CHAPTER 1

HAR MODELING FOR REALIZED VOLATILITY FORECASTING

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1.1 INTRODUCTION

The importance of financial market volatility has generated a very large literature in which volatility dynamics has been modelled in order to take into account its most salient features: clustering, slowly decaying auto-correlation, and non-linear responses to previous market information of a different type.

In the literature, these phenomena have typically given rise to models in which volatility is generated by a long memory process, characterized by fractional integration and an hyperbolic decay of the autocorrelation function. However, in this chapter we follow an alternative direction which generates very similar stylized facts for volatility series using the superposition of short memory frequencies. This framework turns out to be easier to handle, with a straightforward economic interpretation and an excellent fit to the data.

Volatility Models and Their Applications. By Bauwens, Hafner, Laurent

Originally, this framework was inspired by the work of [66] and [41]. We view volatility persistence as the result of the aggregation of the heterogeneous components present in the financial market (the so called Heterogeneous Market Hypothesis). Heterogeneity among participants in the financial market may be of a different nature: differences in the endowments, institutional constraints, risk profiles, information, geographical locations, and so on. The proposed model concentrates on the heterogeneity that originates from (or materializes in) the difference in time horizons. Typically, a financial market is composed of participants having a large spectrum of trading frequencies. At one end of the spectrum are dealers, market makers, and intraday speculators with an intraday trading horizon. At the other end, there are institutional investors, such as insurance companies and pension funds trading much less frequently and possibly for larger amounts. The key idea is that agents with different time horizons perceive, react to, and cause different types of volatility components.

In addition, it has been recently observed that volatility over longer time intervals has stronger influence on volatility over shorter time intervals than conversely.[1] This can be economically explained by noticing that for short-term traders the level of long-term volatility matters because it determines the expected future size of trends and risk. The overall pattern that emerges can be statistically described by a cascade of heterogeneous volatility components (generated by the action of market participants of different natures) from low frequencies to high frequencies.

This idea has been pursued in [34], who proposed an additive cascade model of realized volatility aggregated at different time horizons. This cascade of heterogeneous volatility components leads to a simple AR-type model in the realized volatility that considers volatilities realized over different time horizons and is thus called *Heterogeneous Auto-Regressive* (HAR). In spite of its simplicity and the fact that it does not formally belong to the class of long-memory models, the HAR model for realized volatility is able to reproduce the volatility persistence revealed by the empirical analysis on financial markets. The combination of ease of implementation with a very accurate fit of financial volatility time series has made the HAR models very popular in the financial econometrics community.

In this chapter we survey the HAR model for realized volatility forecasting and its extensions. After reviewing some stylized facts of realized volatility we present the derivation and possible interpretations of the heterogeneous structure of the HAR model. We then discuss different extensions of the univariate HAR model aiming at modeling the forecasting power of jumps, leverage effect and structural breaks.

In particular, we provide evidence for the contention that jumps have significant impact on future realized volatility and that the impact of negative returns (the so-called *leverage effect*) is highly persistent and also presents a HAR structure, confirming the view of the existence of an heterogeneous structure in the financial market. Moreover, we also provide empirical evidence of the existence of other non-linear effects of past market information on volatility on the top of the leverage effect

[1] See [66], [4] and [61].

by introducing a flexible HAR-type model able to explicitly take into account structural breaks and regime-switches. Finally, we provide a brief review of multivariate models for realized variance-covariance matrix dynamics.

1.2 STYLIZED FACTS ON REALIZED VOLATILITY

Summarized from the vast literature on the empirical analysis of financial markets, the main characteristics of financial markets volatility are:

1. Long range dependence: (hourly, daily, weekly and monthly) realized volatility displays significant autocorrelations even at very long lags. This property is often ascribed to a long memory data generating process. In this chapter, we take another approach by using a superposition of autoregressive processes with different time scales.

2. Leverage effect: it is empirically observed that returns are negatively correlated with (realized) volatility. In particular, volatility bursts are more likely associated with negative past returns.

3. Jumps: financial prices are subject to abrupt variations. Jumps are not very frequent and practically unpredictable, but they have a strong positive impact on future volatility.

To illustrate these stylized facts of realized volatility (RV_t), let us now consider historical data on the S&P 500 stock index over the period 1982-2009. Figure 1.1 plots $Corr(RV_t, Z_{t-h})$, i.e. the correlation between RV_t and Z_{t-h}, for $h = 1, \ldots, 50$. Z_t corresponds either to RV_t, negative daily returns ($r_t^- = min(r_t, 0)$, where r_t is the return on day t), positive returns ($r_t^+ = max(r_t, 0)$) or jumps (J_t). More details on the data and the estimation of RV_t and J_t are given in Section 1.3. $Corr(RV_t, RV_{t-h})$ is the AutoCorrelation Function (ACF) of RV_t. Figure 1.1 clearly suggests the presence of long-memory in the realized volatility. This figure also suggests that while past positive daily returns (r_{t-h}^+) are not significantly correlated with RV_t, past negative returns (r_{t-h}^-) have a significant impact on futures volatilities, and negative shocks take a long time to die out (which might also be viewed as long-memory). Interestingly, jumps seem also to have a positive impact on future values of RV_t, although their effect decays at a faster rate than RV_t and r_{t-h}^-. This motivates the analysis in the following sections.

1.3 HETEROGENEITY AND VOLATILITY PERSISTENCE

The appearance of long range dependence might be due to a genuine long-memory data generating process or, alternatively, it can be explained as a combination of different short memory processes (as discussed further below). Although a true long memory process requires the aggregation of an infinite number of short memory processes (as shown by [52]), an approximated long memory process (practically

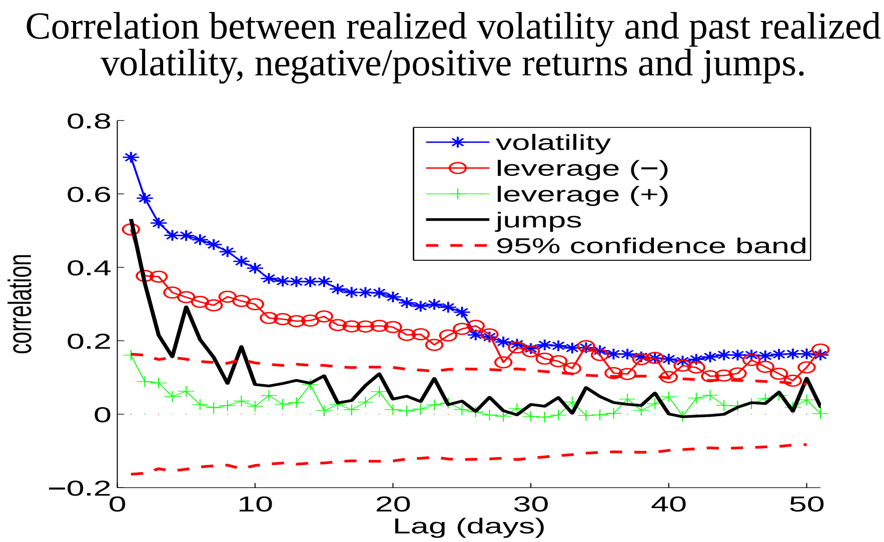

Figure 1.1 $Corr(RV_t, Z_{t-h})$ ($h = 1, \ldots, 50$) for the S&P 500 series for the period January 1990 to February 2009. Z_t corresponds either to RV_t, negative daily returns ($r_t^- = min(r_t, 0)$, where r_t is the return on day t), positive returns ($r_t^+ = max(r_t, 0)$) or jumps (J_t). The displayed 95% confidence bands (dashed lines) are computed with the generalized Bartlett's formula of [46].

indistinguishable from a true one) can be obtained by aggregating only few heterogeneous time scales ([58]).

The need for multiple components in the volatility process has been advocated by (among others) [66], [43], [21], [14], and [26] and has been reconsidered by making use of the concept of an additive cascade of realized volatility aggregated over different time horizons in [34]. In what follows, we briefly review this latter approach.

We assume that the state variable X (typically the log price) is driven by the stochastic process:

$$dX_t = \mu_t dt + \sigma_t dW_t + c_t dN_t, \tag{1.1}$$

where μ_t is predictable, σ_t is cádlág and N_t is a doubly stochastic Poisson process[2] whose intensity is an adapted stochastic process λ_t, the random times of the corresponding jumps are $(\tau_j)_{j=1,\ldots,N_T}$ and c_j are iid adapted random variables measuring the size of the jump at time τ_j. In practice, e.g. for risk management purposes, we

[2]We could also consider a wider class of jumps, such as Lévy, in the case in which they have a finite quadratic variation process.

are interested in forecasting the quadratic variation defined as:

$$\widetilde{\sigma}_t^2 = \int_t^{t+1} \sigma_s^2 ds + \sum_{t \le \tau_j \le t+1} c_{\tau_j}^2,$$

where the time unit is one day.

This quantity is not directly observable and therefore has to be estimated. Let us denote by $\hat{V}_t$ a consistent estimator of $\widetilde{\sigma}_t^2$, that is:

$$\log \widetilde{\sigma}_t^2 = \log \hat{V}_t + \omega_t,$$

where ω_t is iid noise.[3] In the ideal case of no microstructure noise, RV_t is the most natural choice for $\hat{V}_t$. In the presence of microstructure noise, other estimators are preferable such as the two-scale estimator proposed by [74], the realized kernels method of [13], the pre-average approach of [55], or the multi-scales Discrete Sine Transform estimator (DST) of [40]. In our empirical analysis in Section 1.6, we use the DST estimator.

Consider the aggregated values of $\log \widehat{\mathsf{V}}_t$, defined as:

$$\log \widehat{\mathsf{V}}_t^{(n)} = \frac{1}{n} \sum_{j=1}^{n} \log \widehat{\mathsf{V}}_{t-j+1} \tag{1.2}$$

and assume two different time scales, of length n_1 and n_2, with $n_1 > n_2$ (e.g. weekly and daily). For the largest time scale, assume that $\widetilde{\sigma}_t^2$, once aggregated as in (1.2), is determined by:

$$\log \widetilde{\sigma}_{t+n_1}^{2,(n_1)} = c^{(n_1)} + \beta^{(n_1)} \log \widehat{\mathsf{V}}_t^{(n_1)} + \varepsilon_{t+n_1}^{(n_1)} \tag{1.3}$$

where $\varepsilon_t^{(n_1)}$ is an iid random variable with mean zero and unit variance which is independent on the estimation error ω_t, and $c^{(n_1)}$ and $\beta^{(n_1)}$ are unknown parameters.

This can be explained by assuming that the level of short-term volatility does not affect the trading strategies of long-term traders.[4] On the other hand, for short-term traders the level of long-term volatility matters because it determines the expected future size of trends and risk. Hence, the shorter time scale (n_2) is assumed to be influenced by the expected future value of the largest time scale (n_1), so that:

$$\log \widetilde{\sigma}_{t+n_2}^{2,(n_2)} = c^{(n_2)} + \beta^{(n_2)} \log \widehat{\mathsf{V}}_t^{(n_2)} + \delta^{(n_2)} \mathbf{E}_t \left[\log \widetilde{\sigma}_{t+n_1}^{2,(n_1)} \right] + \varepsilon_{t+n_2}^{(n_2)}, \tag{1.4}$$

where $\varepsilon_t^{(n_2)}$ is an iid random variable with mean zero and unit variance, independent on $\varepsilon_t^{(n_1)}$ and ω_t and $\delta^{(n_2)}$ is a constant. The economic interpretation of this mechanism is that each volatility component corresponds to a market component whose

[3] The model can also be specified in terms of $\widehat{\mathsf{V}}_t$ and for $\sqrt{\widehat{\mathsf{V}}_t}$, as in [34] [3] and [38]. However, the log specification has the double advantage of avoiding imposing positivity constraints and making the distribution closer to normality, see e.g. [51].

[4] The HAR model would hold even if we allow the short-term volatility to affect the long-term volatility.

expectation on next period volatility is formed looking at, beyond the current realized volatility value, the forecast on the longer time horizon. The basic idea is that agents with different time horizons perceive, react to, and cause different types of volatility components. By substitution, this gives:

$$\log \widehat{\mathsf{V}}_{t+n_2}^{(n_2)} = c + \beta^{(n_2)} \log \widehat{\mathsf{V}}_t^{(n_2)} + \beta^{(n_1)} \log \widehat{\mathsf{V}}_t^{(n_1)} + \varepsilon_t; \tag{1.5}$$

where ε_t is iid noise depending on $\varepsilon_t^{(n_1)}$, $\varepsilon_t^{(n_2)}$, ω_t. The model (1.5) can be easily extended to d horizons of length $n_1 > n_2 > \ldots > n_d$. Typically, three components are used with length $n_1 = 22$ (monthly), $n_2 = 5$ (weekly), $n_3 = 1$ (daily).

The HAR model is then a parsimonious AR model reparameterized by imposing different sets of restrictions (one for each volatility component) on the autoregressive coefficients of the AR model. Each set of restrictions takes the form of equality constraints among the autoregressive coefficients constituting a given time horizon, so that once combined they lead to a step function for the autoregressive weights. In this sense, the HAR can be related to the MIDAS regression of [47], [48], and [45], although the standard MIDAS with the estimated Beta function lag polynomial cannot reproduce the HAR step function weights.

In practice, the HAR model provides a simple and flexible method to fit the partial autocorrelation function of the empirical data with a step function which has predefined tread depth and estimated (by simple OLS) rise height. More generally, however, nothing prevents the use of different types of kernel in the aggregation of $\widehat{\mathsf{V}}_t$ instead of the rectangular one used in the simple moving average; in this case we would no longer have a step function for the coefficients but a more general function given by a mixture of kernels (e.g. mixture of exponentials for exponentially weighted moving averages) which can still be easily estimated by simple OLS.

Even if the HAR model does not formally belong to the class of long memory processes, it fits the persistence properties of financial data as well as (and potentially better than) true long memory models, such as the fractionally integrated one, which, however, are much more complicated to estimate and to deal with (see the review of [10]). For these reasons, the HAR model has been employed in several applications in the literature, of which an incomplete list is: [47] and [45] compare this model with the MIDAS model; [3] use an extension of this model to forecast the volatility of stock prices, foreign exchange rates and bond prices; [31] implement it for risk management with VaR measures; [20] use it to analyze the risk-return tradeoff; [18] use it to study the relation between intraday serial correlation and volatility.

In the literature dealing with HAR models, it is commonly assumed that the innovations of the log realized volatility are identically and independently distributed. However, volatility clustering in the residuals of the HAR model (as well as in other realized volatility models) are often observed in practical applications. The presence of time-varying conditional distributions in realized volatility models can distort risk assessment and, thus, impair risk management analysis. To account for the observed volatility clustering in realized volatility [37] extend the HAR model by explicitly modelling the volatility of realized volatility. The proposed model adds GARCH type innovations to the standard HAR model, giving rise to an HAR-GARCH(p, q) model which, with the three commonly used frequencies, reads:

$$\log \widehat{\mathsf{V}}^{(1)}_{t+1} = c + \beta^{(1)} \log \widehat{\mathsf{V}}^{(1)}_{t} + \beta^{(5)} \log \widehat{\mathsf{V}}^{(5)}_{t} + \beta^{(22)} \log \widehat{\mathsf{V}}^{(22)}_{t} + \sqrt{h_t}\varepsilon_t \qquad (1.6)$$

$$h_t = \omega + \sum_{j=1}^{q} a_j u^2_{t-j} + \sum_{j=1}^{p} b_j h_{t-j} \qquad (1.7)$$

$$\varepsilon_t|\Omega_{t-1} \sim iid(0,1), \qquad (1.8)$$

where Ω_{t-1} denotes the σ-field generated by all the information available up to time $t-1$ and $u_t = \sqrt{h_t}\varepsilon_t$.

1.3.1 Genuine long memory or superposition of factors?

Assessing whether volatility persistence is generated by a data-generating process with genuine long memory or from a superposition of factors as illustrated above may appear an impossible task. Clearly, the two possibilities might generate very similar empirical features which would make them indistinguishable. In this case, analytical tractability becomes the most important feature to take into account. However, as we discuss here, some specific data generating processes can be ruled out on the basis of the statistical features of the realized volatility time series.

Such an investigation is carried out in [39]. They propose two competing continuous-time models for the volatility dynamics which belong to the class (1.1). The first one is a genuine long-memory model with constant volatility-of-volatility:

$$d\log\sigma_t = k(\omega - \log\sigma_t)dt + \eta dW^{(d)}_t, \qquad (1.9)$$

where $dW^{(d)}_t$ is a *fractional Brownian motion* with memory parameter $d \in [0, 0.5]$, see [33]. The value $d = 0$ corresponds to the standard Brownian motion, while higher d correspond to higher memory in the time series. Model (1.9) (or its discrete counterpart) is usually advocated as the source of long memory in volatility, even if it is very difficult to deal with mathematically and econometrically. It is important to note that in this model persistence comes both from the mean-reverting term $k(\omega - \log\sigma_t)$ and from the fractional Brownian motion $dW^{(d)}_t$. [39] estimate model (1.9) via indirect inference, using the HAR model as auxiliary model. The advantage of indirect inference is that, beyond providing an estimate of the parameters k, ω, η, and d, it provides overall statistics of the goodness-of-fit of the model. They find unambiguously that the model (1.9) is unable to reproduce the time series of volatilities in the S&P500 index.

The second model they test is an affine two-factor model:

$$\begin{aligned} \sigma^2_t &= V^1_t + V^2_t \\ dV^1_t &= \kappa_1(\omega_1 - V^1_t) + \eta_1\sqrt{V^1_t}dW^1_t \\ dV^2_t &= \kappa_2(\omega_2 - V^2_t) + \eta_2\sqrt{V^2_t}dW^2_t, \end{aligned} \qquad (1.10)$$

where W^1 and W^2 are two independent Brownian motions. In this case, even imposing the restriction $\omega_1 = \omega_2$ to identify the model[5], the two-factor model is perfectly able to reproduce the statistical features of the volatility of the S&P500 index. The obtained estimates of $\hat{\kappa}_1 = 2.138$ and $\hat{\kappa}_2 = 0.006$ imply the presence of a fast mean-reverting factor and a slowly mean-reverting factor with a half-life of nearly 166 days, which is usually suggested in the empirical literature on stochastic volatility and option pricing.

Clearly, a more complicated long-memory model (e.g. with two factors) might also reproduce the volatility time series, so it would be wrong to conclude that these results rule out the presence of genuine long memory in the volatility series. However, these results show that the superposition of volatility factors is able to reproduce the long range dependence displayed by realized volatility, for which a genuine long memory data generating process is unnecessary (and certainly not mathematically convenient).

These results can also help explaining the good performance of multi-factor model in the option pricing, see e.g. [16]. They also suggest that two factors might be unnecessary if the volatility dynamics is specified directly with a model similar to HAR: an attempt in this direction is the study proposed by [36] where a realized volatility option-pricing model is developed based on the HAR structure. Such a model is found to provide good pricing performances.

1.4 HAR EXTENSIONS

1.4.1 Jumps measures and their volatility impact

The importance of jumps in financial econometrics is rapidly growing. Recent research focusing on jump detection and volatility measuring in presence of jumps includes [12], [62], [59], [56], [2], [1], [29], [63] and [24]. [3] suggested that the continuous volatility and jump component have different dynamics and should thus be modelled separately. In this section, we closely follow [38] using the $\mathsf{C\text{-}Tz}$ test[6] for jumps detection, and TBPV_t, i.e. the threshold bipower variation, to estimate the

[5]The structural model (1.10) has 6 free parameters while, the auxiliary three components HAR model has 5 (including the parameter of the variance of the innovations).

[6]The $\mathsf{C\text{-}Tz}$ statistics is defined as:

$$\mathsf{C\text{-}Tz}_t = \delta^{-\frac{1}{2}} \frac{(\mathsf{RV}_t - \mathsf{C\text{-}TBPV}_t) \cdot \mathsf{RV}_t^{-1}}{\sqrt{\left(\frac{\pi^2}{4} + \pi - 5\right) \max\left\{1, \frac{\mathsf{C\text{-}TTriPV}_t}{(\mathsf{TBPV}_t)^2}\right\}}}, \tag{1.11}$$

where δ is the time between high-frequency observations, $\mathsf{C\text{-}TBPV}_t$ is a correction of (1.12) devised to be unbiased under the null and $\mathsf{C\text{-}TTriPV}$ is a similar estimator of integrated quarticity $\int_t^{t+1} \sigma_s^4 ds$; see [38] for details.

continuous part of integrated volatility, defined as:

$$\mathsf{TBPV}_t = \frac{\pi}{2} \sum_{j=0}^{n-2} |\Delta_{t,j} X| \cdot |\Delta_{t,j+1} X| I_{\{|\Delta_{t,j}X|^2 \le \vartheta_{j-1}\}} I_{\{|\Delta_{t,j+1}X|^2 \le \vartheta_j\}}, \quad (1.12)$$

where $I_{\{\cdot\}}$ is the indicator function and ϑ_t is a threshold function which we estimate as in [38]. It can be proved that, under model (1.1), $\mathsf{TBPV}_t \to \int_t^{t+1} \sigma_s^2 ds$ as the interval between observations goes to zero. This continuous volatility estimator has much better finite sample properties than standard bipower variation and provides more accurate jump tests, which allows for a corrected separation of continuous and jump components. For this purpose, we set a confidence level α and estimate the jump component as:

$$\mathsf{J}_t = I_{\{\mathsf{C\text{-}Tz} > \Phi_\alpha\}} \cdot \left(\widehat{\mathsf{V}}_t - \mathsf{TBPV}_t\right)^+, \quad (1.13)$$

where Φ_α is the value of the standard Normal distribution corresponding to the confidence level α, and $x^+ = \max(x, 0)$. The corresponding continuous component is defined as:

$$\mathsf{C}_t = \widehat{\mathsf{V}}_t - \mathsf{J}_t, \quad (1.14)$$

which is equal to $\widehat{\mathsf{V}}_t$ if there are no jumps in the trajectory, while it is equal to TBPV_t if a jump is detected by the $\mathsf{C\text{-}Tz}$ statistics.

As for $\log \widehat{\mathsf{V}}_t$ we define aggregated values of $\log \mathsf{C}_t$ as

$$\log \mathsf{C}_t^{(n)} = \frac{1}{n} \sum_{j=1}^{n} \log \mathsf{C}_{t-j+1} \,.$$

For the aggregation of jumps, given the presence of a large number of zeros in the series, we prefer to simply take the sum of the jumps over the window h instead of the average, i.e.:

$$\mathsf{J}_t^{(n)} = \sum_{j=1}^{n} \mathsf{J}_{t-j+1} \,.$$

Consistent with the above section, in the volatility cascade we assume that C_t and J_t enter separately at each level of the cascade, that is:

$$\begin{aligned}
\log \widetilde{\sigma}^{2,(n_1)}_{t+n_1} = c^{(n_1)} \quad &+ \quad \alpha^{(n_1)} \log(1 + \mathsf{J}_t^{(n_1)}) + \beta^{(n_1)} \log \mathsf{C}_t^{(n_1)} + \varepsilon^{(n_1)}_{t+n_1} \\
\log \widetilde{\sigma}^{2,(n_2)}_{t+n_2} = c^{(n_2)} \quad &+ \quad \alpha^{(n_2)} \log(1 + \mathsf{J}_t^{(n_2)}) + \beta^{(n_2)} \log \mathsf{C}_t^{(n_2)} \\
&+ \quad \delta^{(n_2)} \mathbf{E}_t \left[\log \widetilde{\sigma}^{2,(n_1)}_{t+1}\right] + \varepsilon^{(n_2)}_{t+n_2}
\end{aligned}$$

originating the model:

$$\begin{aligned}
\log \widehat{\mathsf{V}}^{(n_2)}_{t+n_2} \; = \; c \quad &+ \quad \alpha^{(n_1)} \log(1 + \mathsf{J}_t^{(n_1)}) + \alpha^{(n_2)} \log(1 + \mathsf{J}_t^{(n_2)}) \\
&+ \quad \beta^{(n_2)} \log \mathsf{C}_t^{(n_2)} + \beta^{(n_1)} \log \mathsf{C}_t^{(n_1)} + \varepsilon_t.
\end{aligned} \quad (1.15)$$

Note that we use $\log(1 + \mathsf{J}_t)$ instead of $\log \mathsf{J}_t$ since J_t can be zero. This model has been introduced as the HAR-CJ model by [3].

1.4.2 Leverage effects

It is well known that volatility tends to increase more after a negative shock than after a positive shock of the same magnitude: this is the so-called leverage effect (see [30, 27, 50] and more recently [19]).

Given the stylized facts presented in Section 1.2, it is then natural to extend the Heterogeneous Market Hypothesis approach to leverage effects. We assume that realized volatility reacts asymmetrically not only to previous daily returns but also to past weekly and monthly returns. We model such heterogeneous leverage effects by introducing asymmetric return-volatility dependence at each level of the cascade considered in the above section. Define daily returns $r_t = X_t - X_{t-1}$ and aggregated returns as:

$$r_t^{(n)} = \frac{1}{n}\sum_{j=1}^{n} r_{t-j+1}.$$

To model the leverage effect at different frequencies, we define $r_t^{(n)-} = \min(r_t^{(n)}, 0)$. We assume that integrated volatility is determined by the following cascade:

$$\begin{aligned}
\log \widetilde{\sigma}^{2,(n_1)}_{t+n_1} &= c^{(n_1)} + \beta^{(n_1)} \log \widehat{\mathsf{V}}_t^{(n_1)} + \gamma^{(n_1)} r_t^{(n_1)-} + \varepsilon^{(n_1)}_{t+n_1} \\
\log \widetilde{\sigma}^{2,(n_2)}_{t+n_2} &= c^{(n_2)} + \beta^{(n_2)} \log \widehat{\mathsf{V}}_t^{(n_1)} + \gamma^{(n_2)} r_t^{(n_2)-} + \delta^{(n_2)} \mathbf{E}_t \left[\log \widetilde{\sigma}^{2,(n_1)}_{t+n_1} \right] + \varepsilon^{(n_2)}_{t+n_2},
\end{aligned}$$

where $\gamma^{(n_{1,2})}$ are constants. This now gives:

$$\log \widehat{\mathsf{V}}^{(n_2)}_{t+n_2} = c + \beta^{(n_2)} \log \widehat{\mathsf{V}}_t^{(n_2)} + \beta^{(n_1)} \log \widehat{\mathsf{V}}_t^{(n_1)} + \gamma^{(n_2)} r_t^{(n_2)-} + \gamma^{(n_1)} r_t^{(n_1)-} + \tilde{\varepsilon}_t. \tag{1.16}$$

We then postulate that leverage effects influence each market component separately, and that they appear aggregated at different horizons in the volatility dynamics.

Combining heterogeneity in realized volatility, leverage, and jumps, we construct the *Leverage Heterogeneous Auto-Regressive with Continuous volatility and Jumps* (LHAR-CJ) model. As is common in practice, we use three components for the volatility cascade: daily, weekly and monthly. Hence, the proposed model reads:

$$\begin{aligned}
\log \widehat{\mathsf{V}}^{(h)}_{t+h} = c \quad &+ \quad \beta^{(d)} \log \mathsf{C}_t + \beta^{(w)} \log \mathsf{C}_t^{(5)} + \beta^{(m)} \log \mathsf{C}_t^{(22)} \\
&+ \quad \alpha^{(d)} \log(1 + \mathsf{J}_t) + \alpha^{(w)} \log(1 + \mathsf{J}_t^{(5)}) + \alpha^{(m)} \log(1 + \mathsf{J}_t^{(22)}) \\
&+ \quad \gamma^{(d)} r_t^{-} + \gamma^{(w)} r_t^{(5)-} + \gamma^{(m)} r_t^{(22)-} + \varepsilon_t^{(h)}.
\end{aligned} \tag{1.17}$$

Model (1.17) nests the other models introduced in the chapter. When $\alpha^{(d,w,m)} = \gamma^{(d,w,m)} = 0$ and $\mathsf{C}_t = \widehat{\mathsf{V}}_t$, the model reduces to the HAR model (1.5). When $\gamma^{(d,w,m)} = 0$, we get the HAR-CJ model (1.15).

Model (1.17) can be estimated by OLS with the Newey-West covariance correction for serial correlation. In order to make multiperiod predictions, we will estimate the model considering the aggregated dependent variable $\log \widehat{\mathsf{V}}^{(h)}_{t+h}$ with h ranging from 1 to 22, i.e. from one day to one month.

1.4.3 General non-linear effects in volatility

Another question of interest is to investigate whether the leverage effects introduced in the previous section are the only relevant non-linear (in that case asymmetric) behaviors present in the realized volatility dynamics in response to past shocks in the market and, more in general, in the whole (macro)economy. In fact, in the last five years several empirical studies published in the literature applied different (parametric and non-parametric) methodologies to the problem of estimating and forecasting realized volatilities, covariances, and correlations dynamics. These showed that they are subject to structural breaks and regime-switches driven by shocks of a different nature: see, among others, [65], [69], and [7].

To investigate this, we generalize the LHAR-CJ model introduced in (1.17) to estimate leverage effects. We propose a tree-structured local HAR-CJ model (Tree HAR-CJ) which is able to take into account both long-memory and possible general non-linear effects in the (log-) realized volatility dynamics. Tree-structured models belong to the class of threshold regime models, where regimes are characterized by some threshold for the relevant predictor variables. The class of tree-structured GARCH models was introduced by [5] in the financial volatility literature, and was generalized recently to capture simultaneous regime shifts in the first and second conditional moment dynamics of returns series (see, for example, [8]). The proposed model reads:

$$\log \widehat{\mathsf{V}}_{t+h}^{(h)} = \mathbf{E}_t[\log \widehat{\mathsf{V}}_{t+h}^{(h)}] + \varepsilon_t^{(h)}, \tag{1.18}$$

where $\mathbf{E}_t[\cdot]$ denotes (as usual) the conditional expectation given the information up to time t. The conditional dynamics of the realized (log-) volatilities are given by:

$$\begin{aligned}
\mathbb{E}_t[\log \widehat{\mathsf{V}}_{t+h}^{(h)}] \quad = & \\
\sum_{j=1}^{k} \Big[c_j \quad & +\beta_j^{(d)} \log \mathsf{C}_t + \beta_j^{(w)} \log \mathsf{C}_t^{(5)} + \beta_j^{(m)} \log \mathsf{C}_t^{(22)} \\
& +\alpha_j^{(d)} \log(1+\mathsf{J}_t) + \alpha_j^{(w)} \log(1+\mathsf{J}_t^{(5)}) + \alpha_j^{(m)} \log(1+\mathsf{J}_t^{(22)}) \\
& +\gamma_j^{(d)} r_t + \gamma_j^{(w)} r_t^{(5)} + \gamma_j^{(m)} r_t^{(22)} \Big] I_{[\mathbf{X}_t^{\text{pred}} \in \mathcal{R}_j]},
\end{aligned} \tag{1.19}$$

where $\theta = (c_j, \alpha_j^{(d,w,m)}, \beta_j^{(d,w,m)}, \gamma_j^{(d,w,m)}, j = 1, \ldots, k)$ is a parameter vector which parameterizes the local HAR-CJ dynamics in the different regimes, k is the number of regimes (endogenously estimated from the data), and $I_{[\cdot]}$ is the identity function that defines regime-shifts.[7]

The regimes are characterized by partition cells $\mathcal{R}_j$ of the relevant predictor space G of $\mathbf{X}_t^{\text{pred}}$:

$$G = \bigcup_{j=1}^{k} \mathcal{R}_j,\ \mathcal{R}_i \cap \mathcal{R}_j = \emptyset\ (i \neq j).$$

[7]The drastic 0-1 rule to define regime-switches can be relaxed to allow for more smooth regime transitions using, for example, a logistic function instead of the identity function; see [65].

For modeling (log-)realized volatilities, the relevant predictor variables in $\mathbf{X}_t^{\text{pred}}$ are past-lagged realized volatilities (considering the estimated ones, as well as the continuous and the jump parts alone), and past-lagged returns of the underlying instrument under investigation to allow explicitly for leverage effects. In taking volatility cascades into account, all such predictor variables are considered at three different time horizons: daily, weekly, and monthly. We also consider time as an additional predictor variable to investigate the relevance of structural breaks in time.[8]

To completely specify the conditional dynamics given in (1.19) of the realized volatilities, we determine the shape of the partition cells $\mathcal{R}_j$, which are admissible in the Tree HAR-CJ model. Similar to the standard classification and regression trees (CART) procedure (see [25]), the only restriction we impose is that regimes must be characterized by (possibly high-dimensional) rectangular cells of the predictor space, with edges determined by thresholds on the predictor variables. Such partition cells are practically constructed using the idea of binary trees. Introducing this restriction has two major advantages: it allows a clear interpretation of the regimes in terms of relevant predictor variables, and it also allows an estimation of the model using large-dimensional predictor spaces G.

The Tree HAR-CJ model introduced above can be estimated for any fixed sequence of partition cells using quasi-maximum likelihood (QML). The choice of the best partition cells (that is, splitting variables and threshold values) involves a model choice procedure for non-nested hypotheses. Similar to CART, the model selection of the splitting variables and threshold values can be performed using the idea of binary trees (for all details, see [8], Section 2.3 and Appendix A). Within any data-determined tree structure, the best model is selected using information criteria or a more formal sequence of statistical tests to circumvent identification problems (see [65]).

1.5 MULTIVARIATE MODELS

We now turn to a multivariate setting, in which a $\mathbb{R}^N$-valued stochastic process X_t evolves over time according to the dynamics:

$$dX_t = \mu_t dt + \Sigma_t dW_t + dJ_t$$

where μ_t is an $\mathbb{R}^N$-valued predictable process, Σ_t an $\mathbb{R}^{N\times N}$-valued cádlág process, $W_1, \ldots, W_N$ is an $N-$dimensional Brownian motion and dJ_t is a $\mathbb{R}^N$ valued jump process. Modeling and forecasting asset returns (conditional) covariance matrix Σ_t is pivotal to many prominent financial problems such as asset allocation, risk management and option pricing. However, the multivariate extensions of the realized volatility approach pose a series of difficult challenges that are still the subject of active research.

[8]The predictor set can be easily expanded to incorporate information included in any other relevant (endogenous or exogenous) explanatory variable.

First, in addition to the common microstructure effect biasing realized volatility measures (i.e. bid-ask spread, price discreteness, etc.), the so called non-synchronous trading effect ([60]) strongly affects the estimation of the realized covariance and correlation measures. In fact, since the sampling from the underlying stochastic process is different for different assets, assuming that two time series are sampled simultaneously when, indeed, the sampling is non-synchronous gives rise to the non-synchronous trading effect. As a result, standard covariance and correlation measures constructed by imposing an artificially regularly spaced time series of high frequency data will possess a bias toward zero which increases as the sampling frequency increases.[9] This effect of a consistent drop of the absolute value of correlations when increasing the sampling frequency was first reported by [44] and hence called the Epps effect. To solve this problem, various approaches have been proposed in the literature: incorporate lead and lag cross returns in the estimator ([70], [32],[22], [9]), avoid any synchronization by directly using tick-by-tick data ([42],[54],[53],[67],[71],[72],[35]), multivariate realized kernel ([11]), and the multivariate Fourier method ([68, 64]). Given the high level of persistence presents in both realized covariances and correlations, the HAR model has also been employed to model the univariate time series dynamics of realized correlations as in [7].

Second, when realized volatility and covariance measures apply any kind of correction for microstructure effects, the resulting variance-covariance matrix is not guaranteed to be positive semi-definite (psd). Exceptions are the multivariate realized kernel with refresh time of [11] and the multivariate Fourier method of [64]. In both cases, however, the frequency at which all the realized variance-covariance estimates are computed are dictated by the asset having the lowest liquidity, hence discarding, in practice, a considerable amount of information especially for the most liquid assets.

Third, in order to have a valid multivariate forecasting model, it is necessary to construct a dynamic specification for the stochastic process of the realized covariance matrix which produces symmetric and psd covariance matrix predictions. In the still relatively scarce but growing literature on multivariate modeling of realized volatilities, three types of approaches have been proposed thus far: modeling the Cholesky factorization of Σ ([28]), its matrix log transformation ([17]), and directly modeling the dynamics of Σ as a Wishart Autoregressive model (WAR) ([23] and [57]).

Fourth, as with all other types of multivariate models, the multivariate modeling of realized volatilities is prone to the curse of dimensionality in the number of parameters of the model. This problem is made particularly severe by the high persistence of the variance-covariance processes, which requires consideration of a large number of variance-covariance elements in the conditioning set. To precisely deal with this

[9] This is because, in addition to the problem of zero returns, any difference in the time stamps between the last ticks for the two assets in each regularly spaced interval will correspond to a portion of the cross product returns that will not be accounted for in the computation of the covariance. This is itself due to the fact that the returns corresponding to this time difference will be ascribed to two different time intervals and hence no longer matched in the cross product summation.

problem, the HAR modelling approach has been also adopted in the multivariate framework and, because of its simplicity, is often preferred to multivariate long memory models.

For instance, after decomposing the realized covariance matrix into Cholesky factors P_t, where

$$P_t^{'} P_t = \Sigma_t,$$

[28] apply both a vector fractionally integrated model (where the same fractional difference parameter is imposed) and an HAR specification with scalar coefficients to the vector of the lower triangular elements of the Cholesky factorization (i.e. to $U_t = vech(P_t)$). In their HAR specification, they also include the biweekly frequency, in addition to the commonly used daily, weekly, and monthly frequencies. The authors find that, in comparison with the more involved vector fractionally integrated model, "the HAR specification shows very good forecasting ability".[10]

For Σ_t, [17] chose the bi-power covariance of [15], but the same principle can be applied to any other covariance estimators. Then they apply a multivariate extension of the HAR-RV model to the principal components of $\mathsf{logm}(\Sigma_t)$. [11] They also include negative past returns to model asymmetric responses and other prediction variables that have been shown to forecast stock returns (such as interest rates, dividend yields, and credit spreads). In their empirical application they find that "lagged principal components of realized weekly and monthly bi-power covariation have a strong predictive power" on the covariance matrix dynamics of size-sorted stock returns.

[23] propose capturing the persistence properties in the realized variances and covariances with a Wishart-based generalization of the HAR model. The HAR structure is then obtained by direct temporal aggregation of the daily covariance matrices over different window lengths. The authors propose a restricted parametrization of their Wishart HAR-type model that is able to deal with large asset cross-section dimensions. In a four dimensional application using two US treasury bills and two exchange rates they show that the restricted specification of the model provides results similar to the fully parameterized model for variance forecasting and risk evaluation.

In the same direction, [57] propose a Wishart specification having HAR type components (i.e. defined as sample averages of past realized covariance matrices). Two types of time-varying Wishart models are considered by the authors: one in which the components affect the scale matrix of the Wishart distribution in a multiplicative way and the second with the components entering in an additive way. Both models are estimated using standard Bayesian techniques with Markov Chain Monte Carlo (MCMC) methods for posterior simulation given that the posterior distribution is unknown. In their empirical analysis on five assets stock prices, the additive spec-

[10] The authors find a slightly superior performance of the fractionally integrated model at a longer horizon. However, this result could be due to the authors' choice to neglect, in the long horizon direct forecast, the forecasting contribution coming from the higher frequency volatility components.

[11] If Σ_t is a $(N \times N)$ psd matrix, we have by the spectral decomposition theorem that $\Sigma_t = E_t \Lambda_t E_t'$, where the columns of the $(N \times N)$ orthonormal matrix E_t correspond to the eigenvectors of Σ_t and Λ_t is a $(N \times N)$ diagonal matrix whose diagonal elements are equal to the N eigenvalues of Σ_t. Then the matrix logarithm of Σ_t, denoted $\mathsf{logm}(\Sigma_t)$, is defined by $\mathsf{logm}(\Sigma_t) = E_t \log(\Lambda_t) E_t'$. Recall that the logarithm of a diagonal matrix is a diagonal matrix whose diagonal elements are taken in $\log$.

ification showed better performance in terms of density forecasts of returns up to 3 months ahead.

1.6 APPLICATIONS

The purpose of this section is first to empirically analyze the performance of the LHAR-CJ model (1.17) and then investigate the presence of other non-linear effects in the dynamics of the S&P500 futures volatilities in addition to the leverage effects.

Our data set covers a long time span of almost 20 years of high frequency data for the S&P 500 futures from January 1990 to February 2009, for a total of 4,766 daily observations. In order to reduce the impact of microstructure effects, the estimator for the daily volatility $\widehat{\mathsf{V}}_t$ is computed with the multi-scales DST estimator of [40]. The multi-scales DST estimator combines the DST orthogonalization of the volatility signal from the microstructure noise with a multi-scales estimator similar to that proposed by [73][12] but constructed with a simple regression based approach.

The (significant) jump component J_t in (1.13) and the continuous volatility C_t in (1.14) are computed at the 5-minute sampling frequency (corresponding to 84 returns per day). The confidence level α in (1.13) is set to 99.9%. All the quantities of interest are computed on an annualized base.

The results of the estimation of the LHAR-CJ on the S&P500 sample from January 1990 to February 2009, with $h = 1, 5, 10, 22$ are reported in Table 1.1, together with their statistical significance, evaluated with the Newey-West robust t-statistic with 44 lags.

As usual, all the coefficients of the three continuous volatility components are positive and highly significant. We observe that the coefficient measuring the impact of monthly volatility on future daily volatility (i.e. 0.203) is more than twice as big as the one of daily volatility on future monthly volatility (i.e. 0.105). This finding is consistent with the hierarchical asymmetric propagation of the volatility cascade formalized in Section 1.3.

A similar hierarchical structure, although less pronounced, is present in the impact of jumps on future volatility. The daily and weekly jump components remain highly significant and positive especially when modelling realized volatility at short horizons. In addition, their impact declines when the frequency at which RV is modelled declines. The jumps aggregated at the monthly level, however, turn out to be insignificant on the considered data set.

Interestingly, estimation results for model (1.17) reveal the strong significance (with the economically expected negative sign) of the negative returns at (almost) all frequencies, which unveils the presence of a heterogeneous structure in the leverage effect as well. In fact, the daily volatility is significantly affected, not only by the daily negative return of the day before (the well know leverage effect) but also of the week and of the month before. This result suggests that the market aggregates

[12] A generalization of the two-scales estimator of [74] to many realized volatilities computed at different frequencies.

S&P500 LHAR in-sample regression

Variable	One day	One week	Two weeks	One month
c	0.765*	0.847*	0.954*	1.096*
	(11.416)	(7.888)	(6.327)	(4.941)
C	0.248*	0.172*	0.132*	0.105*
	(13.169)	(11.720)	(10.182)	(8.215)
$\mathsf{C}^{(5)}$	0.317*	0.299*	0.285*	0.243*
	(11.210)	(8.516)	(7.027)	(5.110)
$\mathsf{C}^{(22)}$	0.230*	0.315*	0.361*	0.398*
	(8.577)	(7.951)	(6.720)	(5.497)
J	0.016*	0.012*	0.012*	0.010*
	(3.135)	(2.914)	(3.606)	(2.654)
$\mathsf{J}^{(5)}$	0.058*	0.055*	0.047*	0.027
	(4.573)	(3.330)	(2.282)	(1.171)
$\mathsf{J}^{(22)}$	0.010	0.011	0.008	0.028
	(0.544)	(0.413)	(0.222)	(0.522)
r^{-}	-0.736*	-0.526*	-0.411*	-0.337*
	(-8.620)	(-10.154)	(-8.226)	(-5.436)
$r^{(5)-}$	-1.070*	-0.685*	-0.739*	-0.644*
	(-4.602)	(-3.054)	(-3.491)	(-2.685)
$r^{(22)-}$	-0.899*	-1.111	-0.985	-0.668
	(-2.116)	(-1.809)	(-1.411)	(-0.778)

Table 1.1 OLS estimates of the LHAR-CJ model (1.17), for S&P500 futures from January 1990 to February 2009, ($4,766$ observations). The LHAR-CJ model is estimated with $h = 1$ (one day), $h = 5$ (one week), $h = 10$ (two weeks) and $h = 22$ (one month). The significant jumps are computed using a critical value of $\alpha = 99.9\%$. Reported in parenthesis are t-statistics based on Newey-West correction.

information at the daily, weekly and monthly levels and reacts to shocks happening at these three levels/frequencies. These findings thus further confirm the views of the Heterogeneous Market Hypothesis.

To evaluate the performance of the LHAR-CJ model, we compare it with the standard HAR (with only heterogeneous volatility) and the HAR-CJ model (with heterogeneous jumps) on the basis of a genuine out-of-sample analysis. For the out-of-sample forecast of $\widehat{\mathsf{V}}_t$ on the $[t, t+h]$ interval we keep the same forecasting horizons (one day, one week, two weeks and one month) and re-estimate the model at each day t on a moving window of length 2500 days. Table 1.2 reports the out-of-sample forecasts of the different models evaluated on the basis of the R^2 of Mincer-Zarnowitz forecasting regressions and the Diebold-Mariano test for the out-of-sample Root Mean Square Error (RMSE).[13]

The superiority of the HAR-CJ model over the HAR model is mild, since it has to be ascribed preeminently to days which follow a jump, and thus on a very small sample; conditioning on days following the occurrence of a jump would show a sharper improvement (as shown in [38]). However, the superiority of the LHAR-CJ model at all horizons, with respect to the HAR (and the HAR-CJ model) is much stronger, validating the importance of including both the heterogeneous leverage effects and jumps in the forecasting model.

In the second part of our empirical analysis, we estimate the Tree HAR-CJ model introduced in (1.19) to investigate whether additional non-linear effects are present in the dynamics of the S&P500 futures volatilities on the top of the leverage effect and whether the explicit modeling of structural breaks and regime-shifts is able to improve the accuracy of the estimates and forecasts. To simplify the interpretations and reduce the number of parameters in the model, we assume that the cascade is present only in the volatility continuous component C_t (i.e. we set the parameters $\alpha_j^{(w,m)}$ and $\gamma_j^{(w,m)}, j = 1, \ldots, k$, to zero). Estimated coefficients, as well as the estimated regimes, are reported in Table 1.3 for $h = 1$. Classical model-based bootstrapped standard errors are given in parentheses.

Table 1.3 shows that almost all coefficients in the local dynamics of realized volatilities are highly significant, with a couple of interesting exceptions. As discussed previously, the leverage effect is found to be the most important asymmetry and yields the first binary split in the procedure. The optimal threshold is found to be around zero, highlighting the different reaction of realized volatilities to past positive and negative S&P 500 returns. A second relevant non-linear behavior of realized volatility dynamics is found in response to past low and moderate vs. high (continuous part) volatilities when past S&P 500 returns are negative. In fact, the threshold value $d_2 = 5.34$ corresponds to the 70% quantile of the estimated $\log C_t$ series.

In these three regimes, local volatility dynamics show significant differences. In particular, it is worth mentioning the following two results: First, past lagged S&P

[13]Diebold-Mariano test should be applied with care when competing models are nested, however, [49] showed that if the window size is bounded (e.g., computed over a fixed moving window as in our setting) the test is still valid.

S&P500 out-of-sample performances

Variable	One day	One week	Two weeks	One month
HAR	0.8073	0.8351	0.8162	0.7573
HAR-CJ	0.8107	0.8397	0.8188	0.7597
	(1.994)	(1.808)	(0.835)	(0.115)
LHAR-CJ	0.8238	0.8487	0.8279	0.7651
	(4.663)	(2.854)	(2.023)	(1.169)

Table 1.2 R^2 of Mincer-Zarnowitz regressions for out-of-sample forecasts for horizons $h = 1$ (one day), $h = 5$ (one week), $h = 10$ (two weeks) and $h = 22$ (one month) of the S&P500 from January 1990 to February 2009 (4,766 observations, the first 2500 observations are used to initialize the models). The forecasting models are the standard HAR, the HAR-CJ and the LHAR-CJ model. In parentheses is reported the Diebold-Mariano test for the out-of-sample RMSE with respect to the standard HAR model.

Tree HAR-CJ estimates and regimes

Regime structure	Local parameters					
$\mathcal{R}_j$	c_j	$\alpha_j^{(d)}$	$\beta_j^{(d)}$	$\beta_j^{(w)}$	$\beta_j^{(m)}$	$\gamma_j^{(d)}$
$r_t \leq 0.05$,	0.6577	0.0574	0.1864	0.4060	0.2578	-0.1987
$\log C_t \leq 5.34$	(0.0411)	(0.0115)	(0.0345)	(0.0428)	(0.0373)	(0.0253)
$r_t \leq 0.05$,	0.5627	-0.0095	0.3924	0.4187	0.0783	-0.1330
$\log C_t > 5.34$	(0.0299)	(0.0133)	(0.0382)	(0.0420)	(0.0387)	(0.0136)
$r_t > 0.05$	0.1854	0.0604	0.3260	0.3962	0.2335	-0.0055
	(0.0516)	(0.0076)	(0.0271)	(0.0404)	(0.0297)	(0.0115)

Table 1.3 Tree HAR-CJ estimated parameters and regimes for the S&P 500 realized (log-) volatilities with $h = 1$. The sample period is from January 1990 to February 2009, for a total of 4,766 daily observations. r_t and $\log C_t$ denote the past-lagged daily S&P 500 return and past-lagged daily (log-) continuous components of the realized volatility, respectively. Model-based bootstrap standard errors computed using 1,000 replications are given in parentheses.

500 returns are significant only in the regimes where they are negative, yielding to an increase in the realized volatilities. When past lagged S&P 500 returns are positive (last regime) their impact in estimating future volatility dynamics is negligible. Second, the impact of jumps highly changes depending on the regime in which they occur: it is positive and significant in regimes characterized by (somehow) stable financial markets (regimes 1 and 3), yielding to an increase of realized volatility. By contrast, in times of market turbulence (measured by past negative returns and high past volatilities), jumps are found to have no particular impact in driving future realized volatility dynamics. These interesting results confirm and extend previous empirical findings shown in this section.

Similarly to what has been shown above for the LHAR-CJ model, in a preliminary series of forecasting experiments for h equal to one, the Tree HAR-CJ model has been found to be able to significantly improve the out-of-sample performance of the classical HAR and HAR-CJ models. A more detailed and complete investigation of how the introduction of regimes (threshold-based or of a Markovian type) may improve predictions in a general HAR setting is left for the future.

1.7 CONCLUDING REMARKS AND AREAS FOR FUTURE RESEARCH

By projecting a dynamic process on its own past values aggregated over different time horizons, the HAR model is a general and flexible approach to fit the autocorrelation function of any persistent process in a very simple and tractable way. In this chapter we have briefly surveyed the nature, construction, and properties of the HAR class of models for realized volatility estimation and prediction. We discussed some of the extensions of the standard HAR model that have been recently proposed to explicitly take into account the predictive power of jumps, leverage effects, and other non-linearities (i.e. structural breaks and regime switches driven by the different sources acting on the financial market) for the time-varying dynamics of realized volatilities. We also reviewed some recent studies generalizing the HAR model for predicting univariate realized volatilities to the multivariate setting of realized covariance matrices. This is a fast-growing field and the list of references will no doubt need updating in the near future.

In our review of the extant literature on HAR models a number of topics stand out as possible avenues for future research. The most obvious, and perhaps difficult, is to generalize the univariate flexible HAR model with jumps, leverage effects, and other non-linear behaviors due to regime changes to the multivariate context. Existing models do not take these effects into account and are not well-designed to deal with (possibly) high-dimensional realized covariance matrices. What is needed are flexible yet parsimonious multivariate HAR-type extensions that remain computationally feasible in large dimensions. This task may be accomplished using recent techniques coming from the computational statistics community, similar to what was done ten years ago in [6] for the estimation of a flexible volatility matrix in a multivariate GARCH setting.

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