

From (6.2.2) and (6.2.3) we can write

$$\theta(\sum_i c_i m_{i,0,0,\ldots,0}) = \sum_i c_i \sum_{\beta_1,\beta_2,\ldots,\beta_h} s_{\beta_1} \times s_{\beta_2} \times \ldots \times s_{\beta_h} n_{i,\beta_1,\beta_2,\ldots,\beta_h}, \quad (6.2.4)$$

Let g be a generator of type(i) with all same colour j propagating lines. If we multiply (6.2.4) by g we get zero when $a_i \neq 0$ and $a_j \neq 0$ for some value of $i, j \in \{1, \ldots, h\}$ because colour sequence do not match. On the other hand, if $a_i \neq 0$ and $a_j = 0$ for all $j \in \{1, \ldots, h\}$ except i then modules $\Delta_n(a_1, a_2, \ldots, a_h)$ and $\Delta_n(b_1, b_2, \ldots, b_h)$ are Temperley-Lieb algebra modules. In this situation g behave as identity. Therefore, homomorphism condition

$$\theta(mg) = \theta(m)g$$

is satisfied.

Now let $g^{j',0,0,\ldots,0}_{j,0,0,\ldots,0}$ be a generator of the algebra of type(ii), with colour k and colour k' crossing. If we multiply (6.2.4) by $g^{j',0,0,\ldots,0}_{j,0,0,\ldots,0}$ we get

$$\begin{aligned}\theta(\sum_i c_i m_{i,0,0,\ldots,0}) g^{j',0,0,\ldots,0}_{j,0,0,\ldots,0} &= \sum_i c_i \sum_{\beta_1,\beta_2,\ldots,\beta_h} s_{\beta_1} \times s_{\beta_2} \times \ldots \times s_{\beta_h} n_{i,\beta_1,\beta_2,\ldots,\beta_h} g^{j',0,0,\ldots,0}_{j,0,0,\ldots,0} \\ &= c_j \sum_{\beta_1,\beta_2,\ldots,\beta_h} s_{\beta_1} \times s_{\beta_2} \times \ldots \times s_{\beta_h} n_{j,\beta_1,\beta_2,\ldots,\beta_h} g^{j',0,0,\ldots,0}_{j,0,0,\ldots,0} \\ &= c_j \sum_{\beta_1,\beta_2,\ldots,\beta_h} s_{\beta_1} \times s_{\beta_2} \times \ldots \times s_{\beta_h} n_{j',\beta_1,\beta_2,\ldots,\beta_h}.\end{aligned}$$

Multiplication of $n_{j,\beta_1,\beta_2,\ldots,\beta_h}$ by $g^{j',0,0,\ldots,0}_{j,0,0,\ldots,0}$, generators with all propagating lines with one cross , gives us $n_{j',\beta_1,\beta_2,\ldots,\beta_h}$ because this multiplication does not change the labels of each colour(crossing of same colour not allowed) but, the colour sequence j of $n_{j,\beta_1,\beta_2,\ldots,\beta_h}$ is changed to j'.

On the other hand

$$\begin{aligned}\left(\sum_i c_i m_{i,0,0,\ldots,0}\right) g^{j',0,0,\ldots,0}_{j,0,0,\ldots,0} &= c_j m_{j,0,0,\ldots,0} g^{j',0,0,\ldots,0}_{j,0,0,\ldots,0} \\ &= c_j m_{j',0,0,\ldots,0}.\end{aligned}$$

Therefore,

$$\begin{aligned}\theta\left((\sum_i c_i m_{i,0,0,\ldots,0}) g^{j',0,0,\ldots,0}_{j,0,0,\ldots,0}\right) &= \theta(c_j m_{j',0,0,\ldots,0}) \\ &= c_j \theta(m_{j',0,0,\ldots,0}) \\ &= c_j \sum_{\beta_1,\beta_2,\ldots,\beta_h} s_{\beta_1} \times s_{\beta_2} \times \ldots \times s_{\beta_h} n_{j',\beta_1,\beta_2,\ldots,\beta_h}.\end{aligned}$$

From these we can say

$$\theta\left((\sum_i c_i m_{i,0,0,\ldots,0})\right) g^{j',0,0,\ldots,0}_{j,0,0,\ldots,0} = \theta\left((\sum_i c_i m_{i,0,0,\ldots,0}) g^{j',0,0,\ldots,0}_{j,0,0,\ldots,0}\right).$$

That means homomorphism condition is satisfied.

Now let us take a generator of the algebra of type(iii), generator with a pair of colour C_k arcs at the northern edge and southern edge. We call it as $g^{j,0,\ldots,0,\gamma_k,0,\ldots,0}_{j,0,\ldots,0,\gamma_k,0,\ldots,0}$. For our convenient we define

$$g_{j,e_1,\ldots,e_{k-1},x_k,e_{k+1},\ldots,e_h} = g^{j,0,\ldots,0,\gamma_k,0,\ldots,0}_{j,0,\ldots,0,\gamma_k,0,\ldots,0}$$

where e_α has all propagating lines for α takes the values 1 to h except k and e_α's dimension can be obtained from the colour sequence j and x_k is a diagram(generator of the algebra) in $TL_{n_{C_k}}$ and it has the northern and southern edge half diagrams γ_k. Let m be $\sum_i c_i m_{i,0,0,\ldots,0}$. Therefore

$$\begin{aligned}
&\theta(m) g_{j,e_1,\ldots,e_{k-1},x_k,e_{k+1},\ldots,e_h}\\
&= \left(\sum_i c_i \sum_{\beta_1,\beta_2,\ldots,\beta_h} s_{\beta_1}\times s_{\beta_2}\times\ldots\times s_{\beta_h} n_{i,\beta_1,\beta_2,\ldots,\beta_h}\right) g_{j,e_1,\ldots,e_{k-1},x_k,e_{k+1},\ldots,e_h}\\
&= c_j \sum_{\beta_1,\beta_2,\ldots,\beta_h} s_{\beta_1}\times s_{\beta_2}\times\ldots\times s_{\beta_h} n_{j,\beta_1,\beta_2,\ldots,\beta_h} g_{j,e_1,\ldots,e_{k-1},x_k,e_{k+1},\ldots,e_h}\\
&= c_j \sum_{\beta_1,\ldots,\beta_{k-1},\beta_{k+1},\ldots,\beta_h} s_{\beta_1}\times\ldots\times s_{\beta_{k-1}}\times s_{\beta_{k+1}}\times\ldots\times s_{\beta_h}\\
&\qquad \sum_{\beta_k} s_{\beta_k} n_{j,\beta_1,\beta_2,\ldots,\beta_h} g_{j,e_1,\ldots,e_{k-1},x_k,e_{k+1},\ldots,e_h}\\
&= c_j \sum_{\beta_1,\ldots,\beta_{k-1},\beta_{k+1},\ldots,\beta_h} s_{\beta_1}\times\ldots\times s_{\beta_{k-1}}\times s_{\beta_{k+1}}\times\ldots\times s_{\beta_h}\\
&\qquad \sum_{\beta_k} s_{\beta_k} n_{j,\beta_1 e_1,\ldots,\beta_{k-1}e_{k-1},\beta_k x_k,\beta_{k+1}e_{k+1},\ldots,\beta_h e_h}.
\end{aligned}$$

Here we say

$$n_{\beta x} = \begin{cases} 0 & \text{if} \beta x \text{ has wrong number of propagating lines} \\ \lambda n_{\beta'} & \text{if } \beta x = \lambda\beta' \text{ and } \beta' \text{ has correct number of propagating lines.} \end{cases}$$

Therefore we can say

$$\begin{aligned}\theta(m)&g_{j,e_1,\ldots,e_{k-1},x_k,e_{k+1},\ldots,e_h}\\ &= c_j \sum_{\beta_1,\ldots,\beta_{k-1},\beta_{k+1},\ldots,\beta_h} s_{\beta_1}\times\ldots\times s_{\beta_{k-1}}\times s_{\beta_{k+1}}\times\ldots\times s_{\beta_h} \\ &\qquad\qquad \sum_{\beta_k} s_{\beta_k} n_{j,\beta_1,\ldots,\beta_{k-1},\beta_k x_k,\beta_{k+1},\ldots,\beta_h}.\end{aligned} \tag{6.2.5}$$

But we know there is a homomorphism from $\Delta_{n_{C_k}}(a_k)$ to $\Delta_{n_{C_k}}(b_k)$ as k varies from 1 to h, and from (6.2.1)

$$\theta_{C_k}(\overset{(k)}{m_0}) = \sum_{\beta_k} s_{\beta_k} n_{\beta_k}.$$

Our x_k is a generator of $TL_{n_{C_k}}$ and it has a pair of arcs at the northern and southern edge. Therefore $\overset{(k)}{m_0}x_k = 0$. From this we can say $\theta_{n_{C_k}}(\overset{(k)}{m_0}x_k) = 0$. But θ_{C_k} is a homomorphism. Therefore

$$\begin{aligned}\theta_{n_{C_k}}(\overset{(k)}{m_0}x_k) &= \theta_{n_{C_k}}(\overset{(k)}{m_0})x_k\\ 0 &= \sum_{\beta_k} s_{\beta_k} n_{\beta_k} x_k\\ 0 &= \sum_{\beta_k} s_{\beta_k} n_{\beta_k x_k}.\end{aligned}$$

But $\sum_{\beta_k} s_{\beta_k} n_{\beta_k x_k}$ and $\sum_{\beta_k} s_{\beta_k} n_{j,\beta_1,\ldots,\beta_{k-1},\beta_k x_k,\beta_{k+1},\ldots,\beta_h}$ has one to one correspondence. From this we can say

$$\sum_{\beta_k} s_{\beta_k} n_{j,\beta_1,\ldots,\beta_{k-1},\beta_k x_k,\beta_{k+1},\ldots,\beta_h} = 0.$$

If we substitute this in (6.2.5) gives us

$$\theta(m) g_{j,e_1,\ldots,e_{k-1},x_k,e_{k+1},\ldots,e_h} = 0. \tag{6.2.6}$$

If we find

$$\begin{aligned} mg_{j,e_1,\ldots,e_{k-1},x_k,e_{k+1},\ldots,e_h} &= \sum_i c_i m_{i,0,0,\ldots,0} g_{j,e_1,\ldots,e_{k-1},x_k,e_{k+1},\ldots,e_h} \\ &= c_j m_{j,0,0,\ldots,0} g_{j,e_1,\ldots,e_{k-1},x_k,e_{k+1},\ldots,e_h} \end{aligned}$$

Our generator $g_{j,e_1,\ldots,e_{k-1},x_k,e_{k+1},\ldots,e_h}$ has one pair of colour C_k arcs. But $m_{j,0,0,\ldots,0}$ has all propagating lines. Therefore product of these two will give wrong number of propagating lines. Therefore

$$mg_{j,e_1,\ldots,e_{k-1},x_k,e_{k+1},\ldots,e_h} = 0.$$

Therefore

$$\theta(mg_{j,e_1,\ldots,e_{k-1},x_k,e_{k+1},\ldots,e_h}) = 0.$$

From this we can say

$$\theta(mg_{j,e_1,\ldots,e_{k-1},x_k,e_{k+1},\ldots,e_h}) = \theta(m) g_{j,e_1,\ldots,e_{k-1},x_k,e_{k+1},\ldots,e_h}.$$

Therefore, homomorphism condition is satisfied.

At last we take a generator of the algebra with a pair of arcs (northern edge colour C_k and southern edge colour $C_{k'}$). Let the generator as $g^{j',0,\ldots,0,\gamma_{k'},0,\ldots,0}_{j,0,\ldots,0,\gamma_k,0,\ldots,0}$. We can write

$$g^{j',0,\ldots,0,\gamma_{k'},0,\ldots,0}_{j,0,\ldots,0,\gamma_k,0,\ldots,0} = \frac{1}{\delta_{C_k}} g_{j,e_1,\ldots,e_{k-1},x_k,e_{k+1},\ldots,e_h} g^{j',0,\ldots,0,\gamma_{k'},0,\ldots,0}_{j,0,\ldots,0,\gamma_k,0,\ldots,0}.$$

If we find

$$\begin{aligned} &\theta(m) g^{j',0,\ldots,0,\gamma_{k'},0,\ldots,0}_{j,0,\ldots,0,\gamma_k,0,\ldots,0} \\ &= \theta(m) \frac{1}{\delta_{C_k}} g_{j,e_1,\ldots,e_{k-1},x_k,e_{k+1},\ldots,e_h} g^{j',0,\ldots,0,\gamma_{k'},0,\ldots,0}_{j,0,\ldots,0,\gamma_k,0,\ldots,0} \\ &= \frac{1}{\delta_{C_k}} \theta(m) g_{j,e_1,\ldots,e_{k-1},x_k,e_{k+1},\ldots,e_h} g^{j',0,\ldots,0,\gamma_{k'},0,\ldots,0}_{j,0,\ldots,0,\gamma_k,0,\ldots,0} \end{aligned}$$

From (6.2.6) we can say

$$
\begin{aligned}
&\theta(m)g^{j',0,\ldots,0,\gamma_{k'},0,\ldots,0}_{j,0,\ldots,0,\gamma_k,0,\ldots,0} \\
&= \frac{1}{\delta_{C_k}} \times 0 \times g^{j',0,\ldots,0,\gamma_{k'},0,\ldots,0}_{j,0,\ldots,0,\gamma_k,0,\ldots,0} \\
&= 0.
\end{aligned}
$$

Now we find

$$
\begin{aligned}
&mg^{j',0,\ldots,0,\gamma_{k'},0,\ldots,0}_{j,0,\ldots,0,\gamma_k,0,\ldots,0} \\
&= m\frac{1}{\delta_{C_k}} g_{j,e_1,\ldots,e_{k-1},x_k,e_{k+1},\ldots,e_h} g^{j',0,\ldots,0,\gamma_{k'},0,\ldots,0}_{j,0,\ldots,0,\gamma_k,0,\ldots,0} \\
&= \frac{1}{\delta_{C_k}} m g_{j,e_1,\ldots,e_{k-1},x_k,e_{k+1},\ldots,e_h} g^{j',0,\ldots,0,\gamma_{k'},0,\ldots,0}_{j,0,\ldots,0,\gamma_k,0,\ldots,0} \\
&= \frac{1}{\delta_{C_k}} \times 0 \times g^{j',0,\ldots,0,\gamma_{k'},0,\ldots,0}_{j,0,\ldots,0,\gamma_k,0,\ldots,0} \\
&= 0.
\end{aligned}
$$

Therefore

$$\theta(mg^{j',0,\ldots,0,\gamma_{k'},0,\ldots,0}_{j,0,\ldots,0,\gamma_k,0,\ldots,0}) = 0.$$

These implies

$$\theta(mg^{j',0,\ldots,0,\gamma_{k'},0,\ldots,0}_{j,0,\ldots,0,\gamma_k,0,\ldots,0}) = \theta(m)g^{j',0,\ldots,0,\gamma_{k'},0,\ldots,0}_{j,0,\ldots,0,\gamma_k,0,\ldots,0}.$$

That is, homomorphism condition satisfied.

Therefore, all four types of generators of the algebra satisfy the homomorphism condition. This implies that $\theta(mx) = \theta(m)x$ for all module element m and the algebra element x. Hence we have proved the theorem. □

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