Efficient pricing of Ratchet Equity Indexed Annuities in a VG economy

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Abstract

In this paper we propose a new method for approximating the price of arithmetic Asian options in a VG economy which is then applied to the problem of pricing Equity Indexed Annuity contracts. The proposed procedure is an extension to the case of a VG-based model of the moment-matching method developed by Turnbull and Wakeman (1991) and Levy (1992) for the pricing of this class of path-dependent options in the traditional Black-Scholes setting. The accuracy of the approximation is analysed against RQMC estimates for the case of ratchet Equity Indexed Annuities with index averaging.

Keywords: Asian options, Equity Indexed Annuities, risk neutral valuation, Randomized Quasi-Monte Carlo, variance reduction techniques.

JEL Classification: C63, G13

1 Introduction

The aim of this contribution is to discuss efficient numerical methods for the market consistent pricing of ratchet equity-indexed annuities (EIAs) in a Variance-Gamma (VG) economy. The payoff structure of this class of contracts reveals, as pointed out by Lin and Tan (2003), that the contract can be synthetically replicated by a portfolio composed of a risk free bond (the minimum guarantee), a bull spread of European call options, and a purely path-dependent option representing the so called minimum value contract. The European options forming the bull spread can be either of vanilla type or path-dependent according to the contract design. The more popular EIA designs are reviewed in section 2.

General pricing frameworks for these products have been proposed by Lin and Tan (2003) and Kijima and Wong (2007); both these contributions are based on the assumption that the log-returns of the equity index, on which the payoff of EIAs is calculated, are driven by a Brownian motion, i.e. follow a Gaussian distribution. However, the limitations of the Brownian motion for the purpose of financial modelling, in particular its inability to capture asymmetry and excess kurtosis, are well known since the early '60s. An alternative process which has been introduced in the literature by Madan et al. (1990, 1991, 1998) is the Variance Gamma (VG) process; this is a Lévy process, i.e. a process with independent and stationary increments, obtained by observing a Brownian motion on a time scale governed by a stochastic clock independent of the Brownian motion itself. This construction has a number of appealing properties. In first place, it recognizes that stock prices are largely driven by news; further, the time between one piece of news and the next one varies quite randomly, as does the precise impact that a news has on stock prices. Hence, the VG process incorporates both effects, i.e. the time at which news come in (the so called business time) and the change in the stock price. Finally, the VG process offers a high degree of mathematical tractability as, once we operate under the business time (i.e. we condition on the stochastic clock), the log-returns are once again Gaussian and therefore all the results derived for the Black-Scholes model still hold. This aspect in particular has also allowed a number of implementations of the VG model for the financial industry (see, for example, Carr et al., 2007).

Applications of the VG model in life insurance have been explored by Ballotta (2009) for the pricing of participating contracts with minimum guarantee and the calculation of the corresponding capital requirements; for the case of EIAs, instead, Jaimungal (2004) has shown how to price analytically a number of EIA contracts, using the fact that (semi-)closed form solutions are available for European vanilla options under the VG economy (Madan et al., 1990, 1991, 1998). However, as smoothing is introduced and more complex options are involved in the EIA design, closed form solutions are no longer attainable, and numerical approximations need to be sought for. For the case of the minimum value contract, instead, pricing can be done only numerically due to the high complexity of the terminal payoff.

In the light of this discussion, in this paper we focus on the problem of implementing efficient numerical methods for ratchet EIAs which use index averaging as smoothing mechanism. As in this case the embedded options are arithmetic Asian options, in section 3 we develop an approximating procedure for this type of exotic derivatives based on the moment matching-approach. The performance of the proposed method is analyzed in section 4; section 5 concludes.

2 Pricing Ratchet EIA in a VG economy

An equity-indexed annuity (EIA) contract is a fixed annuity earning a minimum rate of interest and offering a potential gain which is tied to the performance of a prespecified equity index, S. EIAs come in many different forms, like the Point-to-Point or the Ratchet EIAs. Amongst the Point-to-Point EIAs, we can further distinguish the term-end design, the Asian-end design and the high watermark design; ratchet EIAs can be simple or compound, with or without index averaging.

The performance of the equity index is in general measured on the basis of the returns generated by the index over the lifetime of the contract. Thus, Point-to-Point EIAs are constructed using the index return between two time points (in general, inception and maturity). Specifically, the term-end design uses the overall return over the term of the contract; the Asian-end design instead uses an average of the index levels in the year of expiration, whilst the high watermark design depends on the highest realized index level over the term of the contract. In the case of ratchet EIAs, the index return is measured each year; an averaging scheme is often used to calculate the index growth in order to reduce the costs of the guarantees.

A complete description of the payoff of each of these contracts is offered by Lin and Tan (2003), who also provide a pricing framework based on a risk minimizing approach with stochastic interest rates. Kijima and Wong (2007) adopt the same market model as in Lin and Tan (2003), but use risk neutral valuation to price a number of EIAs contracts. Both contributions use a geometric Brownian motion to model the evolution of the equity index. Given the known limitation of this process in capturing stylized features of the real market dynamics like skewness and excess kurtosis, Jaimungal (2004) explores the pricing and hedging of Point-to-Point and (compound) ratchet EIAs (without index averaging) in a VG economy.

In the following of this paper, we use the VG process to model the evolution of the equity index as in Jaimungal (2004); however, we focus on ratchet EIAs with index averaging, for which prices formulae are not available. Ratchet EIAs are the most popular contract design in the North American market in terms of their sales volume (see Lin and Tan, 2003, and references therein). We distinguish between the simple and the compound ratchet EIAs, whose corresponding payoff functions at maturity $T = 1, 2, ...$ years are given by

$$B_T^{(s)} = 1 + \sum_{t=1}^{T} \min\left(\max\left(\alpha R_t, f\right), c\right), \tag{1}$$

$$B_T^{(c)} = \prod_{t=1}^{T} \left[1 + \min\left(\max\left(\alpha R_t, f\right), c\right)\right], \tag{2}$$

where $\alpha \in (0,1)$ is the participation rate, $f \in (0,1)$ is the floor rate providing the minimum guarantee, and $c \in (0,1)$, with $c > f$, is the cap rate which places an upper bound to the rate of return of the contract. R_t measures the appreciation of the equity index in year t and is given by

$$R_t = \frac{1}{N} \frac{\sum_{i=0}^{N-1} S_{t-\frac{i}{N}}}{S_{t-1}} - 1 \qquad \forall t = 1, ..., T,$$

where N is the number of reset dates. The average frequency can be daily, weekly or monthly. If $N = 1$, only the annual return in year t matters in the payoff construction and therefore no smoothing is applied.

A close inspection of the payoff functions reveals that equations (1) and (2) can be more conveniently rewritten as

$$B_T^{(s)} = 1 + Tf + \sum_{t=1}^{T} \left[(\alpha R_t - f)^+ - (\alpha R_t - c)^+ \right], \tag{3}$$

$$B_T^{(c)} = \prod_{t=1}^{T} \left[1 + f + (\alpha R_t - f)^+ - (\alpha R_t - c)^+ \right]. \tag{4}$$

Hence, the ratchet EIA can be considered as a portfolio composed by a risk free component given by the minimum guarantee, and a sequence of bull spreads formed by European call options, which can be either Asian or vanilla options according to whether smoothing is applied or not.

For the purpose of obtaining the market consistent price of these contracts, consider a frictionless market with infinitely divisible securities, in which the risk free rate of interest is constant, i.e. $r \in \mathbb{R}^{++}$, and the equity index is given by

$$S_t = S_0 e^{(r+\varpi)t + X_t} \tag{5}$$

under some risk neutral martingale measure[1] $\hat{\mathbb{P}}$. $X = (X_t : t \geq 0)$ is assumed to follow a VG process, which is obtained by time changing a Brownian motion by a gamma subordinator. Hence, given a standard $\hat{\mathbb{P}}$-Brownian motion $W = (W_t : t \geq 0)$ and a $\hat{\mathbb{P}}$-gamma process $\tau = (\tau_t : t \geq 0)$ with time scale $a \in \mathbb{R}^{++}$ and decay rate $b \in \mathbb{R}^{++}$, then

$$X_t = \theta \tau_t + \sigma W_{\tau_t}, \qquad \theta \in \mathbb{R}, \sigma \in \mathbb{R}^{++}.$$

In particular, we fix $a = b = \frac{1}{k}$, with $k \in \mathbb{R}^{++}$ representing the variance rate of the gamma subordinator; this last assumption guarantees that the stochastic clock represented by the gamma process is an unbiased reflection of calendar time, so that $\hat{\mathbb{E}}(\tau_t) = t$ (see, for example, Madan et al., 1998). The parameter ϖ in equation (5) is given by

$$\varpi = \frac{1}{k} \ln \left(1 - \theta k - \frac{\sigma^2}{2} k \right),$$

and represents the exponential compensator of the VG process X_t, so that the discounted stock price $e^{-rt} S_t$ is a $\hat{\mathbb{P}}$-martingale.

The VG process is an example of a Lévy process, i.e. of a process with independent and stationary increments. The characteristic function of X_t is given by

$$\phi_X(z; t) = \left(1 - z\theta k - z^2 \frac{\sigma^2}{2} k \right)^{-\frac{t}{k}}, \qquad z \in \mathbb{C}, \tag{6}$$

which exists for

$$\frac{-\theta - \sqrt{\theta^2 + \frac{2\sigma^2}{k}}}{\sigma^2} < \Re(z) < \frac{-\theta + \sqrt{\theta^2 + \frac{2\sigma^2}{k}}}{\sigma^2}.$$

[1] We note that the given market is incomplete and, consequently, there are infinitely many risk neutral martingale measures. However, we do not explore here the issues related to the choice of a suitable pricing probability measure, and we assume that the model is calibrated to the market.

From equation (6), it follows

$$\begin{aligned}
\mathbb{E}X(t) &= \theta t, & \mathbb{V}ar(X(t)) = \left(\sigma^2+\theta^2 k\right)t,\\
\gamma_1(t) &= \frac{\left(3\sigma^2+2\theta^2 k\right)\theta k}{\left(\sigma^2+\theta^2 k\right)^{3/2}\sqrt{t}}, & \gamma_2(t) = \frac{\left(3\sigma^4+12\sigma^2\theta^2 k+6\theta^4 k^2\right)k}{\left(\sigma^2+\theta^2 k\right)^2 t},
\end{aligned}$$

where γ_1 and γ_2 denote respectively the Pearson index of skewness and the index of excess kurtosis. It follows that the distribution of the VG process is fully described by the three parameters (θ, σ, k), which control the skewness, the variance and the excess kurtosis respectively.

Because of the independent and stationary increments properties of Lévy processes, it follows that

$$\frac{\sum_{i=0}^{N-1} S_{t-\frac{i}{N}}}{S_{t-1}} \overset{\mathcal{D}}{=} \sum_{i=0}^{N-1} e^{(r+\varpi)\left(1-\frac{i}{N}\right)+X'_{1-\frac{i}{N}}},$$

where X' is an independent copy of the VG process X. Therefore, the no-arbitrage price of the ratchet EIAs is given by

$$\begin{aligned}
B_0^{(s)} &= e^{-rT}(1+Tf)+Te^{-r(T-1)}\left[\hat{\mathbb{E}}\left(e^{-r}\left(\frac{\alpha}{N}\sum_{i=0}^{N-1} e^{(r+\varpi)\left(1-\frac{i}{N}\right)+X'_{1-\frac{i}{N}}}-K_f\right)^+\right)\right.\\
&\quad \left.-\hat{\mathbb{E}}\left(e^{-r}\left(\frac{\alpha}{N}\sum_{i=0}^{N-1} e^{(r+\varpi)\left(1-\frac{i}{N}\right)+X'_{1-\frac{i}{N}}}-K_c\right)^+\right)\right], \qquad (7)\\
B_0^{(c)} &= \left[e^{-r}(1+f)+\hat{\mathbb{E}}\left(e^{-r}\left(\frac{\alpha}{N}\sum_{i=0}^{N-1} e^{(r+\varpi)\left(1-\frac{i}{N}\right)+X'_{1-\frac{i}{N}}}-K_f\right)^+\right)\right.\\
&\quad \left.-\hat{\mathbb{E}}\left(e^{-r}\left(\frac{\alpha}{N}\sum_{i=0}^{N-1} e^{(r+\varpi)\left(1-\frac{i}{N}\right)+X'_{1-\frac{i}{N}}}-K_c\right)^+\right)\right]^T. \qquad (8)
\end{aligned}$$

Hence, the options embedded in the EIAs have 1 year to maturity, are written on an underlying asset with spot price equal to the participation rate α, and have strike prices $K_f = \alpha + f$ and $K_c = \alpha + c$.

If $N = 1$, the options embedded in the EIAs are European vanilla calls for which a (semi-) analytical formula of the price is available (see Madan et al., 1990, 1991, 1998) and given by

$$C_0(\alpha, K, 1) = \alpha\Psi\left(d, \frac{\theta+\sigma^2}{\sigma}, \frac{1}{k}, \frac{1-s}{k}\right) - e^{-r}K\Psi\left(d, \frac{\theta}{\sigma}, \frac{1}{k}, \frac{1}{k}\right), \qquad (9)$$

where K is equal to either K_f or K_c, and

$$\begin{aligned}
\Psi(\xi, \chi, a, b) &= \int_0^\infty \Phi\left(\frac{\xi}{\sqrt{\tau}}+\chi\sqrt{\tau}\right)\frac{b^a\tau^{a-1}e^{-b\tau}}{\Gamma(a)}d\tau; \qquad (10)\\
d &= \frac{\ln\frac{\alpha}{K}+r+\varpi}{\sigma}, \quad s = k\left(\theta+\frac{\sigma^2}{2}\right), \qquad (11)
\end{aligned}$$

with Φ denoting the standard normal distribution (an alternative proof is provided in Appendix A).

If $N > 1$, the options involved are arithmetic Asian calls with fixed strike price. As the distribution of the arithmetic average is in general not known (even in the Black-Scholes economy), we need to resort to numerics. This is discussed in the next section.

3 Numerical methods for Asian option pricing

The pricing of arithmetic Asian options requires the implementation of numerical schemes due to the fact that the distribution of the arithmetic average of stock prices is in general not known. As Asian options are path-dependent contracts, they represent the typical example for the application of Monte Carlo simulation. However, well known shortcomings of Monte Carlo simulation in terms of efficiency and rate of convergence have motivated research for the development of alternative methods for the numerical approximation of option prices, such as Fourier transform inversion, quadrature methods and schemes for P(I)DEs. For an extensive overview of the techniques available for pricing vanilla and exotic options in a Lévy economy we refer, for example, to Cont and Tankov (2004), Schoutens (2004) and the references therein.

In this paper, we focus on the development of analytical formulae for the approximation of the price of the arithmetic Asian options embedded in EIA contracts, which are based on the moment-matching approach. The basic idea is to calculate the first m moments of the sum

$$\bar{S}_t^{(N)} \doteq \frac{\sum_{i=0}^{N-1} S_{t-\frac{i}{N}}}{S_{t-1}},$$

and then use this partial information to approximate the unobserved distribution with an alternate, more tractable one, whose parameters are chosen to match the original moments. The use of this approximation procedure in the option pricing literature dates back to Jarrow and Rudd (1982), in the context of ameliorating the Black-Scholes-Merton option pricing formula. Turnbull and Wakeman (1991) and Levy (1992) have then used the same idea to obtain an efficient algorithm to price Asian options in the Black-Scholes economy.

In the context of the market model introduced in section 2, the moments of the dependent sum $\sum_{i=1}^{N} S_{t_i}$ can be calculated using the basic properties of the VG process as a Lévy process. Hence, set

$$\begin{cases} R_i = \frac{S_{t_i}}{S_{t_{i-1}}} & i = 1, ..., N \\ L_N = 1 & \\ L_{i-1} = 1 + L_i R_i & i = 2, ..., N. \end{cases}$$

Then

$$\sum_{i=1}^{N} S_{t_i} = S_0 \left(R_1 + R_1 R_2 + ... + R_1 R_2 \cdots R_N \right) = S_0 L_1 R_1.$$

Due to the independence and stationarity of the increments of a Lévy process, it follows that

$$\hat{\mathbb{E}} \left[\left(\sum_{i=1}^{N} S_{t_i} \right)^m \right] = S_0^m \hat{\mathbb{E}} \left(L_1^m \right) \hat{\mathbb{E}} \left(R_1^m \right), \tag{12}$$

with

$$\hat{\mathbb{E}} \left(L_{i-1}^m \right) = \hat{\mathbb{E}} \left[(1 + L_i R_i)^m \right] = \sum_{j=0}^{m} \binom{m}{j} \hat{\mathbb{E}} \left(L_i^j \right) \hat{\mathbb{E}} \left(R_i^j \right), \tag{13}$$

in virtue of the Binomial theorem, and

$$\hat{\mathbb{E}} \left(R_i^j \right) = e^{j(r+\varpi)(t_i - t_{i-1})} \left(1 - j\theta k - \frac{j^2 \sigma^2}{2} k \right)^{-\frac{t_i - t_{i-1}}{k}} \qquad \forall j, \tag{14}$$

where the last equality follows from equation (6).

In order to apply this procedure to the sum $\bar{S}_t^{(N)}$, we observe that

$$\bar{S}_t^{(N)} = L_1 R_1;$$

hence, equation (12) applies for $S_0 = 1$ and $t_i = t - 1 + 1/N$. Consequently, the first m moments of $\bar{S}_t^{(N)}$ can be obtained by means of recursion (13) and equation (14).

The next step in the construction of the numerical procedure involves the choice of the alternate distribution required for the approximation. For the case of the Black-Scholes economy, Turnbull and Wakeman (1991) and Levy (1992) have chosen the log-normal distribution; in the context of the VG economy, Albrecher and Predota (2002) have used another VG distribution. However, this choice does not guarantee that the resulting values approximating the sum of the stock prices are positive almost surely, as it should be indeed the case. Hence, based on this observation, we choose an approximating distribution of the same class as e^Y, where Y is the driving process.

Thus, in order to approximate the stochastic process of the sum of the stock prices, we use an exponential VG process of the form

$$\begin{aligned} Z_t &= Z_0 e^{at+X_t}, \qquad Z_0 = \alpha, \\ X_t &= b\tau_t + \gamma W_{\tau_t}, \qquad b \in \mathbb{R}, \gamma \in \mathbb{R}^{++}, \end{aligned}$$

where τ_t is a $\hat{\mathbb{P}}$-gamma process with variance rate $v \in \mathbb{R}^{++}$. The parameter set (a, b, γ, v) is obtained as the (numerical) solution of the moment-matching problem based on equations (12)-(14). Since we require four parameters to be estimated, we match the first four moments of the distributions involved; therefore, the approximating price of the arithmetic Asian option for the problem set in section 2 is

$$\begin{aligned} A_0(\alpha, K, 1) &= \alpha e^{a-r-\varsigma}\Psi\left(d', \frac{b+\gamma^2}{\gamma}, \frac{1}{v}, \frac{1-s'}{v}\right) - e^{-r}K\Psi\left(d', \frac{b}{\gamma}, \frac{1}{v}, \frac{1}{v}\right), \\ d' &= \frac{\ln\frac{\alpha}{K} + a}{\gamma}, \quad s' = v\left(b + \frac{\gamma^2}{2}\right), \quad \varsigma = \frac{\ln(1-s')}{v}, \end{aligned}$$

where K is equal to either K_f or K_c and the function Ψ is as defined in equation (10).

4 Results

The accuracy of the numerical procedure developed in section 3 is tested in the case of the ratchet EIA introduced in section 2. The analysis is organized as follows. Firstly, we test the accuracy of the approximation by comparing the sample distribution of the process $\bar{S}_t^{(N)}/N$, i.e. the daily average of the index returns over 1 year forming the underlying asset of the EIA contract, with the one of the process Z over the same period of time. Secondly, we analyze the accuracy of the prices obtained for the single Asian options forming the bull spread component of the contract, i.e. the Asian floor option $A_0(\alpha, K_f, 1)$ and the Asian cap option $A_0(\alpha, K_c, 1)$, for a range of feasible values of the floor/cap rates, i.e. the strike prices. The comparison is performed against Monte Carlo estimates. Finally, we analyze the behaviour of the price of the overall ratchet EIA contract.

Unless otherwise stated, the base parameter set is

$$\begin{aligned} &\alpha = 40\%; \quad f = 3\%; \quad c = 10\%; \quad T = 10 \text{ years}; \quad N = 365 \text{ (daily average)}; \\ &r = 5\%; \quad \theta = -0.2; \quad \sigma = 0.2; \quad k = 0.25. \end{aligned}$$

We further assume that the equity index pays a dividend yield $q = 2\%$ p.a. continuously compounded. This is the same parameter set used by Jaimungal (2004).

Testing the approximating distribution

Table 1 reports the moments of the average process $\bar{S}_t^{(N)}/N$ and the approximating process Z, which are calculated using the available analytical formulae. The corresponding QQ plot of the (simulated) samples of the two processes is shown in Figure 1. The results in Table 1 show that the matching procedure, which can only be performed numerically, provides accuracy up to the first 8 decimal places for the first moment, 7 decimal places for the variance, and 3-4 decimal places for the index of skewness and excess kurtosis. The linearity of the QQ plot represented in Figure 1 confirms the similarity of the two distributions, although the accuracy of the approximation tends to deteriorate at the far end of the tails. This is expected as, in general, the first four moments do not provide the full characterization of a probability distribution.

	$\bar{\mathbf{S}}_t^{(N)}/N$	$\mathbf{Z}_T$
mean	0.4060771404	0.4060771455
variance	0.0024904894	0.0024905044
index of skewness	-0.1435419798	-0.1435237099
excess kurtosis	0.7830966150	0.7828234496

Table 1: Moment matching: the first four moments of the distribution of the average process $\bar{S}_t^{(N)}/N$ and the approximating process Z_T.

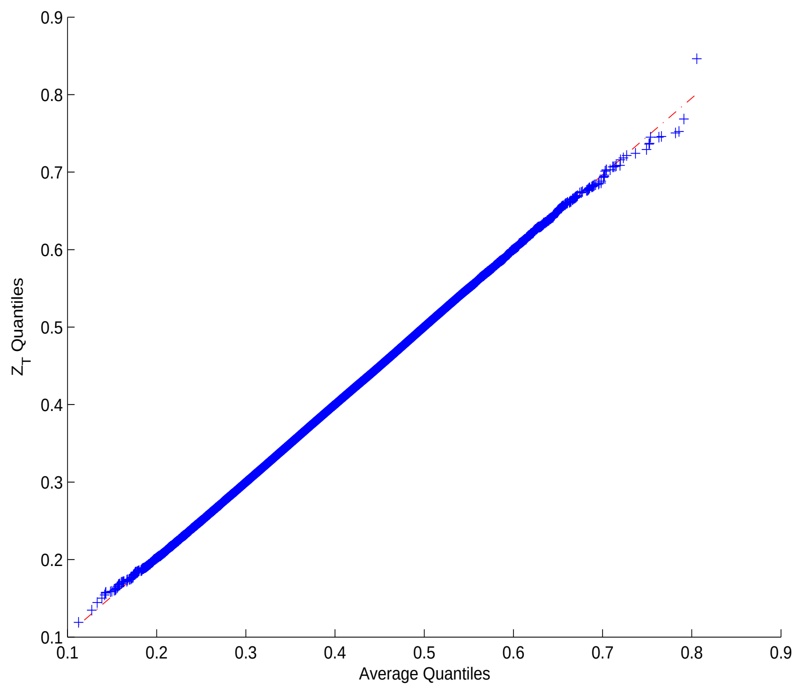

Figure 1: QQ plot of a Monte Carlo sample of the average process $\bar{S}_t^{(N)}/N$ and the approximating process Z_T. Monte Carlo simulation based on 1,000,000 iterations.

	European Floor	European Cap	Simple Ratchet	Compound Ratchet
RQMC estimate	0.025179	0.007146	0.903376	0.978059
sample std. err.	4.09E-05	5.01E-05	6.22E-05	8.18E-05
(%)	0.1626%	0.7010%	0.0069%	0.0084%
Benchmark				
Closed form	0.025169	0.007149	0.903391	0.978084
Monte Carlo error (%)	0.0397%	-0.0419%	-0.0016%	-0.0025%
	Asian Floor	**Asian Cap**	**Simple Ratchet**	**Compound Ratchet**
RQMC estimate	0.008914	0.000651	0.841190	0.886559
sample std. err.	1.19E-04	5.22E-05	6.02E-04	8.53E-04
(% of the price)	1.3342%	8.0205%	0.0716%	0.0962%
Control Variate				
estimate	0.008919	0.000645	0.841250	0.886643
sample std. err.	9.92E-06	3.82E-06	5.59E-05	7.87E-05
(%)	0.1113%	0.5927%	0.0066%	0.0089%

Table 2: Benchmarking the RQMC procedure and the control variate technique. The Randomized Quasi-Monte Carlo procedure makes use of a combination of low-discrepancy and ordinary sampling. The points in the Sobol' sequence used for the low-discrepancy sampling have been randomized using the random shift rules. Since this randomization causes the points in the Sobol' sequence to lose their independence, the computation of the simulation error is based on 100 batches of independent stratified samples of 10,000 replications, for an overall budget of 1,000,000 iterations (see Boyle et al., 1991). Estimates obtained in correspondence of the benchmark parameter set.

Benchmarking to Monte Carlo procedures

In order to assess the quality of the approximation, we need a reliable benchmark; following the example of other contributions in the literature, such as Albrecher and Predota (2002, 2004), Ju (2002) and Turnbull and Wakeman (1991) for example, we use Monte Carlo simulation to generate these benchmark values. To reduce the standard error of the Monte Carlo estimate, we adopt stratification which naturally leads to the development of the simulation procedure in RQMC environment. Specifically, we adopt the Brownian-Gamma bridge algorithm of Ribeiro and Webber (2004) for the generation of the trajectories of the VG process; the actual implementation is the same as in Ballotta (2009). The Monte Carlo procedure is then benchmarked against the analytical formulae developed in section 2 for the ratchet EIA without index averaging (equations 9-11). As shown in Table 2, the relative error of the Monte Carlo estimate of the European vanilla options forming the ratchet EIA without index averaging is around 0.04%. Table 2 also reports the resulting prices of the Asian floor and cap options for the base set of parameters, together with the Monte Carlo standard error. We note that the size of the standard error (expressed as percentage of the estimated price) ranges from 1.33% for the case of the floor option to 8.02% for the case of the cap option.

In the attempt to make the analysis of the approximation error more robust, we try to reduce the error of the Monte Carlo estimates by coupling stratification with the control variate technique (see, for example, Boyle et al., 1997). As a control for the price of the Asian option, we use the price of the European vanilla calls given in equations (9)-(11); as shown in Table 2, the control variate helps to reduce the Monte Carlo error to 0.1113% for the case of the Asian floor option and 0.5927% for the case of the Asian cap option.

	Rates	**Strike**	A_0^{RQMC}	A_0^a	$R.E.$ %	A_0^{RQMCCV}	$R.E.$ %
	0.010	0.410	0.0165417353	0.0165291807	-0.07590%	0.0165424879	-0.08044%
	0.015	0.415	0.0143037378	0.0143119947	0.05773%	0.0143061920	0.04056%
	0.020	0.420	0.0123258946	0.0123051420	-0.16837%	0.0123241628	-0.15434%
	0.025	0.425	0.0105591449	0.0105079505	-0.48483%	0.0105589289	-0.48280%
Floor	0.030	0.430	0.0089201632	0.0089163071	-0.04323%	0.0089199701	-0.04107%
	0.035	0.435	0.0075502327	0.0075223906	-0.36876%	0.0075489218	-0.35146%
	0.040	0.440	0.0063249068	0.0063146488	-0.16218%	0.0063235953	-0.14148%
	0.045	0.445	0.0052979136	0.0053050479	0.13466%	0.0052979066	0.13480%
	0.050	0.450	0.0043975951	0.0043979833	0.00883%	0.0043974857	0.01132%
	0.055	0.455	0.0036473922	0.0036507730	0.09269%	0.0036482317	0.06719%
	0.060	0.460	0.0030206864	0.0030244079	0.12320%	0.0030212999	0.10287%
	0.065	0.465	0.0024804855	0.0025005961	0.81075%	0.0024805773	0.80702%
	0.070	0.470	0.0020667498	0.0020643255	-0.11730%	0.0020667107	-0.11541%
	0.075	0.475	0.0016952219	0.0017021226	0.40707%	0.0016947039	0.43776%
Cap	0.080	0.480	0.0013914541	0.0014022055	0.77268%	0.0013916630	0.75754%
	0.085	0.485	0.0011577470	0.0011543914	-0.28984%	0.0011576764	-0.28376%
	0.090	0.490	0.0009547590	0.0009499851	-0.50001%	0.0009542064	-0.44238%
	0.095	0.495	0.0007874575	0.0007926474	0.65908%	0.0007871813	0.69439%
	0.100	0.500	0.0006506186	0.0006534596	0.43666%	0.0006506890	0.42579%
	0.105	0.505	0.0005334706	0.0005290450	-0.82958%	0.0005334165	-0.81953%
	0.110	0.510	0.0004436795	0.0004443800	0.15789%	0.0004439259	0.10228%
	0.115	0.515	0.0003723934	0.0003752524	0.76774%	0.0003724741	0.74591%
	0.120	0.520	0.0003069391	0.0003104771	1.15268%	0.0003071021	1.09899%
	0.125	0.525	0.0002465731	0.0002432154	-1.36176%	0.0002465869	-1.36725%
	0.130	0.530	0.0002091782	0.0002065847	-1.23985%	0.0002089551	-1.13442%
	0.135	0.535	0.0001698374	0.0001708454	0.59352%	0.0001699000	0.55645%
	0.140	0.540	0.0001462038	0.0001464510	0.16909%	0.0001459839	0.31995%
	0.145	0.545	0.0001218233	0.0001216301	-0.15857%	0.0001216725	-0.03481%
	0.150	0.550	0.0001031555	0.0001031941	0.03744%	0.0001030740	0.11656%

Table 3: Pricing of the Asian Floor and Asian Cap options for the given parameter set. The moment-matching based estimate, the RQMC estimate and the estimate obtained by RQMC with control variate are labelled respectively $A^a, A^{RQMC}, A^{RQMCCV}$.

	Simple Ratchet			Compound Ratchet		
Floor	RQMC & CV	Approximation	*R.E.* %	RQMC & CV	Approximation	*R.E.* %
0.010	0.7685412049	0.7684117927	-0.01684%	0.7894688653	0.7893048181	-0.02078%
0.015	0.7846236425	0.7846009236	-0.00290%	0.8100959797	0.8100665018	-0.00364%
0.020	0.8022877298	0.8021311988	-0.01951%	0.8333095692	0.8331012563	-0.02500%
0.025	0.8213073101	0.8209983328	-0.03762%	0.8589723327	0.8585498157	-0.04919%
0.030	0.8412504055	0.8411760994	-0.00883%	0.8866430675	0.8865384973	-0.01179%
0.035	0.8627927602	0.8626146284	-0.02065%	0.9174317949	0.9171733098	-0.02817%
0.040	0.8853235758	0.8852402597	-0.00941%	0.9506595939	0.9505347526	-0.01313%
0.045	0.9091014639	0.9091292931	0.00306%	0.9868976238	0.9869407546	0.00437%
0.050	0.9336902412	0.9336721268	0.00194%	1.0256744988	1.0256454346	-0.00283%
Cap	RQMC & CV	Approximation	*R.E.* %	RQMC & CV	Approximation	*R.E.* %
0.050	0.8174270891	0.8172999620	0.01555%	0.8536798061	0.8535069055	0.02025%
0.055	0.8221915991	0.8220643854	-0.01547%	0.8601826053	0.8600084011	-0.02025%
0.060	0.8261249449	0.8260582656	-0.00807%	0.8655845534	0.8654927242	-0.01061%
0.065	0.8295829956	0.8293982371	-0.02227%	0.8703589443	0.8701032576	-0.02938%
0.070	0.8322615336	0.8321800213	-0.00979%	0.8740733684	0.8739601222	-0.01296%
0.075	0.8346187602	0.8344895289	-0.01548%	0.8773540141	0.8771738711	-0.02053%
0.080	0.8364588352	0.8364018848	-0.00681%	0.8799226166	0.8798430169	-0.00905%
0.085	0.8380285969	0.8379820172	-0.00556%	0.8821192302	0.8820539789	-0.00740%
0.090	0.8393694858	0.8392853693	-0.01002%	0.8839994823	0.8838814246	-0.01335%
0.095	0.8403649285	0.8402885988	-0.00908%	0.8853976669	0.8852903848	-0.01212%
0.100	0.8413534283	0.8411760994	-0.02108%	0.8867880687	0.8865384973	-0.02814%
0.105	0.8419007108	0.8419694019	0.00816%	0.8875587090	0.8876554770	0.01090%
0.110	0.8424954695	0.8425092498	0.00164%	0.8883968849	0.8884163135	0.00219%
0.115	0.8431705393	0.8429500268	-0.02615%	0.8893491041	0.8890379594	-0.03499%
0.120	0.8434473196	0.8433630524	-0.00999%	0.8897397817	0.8896208215	-0.01337%
0.125	0.8437684725	0.8437919319	0.00278%	0.8901932852	0.8902264207	0.00372%
0.130	0.8439968382	0.8440254996	0.00340%	0.8905158894	0.8905563858	0.00455%
0.135	0.8442122959	0.8442533834	0.00487%	0.8908203556	0.8908784274	0.00652%
0.140	0.8444981726	0.8443915275	-0.01263%	0.8912244761	0.8910737013	-0.01692%
0.145	0.8446264406	0.8445671940	-0.00701%	0.8914058518	0.8913220709	-0.00940%
0.150	0.8447550978	0.8446847471	-0.00833%	0.8915878112	0.8914883104	-0.01116%

Table 4: The price of the simple and compound ratchet Equity Indexed Annuity for the given parameter set. The value of the EIA is calculated using the estimated Asian option prices reported in Table 3 and equations (7)-(8). The table shows only the Monte Carlo estimates obtained by RQMC with control variate.

		Asian Option	**Simple Ratchet**	**Compound Ratchet**
RQMC	**RMSE**	0.001307%	0.011746%	0.016348%
	RMSRE	0.566349%	0.014133%	0.018698%
	MAE	0.00512%	0.03103%	0.04247%
	MRAE	1.36176%	0.03777%	0.04944%
RQMC & CV	**RMSE**	0.001277%	0.011655%	0.016188%
	RMSRE	0.552199%	0.014030%	0.018526%
	MAE	0.00510%	0.03090%	0.04225%
	MRAE	1.36725%	0.03762%	0.04919%

Table 5: Accuracy of the moment matching approximation procedure. Figures of merit: Root Mean Square Error (RMSE), RMS Relative Error (RMSRE), Maximum Absolute Error (MAE), Maximum Relative Absolute Error (MRAE).

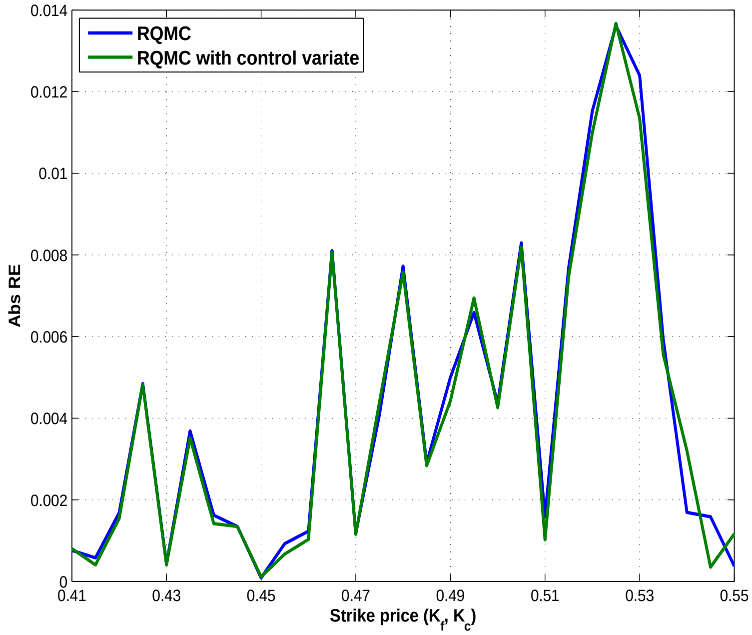

Figure 2: Asian option pricing problem. Relative error of the moment-matching procedure compared to the Monte Carlo method. The plot shows the absolute value of the index $R.E.$, emphasizing the magnitude of the error.

Pricing of the ratchet Equity Indexed Annuity

The moment-matching procedure developed in section 3 is tested by performing a sensitivity analysis of the Asian option prices with respect to the strike price (i.e. the floor and the cap rates). This choice is motivated by the previous observation that the approximating process Z does not fully capture the tail behaviour of the original returns average process, which could potentially affect the accuracy of deep in/out-of the money options. Due to obvious restrictions on the range of variation of the floor and cap rates, we assume that f varies between 1% and 5% with increments of 0.5%; similarly, c varies between 5% and 15% with increments of 0.5%. The corresponding Asian option prices obtained by moment-matching and by RQMC (pure and with control variate) are presented in Table 3, together with the relative error

$$R.E. = \frac{A_0^a - A_0^{RQMC}}{A_0^{RQMC}}.$$

The corresponding EIAs prices are reported in Table 4, whilst the comparison between the resulting relative errors (in absolute terms) is shown in Figure 2.

The results show an irregular pattern of the error of the proposed approximation procedure when compared to the RQMC estimate, although, in general, the accuracy seems to deteriorate in the case of out-of-the-money contracts. This is consistent with the previous findings regarding the accuracy of the chosen approximating process which worsens in the tails of the underlying distribution. Further, the error of the moment-matching approximation with respect to the RQMC estimate obtained using the control variate technique, is almost indistinguishable from the one generated by pure RQMC. This confirms the robustness of the approximating procedure developed in this paper. The average magnitude of the error for the case of the arithmetic Asian option is 0.41%, with a minimum value of 0.011% and a maximum value of 1.37%. The analysis of the overall ratchet EIA contract, shown in Table 4, shows instead that the relative magnitude of the approximation error is significantly smaller than in the case of the single Asian option component;

this feature is certainly due to a "compensation" effect occurring in the construction of the overall portfolio. The average size of the error for the simple ratchet and the compound ratchet are, in fact, 0.011% and 0.015% respectively, whilst the corresponding minimum and maximum values are $(0.0016\%, 0.0022\%)$ and $(0.038\%, 0.049\%)$. Further statistics confirming the accuracy of the moment matching approach are reported in Table 5; in particular, following Ju (2002), we report the root mean square error

$$RMSE = \sqrt{\frac{1}{m}\sum_{i=1}^{m}\left(A_0^a(i) - \hat{A}_0(i)\right)^2},$$

and the RMS relative error

$$RMSRE = \sqrt{\frac{1}{m}\sum_{i=1}^{m}\left(\frac{A_0^a(i) - \hat{A}_0(i)}{\hat{A}_0(i)}\right)^2},$$

where m denotes the number of option prices used and $\hat{A}_0(i)$ denotes the corresponding RQMC/RQMC with control variate estimate.

5 Conclusions

In this note, we developed a new approximation procedure for the pricing of arithmetic Asian options in a VG economy based on the moment-matching approach with an exponential VG distribution. The numerical evidence presented in section 4 shows that this algorithm represents a quick and fairly accurate alternative to Monte Carlo simulation-based pricing techniques. Alternative implementations of the Brownian-Gamma bridge have been explored for example by Avramidis and L'Ecuyer (2006); however, the efficiency of these methods depends on the particular construction of the time partition required for the generation of the trajectory of the driving process, and/or the number of time points in this partition. This might be a non suitable restriction for the pricing of contracts like the EIA, for which the number of reset dates is usually limited and, in general, not coincident, for example, with a dyadic partition as the one implemented by Avramidis and L'Ecuyer (2006).

In the light of the above discussion, current research work focuses on the assessment of the performance of the approximation method proposed in section 3, using as a benchmark other "non-random" pricing techniques, like approaches based on the Fourier transform and its inversion. Further research is currently also directed at testing the moment-matching procedure for in-the-money options as well, extend it to other Lévy processes and compare it against the approximation procedure developed by Albrecher and Predota (2002, 2004).

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A Pricing of the European call option in the VG economy

Consider a European call option with strike price K and maturity at time T, written on a stock S, whose dynamics under any risk neutral probability measure $\hat{\mathbb{P}}$ is described by equation (5). By risk neutral valuation, the price of this contracts is

$$C_0(S_0, K, T) = S_0 \mathbb{P}^{(S)}(S_T > K) - e^{-rT} K \hat{\mathbb{P}}(S_T > K),$$

where $\mathbb{P}^{(S)}$ is the stock-risk-adjusted probability measure defined by the density process

$$\gamma_t^{(S)} \doteq \frac{S_t}{e^{rt} S_0} = e^{\varpi t + X_t},$$

(see Geman et al., 1995). Using equation (5) and the properties of the VG process, it follows that

$$\begin{aligned}
\hat{\mathbb{P}}(S_T > K) &= \int_0^\infty \hat{\mathbb{P}}\left(X_T > \ln \frac{K}{S_0} - (r+\varpi) T \middle| \tau_T = g \right) \frac{g^{\frac{T}{k}-1} e^{-\frac{g}{k}}}{k^{\frac{T}{k}} \Gamma\left(\frac{T}{k}\right)} dg \\
&= \int_0^\infty \Phi\left(\frac{\ln \frac{S_0}{K} + (r+\varpi) T + \theta g}{\sigma \sqrt{g}} \right) \frac{g^{\frac{T}{k}-1} e^{-\frac{g}{k}}}{k^{\frac{T}{k}} \Gamma\left(\frac{T}{k}\right)} dg.
\end{aligned}$$

Let

$$\Psi(\xi, \chi, a, b) \doteq \int_0^\infty \Phi\left(\frac{\xi}{\sqrt{g}} + \chi \sqrt{g} \right) \frac{b^a g^{a-1} e^{-bg}}{\Gamma(a)} dg,$$

then

$$\begin{aligned}
\hat{\mathbb{P}}(S_T > K) &= \Psi\left(d, \frac{\theta}{\sigma}, \frac{T}{k}, \frac{1}{k} \right), \\
d &= \frac{\ln \frac{S_0}{K} + (r+\varpi) T}{\sigma}.
\end{aligned}$$

Further, we note that, under the probability measure $\mathbb{P}^{(S)}$, the characteristic function of the VG process X_t is given by

$$\begin{aligned}
\phi_X^{(S)}(u; t) &= e^{\varpi t} \hat{\mathbb{E}}\left(e^{(iu+1) X_t} \right) \\
&= \left(1 - iu \frac{(\theta + \sigma^2) k}{1 - \theta k - \frac{\sigma^2}{2} k} + u^2 \frac{\sigma^2}{2} \frac{k}{1 - \theta k - \frac{\sigma^2}{2} k} \right)^{-\frac{t}{k}},
\end{aligned}$$

which implies that

$$\begin{aligned}
X_t &= \theta^{(S)} \tau_t + \sigma W_{\tau_t}, \\
\theta^{(S)} &= \theta + \sigma^2,
\end{aligned}$$

where W_t is a $\mathbb{P}^{(S)}$-standard Brownian motion and τ_t is a $\mathbb{P}^{(S)}$-Gamma process with time scale $1/k$ and decay rate

$$k^{(S)} = \frac{k}{1 - \theta k - \frac{\sigma^2}{2} k}.$$

These results are consistent with Ballotta (2009) and Hubalek and Sgarra (2006), as the spot probability measure is an Esscher measure with unit parameter. Hence

$$\begin{aligned}
\mathbb{P}^{(S)}(S_T > K) &= \int_0^\infty \Phi\left(\frac{\ln \frac{S_0}{K} + (r+\varpi) T + \theta^{(S)} g}{\sigma \sqrt{g}} \right) \frac{g^{\frac{T}{k}-1} e^{-\frac{g}{k^{(S)}}}}{\left(k^{(S)}\right)^{\frac{T}{k}} \Gamma\left(\frac{T}{k}\right)} dg \\
&= \Psi\left(d, \frac{\theta + \sigma^2}{\sigma}, \frac{T}{k}, \frac{1-s}{k} \right),
\end{aligned}$$

for $s = k\left(\theta + \frac{\sigma^2}{2}\right)$. Therefore, the price of the given option is

$$C_0\left(S_0, K, T\right) = S_0 \Psi\left(d, \frac{\theta + \sigma^2}{\sigma}, \frac{T}{k}, \frac{1-s}{k}\right) - e^{-rT} K \Psi\left(d, \frac{\theta}{\sigma}, \frac{T}{k}, \frac{1}{k}\right). \tag{A.1}$$