Convertible bonds valuation in a jump diffusion setting with stochastic interest rates

Laura Ballotta† and Ioannis Kyriakou‡

12 August 2014

This paper proposes an integrated pricing framework for convertible bonds, which comprises firm value evolving as an exponential jump diffusion, correlated stochastic interest rates movements and an efficient numerical pricing scheme. By construction, the proposed stochastic model fits in the framework of affine jump diffusion processes of Duffie, Pan and Singleton [Duffie, D., Pan, J. and Singleton, K., Transform analysis and asset pricing for affine jump-diffusions. Econometrica, 2000, 68, 1343–1376] with tractable behaviour. We define the firm’s optimal call policy and investigate its impact on the computed convertible bond prices. We illustrate the performance of the numerical scheme and highlight the effects originated by the inclusion of jumps, stochastic interest rates and a non-zero correlation structure between firm value and interest rates.

Keywords: Convertible bonds pricing; Stochastic interest rates; Affine jump diffusion model; Optimal call strategy

JEL Classification: G12; G13; C63

1 Introduction

In this communication, we introduce a valuation framework for convertible bonds (CBs) based on a structural default model including jumps in the firm value dynamics and correlated stochastic interest rates which fits in the class of affine jump diffusions of Duffie et al. (2000). The proposed framework also includes a pricing algorithm based on a backward convolution approach, which is facilitated by the fact that the characteristic function of the relevant driving risk factors is known in analytical form.

CBs are corporate bonds that can be converted into shares of the issuing firm at the bondholder’s discretion; these instruments represent a pricing challenge because of their hybrid nature and their complex design. Firstly, they depend on variables related to the underlying firm value (or stock), the fixed income part, which includes both interest rates and default risk, and the interaction between these components. Secondly, CBs usually carry call provisions giving the issuer the right to demand premature redemption in exchange for the current call price. A put provision, which allows the investor to force the issuing firm to prematurely repurchase the CB for a pre-specified price, is also sometimes met. Due to these early-exercise features, CBs valuation requires an accurate numerical scheme capable of handling efficiently the complexity of the contract.

We thank Stewart Hodges and Gianluca Fusai for interesting comments to a previous version of the paper. Usual caveat applies.

†Cass Business School, City University London, 106 Bunhill Row, London EC1Y 8TZ, UK. E-mail: L.Ballotta@city.ac.uk, tel.: +44 (0)20 7040 8954, fax: +44 (0)20 7040 8881
‡Corresponding author. Cass Business School, City University London, 106 Bunhill Row, London EC1Y 8TZ, UK. E-mail: Ioannis.Kyriakou@city.ac.uk, tel.: +44 (0)20 7040 8738, fax: +44 (0)20 7040 8881
Closed form solutions for the price of the CB in the standard Black–Scholes–Merton economy have been obtained by Ingersoll (1977a) for the case of non-callable/callable products; however, the introduction in the valuation model of a more realistic specification including, e.g., discretely payable coupons, dividends on the underlying stock, hard call constraints (which protect the CB holder from the issuer calling the bond soon after issue) and soft call provisions (which vary from a notice period before the CB can be called, to stock price trigger and/or make-whole features) prevent the derivation of explicit pricing formulae. For these reasons, various techniques have been considered in the literature such as numerical solutions to partial differential equations/inequalities adopted by, e.g., Brennan and Schwartz (1977, 1980), Carayannopoulos (1996), Tsiveriotis and Fernandes (1998), Takahashi et al. (2001), Barone-Adesi et al. (2003) and Kovalov and Linetsky (2008); lattice methods developed, amongst others, by Derman (1994) and Takahashi et al. (2001); and Monte Carlo simulation, see Lvov et al. (2004) and Ammann et al. (2008).

Early contributions by Ingersoll (1977a), Brennan and Schwartz (1977, 1980), and Carayannopoulos (1996) fall in the structural class of default risk where the relevant state variable is the firm value. More recently the focus has been on reduced form equity default models with/out stochastic interest rates as in Tsiveriotis and Fernandes (1998), Takahashi et al. (2001), Barone-Adesi et al. (2003), Carayannopoulos and Kalimipalli (2003), Yiğitbaşioğlu and Alexander (2006), and Kovalov and Linetsky (2008) (who also include stochastic stock price volatility and default intensity factor).\(^1\) The shift away from structural models had been mainly due to the fact that the spreads generated by these models were not in line with the observed ones (see, e.g., Eom et al. 2004). However, recent evidence provided by Brigo and Morini (2006), Cariboni and Schoutens (2007), Lipton and Sepp (2009), Fang et al. (2010), Fiorani et al. (2010) and Dao and Jeanblanc (2012), amongst others, shows that enhanced structural models, which incorporate, for example, time varying volatility and/or jumps, exhibit high fit to market quotes and, under suitable constructions, high tractability as well. This implies that many of the documented shortcomings are to be attributed to the pure diffusion form of the traditional models.

In light of the previous discussion, in this paper we adopt a jump diffusion model for the firm value with correlated stochastic interest rates, hence comprising four risk factors: the Brownian motion governing the diffusion part of the firm value, the compound Poisson process with random jump sizes modelling the shocks of the firm value, and the Brownian motion driving the short rate of interest. Correlation between interest rates and firm value is imposed on the diffusion components of the two processes. We consider both the cases of Merton’s jump diffusion and Kou’s double exponential jump diffusion for the log-increments of the underlying. In this respect, our setup generalizes the firm value approach of Bermúdez and Webber (2004), in which the default triggering event is represented by a single jump of fixed severity. Further, because of the hybrid nature of CBs, the flexibility to correlate the underlying firm value with the term structure of interest rates yields a realistic model of practical importance for pricing this contract. Finally, we propose a pricing algorithm which belongs to the class of backward price recursion techniques for contracts with early exercise and/or path-dependence (see, e.g., Lord et al. 2008, Fang and Oosterlee 2009, Černý and Kyriakou 2011 and Ruijter and Oosterlee 2012), aimed at handling effectively both the number of risk factors and real-world CB specifications, including discrete cash flows and conversion which either takes place voluntarily at the holders’ choice, or can be forced by a call on notice by the issuer.

As the proposed numerical approach is shown to be flexible enough to handle the dimensionality imposed by the proposed market model, while remaining smoothly convergent and precise, we study the impact of model error on CBs prices, where by model error we mean the error induced by misspecifying the dynamics of either the firm value or the interest rates or both. In particular, we examine the discrepancy between the prices generated by the two jump diffusion processes and the traditional lognormal framework as a function of the model parameter values

\(^1\)For a more detailed review, we refer to Yiğitbaşioğlu and Alexander (2006) and references therein.
and the moneyness (measure of the likelihood of conversion) of the convertible bond. We also explore the effects of modelling explicitly the uncertainty in the term structure of interest rates and the effects of a non-zero correlation structure between the firm value and interest rate processes on CB pricing. Finally, we study the implications of coupons payable to the CB holders, dividends distributed to the current stock holders, and the impact of a varying call policy on the computed prices.

The remainder of the paper is organized as follows. Section 2 presents our assumptions for the firm value and interest rate processes. In Section 3, we introduce the basic notation and describe the CB design focusing on the optimal call strategy assumed for the issuing firm. In Section 4, we develop the theoretical ground for the backward pricing algorithm. The impact of model error and the effects of additional CBs features are analyzed in Section 5, whilst Section 6 concludes.

### 2 Market model

From a valuation perspective, a pricing model for CBs requires assumptions on the dynamics followed by the asset underlying the conversion, the firm’s default-driving mechanism and the term structure of interest rates.

We start from the underlying of the contract and the default-triggering event. We adopt a structural approach to default and assume that the dynamics of the firm value is driven by a jump diffusion process, so that most of the time the process follows a Brownian motion portraying marginal changes due to temporary imbalances between demand and supply, whilst market shocks induced by the arrival of important new information are captured by a compound Poisson process. In details, let \( (\Omega, \mathcal{F}, \mathbb{P} := (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}) \) be a complete filtered probability space, where \( \mathbb{P} \) is a risk neutral probability measure. Under the measure \( \mathbb{P} \), we assume that the firm value \( V \) is given by

\[
V(t) = e^{Y(t)}, \quad t \geq 0,
\]

where \( Y \) follows a jump diffusion process of the form

\[
Y(t) = \bar{Y}(t) + \sum_{k=1}^{N(t)} X_k - \lambda \left( e^X - 1 \right) t
\]

with

\[
\bar{Y}(t) = Y(0) + \int_0^t \left( r(s) - \frac{\sigma^2}{2} \right) ds + \sigma W(t)
\]

and \( Y(0) = \ln V(0) \). \( r \) is the short rate process, \( W \) is a \( \mathbb{F} \)-adapted standard Brownian motion, \( \sigma \geq 0 \) is the diffusion coefficient, \( N \) is a time-homogeneous Poisson process with constant arrival rate \( \lambda > 0 \) and jumps of size \( X \), which is modelled by a sequence of i.i.d. random variables with characteristic function \( \phi_X(u) := E(e^{iuX}) \). The random factors \( W, N \) and \( X \) are assumed to be mutually independent. Two popular choices in the literature for the distribution of \( X \) are the normal distribution (see Merton 1976) with characteristic function

\[
\phi_X(u) = e^{i\mu_X u - \frac{1}{2} u^2 \sigma_X^2}, \quad \mu_X \in \mathbb{R}, \sigma_X \geq 0,
\]

Due to the presence of jumps, the market is incomplete in the sense that there are infinitely many risk neutral pricing measures. Incompleteness can be resolved by calibration of the model to market quotes of traded instruments and credit spreads.
and the double exponential distribution (see Kou 2002) with characteristic function

\[ \phi_X(u) = \frac{pq}{\eta_1 - iu} + \frac{pq}{\eta_2 + iu}, \]

where \( \eta_1 > 1, \eta_2 > 0, \) and \( p, q \geq 0 \) represent the probabilities of an upward and a downward jump respectively, such that \( p + q = 1, \) under the measure \( P. \) We note at this stage that both models generate leptokurtic distributions of the firm (log) value process \( Y(t) \) (see Appendix A).

The proposed construction belongs to the family of the so-called (structural) credit barrier models inspired by the Merton (1976) original contribution and extensions proposed by Black and Cox (1976), Longstaff and Schwartz (1995) (with stochastic interest rates), Leland (1994) and Leland and Toft (1996), to mention a few. Structural models offer an economic rationale behind default in that this occurs if the firm value goes below an appropriate threshold, as opposed to reduced form models in which the default event has exogenous components that are independent of market information. Structural models equipped with unforeseeable jumps, which reflect external impacts on the firm value evolution, allow flexible fitting to the observed credit spreads, see, e.g., Lipton and Sepp (2009), Fiorani et al. (2010) and Dao and Jeanblanc (2012). This way, previous limitations due to a pure diffusion driving process, such as vanishing spreads at very short maturities and poor fit over longer maturities (see, e.g., Eom et al. 2004), are circumvented, as shown for example by Brigo and Morini (2006), Cariboni and Schoutens (2007), Brigo et al. (2010), Fang et al. (2010), Fiorani et al. (2010) and Marazzina et al. (2012). The empirical studies reported in these contributions show, in fact, that enhanced structural models can be calibrated exactly to credit spreads using the default probability formulae generated by the chosen models.

The short rate process \( r \) is assumed to evolve according to the Vašíček (1977) model

\[ r(t) = r(0)e^{-\kappa t} + \mu_r \left(1 - e^{-\kappa t}\right) + \sigma_r \int_0^t e^{-\kappa(t-s)}dW_r(s), \]

for \( \kappa, \mu_r, \sigma_r > 0, \) where \( W_r \) is a standard Brownian motion, such that \( W \) and \( W_r \) have constant correlation coefficient \( \rho \in [-1, 1], \) whereas \( W_r \) is independent of \( N \) and \( X. \) (For ease of notation and to distinguish directly from quantities related to the firm (log) value, we will be using hereafter the subscript ‘\( r \)’ on quantities related to the stochastic short rate of interest.) The price \( B(t, \upsilon) \) at \( t \geq 0 \) of a pure-discount bond maturing at \( \upsilon \geq t \) is given by

\[ B(t, \upsilon) = E \left[ e^{-\int_t^\upsilon r(s)ds} \bigg| \mathcal{F}_t \right] = \exp(A_1(\upsilon - t) - A_2(\upsilon - t) r(t)), \quad (1) \]

where

\[ A_1(\tau) = \frac{1}{\kappa^2} \left(A_2(\tau) - \kappa(\tau^2 \mu_r - \frac{\sigma_r^2}{2}) - \frac{\sigma_r^2 A_2^2(\tau)}{4\kappa}, \quad (2) \right. \]

\[ A_2(\tau) = \frac{1}{\kappa} \left(1 - e^{-\kappa \tau} \right) \quad (3) \]

(see Vašíček 1977).

Under the given assumptions, the process \( (r, \tilde{Y})^T \) is affine as the drift and diffusion terms are linear (in the state variables) up to a constant, see Duffie et al. (2000), allowing us to derive a closed form expression for the characteristic function as we show next. We note that the Vašíček model adopted in this paper suffers from the possibility of negative interest rates, as these are Gaussian. However, the analytical tractability implied by the affine structure of \( (r, Y)^T \) does not hold when assuming other sensible dynamics for the short rate process, such as the Cox–Ingersoll–Ross (CIR) model (see Cox et al. 1985), due to the non-zero correlation structure.
assumed between the processes in our setting¹. Hence, the choice between the CIR and Vašíček models is a trade-off between non-negative rates and non-zero correlation. However, negative interest rates implied by the Vašíček model is a well-known limitation to practitioners which can be controlled, e.g., see Rogers (1995). In fact, the Vašíček model (or its generalized version with time-dependent drift term, e.g., see Hull and White 1996) is frequently met in the recent literature, e.g., see van Haastrecht et al. (2009), Grzelak and Oosterlee (2011), Grzelak et al. (2012) with applications in other hybrid products, and Grzelak et al. (2011) for a multi-factor extension.

By affinity of the process \( (r, \tilde{Y})^T \), it follows (e.g., see Filipović 2009) that

\[
E \left[ e^{-\int_t^s r(s) ds + iu_1 r(v) + iu_2 \tilde{Y}(v)} \right] \mathcal{F}_t = \exp(\psi_0 (v-t, u_1, u_2) + \psi_1 (v-t, u_1, u_2) r(t) + \psi_2 (v-t, u_2) \tilde{Y}(t)),
\]

where \( (\psi_0, \psi_1, \psi_2) \) are given by

\[
\psi_0 (\tau, u_1, u_2) = \frac{\sigma_r^2}{2} \left[ \frac{(iu_1)^2}{2\kappa} (1 - e^{-2\kappa \tau}) + \frac{iu_1 (iu_2 - 1)}{\kappa^2} (1 - 2e^{-\kappa \tau} + e^{-2\kappa \tau}) \right.
\]
\[
+ \frac{(iu_2 - 1)^2}{2\kappa^3} (2\kappa \tau - 3 + 4e^{-\kappa \tau} - e^{-2\kappa \tau})
\]
\[
+ (\sigma_r \rho iu_2 + \kappa \mu_r) \left[ iu_1 \frac{1 - e^{-\kappa \tau}}{\kappa} + \frac{iu_2 - 1}{\kappa^2} (\kappa \tau - 1 + e^{-\kappa \tau}) \right]
\]
\[
\left. + \frac{\sigma^2}{2} iu_2 (iu_2 - 1) \tau, \right.
\]
\[
\psi_1 (\tau, u_1, u_2) = iu_1 e^{-\kappa \tau} + (iu_2 - 1) \frac{1 - e^{-\kappa \tau}}{\kappa},
\]
\[
\psi_2 (\tau, u_2) = iu_2.
\]

Further, Eq. (4), the independence between the jump part of process \( Y \), its continuous part \( \tilde{Y} \) and the short rate process \( r \), and the Lévy-Khintchine formula imply that

\[
E \left[ e^{-\int_t^s r(s) ds + iu_1 r(v) + iu_2 \tilde{Y}(v)} \right] \mathcal{F}_t = \exp(\psi_0 (v-t, u_1, u_2) + \psi_1 (v-t, u_1, u_2) r(t) + \psi_2 (v-t, u_2) \tilde{Y}(t) + \psi_3 (v-t, u_2))
\]

with

\[
\psi_3 (v-t, u_2) = \lambda \left( \phi_X (u_2) - 1 - iu_2 (\phi_X (-i) - 1) \right) (v-t).
\]

We conclude this section with few considerations regarding the calibration of the proposed model to market data. In a structural approach, default occurs as soon as the firm value falls below a preset threshold before or at maturity of the debt. Hence, default probabilities can be derived in a straightforward manner from the given model. Fast and efficient numerical schemes for this purpose have been proposed in similar settings by, for example, Fang et al. (2010) and Marazzina et al. (2012). Due to the link between default probabilities and spreads of credit default swaps (CDS), once the interest rate term structure is fitted to LIBOR and swap rates,

¹It is worth noting that Grzelak and Oosterlee (2011) investigate affine approximations for the case of the CIR interest rate dynamics and derive approximate characteristic functions, the use of which is left to future research.
the default probability of the underlying reference entity can be calibrated to quoted market
CDS spreads, using for example the bootstrapping procedure described in O’Kane and Turnbull
(2003) to recover the default probability term structure, and the minimization procedure in Fang
et al. (2010).

3 Convertible bonds

3.1 Contract features

A convertible bond is a corporate bond which offers the investors the option to exchange it for
a predetermined number of shares of the issuing firm at certain points in time. Upon conversion
each investor receives the conversion value \( \gamma V(t) \), where \( t \in (0, T] \), \( T > 0 \) is the maturity time of
the CB, \( \gamma \in (0, 1/m) \) denotes the dilution factor (Ingersoll 1977a), i.e., the fraction of common
stock possessed by each CB holder post-conversion, and \( m \) the number of CBs outstanding.
The CB issue usually offers regular aggregate coupon payments \( C_t \) at times \( t_j \in (0, T] \), i.e., a
payment of \( c_{t_j} := \frac{C_t}{m} \) per bond. If the issue is kept alive to time \( T \), then this is redeemed for
a total face value \( mF \). Instead, the firm’s stock holders receive discrete aggregate dividends \( D_t \)
at the dividend dates \( t_i \in (0, T] \) (see Brennan and Schwartz 1977).

CBs also carry a call provision allowing the issuer to redeem it prematurely in exchange for
the current call price (e.g., see Ingersoll 1977a, Ingersoll 1977b and Brennan and Schwartz 1977).
In general, the issuer announces its decision to call the bond a certain period in advance (call
notice period). Once the CB is called, the investor needs to consider if it is the case to exercise
the conversion option at the end of the call notice period (so-called “forced conversion by call”)
instead of receiving the call price. The existence of a call provision implies that the CB payoff
depends on the optimal exercise strategy adopted by the issuer. We discuss this in more details
in the next section.

3.2 The optimal call strategy

Under the assumption of a market that is not subject to any imperfections, in which the
Modigliani–Miller theorem holds and no call notice applies, Ingersoll (1977a) proves that the
optimal call policy for a callable convertible issue is to call as soon as the firm value \( V(t) \) reaches
the critical level \( K_t/\gamma \), for a call price \( K_t \) which is usually fixed by the firm at the issue of the
contract or is time-dependent. This feature endows the CB with path-dependence, hence implies
the need for frequent monitoring. However, based on the empirical findings of Ingersoll (1977b),
the issuer delays announcing the call when the conversion value is in excess of the call price.

Asquith and Mullins (1991) and Asquith (1995) demonstrate that, in the absence of call
protection, existence of significant cash flow advantages causes delayed calls, i.e., the firm might
be saving cash by delaying the call if, for example, the after-corporate tax coupons on the CB
are less than the dividends payable post-conversion. A good portion of the observed delay is
also explained by the existence of a call notice and a safety premium, i.e., the issuer delays
announcing the call until the conversion value exceeds the call price enough, so that there is
high chance of this to still exceed the call price at the end of the call notice period and, hence,
avoid the bond redemption in cash.

Thus, abstracting from factors such as taxes, we formulate the optimal call policy for the CB
as follows. The firm announces the decision to call as soon as the conversion value exceeds
the call price increased by some safety premium \( \theta \in [0, 1) \); consequently, the firm’s optimal call
announcement is given by the stopping time

\[
\tau^c := \inf \left\{ t \in (0, T) : Y(t) \geq \frac{\ln (1 + \theta) K_t \gamma}{\gamma} \right\}.
\]
In addition, denote by \( s^c \) the call notice period and define the accrued interest on coupons\(^{48} \) \( \text{AccIR} = (\tau^c + s^c - t_j)/(t_{j+1} - t_j) c_{t_{j+1}}, t_j \leq \tau^c + s^c < t_{j+1}. \) At the end of the call notice period, the bondholder chooses whether to convert and receive the conversion value or redeem the bond for the call price accrued by any coupon due. The no-arbitrage value of this option as at the time of call announcement is

\[
Q_{\tau^c}(r(\tau^c), Y(\tau^c)) := E \left[ e^{-\int_{\tau^c}^{\tau^c+s^c} r(s) ds} \max \left\{ \gamma e^{Y(\tau^c+s^c)} K_{\tau^c} + \text{AccIR} \right\} \bigg| F_{\tau^c} \right]. \quad (6)
\]

As we show in the following section, the call payoff (6) is the amount the CB holders will receive when the issuer announces a call.

### 3.3 The payoff function

Because of the early-exercise rights embedded in the CB, we distinguish between the contract payoff function \( H_t \) at any possible decision time \( t \in (0, T] \) and the no-arbitrage (continuation) value \( G_t \). The payoff function is defined as follows.

At maturity, \( t = T \), the investors can choose between converting to common stock (see Brennan and Schwartz 1977) and receiving the face value and the final coupon, provided that the firm can afford the total of this payment. Otherwise, they recover the outstanding firm value at that time. Hence,

\[
H_T(r(T), Y(T)) = \begin{cases} \gamma e^{Y(T)}, & \text{if } Y(T) \geq \ln \frac{F+C_T}{\gamma} \\ F + c_T, & \text{if } \ln (mF + C_T) \leq Y(T) < \ln \frac{F+C_T}{\gamma} \\ e^{Y(T)/m}, & \text{if } Y(T) < \ln (mF + C_T) \end{cases} \quad (7)
\]

At a date where no coupon or dividend payments are due, \( t \neq t_j, t_i \), CB may be forced by a call to conversion \( (t = \tau^c) \) or continue to exist at least until the next decision date \( (t \neq \tau^c) \), i.e.,

\[
H_t(r(t), Y(t)) = \begin{cases} Q_t(r(t), Y(t)), & \text{if } Y(t) \geq \ln \frac{(1+\theta)K_t}{\gamma} \\ G_t(r(t), Y(t)), & \text{if } Y(t) < \ln \frac{(1+\theta)K_t}{\gamma} \end{cases} \quad (8)
\]

where \( Q_t \) is given by Eq. (6).

At a coupon date, \( t = t_j \neq t_i \), the payoff of the CB depends on whether the firm has enough funding to meet the claim. If \( V(t_{j-}) \leq C_t \), the CB defaults, its value is \( G_{t_j} = 0 \) as \( 0 \leq G_t \leq V(t_j) \) by limited liability and the Modigliani–Miller theorem, whilst the investors receive the available assets. If instead \( V(t_{j-}) > C_t \), for as long as the CB is uncalled \( (t_{j-} \neq \tau^c) \), the contract remains in force and the coupons are paid in full. On the other hand, if the CB is called \( (t_{j-} = \tau^c) \), its holders receive both the call payoff (6) and the coupon. Hence,

\[
H_{t_j-}(r(t), Y(t_{j-})) = \begin{cases} Q_{t_j-}(r(t), \ln(e^{Y(t_{j-})} - C_t)) + c_t, & \text{if } Y(t_{j-}) \geq \ln \frac{(1+\theta)K_t}{\gamma} \\ G_{t_j-}(r(t), \ln(e^{Y(t_{j-})} - C_t)) + c_t, & \text{if } \ln C_t < Y(t_{j-}) < \ln \frac{(1+\theta)K_t}{\gamma} \\ e^{Y(t_{j-})/m}, & \text{if } Y(t_{j-}) \leq \ln C_t \end{cases} \quad (9)
\]

Finally, at a dividend date, \( t = t_i \neq t_j \), the existing stock holders are entitled to receive dividends for as long as the firm can afford their payment, providing that it has already met all the other claims ranking above them. Hence, the investors may find optimal to convert prior to the dividend payment (so-called “voluntary conversion”). The following condition, which is proved in Brennan and Schwartz (1977), applies

\[
H_{t_i-}(r(t), Y(t_{i-})) = \max \left\{ G_{t_i-}(r(t), \ln(e^{Y(t_{i-})} - D_t 1_{Y(t_{i-}) > \ln D_t})) + c_t, e^{Y(t_{i-})/m}, \gamma e^{Y(t_{i-})} \right\}. \quad (10)
\]
If, in addition, the coupon and dividend dates coincide, \( t = t_j = t_i \), then (10) is extended to
\[
H_{t_i}(r(t), Y(t_-)) = \left\{ \begin{array}{ll}
\max \{ G_{t_i}(r(t), \ln(e^{Y(t_-)} - C_i - D_i1_{Y(t_-) > \ln(D_i + C_i)}) + c_i, \gamma e^{Y(t_-)}) \}, & \text{if } Y(t_-) > \ln C_i \\
\text{otherwise}. & \text{if } Y(t_-) \leq \ln C_i .
\end{array} \right.
\]
(11)

A call protection provision may further apply which will prohibit any call back by the issuer typically for 2 or 3 years after issue. Hence, when pricing the convertible, the call feature must be disabled at the dates falling within the period of call protection. In addition, we may take into account a put provision, i.e., a right which allows the holder of the convertible to sell it back to the issuer for some cash amount \( R \) (which can be time-dependent). Upon active put provision, we need to modify the payoff function as follows: in (8), use \( \max \{ G_t(r(t), Y(t)), R + \text{AccIR} \} \) for \( Y(t) < \ln((1 + \vartheta) K_i/\gamma) \); in (9), use \( \max \{ G_t(r(t), \ln(e^{Y(t_-)} - C_i)) + c_i, R + \text{AccIR} \} \) for \( \ln C_i < Y(t_-) < \ln((1 + \vartheta) K_i/\gamma) \); in (10), use \( \max \{ G_t(r(t), \ln(e^{Y(t_-)} - D_i1_{Y(t_-) > \ln D_i}), \gamma e^{Y(t_-)}, R + \text{AccIR} \} \) (payoff 11 is modified accordingly). The terminal payoff (7) remains unaffected. In this paper, we only account for call notice features, while a call protection period can be very easily incorporated if necessary. We also do not account for put features which are less common than call features.

4 Price convolution for convertible bonds

Consider the time partition \( T := \{ t_k \}_{k=0}^n, n \in \mathbb{N}, \) of the contract’s term \([0, T]\) denoting the set of decision dates. For ease of exposition, assume that these dates are equidistant so that \( t_k - t_{k-1} =: \delta t \) for \( 0 < k \leq n \), with \( t_0 := 0 \) and \( t_n := T \). With these assumptions in mind, the price of the CB is the solution to the dynamic programming problem described next.

We write \( G_{t_k} \) (resp. \( H_{t_k} \)) for the continuation value \( G_{t_{k-1}} \) (resp. payoff function \( H_{t_{k-1}} \)) of the CB as at \( t_k, 0 \leq k \leq n \). (We use the convention \( 0_- \equiv 0 \) and \( n_- \equiv n \) in the subscript.) Based on the fundamental theorem of asset pricing, we write for \( 0 < k \leq n \) the iteration
\[
G_{k-1}(r(t_{k-1}), Y(t_{k-1})) = E \left[ e^{-\int_{t_{k-1}}^{t_k} r(s) \, ds} H_k(r(t_k), Y(t_k)) \mid F_{t_{k-1}} \right] = e^{A_1(\delta t) - A_2(\delta t)r(t_{k-1})} E^* \left[ H_k(r(t_k), Y(t_k)) \mid F_{t_{k-1}} \right],
\]
(12)
where the second equality follows by a change to the \( t_k \)-forward measure \( P^*_k \) induced by taking as numéraire the price \( B(t, t_k) \) of a pure-discount bond maturing at \( t_k \) as at time \( t \in [t_{k-1}, t_k], \) see Eq. (1). (For more details on probability measure changes, we refer to Geman et al. 1995.)

In order to analyze (12) further, it is essential that we introduce some key functions and random variables. First, we define the functions
\[
l_k(y) = \left\{ \begin{array}{ll}
y, & \text{if } k \neq i, j \\
\ln(e^y - C_k), & \text{if } k = j \neq i, \\
\ln(e^y - D_k1_{y > \ln D_k}), & \text{if } k = i \neq j, \\
\ln(e^y - C_k - D_k1_{y > \ln(D_k + C_k)}), & \text{if } k = i = j, \\
y > \ln C_k & \text{y > } \ln C_k
\end{array} \right.
\]
(13)
\[g_k(y_r, y) = l_k(y) - A_1(\delta t) + A_2(\delta t)y_r, \]
(14)
\[g_r, k(y_r) = y_r e^{-\kappa \delta t}, \forall k, \]
(15)
where the functions \( A_1, A_2 \) are given by Eqs. (2)–(3). From Eqs. (13)–(15), we obtain \( g_{r, k-1}(r(t_{k-1})) = r(t_{k-1}) e^{-\kappa \delta t} \) and \( g_{k-1}(r(t_{k-1}), Y(t_{k-1})) = Y(t_{k-1}) - A_1(\delta t) + A_2(\delta t) r(t_{k-1}) \) as, by definition, \( l_{k-1}(Y(t_{k-1})) = Y(t_{k-1}) \), that is, the firm log-value at the beginning of the
sub-period \([t_{k-1}, t_k]\) after any discrete payment has taken place: no payment if \(t_{k-1} \neq t_j, t_i\), coupon payment if \(t_{k-1} = t_j \neq t_i\), dividend payment if \(t_{k-1} = t_i \neq t_j\), both coupon and dividend payments if \(t_{k-1} = t_j = t_i\). In addition, we define the pairs

\[
(Z_{r,k}, Z_k) = (r(t_k) - g_{r,k-1}(r(t_{k-1})), Y(t_{k-1}) - g_{k-1}(r(t_{k-1}), Y(t_{k-1}))) \quad 0 < k \leq n.
\] (16)

This allows us to rewrite Eq. (12) as

\[
G_{k-1}(r(t_{k-1}), Y(t_{k-1})) = e^{A_1(\delta t) - A_2(\delta t) r(t_{k-1})} \times G_{k-1}(g_{r,k-1}(r(t_{k-1})), g_{k-1}(r(t_{k-1}), Y(t_{k-1}))),
\] (17)

where

\[
G_{k-1}(g_{r,k-1}(r(t_{k-1})), g_{k-1}(r(t_{k-1}), Y(t_{k-1}))),
\] (18)

\[
= \int_{\mathbb{R}^2} H_k(g_{r,k-1}(r(t_{k-1}))) + z_r, g_{k-1}(r(t_{k-1}), Y(t_{k-1}))) + z) f_k^* (z_r, z) \, d(z_r, z)
\] (19)

with \(f_k^*\) denoting the \(P_t^t\)-density function of \((Z_{r,k}, Z_k)\). In fact, the distribution law of \((Z_{r,k}, Z_k)\) is known via its characteristic function which follows from Eqs. (5) and (16)

\[
\varphi(\delta t, u_1, u_2) := E^* \left[ e^{iu_1 Z_{r,k} + iu_2 Z_k} \big| F_{t_{k-1}} \right] = e^{-A_1(\delta t) + A_2(\delta t) r(t_{k-1})} E^* \left[ e^{-f_k^* (r(s) ds + iu_1 Z_{r,k} + iu_2 Z_k)} \big| F_{t_{k-1}} \right]
\]

\[
= \exp(\psi_0(\delta t, u_1, u_2) + (i u_2 - 1) A_1(\delta t) + \psi_3(\delta t, u_2))
\] (20)

for all \(k\). Clearly, our choice of functions \(g_r\) and \(g\) ensures that (20) does not depend on the state variables \(r\) and \(Y\), facilitating the numerical computation of the convolution (19) as described in Appendix B. Few comments are in order. Recursion (19) is initialized by (7). Evaluating then the function \(\tilde{G}_{k-1}\) in (18) at \((g_{r,k-1}, g_{k-1})\) gives us access to \(G_{k-1}\) in (17), i.e., the continuation value of the CB at the beginning of the sub-period \([t_{k-1}, t_k]\), for all \(k\), prior to any discrete payment taking place. For this reason the function \(g\) in Eq. (14) (see also Eq. 13) is defined for firm values higher than the coupon level over which the CB does not default (continues to exist). The continuation value \(G_{k-1}\) is used to compute the new payoff function \(H_{k-1}\) consistently with (8)–(11), which is applied in the subsequent iteration. Ultimately, the price of the CB at inception is \(G_0\).

In addition, upon a call of the CB by the issuing firm, we need to compute the payoff to the CB holders (see Eqs. 8–9). From (6),

\[
Q_{r^c}(r(t^e), Y(t^e)) = e^{A_1(s^c) - A_2(s^c) r(t^e)} E^* \left[ \max \left\{ \gamma e^{Y(t^e + s^c)} K_{r^e} + \text{AccIR} \right\} \big| F_{t^e} \right],
\]

where the expectation is taken under the \(P_{r^e + s^c}\) measure. Similarly to \(g\) in (14) we define the function

\[
g_{r^c}^Z(y, r) = \begin{cases} 
    y - A_1(s^c) + A_2(s^c) y_r, & \text{if } r^c \neq t_j, t_i \\
    \ln(e^y - C_{r^e}) - A_1(s^c) + A_2(s^c) y_r, & \text{if } r^c = t_j, \neq t_i, \neq t_i, \\
    y > \ln C_{r^e} & \text{if } \tau \neq t_j, t_i,
\end{cases}
\]

and then the random variable

\[
Z_{r^e + s^c} = Y(t^e + s^c) - g_{r^c}^Z(r(t^e), Y(t^e))
\]
whose probability distribution is assumed to have density function \( q^c_{\tau^c+s^c} \) under the measure \( P^c_{\tau^c+s^c} \). Finally, we obtain

\[
Q_{\tau^c}(r(\tau^c), Y(\tau^c)) = e^{A_1(s^c) - A_2(s^c)r(\tau^c)} \int_{\mathbb{R}} \max \left\{ \gamma e^{\theta v}(r(\tau^c), Y(\tau^c) + z), K_{\tau^c} + \text{AccIR} \right\} q^c_{\tau^c+s^c}(z)dz.
\]

The characteristic function of \( Z^c_{\tau^c+s^c} \) under the measure \( P^c_{\tau^c+s^c} \) is given by

\[
\varphi^c(s^c, u) := E^c \left[ e^{iuZ^c_{\tau^c+s^c}} \mid \mathcal{F}_{\tau^c} \right] = \exp(\psi_0(s^c, 0, u) + (iu - 1) A_1(s^c) + \psi_3(s^c, u)).
\]

5 Numerical study

In this section, we analyze the performance of the proposed pricing algorithm and study the behaviour of the CB price under different modelling assumptions and for different payoff structures. In first place, we study the convergence of the numerical pricing scheme in the case of constant interest rates and lognormal firm value, as to benchmark our algorithm to the exact pricing formula of Ingersoll (1977a). Results are presented in Section 5.1. Secondly, we introduce jumps and stochastic interest rates; we compare the results with the CB prices obtained under the standard Gaussian economy and constant interest rates in an attempt to quantify model error, meant as the error originated by the misspecification of the relevant driving risk factors. The effect of jumps only in presence of constant interest rates is discussed in Section 5.2, whilst the impact of stochastic interest rates and the complex interaction with firm value and CB prices is shown in Section 5.3. Finally we analyze the impact of different contract specifications: the introduction of coupons and dividends is examined in Section 5.4, whilst alternative call policies (in terms of different call notice periods and safety premia) are presented in Section 5.5.

5.1 Convergence of the pricing scheme

In the following, we illustrate the numerical performance of the proposed pricing scheme under various CB payoff structures. In particular, we consider callable CBs with: (a) \( K = 40, s^c = \vartheta = 0, C = D = 0 \); (b) \( K = 40, s^c = \vartheta = 0, C = 1 \) (payable semiannually), \( D = 2 \) (payable at the first quarter and third quarter of the year); (c) \( K = 40, s^c = 1/12, \vartheta = 0.2, C = D = 0 \); (d) \( K = 40, s^c = 1/12, \vartheta = 0.2, C = 1, D = 0 \). In order to benchmark our numerical scheme, we compute prices for 5-year CBs with daily sampling, assuming constant interest rates and firm value driven by the lognormal model, as in this case we can compare against the prices obtained using the exact analytical formula of Ingersoll (1977a) for continuously (infinitely sampled) callable CBs.

Let \( \alpha^N_N \) be the CB price calculated using the numerical scheme with \( N \) grid points (see Appendix B) and \( n \) sampling dates. For fixed \( n \), we compute the absolute price differences \( \beta^N_N := |\alpha^N_N - \alpha^{n/2}_N| \) for \( N = 2^{12}, 2^{13}, \ldots, 2^{16} \) and plot these on a log-log scale in Fig. 1. The solid lines in Fig. 1 correspond to regular linear convergence in the number of grid points, which clearly remains unaffected under the various payoff structures. Smooth convergence further enables the use of Richardson extrapolation as a way to accelerate convergence (e.g., see Andricopoulos et al. 2003). To this end, we obtain CB prices \( \alpha^N_{RE,N} \) from Richardson extrapolation of the pairs of prices \( (\alpha^N_N, \alpha^{n/2}_N) \) and calculate successive differences \( \beta^N_{RE,N} := |\alpha^N_{RE,N} - \alpha^{n/2}_{RE,N/2}| \) for \( N = 2^{13}, \ldots, 2^{16} \) which exhibit quadratic convergence in \( N \) (see dotted lines in Fig. 1).

Richardson extrapolation can also be applied on pairs of callable CB prices obtained for different number of sampling dates \( (\alpha^N_N, \alpha^{n/2}_N) \), \( N \) held fixed and varying \( n = 2^5, 2^6, \ldots, 2^{11} \), to price an otherwise equivalent continuously callable CB. Results are shown in Table 1 for a callable CB with \( K = 40 \), where we denote by \( \alpha^N_{RE,n} \) the extrapolated continuously callable CB.
Figure 1. Convergence of CB pricing scheme in number of grid points $N$. Cases: (i) $\alpha^N_n$ denotes callable CB price calculated directly using the numerical scheme in Appendix B with $N$ grid points and $n$ sampling dates. $\beta^N_n = |\alpha^N_n - \alpha^{N/2}_n|$ denotes absolute price differences for $N = 2^1, 2^2, \ldots, 2^6$. (ii) $\alpha_{RE,N}^n$ denotes callable CB price obtained from Richardson extrapolation of $(\alpha^N_n, \alpha^{N/2}_n)$. $\beta_{RE,N}^n = |\alpha_{RE,N}^n - \alpha_{RE,N/2}^n|$ denotes absolute differences of extrapolated prices for $N = 2^1, 2^2, \ldots, 2^6$. Solid lines correspond to case (i) indicating linear convergence in $N$, dotted lines correspond to case (ii) indicating quadratic convergence in $N$. CB specifications: (a) $K = 40$, $s' = \vartheta = 0$, $C = D = 0$; (b) $K = 40$, $s' = \vartheta = 0$, $C = 1$ (payable semiannually), $D = 2$ (payable at the first and third quarters of the year); (c) $K = 40$, $s' = 1/12$, $\vartheta = 0.2$, $C = D = 0$; (d) $K = 40$, $s' = 1/12$, $\vartheta = 0.2$, $C = 1$, $D = 0$. Other CB parameters: $T = 5$, $n = 1250$, $F = 40$, $m = 1$, $\gamma = 0.2$. Firm value parameters: $V(0) = 100$, $\sigma = 0.25$. Constant interest rate: $r = 0.04$. 

price based on pair $(\alpha^N_n, \alpha^{N/2}_n)$. Obviously, with increasing $n$, the numerical CB price approaches smoothly the one of the continuously callable CB computed using the exact price result of Ingersoll (1977a) which we denote by $\alpha^\infty$. Richardson extrapolation speeds up convergence to the exact price.

Note that the convergence of the numerical scheme is not affected when we introduce stochastic interest rates and/or jumps in the firm value dynamics (in the interest of space, results are omitted and available by the authors upon request). However, changing from a constant interest rates setting to a stochastic interest rates one increases the computing time (by an average factor of 14) due to the change from one to two state variables and double price integrals. The computing time also increases (by an average factor of 1.5) from the case of a simply callable CB to a callable CB with coupons (and a dividend-paying stock) because of the increasingly power-demanding surface-fitting procedure in the numerical implementation of the pricing algorithm (see Step 2 in the description given in Appendix B). In the sections to follow, we produce and make use in our analysis CB price results which are precise to the fifth and third decimal place, respectively, in the case of constant and stochastic interest rates, aiming for the lowest possible computing time for given interest rate modelling assumptions, while achieving sufficient accuracy for practical applications.
5.2 Model error: jumps

In this section, we examine the impact of introducing jumps; to this end, we ignore for convenience and without loss of generality the case of stochastic interest rates due to their independence of the jump component.

For illustration purposes, in this analysis we use the same values as in Dao and Jeanblanc (2012) for the diffusion coefficient, i.e., $\sigma = 0.2$, and the parameters of the jump part of the double exponential jump diffusion (DEJD) model. Therefore, we assume probability of an upward jump $p = 30\%$ (hence, probability of a downward jump $q = 70\%$); further, the jumps have upward mean size $1/\eta_1 = 0.02$ and downward mean size $1/\eta_2 = 0.03$, whilst the jump arrival rate is $\lambda = 3$. The emphasis, therefore, is on downward jump risk. The parameters of Merton’s jump diffusion (MJD) model, i.e., $\mu_X, \sigma_X$, are matched to the mean and standard deviation of the jump size distribution of the DEJD model (see Eqs. A5, A6, A7).\footnote{Note that in the MJD model the parameters $\mu_X, \sigma_X$ coincide with the mean and standard deviation of the jump size distribution (see Eqs. A5, A7), whilst this is not the case with the DEJD model (see Eqs. A6, A7).} By assuming common jump arrival rate for both models, the mean and the volatility of the MJD and DEJD firm log-value processes (see Eqs. A1–A2) are also matched. The parameters are reported in Table 2 together with the resulting coefficients of skewness and excess kurtosis (see Eqs. A3–A4). In addition, Table 2 reports the parameters and resulting distribution features for the different cases I–VI used to analyze the impact of jumps on the CB prices. These cases are obtained by changing the parameters of the jump process in order to change the level of skewness and excess kurtosis in the distribution. In particular, we consider cases I–II of higher/lower frequency of occurrence of jumps; cases III–IV (cases V–VI) in which the mean jump size (standard deviation of the jump size) are either doubled or halved compared to the base case for both models. Similarly to the base case, assuming common jump arrival rate for both models, we obtain the remaining DEJD parameters by matching the mean and the volatility of the MJD and DEJD firm log-value processes.

As reported in Table 2, the DEJD distribution shows consistently higher left skewness and excess kurtosis than the MJD distribution. In order to highlight the effect of these features on the CB prices, in Table 3 we report the differences in the prices generated by the MJD, DEJD models and the Gaussian model (i.e., with the jump part ignored) in correspondence

<table>
<thead>
<tr>
<th>log$_2 n$</th>
<th>$\alpha^\alpha_N$</th>
<th>$\alpha^\text{RE},n_N$</th>
<th>% error</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>32.99098</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>6</td>
<td>33.03468</td>
<td>33.14018</td>
<td>0.0127</td>
</tr>
<tr>
<td>7</td>
<td>33.06626</td>
<td>33.14250</td>
<td>0.0057</td>
</tr>
<tr>
<td>8</td>
<td>33.08890</td>
<td>33.14356</td>
<td>0.0025</td>
</tr>
<tr>
<td>9</td>
<td>33.10504</td>
<td>33.14400</td>
<td>0.0012</td>
</tr>
<tr>
<td>10</td>
<td>33.11652</td>
<td>33.14424</td>
<td>0.0005</td>
</tr>
<tr>
<td>11</td>
<td>33.12465</td>
<td>33.14428</td>
<td>0.0004</td>
</tr>
</tbody>
</table>
Table 2. Model parameters.

<table>
<thead>
<tr>
<th>case</th>
<th>MJD &amp; DEJD</th>
<th>MJD</th>
<th>DEJD</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\lambda$</td>
<td>$\mu$</td>
<td>$\sigma$</td>
</tr>
<tr>
<td>base</td>
<td>3</td>
<td>0.0178</td>
<td>0.2110</td>
</tr>
<tr>
<td>I</td>
<td>5</td>
<td>0.0163</td>
<td>0.2179</td>
</tr>
<tr>
<td>II</td>
<td>1</td>
<td>0.0193</td>
<td>0.2037</td>
</tr>
<tr>
<td>III</td>
<td>3</td>
<td>0.0168</td>
<td>0.2157</td>
</tr>
<tr>
<td>IV</td>
<td>3</td>
<td>0.0163</td>
<td>0.2179</td>
</tr>
<tr>
<td>V</td>
<td>3</td>
<td>0.0193</td>
<td>0.2037</td>
</tr>
<tr>
<td>VI</td>
<td>3</td>
<td>0.0163</td>
<td>0.2179</td>
</tr>
</tbody>
</table>

“mean”, “vol”, “skew” and “exc kurt” denote respectively mean, volatility, skewness coefficient and excess kurtosis of $\ln \left( \frac{V(1)}{V(0)} \right)$ (see Eqs. A1, A2, A3, A4). Cases I–VI are consistent with departures of $\lambda$, mean jump size $E(X)$ (see Eqs. A5, A6) and standard deviation of jump sizes $\text{Var}(X)$ (see Eq. A7) from base values, with corresponding (scale-free, where applicable) cumulants and fitted DEJD model parameters so that mean and standard deviation of jump sizes are matched. Other parameters (fixed in all cases): $r = 0.04$, $\sigma = 0.2$.

Table 3. Model error: MJD vs DEJD & Gaussian vs DEJD.

<table>
<thead>
<tr>
<th>case</th>
<th>MJD vs DEJD price diff.</th>
<th>Gaussian vs DEJD price diff.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>CB moneyness (0.6,1.0) [1.0,1.2) [1.2,1.4)</td>
<td>CB moneyness (0.6,1.0) [1.0,1.2) [1.2,1.4)</td>
</tr>
<tr>
<td>base</td>
<td>0.006</td>
<td>0.004</td>
</tr>
<tr>
<td>I</td>
<td>0.009</td>
<td>0.006</td>
</tr>
<tr>
<td>II</td>
<td>0.002</td>
<td>0.001</td>
</tr>
<tr>
<td>III</td>
<td>0.010</td>
<td>0.021</td>
</tr>
<tr>
<td>IV</td>
<td>0.003</td>
<td>0.006</td>
</tr>
<tr>
<td>V</td>
<td>0.023</td>
<td>0.050</td>
</tr>
<tr>
<td>VI</td>
<td>0.001</td>
<td>0.003</td>
</tr>
</tbody>
</table>

Estimated average (across a range of $V(0)$ values) price difference in correspondence of the parameters combinations in Table 2 (volatility of Gaussian model matched to that of MJD and DEJD models), and CB moneyness in the regions (0.6,1.0), [1.0,1.2), [1.2,1.4). Moneyness calculated as ratio between conversion and investment values; investment value defined as hypothetical bond value in the absence of conversion option and credit risk (definition follows Ammann et al. 2008). CB parameters: $T = 5$, $n = 1250$, $F = 40$, $K = 50$, $C = D = 0$, $m = 1$, $\gamma = 0.2$, $\delta = 0$, $s^c = 0$. Constant interest rate: $r = 0.04$. CB prices considered are for $\ln V(0) = \ln 100 + k\delta v \in [\ln 100, \ln(K/\gamma)]$, for $k = 0, \ldots, 15534$, $\delta v = 5 \times 10^{-5}$. For deep in-the-money CBs (moneyness $\geq 1.4$), average price difference tends practically to zero level due to the firm’s highly likely call (excluded from the table).
Table 4. Model error: constant interest rates vs stochastic interest rates.

<table>
<thead>
<tr>
<th>r</th>
<th>model prices (const. interest rates)</th>
<th>model prices (stoch. interest rates)</th>
<th>price differences (const. vs stoch. interest rates)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>r</td>
<td>σ_r</td>
<td>MJD</td>
</tr>
<tr>
<td>3.90%</td>
<td>33.55010</td>
<td>33.54485</td>
<td>-0.110</td>
</tr>
<tr>
<td>2.42%</td>
<td>35.83379</td>
<td>35.82866</td>
<td>-0.262</td>
</tr>
<tr>
<td>0.45%</td>
<td>39.19397</td>
<td>39.18902</td>
<td>0.886</td>
</tr>
</tbody>
</table>

Prices and price differences under stochastic and constant interest rates in correspondence of the Vas’cˇek model with varying interest rate volatility σ_r and constant interest rates r set equal to matched 5-year yields to maturity generated by the Vas’cˇek model. Remaining Vas’cˇek model parameters: r(0) = μ_r = 0.04, κ = 0.858. Correlation coefficient: ρ = 0.2. Firm value parameters: V(0) = 100, base set (see Table 2). CB parameters: T = 5, n = 1250, F = 40, K = 50, C = D = 0, m = 1, γ = 0.2, s’ = 0, ϑ = 0.

Moreover, in all cases, for a deep in-the-money CB, all the models’ prices converge to the call price as the CB is then forced to conversion by call. From Table 3, in case III the MJD versus DEJD price difference peaks when the CB is close to the money because of high default probability, while in case IV of lower left skewness and excess kurtosis the peak occurs only when the CB moves in the money. A similar pattern is noted respectively in cases V and VI.

5.3 Model error: interest rates

The impact on the callable CB prices of alternative interest rates modelling assumptions is examined in Table 4 and Fig. 2. The parameters of the Vas’cˇek model are r(0) = μ_r = 0.04, κ = 0.858; the interest rate volatility σ_r is set to generate a 5-year yield to maturity which decreases from 3.9% to 2.42% and 0.45% (the latter being more in line with recent market quotes). These values of σ_r are reported in Table 4, together with CB prices obtained under the Vas’cˇek model assuming correlation coefficient ρ = 0.2 between the Brownian motions driving the interest rate process and the continuous part of the firm value process. In Table 4, we also report the CB prices calculated for constant interest rates set equal to the different yields to maturity generated by the Vas’cˇek model, and the resulting price differences. Prices are reported to highlight the effects of different assumptions on the yield to maturity; price differences instead emphasize the impact of model misspecification. In addition, in Fig. 2, we illustrate the variation in the CB prices1 obtained under the Vas’cˇek model when we vary the correlation level: in panel (a) we consider the benchmark value of the dilution factor assumed in the paper, i.e., γ = 0.2; in panel (b) we consider a higher value γ = 0.45 in order to assess the CB behaviour when the bond component has smaller weight in the contract structure.

From Table 4, we note that, in line with the results of Brennan and Schwartz (1980), the CB prices decrease as the yield to maturity increases. However, differently from Brennan and Schwartz (1980), we observe that the difference between the prices originated by the two interest rates modelling assumptions increases as the bond component of the contract becomes more valuable. In fact, the (absolute) price difference - expressed as percentage of the price in the case of stochastic interest rates - is only marginal (0.33%) when the 5-year yield to maturity is at its highest level, whilst it increases up to 2.31% in the case of a 0.45% 5-year yield to maturity, independently of the chosen firm value model. This price difference induced by ‘interest rate model error’ increases with the correlation level and can reach up to 5.09%. This is due to the fact that when ρ increases, the bond component becomes more sensitive to variations in the firm value and the firm’s credit quality. The effects on the CB price of the interaction between interest rates and firm value can, in fact, be quite complex. The interest rate affects CB prices as it reflects the discounting applied to any (certain) future cash flows. Changes in the firm value, on the other hand, impact the value of the conversion option but also the probability of default and, consequently, the straight debt component. In presence of (positive) correlation between

---

1Prices are not reported in the interest of space and are available by the authors upon request.
the risk drivers, interest rates and firm value are expected to move in the same direction most of times, therefore their impact on the CB prices is amplified; a feature which would be missed if the rate of interest were to be assumed constant.

The ramifications of the complex interactions between interest rates, firm value and CB prices are illustrated in Fig. 2, where we plot CB prices for increasing levels of $\rho > 0$ for different yields to maturity and dilution factors. When $\rho > 0$, interest rates and firm value are more likely to either increase or decrease together. If they increase, the conversion option is more valuable and it offsets the loss in the bond value. If interest rates and firm value decrease, the increase in the bond value counteracts the price reduction of the conversion rights; however the probability of default deteriorates with a clear negative impact on the CB prices. Which of these effects prevails on the CB price depends, as shown in Fig. 2, on the current market conditions and the contract specification concerning the dilution factor. In presence of a low dilution factor (see Fig. 2, panels a.i–a.ii), the CB prices decrease with increasing $\rho$ in a regime of low yields to maturity. This is due to the higher weight given to the bond position with respect to the conversion option and a probability of default high enough to offset any increase in the bond value when interest rates reduce. On the other hand, if the 5-year yield to maturity is set at a very high level, the bondholders will find the option to convert comparatively more attractive even when the interest rates fall, as by converting they can exit the now riskier contract and sell any acquired shares on the market, eliminating this way their exposure to the firm’s default risk. In these circumstances the CB prices increase with $\rho$, although marginally. In presence of a higher dilution factor (see Fig. 2, panels b.i–b.ii), instead, the straight debt component has a relatively lower weight in the contract, hence the bondholders will always benefit of the downside protections provided by the

Figure 2. Impact of 5-year yield to maturity (5-yr YTM), correlation level $\rho$ and dilution factor $\gamma$ on callable CB prices. Stochastic interest rates parameters: $r(0) = \mu_r = 0.04$, $\kappa = 0.858$, $\sigma_r \in \{0.047, 0.188, 0.282\}$. 5-yr YTM $\in \{3.90\%, 2.42\%, 0.45\%\}$ generated by the Vašíček model in correspondence of the different $\sigma_r$ levels. Firm value parameters: $V(0) = 100$, base set (see Table 2). CB parameters: $T = 5$, $n = 1250$, $F = 40$, $K = 50$, $C = D = 0$, $m = 1$, $s^* = 0$, $\vartheta = 0$. 

- 5-yr YTM = 3.90% 
- 5-yr YTM = 2.42% 
- 5-yr YTM = 0.45%
Table 5. CB prices for different contract specifications.

<table>
<thead>
<tr>
<th>CB specification</th>
<th>model prices (const. inter. rate)</th>
<th>model prices (stoch. inter. rates)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>MJD</td>
<td>DEJD</td>
</tr>
<tr>
<td>call</td>
<td>33.40333</td>
<td>33.39807</td>
</tr>
<tr>
<td>call, coupons</td>
<td>41.91545</td>
<td>41.91100</td>
</tr>
<tr>
<td>call, coup., div.</td>
<td>40.76062</td>
<td>40.75578</td>
</tr>
</tbody>
</table>

Callable CB, callable CB with coupon payments, callable CB with coupon and dividend payments. Firm value parameters: \( V(0) = 100 \), base set (see Table 2). Case of constant interest rate: \( r = 0.04 \). Case of stochastic interest rates: \( r(0) = \mu_r = 0.04, \kappa = 0.858, \sigma_r = 0.047, \rho = 0.2 \). CB parameters: \( T = 5, n = 1250, F = 40, K = 50, C = 1 \) (where applicable), \( D = 2 \) (where applicable), \( m = 1, \gamma = 0.2, s^c = 0, \vartheta = 0 \).

CB, and its price increases as \( \rho \) increases regardless of the level of the yield to maturity.

5.4 Effects of discrete coupon and dividend payments

In the following, we explore the impact of adding discrete coupons and dividends to the valuation framework. Table 5 reports prices for 5-year callable CBs with daily sampling under constant and stochastic interest rates. In the case of stochastic interest rates, we choose parameter values \( r(0) = \mu_r = 0.04, \kappa = 0.858, \sigma_r = 0.047 \) and \( \rho = 0.2 \); for the constant interest rates assumption instead we set \( r = r(0) \). We employ the base set of parameters as in Table 2 for the firm value process, and further assume \( V(0) = 100, F = 40, K = 50, C = 1 \) (payable semiannually), \( D = 2 \) (payable at the first and third quarters of the year), \( m = 1, \gamma = 0.2, s^c = 0, \vartheta = 0 \).

Few comments are in order. Adding coupons to the bond indenture raises substantially the payoff to the investors and, consequently, the CB value, while the default event becomes more likely negatively affecting the CB value. Nevertheless, the first effect offsets the second one, justifying the overall increase in the CB value observed. We also note that, for the case of a dividend-paying common stock, the contract’s price increases less compared to the case of a non-dividend-paying stock. This occurs because the dividends are not payable to the CB holders pre-conversion, although they affect the market perception of the firm’s liquidity and therefore the chances of a future default. Further, the MJD model prices remain higher than the DEJD model prices. The price discrepancies are smaller in the case of a coupon-bearing CB, as the coupons have a primary positive upshot on the value of the bond.

5.5 Effects of call policy

In this section, we investigate the impact of the adopted call strategy by studying how changing the call notice length and the safety premium affects the model prices produced under the base and case V sets of parameters, which corresponds to the more significant departure from the Gaussian model (see Section 5.2). We compute prices for 5-year callable CBs with daily sampling, \( V(0) = 100, F = 40, K = 50, D = 0, m = 1, \gamma = 0.2 \). For the interest rates model, we assume the same parameters as in Section 5.4.

We observe from Table 6 that introducing a call notice period and a safety premium affects the CBs prices. In details, under the base set with constant interest rates, the percentage price changes incurred by varying \((s^c, \vartheta)\) from \((0,0)\) range from 0.25% to −0.76% in the absence of coupons, and 0.19% to −0.24% when non-zero coupon payments are assumed in the contract specification. Results are similar under the assumption of stochastic interest rates. Introducing a call notice increases the value of the CB as it yields a higher probability of a significant favourable movement in the firm value, and therefore the conversion value. The effect is particularly evident under the case \( V \) of high volatility of the firm log-value process, as the option to convert becomes
Table 6. Callable CB prices for varying call specification $(s^c, \vartheta)$.

<table>
<thead>
<tr>
<th>$(s^c, \vartheta)$</th>
<th>Model prices (constant interest rate)</th>
<th>Model prices (stochastic interest rates)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>base parameters</td>
<td>case V parameters</td>
</tr>
<tr>
<td></td>
<td>MJD DEJD</td>
<td>MJD DEJD</td>
</tr>
<tr>
<td>$C = 0$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(0.00, 0.00)</td>
<td>33.40333 33.39807</td>
<td>33.56636 33.55806</td>
</tr>
<tr>
<td>(1/12,0.00)</td>
<td>33.48615 33.48020</td>
<td>33.69809 33.69552</td>
</tr>
<tr>
<td>(1/12,0.20)</td>
<td>33.20939 33.20351</td>
<td>33.40978 33.40891</td>
</tr>
<tr>
<td>(1/12,0.25)</td>
<td>33.14899 33.14321</td>
<td>33.33957 33.33896</td>
</tr>
<tr>
<td>(1/24,0.00)</td>
<td>33.46714 33.46131</td>
<td>33.67012 33.66780</td>
</tr>
<tr>
<td>(1/24,0.20)</td>
<td>33.20935 33.20345</td>
<td>33.40940 33.40846</td>
</tr>
<tr>
<td>(1/24,0.25)</td>
<td>33.14899 33.14319</td>
<td>33.33946 33.33877</td>
</tr>
<tr>
<td>$C = 1$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(0.00, 0.00)</td>
<td>41.91545 41.91100</td>
<td>41.85611 41.84985</td>
</tr>
<tr>
<td>(1/12,0.00)</td>
<td>41.99472 41.98956</td>
<td>41.98329 41.98102</td>
</tr>
<tr>
<td>(1/12,0.20)</td>
<td>41.85176 41.84633</td>
<td>41.85319 41.85248</td>
</tr>
<tr>
<td>(1/12,0.25)</td>
<td>41.81551 41.81010</td>
<td>41.81250 41.81196</td>
</tr>
<tr>
<td>(1/24,0.00)</td>
<td>41.97622 41.97120</td>
<td>41.95501 41.95350</td>
</tr>
<tr>
<td>(1/24,0.20)</td>
<td>41.85170 41.84626</td>
<td>41.85276 41.85200</td>
</tr>
<tr>
<td>(1/24,0.25)</td>
<td>41.81550 41.81008</td>
<td>41.81237 41.81178</td>
</tr>
</tbody>
</table>

Firm value parameters: $V(0) = 100$, base and case V sets of parameters (see Table 2). Case of constant interest rate: $r = 0.04$. Case of stochastic interest rates: $r(0) = \mu_r = 0.04$, $\kappa = 0.858$, $\sigma_r = 0.047$, $\rho = 0.2$. CB parameters: $T = 5$, $n = 1250$, $F = 40$, $K = 50$, $D = 0$, $m = 1$, $\gamma = 0.2$.

6 Conclusions

This paper examines the pricing of convertible bonds in a structural setting with jumps and stochastic interest rates correlated to the firm value process. To this aim, we have developed and implemented a pricing algorithm which has proved to be accurate and capable of flexibly handling a number of contract-design features, such as call provisions, discrete coupons and dividends.

As the proposed jump diffusion market setup for pricing CBs is new in the literature, we analyze the behaviour of the contract price under this more complex representation of the firm value. Our numerical analysis shows that misspecifying the underlying process driving the firm value, i.e., falsely specifying a jump diffusion process as a continuous Brownian motion, can substantially overstate the value of the CB due to the higher probability of default generated by the inclusion of market shocks in the model.

The results of our analysis also indicate the importance of an accurate modelling of the term structure of interest rates corroborating the results of Yiğitbaşoğlu and Alexander (2006); the mispricing, in fact, can be significant depending on prevailing market conditions, such as yield to maturity and correlation between risk drivers, and contract specifications, in particular the dilution factor. For practical purposes, we note that the additional complexity induced by a more sophisticated model for the term structure of interest rates can be handled efficiently by the proposed numerical scheme.

Finally, it is shown that ignoring the call notice period and the safety premium when valuing converts yields noticeable pricing biases.

more valuable. The price rise is smaller in the presence of coupons in the bond indenture. On the contrary, the presence of the safety premium, which relates to the firm’s decision on the date of the call announcement, reduces the price of CBs as a successful forced conversion by call is more likely. The reduction is though slightly mitigated by the accrued interests on any coupon still due; this effect is more pronounced under the case V.
References


Lipton, A. and Sepp, A., Credit value adjustment for credit default swaps via the structural default model. Journal of Credit Risk, 2009, 5, 123–146.
O’Kane, D. and Turnbull, S., Valuation of credit default swaps. Fixed income quantitative credit research, Lehman Brothers, 2003.
Appendix A: General properties of a jump diffusion process

A jump diffusion process is a Lévy process, i.e., a process with stationary and independent increments, which can be decomposed as the sum of a Brownian motion and an independent compound (time-homogeneous) Poisson process. It follows from the construction proposed in Section 2 that under the assumption of constant interest rates, the cumulants of the process \( \ln \left( \frac{V(t)}{V(0)} \right) \) under the risk neutral measure are

\[
\begin{align*}
  c_1 &:= E \left[ \ln \frac{V(t)}{V(0)} \right] = \left( r - \frac{\sigma^2}{2} - \lambda E \left( e^X - 1 \right) + \lambda E(X) \right) t, \\
  c_2 &:= \text{Var} \left[ \ln \frac{V(t)}{V(0)} \right] = \left( \sigma^2 + \lambda E(X^2) \right) t, \\
  c_3 &:= E \left[ \left( \ln \frac{V(t)}{V(0)} - E \left( \ln \frac{V(t)}{V(0)} \right) \right)^3 \right] = \lambda E(X^3) t, \\
  c_4 &:= E \left[ \left( \ln \frac{V(t)}{V(0)} - E \left( \ln \frac{V(t)}{V(0)} \right) \right)^4 \right] - 3c_2 = \lambda E(X^4) t.
\end{align*}
\]

The skewness coefficient and excess kurtosis are given, respectively, by

\[
\begin{align*}
  \frac{c_3}{c_2^{3/2}} &= \frac{\lambda E(X^3) t^{-1/2}}{(\sigma^2 + \lambda E(X^2))^{3/2}}, \\
  \frac{c_4}{c_2^2} &= \frac{\lambda E(X^4) t^{-1}}{(\sigma^2 + \lambda E(X^2))^2},
\end{align*}
\]

where the formulae for \( E(X^j) \), \( j = 1, 2, 3, 4 \), for the Merton and Kou processes are, respectively,

\[
\begin{align*}
  E(X) &= \mu_X, \quad E(X^2) = \mu_X^2 + \sigma_X^2, \\
  E(X^3) &= \mu_X \left( \mu_X^2 + 3\sigma_X^2 \right), \quad E(X^4) = \mu_X^4 + 6\mu_X^2\sigma_X^2 + 3\sigma_X^4,
\end{align*}
\]

and

\[
\begin{align*}
  E(X) &= p/\eta_1 - q/\eta_2, \quad E(X^2) = 2 \left( p/\eta_1^2 + q/\eta_2^2 \right), \\
  E(X^3) &= 6 \left( p/\eta_1^3 - q/\eta_2^3 \right), \quad E(X^4) = 24 \left( p/\eta_1^4 + q/\eta_2^4 \right).
\end{align*}
\]

In addition, knowledge of \( E(X) \) and \( E(X^2) \) gives us access to

\[
\text{Var}(X) = E(X^2) - E(X)^2.
\]

We note from (A3) that the skewness of the process distribution is governed by the third (raw) moment of the jump size distribution, whilst it follows from (A4) that the process is leptokurtic. Finally, (A3) and (A4) show that both skewness and excess kurtosis vanish over very long time horizons.
Appendix B: Computation of convolutions

For a fast computation of convolution (19), we use the fast Fourier transform (FFT). We present
a discrete approximation to (19) which allows computation by FFT directly (see Eqs. B1–B2);
the derivation follows the one in Lord et al. (2008) for the 1-D case, hence it is omitted here.
Further, we provide implementation details which are specific to our CB price application. We
emphasize that the suggested FFT pricing approach requires knowledge of the distribution law
of the pair of random variables \((Z_{r,k}, Z_k)\) defined in (16) only via the associated characteristic
function which is given by (20).\textsuperscript{1}

\begin{equation}
\varphi(t) = e^{rt} \varphi(0) \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/(2\sigma^2)} \text{ for } x \in \mathbb{R}.
\end{equation}

\section{Implementation of price recursion}

Based on backward recursion, we compute at each sampling date \(t_{k-1}, \ k = n, \ldots, 1\), the con-
tinuation value of the CB and, subsequently, the CB payoff for use in the next iteration. In what
follows, we provide detailed steps of the pricing procedure; a summarized version is given in
Table B1.

\begin{table}[h]
\centering
\begin{tabular}{|l|}
\hline
\textbf{Step 0.a.} Choose grid size \(N\) to be a power of 2 for fast computation of the sums (B1) and (B2)
using the FFT with radix 2 (see Černý 2004).
\textbf{Step 0.b.} For each of the state variables, i.e., the short rate of interest and the firm log-value,
select equally spaced grids \(x_r := \{x_{r,0} + m_1 \delta_r\}_{m_1=0}^{N-1}\) and \(x := \{x_0 + m_2 \delta\}_{m_2=0}^{N-1}\) on
which we will evaluate the continuation value of the CB. The grid ranges are chosen
based on the cumulants of \(Z_{r,k}\) and \(Z_k\) as in Lord et al. (2008): \(x_{r,0} := c_{r,1} - \delta r \sqrt{c_{r,2}}\),
\(x_0 := c - \delta r \sqrt{c_1} + \sqrt{c_4}\) and grid spacings \(\delta_r := 2\delta r \sqrt{c_{r,2}}/N, \ \delta := 2\delta c_2 + \sqrt{c_4}/N,\) where
\(\delta\) is a user-defined proportionality constant, e.g., \(\delta = 5\), \(c_{r,1} = -i \delta \ln \varphi(\delta t, u_2)/\partial u_1\),
\(c_{r,2} = -\delta^2 \ln \varphi(\delta t, u_2)/\partial u_1^2\), \(c_1 = -i \delta \ln \varphi(\delta t, u_1, 0)/\partial u_2\), \(c_2 = -\delta^2 \ln \varphi(\delta t, u_1, 0)/\partial u_2^2\)
and \(c_4 = \delta^4 \ln \varphi(\delta t, u_1, 0)/\partial u_2^4\). Given the closed form expression for the characteristic
function \(\varphi(\delta t, u_1, u_2)\) in (20), the cumulants can be obtained explicitly by straightforward differen-
tiation using any symbolic computation package such as Mathematica. Cumulant \(c_{r,4}\) is included to ensure that the tails of the distribution of \(Z_{r,k}\) are sufficiently captured.
\textbf{Step 0.c.} Select equally spaced grids \(y_r\) and \(y\) on which we will evaluate the CB payoff for the
relevant decision date. For convenience, we choose \(y_r = x_r\) and \(y = x\).
\textbf{Step 0.d.} Select uniform, symmetric grids of size \(N\) with spacings \(\delta_r^u\) and \(\delta^u\), respectively, \(u_r := \{(j_1 - N/2)\delta_r^u\}_{j_1=0}^{N-1}\) and \(u := \{(j_2 - N/2)\delta^u\}_{j_2=0}^{N-1}\). The range of values of the grids
\((u_r, u)\) is determined to ensure that the tail of the absolute value of the characteristic
function \(\varphi(\delta t, u_1, u_2)\) is sufficiently captured. More specifically, given \(N\) from \textbf{Step 0.a}, choose grid spacings \(\delta_r^u, \delta^u\), hence grid end-points \((N/2)\delta_r^u\) and \((N/2)\delta^u\), so that
\(|\varphi(\delta t, (N/2)\delta_r^u, (N/2)\delta^u)| \leq 10^{-6}\) holds, where the characteristic function \(\varphi\) is given in
(20) and the value \(g' \in \mathbb{N}\) is guided by the desired precision, e.g., choose \(g' = 5\) when
computing results to 5 decimal places.
\textbf{Step 0.e.} As required for a FFT implementation, check that the Nyquist relations \(\delta_r^u \delta_r = 2\pi/N,\)
\(\delta^u \delta = 2\pi/N\) hold. If not, adjust accordingly the original grid spacings \((\delta_r, \delta)\) and/or
\((\delta_r^u, \delta^u)\).
\textbf{Step 0.f.} Calculate the values of \(\varphi\) on the grid \((-u_r, -u)\) and denote these by \(\varphi\). Store for use in
all iterations.

\textsuperscript{1}An alternative accurate methodology for 2-D pricing problems based on the use of characteristic functions has been recently
proposed by Ruijter and Oosterlee (2012). However, as the authors show, the full applicability of their method is restricted
in the case of the (non-Lévy) Heston model, even when pricing simpler Bermudan put options. Similarly, the jump diffusion
model with stochastic interest rates assumed in this paper does not belong to the Lévy class. Given this and also the complex
payoff structure of the CB, we anticipate limited improvements in this case. We leave the study of the future methodology for
future research.
Step 1. Denote by $\tilde{G}_{k-1}\{m_1,m_2\}_{m_1=0,m_2=0}^{N-1,N-1}$ the values approximating the function $\tilde{G}_{k-1}$ in (18) on the grid $(x_r,x)$. Compute

$$
\tilde{G}_{k-1,m_1,m_2} := \frac{1}{4\pi^2} e^{i\frac{N-1}{2} \delta_y (\varphi_1 - x) + i \frac{N-1}{2} \delta_x (\varphi_2 - x)} 
\sum_{j_1=0}^{N-1} \sum_{j_2=0}^{N-1} e^{-i \frac{\delta_y}{\delta_y} j_1 m_1 - i \frac{\delta_x}{\delta_x} j_2 m_2} \left( e^{-i u_{r,x} e^{-i u_x} \circ D_{k-1} \circ \varphi} \right)_{j_1,j_2},
$$

(B1)

where we denote by $D_{k-1} := \{D_{k-1,j_1,j_2}\}_{j_1=0,j_2=0}^{N-1,N-1}$ the values approximating the discrete Fourier transform of (19) on the grid $(u_r,u)$ with

$$
D_{k-1,j_1,j_2} := e^{i(u_r^T + \frac{N-1}{2} \delta_r) y_r + i(u_r^T + \frac{N-1}{2} \delta_r) y_r \delta_r \delta_y} 
\sum_{m_1=0}^{N-1} \sum_{m_2=0}^{N-1} e^{i \frac{\delta_y}{\delta_y} m_1 j_1 + i \frac{\delta_x}{\delta_x} m_2 j_2} \left( e^{-i \frac{N-1}{2} \delta_y y_r e^{-i \frac{N-1}{2} \delta_r y_r} \circ H_{k-1} \circ \varphi} \right)_{m_1,m_2},
$$

(B2)

and $H_{k-1} := \{H_{k-1,m_1,m_2}\}_{m_1=0,m_2=0}^{N-1,N-1}$ the payoff function values $H_{k-1}$ on the grid $(y_r,y)$ initialized by (7) (see also Step 4); $\circ$ denotes the Hadamard element-wise product. Finally, $w := \{w_{m_1,m_2}\}_{m_1=0,m_2=0}^{N-1,N-1}$ are the trapezoidal weights, where

$$
w_{m_1,m_2} :=
\begin{cases}
1/4, & \text{if } (m_1,m_2) = (0,0), (0,N-1), (N-1,0), (N-1,N-1) \\
1/2, & \text{if } m_1 = 1,\ldots,N-2, \ m_2 = 0, N-1 \\
1/2, & \text{if } m_1 = 0, N-1, \ m_2 = 1,\ldots,N-2 \\
1, & \text{if } m_1 = 1,\ldots,N-2, \ m_2 = 1,\ldots,N-2
\end{cases}.
$$

(B3)

Step 2. Consider the functions $g$ and $g_r$ given by (14)–(15). By fitting a surface to the nodes $(x_r,x,\tilde{G}_{k-1})$, we evaluate $\tilde{G}_{k-1}$ at $g_r,k-1(y_r) \subseteq x_r$, $g_{k-1}(y_r) \subseteq x$ using cubic interpolation. For $g_{k-1}(y_r,y) \not\subseteq x$ we use linear extrapolation in $e^x$. See Remark B1 for more details.

Step 3. Compute $G_{k-1} := \{G_{k-1,m_1,m_2}\}_{m_1=0,m_2=0}^{N-1,N-1}$ the continuation values of the CB in (17), where

$$
G_{k-1,m_1,m_2} := e^{A_1(\delta t) - A_2(\delta t) y_r,m_1} \tilde{G}_{k-1,g_{r,k-1}(y_r,m_1),g_{k-1}(y_r,m_1,m_2))}.
$$

Step 4. Compute the payoff function values $H_{k-1}$ on the grid $(y_r,y)$ from Eqs. (8)–(11) for the relevant decision date. Continue with Step 1 until $k = 1$.

Remark B1: From Step 1, we obtain the values $\tilde{G}_{k-1}$ of the function $G_{k-1}$ on the grid $(x_r,x)$. However to compute the CB continuation values $G_{k-1}$, as it is obvious from the right-hand side of (17), we require knowledge of $\tilde{G}_{k-1}$ on the non-uniformly spaced grid $(g_{r,k-1}(y_r),g_{k-1}(y_r,y))$. As we have immediate access to $\tilde{G}_{k-1}(x_r,x)$ only, we compute $\tilde{G}_{k-1}(g_{r,k-1}(y_r),g_{k-1}(y_r,y))$ by cubic interpolation (using, for example, the built-in algorithm available in Matlab). In addition, to evaluate $\tilde{G}_{k-1}$ wherever $g_{k-1}(y_r,y)$ lies outside the range of $x$, we use linear extrapolation in the firm value dimension (i.e., in $e^x$).

Remark B2: The 1-D convolution in (21) is computed separately by implementing Steps 1–2 above using, however, a 1-D transform (similarly to a European vanilla option, see Lord et al. 2008). In the absence of accrued interest, this computation is performed once as a preliminary step and stored for use at the relevant decision dates. If we allow for accrued interest, then the
Table B1. CB pricing algorithm.

<table>
<thead>
<tr>
<th>Inputs: grid parameters:</th>
<th>$N, (\delta_x, \delta_y), (\delta_p, \delta^\t)$ (see Steps 0.a–c)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Set grids in spatial domain $x_r \leftarrow {x_{r,0} + m_1 \delta_x}<em>{m_1=0}^{N-1}, y \leftarrow {y_0 + m_2 \delta_y}</em>{m_2=0}^{N-1}$, $y_r \leftarrow y, y_r \leftarrow y$</td>
<td></td>
</tr>
<tr>
<td>2. Set grids in Fourier domain $u_r \leftarrow {(j_1 - N/2) \delta_p}<em>{j_1=0}^{N-1}, u \leftarrow {(j_2 - N/2) \delta^\t}</em>{j_2=0}^{N-1}$</td>
<td></td>
</tr>
<tr>
<td>3. Compute characteristic function (see Eq. 20) on grid $(-u_r, -u)$ for use in all iterations: $\varphi = \varphi(\delta t, -u_r, -u)$</td>
<td></td>
</tr>
<tr>
<td>Iterate for $k = n, \ldots, 1$:</td>
<td></td>
</tr>
<tr>
<td>4. Compute discr. Four. trans. of (19) on grid $(u_r, u)$:</td>
<td></td>
</tr>
<tr>
<td>$D_{k-1} \leftarrow \left{ e^{i(u_r^T y + \frac{\delta p}{2} y)r_{r,0} + i(u^T + \frac{\delta}{2} \delta^\t)w} \right}_{m_1=0, m_2=0}^{N-1, N-1}$</td>
<td></td>
</tr>
<tr>
<td>5. Compute $\tilde{G}_{k-1}$ (see Eq. 18) on grid $(x_r, x)$:</td>
<td></td>
</tr>
<tr>
<td>$\tilde{G}<em>{k-1} \leftarrow \left{ e^{i(\frac{\delta x}{2} y \delta x</em>{r,0} + \frac{\delta^\t}{2} (u^T - x_0))} \right}_{m_1=0, m_2=0}^{N-1, N-1}$</td>
<td></td>
</tr>
<tr>
<td>6. Compute CB continuation values $G_{k-1, r}$ (see Eq. 17):</td>
<td></td>
</tr>
<tr>
<td>a. Evaluate $\tilde{G}<em>{k-1, r}(y_r, k-1(y_r, y); g</em>{k-1}(y_r, y))$ using interpolation/extrapolation (see Step 2)</td>
<td></td>
</tr>
<tr>
<td>b. $G_{k-1, r} \leftarrow \left{ e^{A_1(\delta t) - A_2(\delta t) y_{r, m_1}} \tilde{G}<em>{k-1, r}(y_r, k-1(y_r, m_1); g</em>{k-1}(y_r, m_2; y_m)) \right}_{m_1=0, m_2=0}^{N-1, N-1}$</td>
<td></td>
</tr>
<tr>
<td>7. Compute CB payoff function values $H_{k-1, r}$ on grid $(y_r, y)$ (see Eqs. 8–11)</td>
<td></td>
</tr>
<tr>
<td>8. Continue with next iteration until $k = 1$</td>
<td></td>
</tr>
<tr>
<td>Return CB prices $G_0$ on grid $(y_r, y)$</td>
<td></td>
</tr>
</tbody>
</table>

Note. Inputs: grid spacings are determined as explained in preliminary Steps 0.a–c. Iteration part: initialized by CB payoff (7). Trapezoidal weights $w$ given by (B3). $\circ$ denotes the Hadamard element-wise product. Output: CB prices $G_0$ on uniform 2-D grid of short rate and firm log-values.

computation needs to be performed as many times as the number of sampling dates between two consecutive coupon dates; this is because the accrued interest varies at different sampling dates between two coupon dates.

 Remark B3: At each decision date, we need to identify the critical value of $Y(t_\sigma) = y^*$ at which the CB payout changes. These are obvious for the payoffs (7)–(9), e.g., in (8) at $y^* = \ln((1 + \delta)K_1/\gamma)$ continuation of the CB changes to call by the issuer. However, the optimal conversion point in the payoff function (10) is not fixed and is unknown. We need to locate at each time $t = t_i$ the critical value $Y(t_i) = y^*_i$ which satisfies $G_{i-1}(r(t_i), \ln(e^{y^*_i - D_i} y^*_i > \ln D_i)) - \gamma e^{y^*_i} = 0$. This equation can be solved fast numerically using, for example, the Trust-Region Dogleg algorithm in Matlab with user-defined termination tolerance level and maximum number of iterations. The same procedure can be used for the payoff function (11).

Remark B4: If we assume constant interest rates, 2-D transforms (B1)–(B2) reduce to 1-D transforms based on a single state variable, that is, the firm log-value. (From a numerical implementation perspective, the valuation problem in this case is similar to that of a Bermudan option, as in Lord et al. 2008, based instead on the relevant CB payoff at each decision date.)