Income drawdown schemes for a defined-contribution pension plan

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Abstract

In retirement a pensioner must often decide how much money to withdraw from a pension fund, how to invest the remaining funds, and whether to purchase an annuity. These decisions are addressed here by introducing a number of income drawdown schemes, which are relevant to a defined-contribution personal pension plan. The optimal asset allocation is defined so that it minimises the expected loss of the pensioner as measured by the performance of the pension fund against a benchmark. Two benchmarks are considered: a risk-free investment and the price of an annuity. The fair-value income drawdown rate is defined so that the fund performance is a martingale under the objective measure. Annuitisation is recommended if the expected fair-value drawdown rate falls below the annuity rate available at retirement. As an illustration, the annuitisation age is calculated for a Gompertz mortality distribution function and a power law loss function.

1 Introduction

A defined-contribution (DC) pension scheme provides an income for a pensioner after retirement from a fund built-up from investing a series of contributions during their period of employment. The financial risk is taken by the member of the scheme since the fund is associated with an individual and there is no guarantee of a fixed benefit level at retirement. The pension scheme is split into two phases. During the accumulation (or pre-retirement) phase, the scheme member and/or their employer

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contribute to the pension fund, which is invested in a portfolio of assets with a particular risk profile. In the distribution (or post-retirement) phase the pensioner receives periodic income from the fund in order to provide support in old age. There are a number of mechanisms operating in different countries for distributing the pension fund (Lunnon, 2002).

In some countries, the retirement income is provided by an annuity, which (according to regulations) must be bought at retirement and provides an income for the lifetime of the pensioner. In the U.S. there is no such requirement, and an individual can choose whether to withdraw from the DC pension fund subject to certain restrictions. For example, in the 401(k) DC pension plan the pensioner must begin to withdraw the Required Minimum Distribution (RMD) from the fund by the age of 70.5. In the U.K. the pensioner has the option to defer the purchase of the annuity, and instead receive income direct from the pension fund. This is called the income drawdown option. However, irrespective of the details of distribution phase of a particular DC plan, the pensioner faces the problem of how much money to withdraw in retirement, how to invest the remaining funds and whether to purchase an annuity. These are the problems that we address here.

There is a growing literature on investment decisions in the accumulation phase of the DC pension scheme (Blake *et al.*, 2001, and cited references). There is less literature on the distribution phase and the income drawdown option (Milevsky, 1998; Lunnon, 2002; Blake *et al.*, 2003; Gerrard *et al.*, 2004a,b, 2006).

Milevsky (1998) finds the optimal time to annuitise based on a deterministic model, and a more sophisticated stochastic model incorporating stochastic interest rate, asset and mortality models. For the deterministic model, the optimal time to annuitise is when the fund is unable to provide an income stream comparable to an annuity. For the stochastic model, Milevsky finds the probability that the attainable consumption is greater than the initial consumption. If one sets a threshold for this probability, then one can determine the optimal time to annuitise. Milevsky & Robinson (2000) fix the income drawdown rate and invest the fund in a single risky asset. They find the eventual probability of ruin, and find an approximation in order to determine if ruin occurs before the time of death, that is the pensioner outlives his or her funds. The option to defer the purchase of an annuity can also be considered as a real option (Milevsky & Young, 2002), and its value is the loss of utility arising from being unable to behave optimally.

Gerrard *et al.* (2004b) determine the optimal asset allocation for a fixed income drawdown rate which minimises a quadratic loss function. Using the loss function they specify a target wealth over the drawdown period in order that a more favourable annuity can be purchased at the time of compulsory annuitisation. Consequently, they find that the optimal stock allocation decreases with time for an exponential target function and the distribution of the final annuity rate is similar to a lognormal distribution. In a follow-up paper, Gerrard *et al.* (2004a) use a target drawdown rate based on a quadratic loss function and find that the optimal amount invested in the risky asset decreases with time. The expected consumption is constant over time and they find that introducing mortality risk and a bequest function does not significantly alter the optimal controls. Finally, Gerrard *et al.* (2006) combine both formulations and consider a loss function which is a weighted sum of a target wealth profile and a target drawdown rate. Numerical simulations generally show a decreasing optimal asset allocation and an increasing optimal drawdown rate, although the results do depend sensitively on the weighting used in the loss function.

This work is complementary to Gerrard *et al.* (2004a,b, 2006), but rather than specify a target for the pensioner, we focus on the asset allocation which minimises the shortfall in the pension fund since the pension may be a significant proportion of income post-retirement. Income drawdown is normative and it is formulated in terms of the performance of the pension fund relative to a benchmark. These schemes describe a form of self-annuitisation where the pensioner takes on investment risk whilst tracking a benchmark fund.

The paper is organised as follows. In the following section, we introduce the model of income drawdown ignoring mortality risk. We formulate the problem as an optimal control problem in Section 3. In Section 4, we categorise the loss functions which lead to a separable solution of the HJB equation. Mortality risk is introduced in Section 5, while Section 6 contains conclusions and suggestions for further work.

2 Model

Suppose that the pension fund is of size $X(s)$ at the time of retirement s. If the pensioner buys a level annuity immediately then s/he receives a constant cash flow b_s per unit time for life. Alternatively, we suppose that the pensioner can defer the purchase of an annuity, and invest the fund $X(s)$ in a single riskless asset, which provides a rate of return r. In order to provide a comparable cash flow to the annuity the pensioner withdraws b_s per unit time until time T, at which point we assume that the pensioner is required to purchase an annuity with the remainder of the fund. We ignore any consideration of the effect of mortality at this stage.

If the size of the fund is $F(t)$ at time t then the fund evolves according to

$$\frac{dF}{dt} = rF - b_s, \tag{1}$$

as in Gerrard *et al.* (2004b) who consider the same equation with a terminal boundary condition since they pose $F(t)$ as a natural target for the pensioner. Our interpretation follows Milevsky (1998) and so we integrate (1) and apply the boundary condition $F(s) = X(s)$ to obtain

$$F(t) = \frac{b_s}{r} + \left(X(s) - \frac{b_s}{r} \right) e^{r(t-s)}. \tag{2}$$

We call $F(t)$ the benchmark fund: it is the size of the pension fund if the pensioner makes a riskless investment with return r at time s and subsequently withdraws income at a constant rate b_s. If $b_s < rX(s)$ then the fund grows exponentially and the pension fund always outperforms an annuity. Normally the annuity rate b_s is set so that this condition is not satisfied. If $b_s > rX(s)$ then there is a time t^* at which $F(t^*) = 0$ and the fund is exhausted. We shall assume $t^* > T$ since otherwise the annuity is preferable to income drawdown. Note that we refer to the fund by its value at time t for the sake of brevity.

Suppose now a proportion $y(t)$ of the pension fund is invested in a risky asset at time $t > s$ whilst the remainder is invested in a riskless asset. If the price of the risky asset is lognormally distributed with constant drift λ and constant volatility σ then the change in the value of the fund is

$$dX(t) = [X(t)(y(t)(\lambda - r) + r) - b(t, X(t))]\, dt + X(t)y(t)\sigma\, dW(t), \tag{3}$$

where $b(t, X(t))$ is the income drawdown rate, which can depend on time and the current state of the fund.

We define the performance of the pension fund $X(t)$ relative to the benchmark fund $F(t)$ by

$$Z(t) = \frac{X(t)}{F(t)}, \tag{4}$$

so that $Z(s) = 1$. The relative performance Z is a non-dimensional quantity, which measures the benefit of placing some of the fund in a stock rather than solely in a risk-free asset.

We measure the pensioner's aversion to shortfall by introducing a loss function $\mathcal{L}(t, Z(t))$ (Cairns, 2000; Gerrard *et al.*, 2004b; Boulier *et al.*, 1995) with some basic properties. Boulier *et al.* (1995) and Cairns (2000) use a loss function which is a function of the pensioner's consumption since they consider the accumulation phase of a DC pension scheme. We use the loss function as a way to express the risk preferences of the pensioner during the income drawdown phase so that it is similar to a utility function (Pratt, 1964). We restrict the domain of $\mathcal{L}$ to $Z \geq 0$ and if $Z = 0$ then the pension fund is empty, which leads to a substantial loss, so that we require $\mathcal{L}(t, 0) > 0$. We also require that $\partial\mathcal{L}/\partial Z < 0$ so that the loss function strictly decreases with increasing fund performance. We note that the loss function measures performance preferences as first suggested by Markowitz (1952). We shall study loss functions which are asymmetric about $Z = 1$ in order to exaggerate a shortfall in the pension fund $Z < 1$ in comparison to a surplus $Z > 1$.

In order to determine the optimal asset allocation $y(t)$ for the pensioner (or the company operating the pension scheme) we minimise the expected total discounted loss over the planning horizon:

$$\mathbb{E}_{s,x}\left[\int_s^T e^{-\rho(u-s)}\mathcal{L}(u, Z(u))\, du + \epsilon e^{-\rho(T-s)}\mathcal{L}(T, Z(T))\right], \tag{5}$$

where the subjective discount is ρ and ϵ measures the value of any terminal cost at compulsory retirement. Gerrard *et al.* (2004b) use a similar objective in their analysis. It is clear from the form of this objective that the optimal strategy is unaffected by the addition of a constant to the loss function. Consequently we write

$$\mathcal{L}(t, Z(t)) = L_0 + L(t, Z(t)), \tag{6}$$

where L_0 is chosen so that there is no loss if the current pension fund is equal to the benchmark fund i.e. $\mathcal{L} = 0$ at $Z = 1$. Henceforth, we focus on the loss function L.

3 Optimisation

We use stochastic optimal control theory in order to solve this problem. Define the value function

$$V(t,x) := \min_y \mathbb{E}_{t,x}\left[\int_t^T e^{-\rho(u-s)} L(u, Z(u))\, du + \epsilon e^{-\rho(T-s)} L(T, Z(T))\right]. \tag{7}$$

If V is sufficiently smooth then it satisfies the HJB equation

$$V_t + \min_y \{(x(y(\lambda - r) + r) - b(t,x))V_x + \tfrac{1}{2}(xy\sigma)^2 V_{xx}\} + e^{-\rho(t-s)} L(t,z) = 0, \tag{8}$$

with terminal boundary condition

$$V(T, X(T)) = \epsilon e^{-\rho(T-s)} L(T, Z(T)). \tag{9}$$

The first order condition for the HJB equation is

$$y = -\frac{\beta V_x}{\sigma x V_{xx}}, \tag{10}$$

where the Sharpe ratio is

$$\beta = \frac{\lambda - r}{\sigma}. \tag{11}$$

The second order condition for a local minimum is $V_{xx} > 0$.

Substituting (10) back into the HJB equation (8) yields

$$V_t + (rx - b(t,x))V_x - \tfrac{1}{2}\beta^2 \frac{V_x^2}{V_{xx}} + e^{-\rho(t-s)} L(t,z) = 0. \tag{12}$$

The presence of the advective term $(rx - b(t,x))V_x$ makes it difficult to postulate a form for the value function V even if we were to adopt a particular form for the loss function $L(t,z)$. If $r = 0$ and $b = b(t)$ then it is natural to look at exponential loss functions, while if b is proportional to x then power functions yield analytical solutions. However, in general, the value function depends on x and t and must be calculated numerically: it is then much more difficult to apply the verification theorems of stochastic optimal control theory (Fleming & Rishel, 1975). Instead, we consider the relative performance of the fund since this seems a more suitable state variable for a pensioner.

Consequently, we restrict attention to loss functions which are functions solely of the pension fund performance $L = L(z)$ and rewrite the HJB equation so that $V = V(t,z)$. Under this coordinate transformation

$$V_t \to V_t - \frac{zV_z}{F}\frac{dF}{dt}, \quad V_x \to \frac{V_z}{F}, \quad V_{xx} \to \frac{V_{zz}}{F^2}, \tag{13}$$

and the HJB equation (12) becomes

$$V_t + (b_s z - b(t,z))\,\frac{V_z}{F} - \tfrac{1}{2}\beta^2 \frac{V_z^2}{V_{zz}} + e^{-\rho(t-s)}L(z) = 0, \tag{14}$$

using the definition of the benchmark fund (1). It is easy to calculate the evolution of the performance of the fund for a given asset allocation strategy y:

$$dZ = \frac{dX}{F} - \frac{X(rF - b_s)}{F^2}dt = yZ((\lambda - r)\,dt + \sigma\,dW(t)) + \frac{(b_s Z - b(t,Z))}{F}\,dt. \tag{15}$$

The classical Merton approach (Merton, 1990) to the lifetime investment problem involves two controls: the asset allocation y and the consumption rate b. The optimal consumption rate b^* is derived by balancing the consumption stream with the total utility derived from that consumption. We do not adopt that approach here because it seems inappropriate for the income drawdown phase of a pension scheme. The pension scheme exists to provide an adequate income for the pensioner in old age. With that aim in mind, there are often restrictions on the form of drawdown in order that sufficient funds are likely to exist for the pensioner to purchase an annuity at a future date. But these restrictions are of an ad-hoc form without any firm theoretical foundation. Our consumption rate is not optimal because it is not part of an optimisation problem. However, it is not clear how one measures the "social utility" of maintaining an adequate level of income post-retirement, since it is not just the individual pensioner who benefits from having a pension.

The formulation that we describe here has some parallels with prospect theory (Kahneman & Tversky, 1979). Kahneman & Tversky (1979) describe an alternative to utility maximisation which incorporates the idea of relative value rather than absolute wealth. This is analogous to our performance process Z and the loss function $\mathcal{L}$, which measures the risk preferences of a pensioner with reference to a benchmark. They describe an asymmetric loss function that exaggerates a shortfall in relation to a gain just as here. The certainty effect observed in the experiments of Kahneman & Tversky (1979) describes how people overweight outcomes that are considered certain. The benchmark in this section is the deterministic risk free fund, a fund that is known with certainty provided that interest rates are deterministic. Thus, we measure losses or gains with respect to outcomes that are known with certainty. In Section 5, we modify the theory in order to incorporate another deterministic fund which measures mortality risk.

Sen (1997) describes many problems where the objective of maximising the expected total utility of consumption is violated, and some of the examples which he cites hinge upon the difficulty of defining a utility function for all possible outcomes rather than those that just increase some measure of consumption or wealth. The act of choice is dependent on the identity of the chooser (in our context this is a pensioner) and the range of alternatives on offer. A pensioner is concerned by longevity risk and bequest motives in addition to the maximisation of consumption, and the alternatives on offer in retirement vary from country to country. The choice of income drawdown is strongly influenced by government legislation. Consequently, rather than describe a comprehensive set of outcomes, the utility of consumption of a pensioner, and the set of constraints imposed by government, we adopt a normative model.

3.1 Performance based income drawdown

Both (14) and (15) suggest that we link the amount that the pensioner can withdraw per unit time to the performance of the fund. Thus, we suppose that

$$b(t, Z) = b_s Z, \tag{16}$$

that is the pensioner can withdraw at a rate greater/less than b_s when the fund is performing above/below the benchmark. This scheme is similar to that proposed by Gerrard *et al.* (2006) in the sense that they found that it is optimal to withdraw at a rate that is proportional to the size of the pension fund. However, Gerrard *et al.* (2006) determine the withdrawal rate as part of a linear–quadratic optimisation problem whereas here we specify the drawdown scheme explicitly.

We look for a separable solution of (14) by writing

$$V(t, z) = e^{-\rho(t-s)} G(t) L(z), \tag{17}$$

If such a solution can be found then the optimal asset allocation is from (4) and (10)

$$y^* = -\frac{\beta L'}{\sigma z L''}, \tag{18}$$

and it is independent of G. In general, the optimal strategy depends only on the form of the loss function, the dimensionless risk of the stock

$$\eta := \frac{\beta}{\sigma} = \frac{\lambda - r}{\sigma^2}, \tag{19}$$

(Emms & Haberman, 2007) and the current performance of the fund z.

If we substitute the optimal asset allocation back into the state equation (15) we find

$$dZ = -\frac{L'}{L''}\left(\beta^2\,dt + \beta\,dW(t)\right). \tag{20}$$

Now under the risk-neutral measure (Björk, 1998), the Brownian motion becomes

$$d\tilde{W}(t) = dW(t) + \beta\,dt, \tag{21}$$

where β is the Sharpe ratio, or the market price of risk. Consequently,

$$dZ = -\frac{\beta L'}{L''}\,d\tilde{W}(t),$$

so that both $b(t, Z)$ and Z are local martingales under the risk-neutral measure. In fact, they are both martingales under this measure since the form of the loss function is restricted as we see next.

Substituting (17) into (14) we obtain

$$\frac{G' - \rho G + 1}{G} = \tfrac{1}{2}\beta^2\frac{L'^2}{LL''}, \tag{22}$$

where $'$ denotes the relevant derivative. Since the LHS is a function of t and the RHS is a function of z we must have

$$\frac{L'^2}{LL''} = A = \text{const.}, \tag{23}$$

and we can integrate and apply the boundary condition $G(T) = \epsilon$ to find

$$G(t) = \frac{1}{\rho + \frac{1}{2}A\beta^2} + \left(\epsilon - \frac{1}{\rho + \frac{1}{2}A\beta^2}\right)e^{(\rho + \frac{1}{2}A\beta^2)(t-T)} \geq 0. \tag{24}$$

We can interpret $G(t)$ as the relative expected loss from not annuitising at time t using the optimal asset allocation strategy since

$$G(t) = \frac{V}{e^{-\rho(t-s)}L(Z(t))} = \min_{y}\mathbb{E}_{t,z}\left[\int_t^T e^{-\rho(u-t)}\frac{L(Z(u))}{L(Z(t))}\,du + \epsilon e^{-\rho(T-t)}\frac{L(Z(T))}{L(Z(t))}\right]. \tag{25}$$

Equation (23) can be rewritten

$$\frac{dL'}{dL} = \frac{L'}{AL}, \tag{26}$$

since there is no solution corresponding to $A = 0$. If $A = 1$ then integrating with respect to L followed by z gives $L(z) = Be^{-\alpha z}$ where α, B are constants of integration. If $A \neq 1$ then integrating twice again gives

$$L(z) = \left(\frac{Cz + D}{\gamma}\right)^{\gamma}, \tag{27}$$

where C, D are constants of integration and we have set

$$\gamma = \frac{A}{A-1} \neq 0. \tag{28}$$

If $A = 1$ then $\alpha, B > 0$ because the loss function decreases with the fund performance and must be strictly convex. Without loss of generality we set $B = 1$ since this does not change the optimal asset allocation or the optimal state trajectory. If $A \neq 1$ then we require the loss function to decrease with performance and yield a minimum in the HJB equation:

$$L' = C\left(\frac{Cz+D}{\gamma}\right)^{\gamma-1} < 0, \quad L'' = \frac{C^2(\gamma-1)}{\gamma}\left(\frac{Cz+D}{\gamma}\right)^{\gamma-2} > 0. \tag{29}$$

In addition, we specify a positive loss if the pension fund is empty ($z = 0$) which means $D/\gamma > 0$ assuming that we take the positive root if required. Consequently, there are two cases: either $D, \gamma < 0$ or $D, \gamma > 0$. If $\gamma < 0$ then we set $C = \gamma$ without loss of generality and $a = D/\gamma$. Therefore $L' = \gamma(z+a)^{\gamma-1} < 0$ and $L'' = \gamma(\gamma-1)(z+a)^{\gamma-2}>0$. If $\gamma > 0$ then we set $C = -\gamma$, $c = D/\gamma > 0$. Both conditions in (29) are satisfied if $\gamma > 1$ and $z < c$, which places a restriction on the domain of the loss function.

3.2 Fair-value income drawdown

The performance based rule given by (16) yields a fund performance that is a martingale under the risk-neutral measure. However, there is no replicating portfolio when mortality risk is present, and this is certainly present in the distribution phase of a pension scheme. Consequently, we define the fair-value income drawdown such that the performance of the pension fund is a martingale under the objective measure.

For the value function to be tractable we look for separable solutions of the form (17) and substitute into (14):

$$\frac{G' - \rho G + 1}{G} = \tfrac{1}{2}\beta^2 \frac{L'^2}{L''L} - (b_s z - b(t,z))\frac{L'}{FL}. \tag{30}$$

If income drawdown is of the form

$$b(t, Z) = b_s Z + \phi(Z)F(t), \tag{31}$$

then the LHS of (30) is a function of t and the RHS of (30) is a function of z. Consequently we can write

$$\frac{G' - \rho G + 1}{G} = \tfrac{1}{2}\beta^2 \frac{L'^2}{L''L} + \frac{\phi L'}{L} = \tfrac{1}{2}\beta^2 K, \tag{32}$$

where K is a constant. Since the value function is separable, the optimal asset allocation is given by (18) and from (15) we set

$$\phi(Z) = -\frac{\beta^2 L'}{L''}, \tag{33}$$

so that Z is a martingale under the objective measure. The function $\phi(Z)$ is positive provided the loss function is convex. Thus, there is an additional drawdown over (16), and the exact drawdown rule depends on the loss function. Substituting this expression back into (32) yields

$$\frac{L'^2}{L''L} = -K, \tag{34}$$

which restricts the form of loss function so that there are separable solutions of the HJB equation. The analysis now follows (23) with A replaced by $-K$. Thus the loss functions for the performance-based drawdown (16) and fair-value drawdown (31) which yield separable value functions are of the same form.

4 Loss functions

In summary, the optimal strategy, the optimal state trajectory and the distribution of the pension fund at time T are strongly dependent on the form of the loss function. The three loss functions which satisfy (23) and (34) can be written in canonical form and are the exponential or power functions of the first and second kind respectively:

$$L(z) = e^{-\alpha z}, \quad (z+a)^{\gamma}, \quad (c-z)^{n+1}, \tag{35}$$

where $\alpha > 0$ is the constant risk aversion, $\gamma < 0$ and $n > 0$ since we want the loss to decrease with z. In addition, $a \geq 0$, $c > 0$ in order that the loss is positive at $z = 0$, and $a = 0$ leads to infinite loss should the pension fund reach $X_t = 0$.

If L is exponential then $A = 1$, while if L is of power form then

$$A = \frac{\gamma}{\gamma - 1} \text{ or } A = \frac{n+1}{n}. \tag{36}$$

The form of the three loss functions is shown in Figure 1 with L_0 chosen so that $\mathcal{L} = 0$ at $z = 1$ in each case.

In the following three sections we describe the optimal asset allocation and the corresponding fund performance for each of the canonical loss functions using performance based income drawdown (16) and the fair-value income drawdown (31). In all three cases the optimal asset allocation is the same for both drawdown schemes. For (16) the drift of the fund performance is positive so that the expected fund size is greater than the case that no risky investment is made, whereas for (31) the fund performance is a martingale under the objective measure.

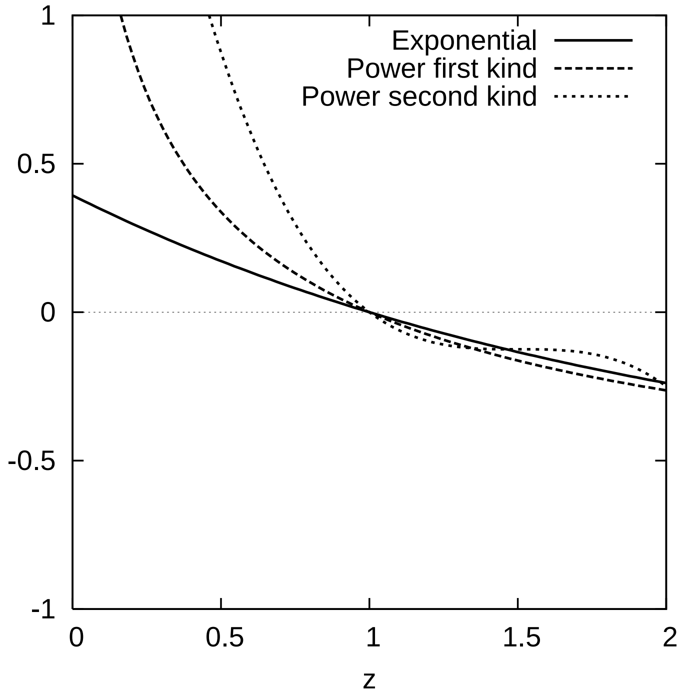

Figure 1: Comparison of the canonical loss functions $\mathcal{L}(z)$ based on the performance of the pension fund z with L_0 set so that $\mathcal{L}(1) = 0$. Here we have set $a = 0.1$, $c = 1.5$, $\gamma = -0.5$, $n = 3$ and the power loss function of the second kind is only valid for $z < c$.

4.1 Exponential

If the loss function is exponential then from (18)

$$y^* = \frac{\eta}{\alpha z}, \tag{37}$$

so that the distribution of $y^* \sim 1/Z$. A risky stock has $\eta \ll 1$ and so the optimal asset allocation is small if the fund performs similarly to the benchmark fund $z \sim 1$. If the fund is performing much better than the benchmark fund $z \gg 1$ and it is optimal to invest a small amount in the stock, and a relatively large withdrawal $b_s z$ is permitted. The investment in the risky asset decreases as the risk aversion of the pensioner α increases.

From (15) the performance of the fund evolves according to

$$dZ = \frac{1}{\alpha}\left(\beta^2\, dt + \beta\, dW(t)\right), \tag{38}$$

using performance based drawdown (16). On an optimal state trajectory Z is normally distributed with positive mean $1 + \beta^2(t-s)/\alpha$ and variance $\beta^2(t-s)/\alpha^2$. Therefore the distribution of the optimal asset allocation has infinite moments for $t > s$. In addition, the fund can go bankrupt before the compulsory purchase of an annuity at time T.

Let us go through the steps in Section 3.2 and derive the optimal control and state trajectory using fair-value income drawdown for this loss function. From (31) and (33) the fair-value income drawdown is

$$b(t, Z) = b_s Z + \frac{\beta^2}{\alpha} F(t) \tag{39}$$

per unit time, where $\beta^2 F(t)/\alpha$ is the additional risk premium. The HJB equation (14) then becomes

$$V_t - \frac{\beta^2}{\alpha} V_z - \tfrac{1}{2}\beta^2 \frac{V_z^2}{V_{zz}} + e^{-\rho(t-s)-\alpha z} = 0. \tag{40}$$

and we can find a separable solution of the form $V = G(t)e^{-\rho(t-s)-\alpha z}$ where now the expected relative loss is

$$G(t) = \frac{1}{\rho - \frac{1}{2}\beta^2} + \left(\epsilon - \frac{1}{\rho - \frac{1}{2}\beta^2}\right) e^{(\rho - \frac{1}{2}\beta^2)(t-T)}, \tag{41}$$

which is just (24) with $K = -A = 1$. The optimal asset allocation strategy is given by (37) while the optimal state trajectory is

$$dZ = \frac{\beta}{\alpha}\, dW(t), \tag{42}$$

using (15). Consequently, Z is a martingale under the objective measure, and on average, the fund will maintain its value with reference to the benchmark fund up until the time T when it is compulsory to purchase an annuity. Notice that income drawdown $b(t, Z)$ is not a martingale under the objective measure.

Using the normal density function, it is easy to determine the probability of ruin $\mathbb{P}[Z(T) < 0]$ or the probability that there is a shortfall in the fund relative to the benchmark at time T: $\mathbb{P}[Z(T) < 1]$.

4.2 Power function of first kind

If the loss function is a power function of the first kind then

$$y^* = \frac{\eta}{1-\gamma}\left(1 + \frac{a}{z}\right), \tag{43}$$

and the optimal performance state trajectory for drawdown scheme (16) is

$$dZ = \left(\frac{Z+a}{1-\gamma}\right)(\beta^2\, dt + \beta\, dW(t)). \tag{44}$$

If $a = 0$ the optimal asset allocation is constant, that is independent of the current performance of the fund, while the performance of the fund is lognormally distributed with mean $e^{\beta^2(t-s)/(1-\gamma)}$ and variance $e^{2\beta^2(t-s)/(1-\gamma)}\left(e^{\beta^2(t-s)/(1-\gamma)^2} - 1\right)$.

This result is consistent with that in Gerrard *et al.* (2004b) where a constant mean asset allocation strategy was observed as being optimal for the natural target function. This loss function is appropriate for a pensioner who wishes to avoid outliving their available funds since $Z > 0$, providing that $X(s) > 0$, i.e. there is a positive amount in the fund at retirement. The probability that there is a shortfall compared to a deterministic fund $\mathbb{P}[Z(T) < 1]$ is easily calculated using the lognormal density function.

If $a > 0$ then the asset allocation decreases with the increasing performance of the fund as in the exponential case (Section 4.1), while $Z + a$ is lognormally distributed.

The fair-value income drawdown scheme for this loss function is

$$b(t, Z) = \left(b_s + \frac{\beta^2 F}{1-\gamma}\right) Z + \frac{a\beta^2 F}{1-\gamma}, \tag{45}$$

since then

$$dZ = \left(\frac{Z+a}{1-\gamma}\right)\beta\, dW(t), \tag{46}$$

and the optimal asset allocation is given by (43). Notice that for this scheme, the income drawdown is bounded from below by $Z = -a$.

4.3 Power function of second kind

If the loss function is a power function of the second kind then the optimal asset allocation is

$$y^* = \frac{\eta}{n}\left(\frac{c-z}{z}\right), \tag{47}$$

so that the allocation is positive only if $0 < z < c$. If this is the case then the second order condition is satisfied. The optimal state trajectory for drawdown scheme (16) is

$$dZ = \left(\frac{c-Z}{n}\right)(\beta^2\,dt + \beta\,dW(t)), \tag{48}$$

so that $c - Z$ is lognormally distributed.

In fact we can think of $Z(t) = c$ as a performance target for the pensioner since this is the value which minimises the loss function over its convex part (see Figure 1). The convexity ensures that the first order condition does yield a minimum in the Bellman equation. As in Gerrard *et al.* (2004b) the target is never attained since $Z(s) = 1$ by definition. Furthermore, the distribution of the final annuity is given by $X(T) = F(T)Z(T)$, where $c - Z(T)$ is lognormally distributed, and so the final annuity is comparable with the plots in Figure 6 of their paper, even though the loss function is asymmetric about $Z = 1$.

The fair-value income drawdown scheme is

$$b(t,Z) = \left(b_s - \frac{\beta^2 F}{n}\right)Z + \frac{c\beta^2 F}{n} \leq b_s c. \tag{49}$$

and under this drawdown the optimal asset allocation is still given by (47). Income drawdown is bounded by $b_s c$ for this loss function for both the performance-based and fair-value schemes since $Z < c$. If $Z < 0$ then the pensioner may be forced to contribute to the pension fund in retirement in order to maintain the value of the final annuity at time T.

5 Mortality risk

The price of an annuity paying continuously a rate of one currency unit per annum for a pensioner of age t is

$$\overline{a}(t) = (1+\theta(t))\int_0^\infty e^{-ru}\,{}_u p_t\,du, \tag{50}$$

where $\theta(t)$ is the annuity loading and ${}_u p_t$ is the probability that a pensioner of age t survives to age $t+u$. The change in the price of an annuity as the age of the pensioner changes is given by

$$\frac{d\overline{a}}{dt} = \frac{\overline{a}}{1+\theta}\frac{d\theta}{dt} + (1+\theta)\int_0^\infty e^{-ru}\frac{d\,{}_u p_t}{dt}\,du. \tag{51}$$

Following Booth *et al.* (1998), the conditional probability of survival ${}_up_t$ is related to the force of mortality $\mu(t)$ by ${}_up_t = \exp\left[-\int_t^{t+u}\mu(\tau)\,d\tau\right]$ so that

$$\frac{d\,{}_up_t}{dt} = {}_up_t(\mu(t) - \mu(t+u)), \tag{52}$$

which is negative if the force of mortality increases with age. Consequently, the price of an annuity decreases with age if the change in loading is sufficiently small. Substituting (52) into (51), using

$$\frac{d\,{}_up_t}{du} = -{}_up_t\mu(t+u), \tag{53}$$

and then integrating by parts yields

$$\frac{d\overline{a}}{dt} = \left(\mu + r + \frac{1}{1+\theta}\frac{d\theta}{dt}\right)\overline{a} - (1+\theta). \tag{54}$$

This expression is given in Bowers *et al.* (1986) without the loading factor: the annuity price increases with age at a rate proportional to sum of the force of mortality, interest rate and change in loading, and decreases at a rate proportional to the cost of the payout.

We now develop the model by using the price of an annuity paying continuously at rate b_s as the benchmark fund. Thus, we set $F(t) = b_s\overline{a}(t)$ with the initial annuity $b_s = X(s)/\overline{a}(s)$ so that the benchmark fund varies according to

$$\frac{dF}{dt} = \left(\mu + r + \frac{1}{1+\theta}\frac{d\theta}{dt}\right)F - (1+\theta)b_s, \tag{55}$$

with initial condition $F(s) = X(s)$ by construction. If the force of mortality $\mu = 0$ and the loading $\theta = 0$ then this is the same equation as (1). Thus, we incorporate the mortality risk in the objective by considering the price of an annuity. We define the performance of the pension fund relative to the price of the annuity:

$$Z(t) = \frac{X(t)}{F(t)}. \tag{56}$$

Let us define the value function by

$$V(t,x) := \min_y \mathbb{E}_{t,x}\left[\int_t^{T\wedge T_D} e^{-\rho(u-s)}L(u,Z(u))\,du + \epsilon e^{-\rho(T-s)}L(T,Z(T))\mathbf{1}_{T_D>T}\right], \tag{57}$$

where T_D is the random time of death. Following Gerrard *et al.* (2006), the corresponding Bellman equation takes the form

$$V_t + \min_y\{(x(y(\lambda-r)+r) - b(t,x))V_x + \tfrac{1}{2}(xy\sigma)^2V_{xx}\} - \delta(t)V + e^{-\rho(t-s)}L(t,z) = 0, \tag{58}$$

if we assume that there is no bequest motive, the drawdown rate is $b(t,x)$, and the subjective force of mortality is $\delta(t)$[1]. The terminal boundary condition is

$$V(T,x) = \epsilon e^{-\rho(T-s)} L(T, Z(T)). \tag{59}$$

The first order condition is given by (10), which on substitution into (58), using the coordinate transformation (13) and adopting a loss function $L = L(z)$ yields

$$V_t + \frac{V_z}{F}\left(z\left(b_s(1+\theta(t)) - \mu(t)F(t) - \frac{F(t)}{1+\theta(t)}\frac{d\theta}{dt}\right) - b(t,z)\right) - \\ \tfrac{1}{2}\beta^2 \frac{V_z^2}{V_{zz}} - \delta(t)V + e^{-\rho(t-s)}L(z) = 0. \tag{60}$$

The form of the advective term in this equation suggests we adopt the drawdown process

$$b(t,Z) = \left(b_s\left(1+\theta(t)\right) - \mu(t)F(t) - \frac{F(t)}{1+\theta(t)}\frac{d\theta}{dt}\right) Z(t), \tag{61}$$

for then we can find three forms of separable solution as before using

$$V = e^{-\rho(t-s)}G(t)L(z), \tag{62}$$

depending on the form of $L(z)$, given by (35). In addition, the optimal asset allocation strategy is given by (18), and the performance of the fund is

$$dZ = yZ((\lambda - r)\,dt + \sigma\,dW(t)), \tag{63}$$

so that the analysis in Section 4 carries through as before. Notice that the subjective force of mortality $\delta(t)$ does not change the optimal asset allocation strategy, but only appears in the time dependent part $G(t)$ of the value function. It is only the objective force of mortality $\mu(t)$ that affects the income drawdown rate because this determines the price of an annuity[2].

5.1 Annuitisation and the value of deferral

In this section we focus on a power loss function of the first kind $L(z) = (z+a)^\gamma$ where the optimal asset allocation is given by (43). At retirement $Z(T) + a$ is lognormally distributed and the fair-value drawdown process is

$$b(t,Z) = \left(b_s\left(1+\theta(t)\right) + \left(\frac{\beta^2}{1-\gamma} - \mu(t)\right)F(t) - \frac{F(t)}{1+\theta(t)}\frac{d\theta}{dt}\right) Z(t) + \frac{a\beta^2 F(t)}{1-\gamma}, \tag{64}$$

[1]The objective force of mortality $\mu(t)$ is calculated from a life table by an insurance company in order to price an annuity, whereas the subjective force of mortality $\delta(t)$ represents the perception of the pensioner with regard to their own longevity at age t.

[2]It easy to show that a logarithmic loss function of the form $\mathcal{L}(z) = -\log\phi z$ leads to a value function of the form $V(t,z) = H_1(t) + H_2(t)\log\phi z$. The optimal asset allocation given by (18) is then constant and so independent of the subjective force of mortality $\delta(t)$.

since then $\mathbb{E}_s [Z(T)] = 1$, that is the expected size of the pension fund is sufficient to buy an annuity which pays b_s for the remainder of the pensioner's life. If the loading is constant then drawdown can be above or below the initial annuity rate b_s depending on the size of the term $\mu(t)F(t)$. We illustrate this point with an example in the next section for a specific choice for the force of mortality.

If $a = 0$ and the loading θ is constant then the expected drawdown rate falls below $b_s(1+\theta)$ if $\mu(t) > \beta^2/(1-\gamma)$, which is similar to the annuitisation condition given in Milevsky & Young (2002). One should note that γ is a parameter in the utility function for consumption in their paper, whereas here it is a parameter in the loss function. However, the interpretation of the condition is the same: the force of mortality can be thought of as the excess return on the annuity, while $\beta^2/(1-\gamma)$ is the drift in the performance of the fund if the pensioner adopts the drawdown scheme given by (61).

Let us define

$$t^a = \inf\{t : s \le t \le T,\ \mathbb{E}_s [b(u, Z(u))] < b_s \quad \forall t \le u \le T\}, \tag{65}$$

with $t^a = \infty$ if the infimum does not exist. We suggest annuitisation at age t^a because the annuity guarantees a greater income stream for the remainder of the drawdown period. Whether such a t^a exists depends on the behaviour of the objective force of mortality as $t \to \infty$. Notice that it is the objective force of mortality, μ, which affects the annuitisation time rather than the subjective force of mortality δ. Thus, a strong feature of the model is that the time for annuitisation is independent of pensioner's view of their own mortality.

The value at retirement of deferring annuitisation is the expected consumption increase over the annuity rate:

$$\begin{aligned} V_d =& \mathbb{E}_s \int_s^{t^a \wedge T \wedge T_D} e^{-\rho(u-s)} (b(u, Z(u)) - b_s)\, du \\ =& \int_s^{t^a \wedge T} e^{-\rho(u-s)} (\mathbb{E}_s[b(u, Z(u))] - b_s)\, {}_{u-s}p_s^S \, du, \end{aligned} \tag{66}$$

where the subjective conditional survival probability is

$${}_{u-s}p_s^S = \exp\left[- \int_s^u \delta(\tau)\, d\tau \right]. \tag{67}$$

In general, this cannot be simplified for given subjective and objective mortality distributions.

5.2 Example: Gompertz distribution

Let us suppose that the loading θ is constant and the power loss function parameter $a = 0$. If we model the objective mortality using a two-parameter Gompertz distribution (Frees *et al.*, 1996) then the conditional survival probability is

$$ {}_u p_t = \frac{\exp\left(e^{-\frac{m}{\phi}}\left(1 - e^{\frac{t+u}{\phi}}\right)\right)}{\exp\left(e^{-\frac{m}{\phi}}\left(1 - e^{\frac{t}{\phi}}\right)\right)}, \tag{68} $$

where m is the mode and ϕ is the scale measure. Using this notation the force of mortality is

$$ \mu(t) = \frac{e^{\frac{t-m}{\phi}}}{\phi}. \tag{69} $$

Substituting this expression into (50) gives the explicit price for a continuous unit annuity as

$$ \overline{a}(t) = (1+\theta)\phi e^{(t-m)r} \exp\left(e^{\frac{t-m}{\phi}}\right) \Gamma\left(-\phi r, e^{\frac{t-m}{\phi}}\right), \tag{70} $$

where the incomplete Gamma function is defined by $\Gamma(w,x) = \int_x^\infty e^{-u} u^{w-1}\, du$.

From (64) the expected fair-value drawdown is

$$ \mathbb{E}_s\left[b(t,Z)\right] = b_s \left(1 + \theta + \overline{a}(t)\left(\frac{\beta^2}{1-\gamma} - \mu(t)\right)\right). \tag{71} $$

The expected drawdown rate relative to that provided by the annuity at retirement is

$$ \begin{aligned} r_d(t) = \frac{\mathbb{E}_s\left[b(t,Z)\right]}{b_s} = \\ (1+\theta)\left(1 + \left(\frac{\phi\beta^2}{1-\gamma} - e^{\frac{t-m}{\phi}}\right) e^{(t-m)r} \exp\left(e^{\frac{t-m}{\phi}}\right) \Gamma\left(-\phi r, e^{\frac{t-m}{\phi}}\right)\right), \end{aligned} \tag{72} $$

using (69),(70) and (71).

Next, we choose a suitable parameter set. First suppose that, for the power loss function, $\gamma = -0.5$. This gives the optimal asset allocation as $y^* = \frac{2}{3}\eta$ from (43), which means it is optimal to maintain a constant proportion of wealth in the risky asset. If $\eta < \frac{3}{2}$ then no borrowing is required, and this condition is satisfied for a sufficiently risky stock. For example, if $\lambda = 0.08$, $r = 0.05$, $\sigma = 0.2$ then $y^* = 0.5$ and no borrowing is required.

Frees *et al.* (1996) use the Gompertz distribution to fit annuity data from a Canadian insurance company. For male policyholders they found $m = 86.4$ years and $\phi = 9.8$ years. If we take $r = 0.05$, $\theta = 0.1$, $s = 60$, $T = 80$ then we can plot $r_d(t)$ as the Sharpe ratio β is varied. The results are shown in Figure 2.

As the riskiness of the stock is increased the pensioner receives greater expected income drawdown than from an annuity. This also leads to greater variation in the final annuity. The income rate decreases over the drawdown period in order to maintain the expected level of the fund at annuitisation. In Figure 2, if $\beta = 0.2$ and

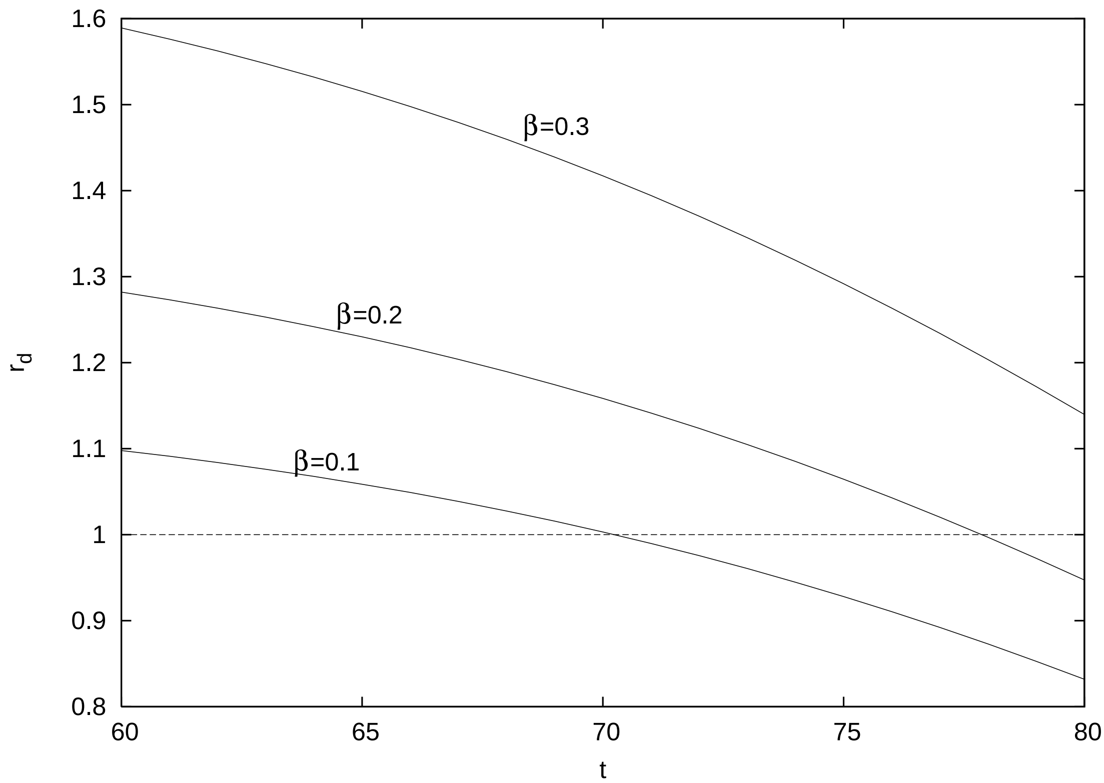

Figure 2: Relative income drawdown rate $r_d(t)$ as the Sharpe ratio β is varied using a Gompertz mortality distribution.

the pensioner wishes to receive expected income above or at the rate of the initial annuity then we suggest annuitisation at $t^a \sim 78$ years.

For this mortality distribution the annuitisation condition (65) becomes

$$e^{(t^a-m)r} \exp\left(e^{\frac{t^a-m}{\phi}}\right) \Gamma\left(-\phi r, e^{\frac{t^a-m}{\phi}}\right) \left(e^{\frac{t^a-m}{\phi}} - \frac{\beta^2\phi}{1-\gamma}\right) = \frac{\theta}{1+\theta}, \tag{73}$$

and if the loading is $\theta = 0$ then this expression reduces to $t^a = m + \phi \log\left(\frac{\beta^2\phi}{1-\gamma}\right)$. If we suppose that the subjective probabilities are equal to the objective probabilities used in the annuity pricing, then the value of deferring annuitisation from (66) is

$$\begin{aligned} V_d(s) &= b_s \int_s^{t^a \wedge T} \left(\theta + e^{-\rho(u-s)} \left(\frac{\beta^2}{1-\gamma} - \mu(u)\right) \overline{a}(u)\right) {}_{u-s}p_s^S \, du \\ &= b_s \exp\left(e^{\frac{s-m}{\phi}}\right) \int_s^{t^a \wedge T} e^{-r(u-s)} \times \\ &\left(\theta \exp\left(-e^{\frac{u-m}{\phi}}\right) + (1+\theta)\left(\frac{\beta^2\phi}{1-\gamma} - e^{\frac{u-m}{\phi}}\right) e^{r(u-m)} \Gamma(-\phi r, e^{\frac{u-m}{\phi}})\right) du. \end{aligned} \tag{74}$$

We plot $V_d(s)$ as β is varied in Figure 3 using the previous parameter set, the annuity rate of $b_s = £10{,}000$ per annum, and $\rho = r = 0.05$. As the Sharpe ratio β increases the value of deferral increases because a greater expected return can be obtained from taking greater risk. The variation about V_d can be computed by Monte-Carlo simulation of the square of the integral in (66).

6 Conclusions and further work

We have proposed an income drawdown scheme (16) which minimises the expected total loss of performance of the fund over the planning horizon by using the optimal asset allocation. Drawdown is proportional to the fund performance measured against a given benchmark, and it is independent of the loss function. Moreover, the expected fund performance increases with time and the rate of expected increase depends on the riskiness of the stock and the loss function. Thus the pensioner can tailor their drawdown strategy so that the pension fund yields a given expected return using an optimal asset allocation strategy. The investment risk taken by the pensioner can be determined analytically for each of the canonical loss functions considered in the paper.

In addition, the analysis motivates the fair-value drawdown schemes each of which depend on the loss function of the pensioner. In these schemes, there is an additional withdrawal such that the fund performance has zero drift, that is, it is a martingale. All of the drawdown schemes lead to an income which is greater than the rate offered by an annuity if the pension fund performs particularly well, whilst at the same time maintaining (or increasing) the expected fund size so that

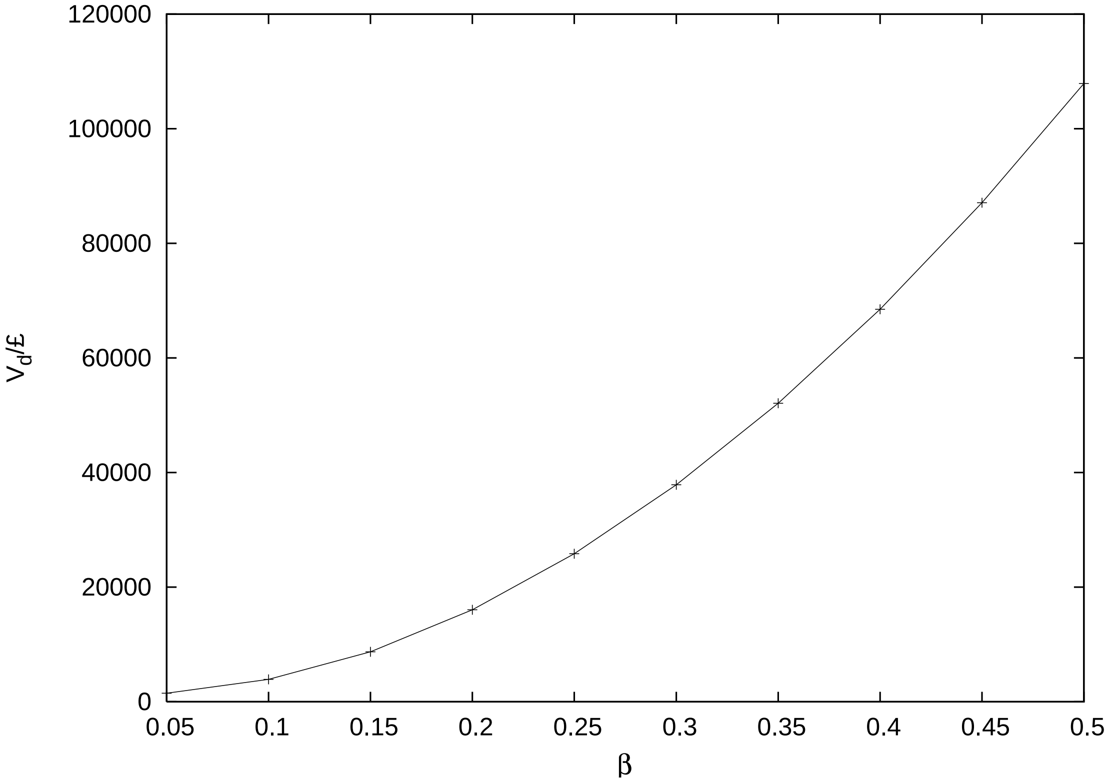

Figure 3: The value of deferral of annuitisation at retirement V_d as the Sharpe ratio β is varied if the annuity rate at retirement is $b_s = £10{,}000$ per annum.

an annuity can be purchased if this is compulsory. Therefore, the schemes allow the pensioner to benefit from investment performance, provide a measure of security for government, and release funds to stimulate economic growth.

The fair-value drawdown schemes provide an answer as to how one can determine the appropriate level of income post-retirement. The merit of the fair-value scheme is its simplicity and the simple form for the distribution of the final annuity should annuitisation occur. One might consider the schemes themselves as a benchmark: if a pensioner withdraws more than the fair-value drawdown rate then they should expect a smaller final annuity. In addition, if the benchmark is the annuity price then only the objective mortality rates are used to calculate fair-value drawdown. It is only the market price of mortality which affects fair-value drawdown rather than the subjective view of the pensioner.

We have studied two benchmark funds: a risk-free investment and the price of an annuity. For the second benchmark fund, the fair-value drawdown scheme fixes income so that the expected fund size allows the purchase of annuity which provides the same income stream as an annuity bought at retirement. Annuitisation should occur if and when the force of mortality exceeds a performance threshold. We have illustrated these results with a Gompertz mortality distribution function. However, the proposed drawdown schemes would break current U.K. legislation, which specifies no minimum withdrawal and a maximum withdrawal of 120% of the retirement annuity rate before age 75. These schemes are a compromise between a lump sum payment on retirement and immediate compulsory annuitisation. In the case of a lump sum payment, we provide guidance on how the pensioner might invest the payment whilst making periodic withdrawals.

More generally we can interpret the benchmark fund $F(t)$ as a target wealth profile for the pension fund. Suppose the target evolves according to

$$\frac{dF}{dt} = H(t, F), \tag{75}$$

for a given function H and $F(s) = X(s)$. Both choices for the benchmark fund in this paper are of this form. If we set the advective term in the HJB equation equal to zero then

$$b(t, Z) = (rF(t) - H(t, F(t)))Z(t). \tag{76}$$

With this choice of drawdown the fund performance evolves according to

$$dZ = \frac{dX}{F} - \frac{XH}{F^2}dt = yZ((\lambda - r)\,dt + \sigma\,dW(t)). \tag{77}$$

The optimal asset allocation strategy depends on the three forms of the loss function, while the fair-value drawdown removes the drift in the above equation. Thus with a fair-value drawdown scheme and an optimal asset allocation strategy the expected pension fund size *is* the target: $\mathbb{E}_s[X(t)] = F(t)$.

Further research may include generalising the model to incorporate a stochastic interest rate and a stochastic force of mortality (Milevsky & Promislow, 2001; Biffis, 2005). If the force of mortality is a stochastic process then the price of an annuity for a pensioner of age t is also stochastic. Thus, the benchmark process is stochastic and then the optimisation problem is similar to that studied by Browne (1999). He considers a variety of objectives related to the fund performance, which may be appropriate for a pensioner choosing the income drawdown option. The idea of maintaining the value of the pension fund is also similar to the value preserving portfolio strategies described by Korn (2000), and this is a further line of research.

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