Optimal capital allocation in a hierarchical corporate structure

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Abstract

We consider capital allocation in a hierarchical corporate structure where stakeholders at two organizational levels (e.g. board members vs line managers) may have conflicting objectives, preferences, and beliefs about risk. Capital allocation is considered as the solution to an optimization problem whereby a quadratic deviation measure between individual losses (at both levels) and allocated capital amounts is minimized. Thus, this paper generalizes the framework of Dhaene et al. [5], by allowing potentially diverging risk preferences in a hierarchical structure. An explicit unique solution to this optimization problem is given. In several examples, it is shown how the optimal capital allocation achieves a compromise between conflicting views of risk within the organization.

Keywords: Capital allocation, Solvency II, Basel II, weighted capital allocation, hierarchical firms.

1 Introduction

Capital allocation is an exercise whereby the total amount of economic capital available to an insurance or financial institution is apportioned to individual sub-portfolios, such as business divisions, lines of business, distinct legal entities (as in the case of an insurance group), or individual contracts. Such allocation of capital may be purely notional or involve an actual transfer of funds, depending on fungibility constraints. The purposes of capital allocation can include performance measurement, assessment of investment opportunities, portfolio management, and even incentive compensation.

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While a host of capital allocation methods are described in the literature, the underlying principle is typically that the capital allocated to a particular risk should in some way reflect the contribution of that risk to the portfolio, often as captured by a risk measure. There are multiple ways of defining such contributions. Marginal cost arguments are used by Tasche [13] in the context of performance measurement, while related game theoretical criteria emphasize principles of fairness and stability in the portfolio (Denault [3], Tsanakas and Barnett [14]). Capital allocation in models with hierarchically structured dependence has been considered by Arbenz et al. [2]. Capital allocation methods derived from various notions of optimality are investigated in Dhaene et al. [4], Laeven and Goovaerts [8], Zaks et al. [16], and Dhaene et al. [5].

These last two papers are most closely related to the present contribution. In particular, Dhaene et al. [5] formulate capital allocation as an optimization problem, where the available capital is exogenously given and the objective function is formed by summing the distances of allocated capital amounts from individual losses. Distance is measured by expected (quadratic or absolute) deviations, after a re-weighting of probabilities, which assigns higher weights to scenarios of higher relevance. It is shown that appropriate choice of such scenario weights, reflecting management preferences for different parts of the portfolio, can generate a wide variety of capital allocations, reproducing most of the allocation methods found in the literature. Zaks [17] generalized such arguments to a situation case where capital is invested in a number of risky assets.

An important practical issue that is generally ignored in the above literature relates to the potentially conflicting objectives, preferences and beliefs at different levels of a financial institution's hierarchy, for example the levels of a company board and line managers. Such conflicts may take different forms, reflecting different organizational structures and cultures. Preferences / scenario weights may be solvency-driven at board level and price sensitive at line-of-business level (or indeed the converse may be true). Boards may be concerned with overall portfolio performance, while line managers focus on the performance of the books they are managing. Even when preferences are consistent, the beliefs about loss probability distributions may differ, for example reflecting the specific expertise that line managers have in relation to the liabilities they are managing. For ways in which organizational design influences the allocation of capital to competing investment projects (a problem indirectly related to what is studied here), see Stein [12] and the references therein.

In this paper, we address the above issues by generalizing the argument of Dhaene et al. [5] in a hierarchical setting. An augmented objective function is proposed, involving quadratic deviations between loss and allocated capital at different levels of the organization's hierarchy.

There is enough flexibility in the selection of possible scenario weights, to reflect divergent risk preferences by the same stakeholder for different parts of the portfolio, as well as by different stakeholders in relation to the same part of the portfolio. Hence the conflicting views of, say, company board and line managers can be accommodated in a single framework.

An explicit unique solution to the above optimization is derived. This leads to explicit formulas for the optimally allocated capital at different (top and bottom) levels of the hierarchy. Capital allocation becomes a two-step procedure. First, capital allocated at the top level under consideration (e.g. to lines of business) is driven by a combination of preferences at both levels. Second, the allocation of those capitals at the bottom level (e.g. to individual policies) is only driven by bottom-level preferences. Thus, while board-level preferences impact on the capital available to line managers, there is no interference from above in the allocation of capital within lines of business.

Special cases where the formulas simplify are considered and a number of detailed examples are given. It is beyond the scope of this paper to complicate the rather exhaustive discussion of Dhaene et al. [5], by considering a multitude of combinations of diverging preferences at different levels. The examples are chosen to highlight particularly pertinent cases of conflicting objectives, preferences and beliefs at different levels of a financial institution's hierarchy, as discussed above. We focus on how the optimal capital allocation derived attempts to resolve these conflicts and achieve a compromise view of risk.

The formal setting and main result are given in Section 2, while special cases and examples are discussed in Section 3. The proof of the main result is presented in Section 4. Finally, brief conclusions are given in Section 5.

2 Optimal capital allocations

2.1 Set-up

We consider a financial institution with n portfolios, where the i^{th} portfolio is in turn divided into n_i sub-portfolios. Several situations fit this hierarchical setting, for example (i) an insurance group consisting of n legal entities, each writing n_i lines of business; (ii) an insurance company active in n lines of business, in each of which n_i (groups of similar) policies are sold; (iii) a financial institution exposed to n types of risk as defined by solvency regulation (market, credit, operational etc), each of which is decomposed into n_i sources of exposure.

The loss arising from the i^{th} portfolio is denoted by the random variable X_i for $i = 1, \ldots, n$. The loss arising from the j^{th} sub-portfolio of the i^{th} portfolio is denoted by the random variable X_{ij} for $j = 1, \ldots, n_i$. Note that we do not in general require that $\sum_{j=1}^{n_i} X_{ij} = X_i$, though the simplifying assumption is used in the examples of Section 3. That allows for the presence of portfolio non-linearities, as well as the inclusion in X_i of deadweight costs or risks to which no capital will be allocated[1].

We assume that an exogenously given total amount of capital K is available. This will be allocated to the n portfolios by $\boldsymbol{K} = (K_1, ..., K_n)$, where the *top-level capitals* add up to the total available capital, $\sum_{i=1}^{n} K_i = K$. In turn, each K_i will be allocated to n_i sub-portfolios via $\boldsymbol{k}_i = (k_{i1}, ..., k_{in_i})$, where the *bottom-level capitals* add up to K_i, i.e. $\sum_{j=1}^{n_i} k_{ij} = K_i$. Denote the $\sum_{i=1}^{n} n_i$-vector of bottom-level capitals as $\boldsymbol{k} = (\boldsymbol{k}_1, \ldots, \boldsymbol{k}_n)$.

Consistently with the arguments of Zaks et al. [16] and Dhaene et al. [5], the capital allocation will be derived from the general principle that *the capital allocated to a risk should be close to it, according a measure of distance that reflects management preferences.* In particular, capital allocation in this paper arises as the solution to the following optimization problem:

$$\begin{cases} \min\limits_{\boldsymbol{K},\boldsymbol{k}} & \left\{ (1-\lambda) \sum\limits_{i=1}^{n} \frac{1}{\nu_i} \mathbb{E}\left[\xi_i \left(K_i - X_i\right)^2\right] + \lambda \sum\limits_{i=1}^{n} \sum\limits_{j=1}^{n_i} \frac{1}{\nu_{ij}} \mathbb{E}\left[\xi_{ij} \left(k_{ij} - X_{ij}\right)^2\right] \right\} \\ s.t. & \sum\limits_{i=1}^{n} K_i = K \\ & \sum\limits_{j=1}^{n_i} k_{ij} = K_i \quad \forall i = 1, \ldots, n, \ j = 1, \ldots, n_i, \end{cases} \tag{1}$$

where the distance measures used are built with the following elements:

- A *quadratic deviation measure*, consistent with the common use of quadratic loss functions in insurance, see e.g. Lemaire [9].
- Measures of *business volume*, $\nu_i > 0,\ \nu_{ij} > 0$ corresponding to $X_i,\ X_{ij}$ respectively.
- *Scenario weights* ξ_i, ξ_{ij}, corresponding to X_i, X_{ij} respectively. Each of ξ_i, ξ_{ij} is a non-negative random variable with $\mathbb{E}[\xi_i] = \mathbb{E}[\xi_{ij}] = 1$. These weights reflect an assessment that certain scenarios (states of the world) may be more relevant as drivers of capital than others. Depending on management preferences, the variables ξ_i, ξ_{ij} may assign a higher weight on scenarios where particular (sub-)portfolios incur high losses or where market conditions are adverse. For a full discussion and several examples see Dhaene et al. [5]. A key difference in this paper is that the ξ_i, ξ_{ij} are generally *not* the same, reflecting differently defined risk preferences at different (top and bottom) levels of the organization's hierarchy.
- A constant $0 < \lambda < 1$ that reflects the balance between top-level preferences (low λ) and bottom-level preferences (high λ).

[1]For example, if capital allocation is used to derive profitability targets, there may be no allocated capital to forms of operational risk, that are not directly associated with profit-making.

2.2 Main result

For the case $\lambda = 0$ the solution to the optimization problem (1) is given by Dhaene et al. [5], with the obvious difference that optimization is over $\boldsymbol{K}$ only and the second constraint does not apply. For completeness and to draw comparisons we state the solution to that simpler problem:

$$K_i^* = \mathbb{E}[\xi_i X_i] + \frac{\nu_i}{\sum_{r=1}^n \nu_r} \left(K - \sum_{r=1}^n \mathbb{E}[\xi_r X_r] \right). \tag{2}$$

Thus, the capital allocated to X_i is equal to its weighted expectation with respect to ξ_i, plus a proportional share, according to business volume, of the excess of available capital K over the weighted expectations of all portfolio losses.

The following result shows how the allocation formula (2) is generalized, when potentially conflicting preferences are present at different levels of the organization's hierarchy. As the proof is lengthy and requires the introduction of multiple notations, it is delegated to Section 4.

Theorem 1. *Let* $0 < \lambda < 1$. *The optimization problem* (1) *has the unique solution*

$$\begin{aligned}
K_i^* &= \tilde{w}_i \mathbb{E}\left[\xi_i X_i\right] + w_i \sum_{j=1}^{n_i} \mathbb{E}\left[\xi_{ij} X_{ij}\right] \\
&\quad + \frac{\nu_i \tilde{w}_i}{\sum_{r=1}^n \nu_r \tilde{w}_r} \left(K - \sum_{r=1}^n \tilde{w}_r \mathbb{E}\left[\xi_r X_r\right] - \sum_{r=1}^n w_r \sum_{j=1}^{n_r} \mathbb{E}\left[\xi_{rj} X_{rj}\right] \right), && (3)\\
k_{ij}^* &= \mathbb{E}\left[\xi_{ij} X_{ij}\right] + \frac{\nu_{ij}}{\sum_{s=1}^{n_i} \nu_{is}} \left(K_i^* - \sum_{s=1}^{n_i} \mathbb{E}\left[\xi_{is} X_{is}\right] \right), && (4)
\end{aligned}$$

where

$$w_i = \frac{\lambda \nu_i}{(1-\lambda) \sum_{s=1}^{n_i} \nu_{is} + \lambda \nu_i}, \quad \tilde{w}_i = 1 - w_i. \tag{5}$$

Glancing at (4) it is seen that the allocated capitals at the bottom level k_{ij}^* are given by a formula conceptually similar to (2): the allocated capital to X_{ij} is given by its weighted expectation $\mathbb{E}[\xi_{ij} X_{ij}]$ plus a proportional share of the excess of the optimal capital K_i^* available for the portfolio that X_{ij} belongs to, over the sum of weighted expected losses in that portfolio. However, a key difference is that K_i^* is now endogenously given by the solution of optimization problem.

K_i^* in (3) is still given by a formula that obeys the general structure of "weighted expectation plus share of excess capital". However, the weighted expectation now reflects *both* the preferences at both the top level (via $\mathbb{E}[\xi_i X_i]$) and the bottom level (via $\sum_{j=1}^{n_i} \mathbb{E}[\xi_{ij} X_{ij}]$). The weights $w_i, \tilde{w}_i$, depending on λ and the business volumes ν_i, ν_{ij}, determine which of the two levels will dominate the capital allocation.

Theorem 1 gives explicit formulas for the capital allocation for any specification of business volumes and scenario weights. In Section 3, two special cases are treated, for which more transparent and easily interpretable formulas can be obtained.

2.3 Limiting cases

Since the value of λ controls the balance between top- and bottom-level preferences, it is useful to consider what happens to the solution of optimization problem (1) in the limiting cases $\lambda = 0$ and $\lambda = 1$, which are not formally covered by Theorem 1.

First consider the case that $\lambda \to 0$, corresponding to the top-level preferences dominating. Then we have

$$\lim_{\lambda \to 0} w_i = 0, \quad \lim_{\lambda \to 0} \tilde{w}_i = 1 \implies \lim_{\lambda \to 0} \frac{\nu_i \tilde{w}_i}{\sum_{r=1}^{n} \nu_r \tilde{w}_r} = \frac{\nu_i}{\sum_{r=1}^{n} \nu_r}. \tag{6}$$

Substituting these quantities into the equation (3), we obtain

$$\lim_{\lambda \to 0} K_i^* = \mathbb{E}[\xi_i X_i] + \frac{\nu_i}{\sum_{r=1}^{n} \nu_r} \left(K - \sum_{r=1}^{n} \mathbb{E}[\xi_r X_r] \right), \tag{7}$$

such that the top-level preferences only are considered in the determination of the optimal capital K_i^*. Of course this is identical to the capital allocation (2) in Dhaene et al. [5]. Notice that equation (4) is unaffected by taking the limit; the allocation of a given K_i^* to k_{ij}^* is still determined from bottom-level preferences only. Thus the capital allocation problems for the two organizational levels are essentially completely decomposed.

In the case that $\lambda \to 1$, with bottom-level preferences dominating, first note that from the definition of $w_i, \tilde{w}_i$ it is $\nu_i \tilde{w}_i/(1-\lambda) = \sum_{s=1}^{n_i} \nu_{is} w_i/\lambda$, such that

$$\frac{\nu_i \tilde{w}_i}{\sum_{r=1}^{n} \nu_r \tilde{w}_r} = \frac{\sum_{s=1}^{n_i} \nu_{is} w_i}{\sum_{r=1}^{n} \sum_{s=1}^{n_r} \nu_{rs} w_r}. \tag{8}$$

Then, we have

$$\lim_{\lambda \to 1} w_i = 1, \quad \lim_{\lambda \to 1} \tilde{w}_i = 0 \implies \lim_{\lambda \to 1} \frac{\sum_{s=1}^{n_i} \nu_{is} w_i}{\sum_{r=1}^{n} \sum_{s=1}^{n_r} \nu_{rs} w_r} = \frac{\sum_{s=1}^{n_i} \nu_{is}}{\sum_{r=1}^{n} \sum_{s=1}^{n_r} \nu_{rs}}, \tag{9}$$

such that (3) now becomes

$$\lim_{\lambda \to 1} K_i^* = \sum_{j=1}^{n_i} \mathbb{E}\left[\xi_{ij} X_{ij}\right] + \frac{\sum_{s=1}^{n_i} \nu_{is}}{\sum_{r=1}^{n} \sum_{s=1}^{n_r} \nu_{rs}} \left(K - \sum_{r=1}^{n} \sum_{j=1}^{n_r} \mathbb{E}\left[\xi_{rj} X_{rj}\right] \right) \tag{10}$$

and (4) again remains unchanged. Hence, the allocation is now completely driven by bottom-up preferences.

3 Special cases and examples

3.1 Consistent business volume measures

A simplification to the formula (3) for K_i^* occurs if we assume that the way that business volume is measured within the organization is consistent across different levels of the hierarchy.

$$\nu_i = \sum_{j=1}^{n_i} \nu_{ij}, \quad \text{for all} \quad i = 1, \ldots, n, \tag{11}$$

and denote $\nu = \sum_{i=1}^{n} \nu_i$. Such consistency is reasonable when business volume is represented by an additive measure such as premium income or expected loss.

Notice that (11) leads to $w_i = \lambda$, $\tilde{w}_i = 1 - \lambda$, for all $i = 1, \ldots, n$. Therefore, the solution of optimization problem (1) becomes

$$\begin{aligned} K_i^* &= (1-\lambda)\mathbb{E}\left[\xi_i X_i\right] + \lambda \sum_{j=1}^{n_i} \mathbb{E}\left[\xi_{ij} X_{ij}\right] \\ &\quad + \frac{\nu_i}{\nu}\left(K - (1-\lambda)\sum_{r=1}^{n} \mathbb{E}\left[\xi_r X_r\right] - \lambda \sum_{r=1}^{n}\sum_{j=1}^{n_r} \mathbb{E}\left[\xi_{rj} X_{rj}\right]\right) \qquad (12) \\ k_{ij}^* &= \mathbb{E}\left[\xi_{ij} X_{ij}\right] + \frac{\nu_{ij}}{\nu_i}\left(K_i^* - \sum_{s=1}^{n_i} \mathbb{E}\left[\xi_{is} X_{is}\right]\right). \qquad (13) \end{aligned}$$

A further simplification occurs when there is consistency between the risk assessment at the top and bottom levels, in the sense that $\sum_{j=1}^{n_i} \mathbb{E}[\xi_{ij} X_{ij}] = \mathbb{E}[\xi_i X_i]$ for each $i = 1, \ldots, n$. A natural sufficient condition for this to let $X_i = \sum_{j=1}^{n_i} X_{ij}$ and $\xi_i = \xi_{ij}$ for all $j = 1, \ldots, n_i$. Then, simple manipulations yield

$$K_i^* = \mathbb{E}\left[\xi_i X_i\right] + \frac{\nu_i}{\nu}\left(K - \sum_{r=1}^{n} \mathbb{E}\left[\xi_r X_r\right]\right), \tag{14}$$

which replicates Dhaene et al. [5].

In the following three examples, different corporate structures, risk preferences and beliefs are considered in the context of a consistent measure of business volume. In all examples we consider additivity of risks in the sense that

$$X_i = \sum_{j=1}^{n_i} X_{ij}, \quad \text{for all} \quad j = 1, \ldots, n_i. \tag{15}$$

We also denote the total risk by $X = \sum_{i=1}^{n} X_i$.

Example 1. Here we consider the case of an insurance group consisting of n legal entities (companies), each exposed to loss X_i, $i = 1, \ldots, n$, and sub-divided into lines of business with losses X_{ij}, $i = 1, \ldots, n$.

We assume that the risk preferences of each company's management are driven by a (possibly regulatory) risk measure. Formally, we consider weights $\xi_i = h_i(X_i)$, with $\mathbb{E}[h_i(X_i)] = 1$. This implies that the scenarios of interest for the i^{th} entity are weighted in terms of outcomes of X_i only. Thus, consistently with Furman and Zitikis [6] and Dhaene et al. [5], $h_i(X_i)$ defines a risk measure ρ_i by

$$\mathbb{E}[\xi_i X_i] = \mathbb{E}[h_i(X_i)X_i] = \rho_i(X_i). \tag{16}$$

Several common risk measures can be constructed in this way. For example, one can let $h_i(X_i) = \frac{1}{1-p_i}\mathbf{1}\{X_i > Q_{p_i}(X_i)\}$, where we denote by $\mathbf{1}\{A\}$ the indicator function of the event A and by $Q_p(Y)$ the $100p^{th}$ percentile of a random variable Y. In that case (assuming continuity of distributions for simplicity), $\rho_i(X_i) = \mathbb{E}[X_i|X_i > Q_{p_i}(X_i)]$ is the *Tail-Value-at-Risk (TVaR)* risk measure of X_i at security level p_i.

On the other hand, we assume that the preferences of each line manager within an entity are market driven, with profitability rather than solvency being a primary concern. This can be modeled by letting $\xi_{ij} = \zeta$ for all $i = 1, \ldots, n,\ j = 1, \ldots, n_i$, where ζ is a market pricing kernel. In particular, it is implied that

$$\begin{aligned} &\mathbb{E}[\xi_{ij}X_{ij}] = \mathbb{E}[\zeta X_{ij}] = \pi(X_{ij}), \\ &\sum_{j=1}^{n_i} \mathbb{E}\left[\xi_{ij}X_{ij}\right] = \mathbb{E}[\zeta X_i] = \pi(X_i) \\ &\sum_{r=1}^{n}\sum_{j=1}^{n_r} \mathbb{E}\left[\xi_{rj}X_{rj}\right] = \mathbb{E}[\zeta X] = \pi(X). \end{aligned} \tag{17}$$

Due to the linearity of the functional $\pi(\cdot)$ it is convenient to define business volumes by

$$\nu_i = \pi(X_i), \quad \nu_{ij} = \pi(X_{ij}). \tag{18}$$

Under the above assumptions, equation (13) immediately becomes

$$\begin{aligned} k_{ij}^* &= \pi(X_{ij}) + \frac{\pi(X_{ij})}{\pi(X_i)}\left(K_i^* - \pi(X_i)\right) \\ \Leftrightarrow \frac{k_{ij}^* - \pi(X_{ij})}{\pi(X_{ij})} &= \frac{K_i^* - \pi(X_i)}{\pi(X_i)} \end{aligned} \tag{19}$$

Therefore, it is implied that for each business line, the *solvency ratio* under the optimal capital allocation is equal to the solvency ratio of the entity as a whole.

After some simple but tedious algebra that is not repeated here, (12) can be written as

$$\frac{K - \pi(X)}{\pi(X)} - \frac{K_i^* - \pi(X_i)}{\pi(X_i)} = (1-\lambda)\left(\frac{\sum_{r=1}^{n}\rho_r(X_r) - \pi(X)}{\pi(X)} - \frac{\rho_i(X_i) - \pi(X_i)}{\pi(X_i)}\right) \tag{20}$$

The left hand side of (20) represents the difference between the solvency ratios at the group and legal entity levels, under the optimal capital allocation. The right hand side involves the difference between the solvency ratios at the group and company levels, if capital is set for each company according to the risk measure that represents the preferences of its management. Since $1-\lambda < 1$, the difference between the solvency ratios under the optimal allocation is always smaller than the difference between the solvency rations calculated using risk measures. For λ approaching 1, the preferences of company management are ignored in favor of the preferences of line managers and $\frac{K-\pi(X)}{\pi(X)}$ approaches $\frac{K_i^*-\pi(X_i)}{\pi(X_i)}$. □

Example 2. So far we have assumed that the total available capital K is exogenously given. However, in several applications the capital is itself driven by a risk measure, as is the case for regulatory capital requirements. To illustrate this point consider the case of a single insurance company, with n lines of business, producing losses $X_i,\ i = 1,\dots,n$. The i^{th} line of business is divided into n_i (groups of) policies, each with loss $X_{ij},\ j = 1,\dots,n_i$.

We assume that the company's board agrees to a set of scenario weights that are of relevance across the company and are driven by the aggregate company loss. Hence it is required that

$$\xi_i = \zeta = h(X) \text{ for all } i = 1,\dots,n. \tag{21}$$

The total capital held is given as the expected total loss under the weighting specified by the board, that is,

$$K = \mathbb{E}[\zeta X]. \tag{22}$$

On the other hand, each line manager also specifies scenario weights that are the same for all policies she is in charge of and are driven by the loss of that line of business only, such that

$$\xi_{ij} = \zeta_i = h_i(X_i) \implies \sum_{j=1}^{n_i} \mathbb{E}[\xi_{ij} X_{ij}] = \mathbb{E}[\zeta_i X_i], \text{ for all } j = 1,\dots,n_i. \tag{23}$$

Under these assumptions, simple manipulations of (12) and (13) yield:

$$K_i^* = (1-\lambda)\mathbb{E}[\zeta X_i] + \lambda \mathbb{E}[\zeta_i X_i] + \lambda \frac{\nu_i}{\nu}\left(\mathbb{E}[\zeta X] - \sum_{r=1}^{n} \mathbb{E}[\zeta_r X_r]\right) \tag{24}$$

$$k_{ij}^* = \mathbb{E}[\zeta_i X_{ij}] + (1-\lambda)\frac{\nu_{ij}}{\nu_i}\left(\mathbb{E}[\zeta X_i] - \mathbb{E}[\zeta_i X_i]\right) + \lambda \frac{\nu_{ij}}{\nu}\left(\mathbb{E}[\zeta X] - \sum_{r=1}^{n} \mathbb{E}[\zeta_r X_r]\right) \tag{25}$$

Hence K_i^* arises as linear combination of the weighted averages of X_i under board and line manager risk preferences, plus an adjustment term reflecting the inconsistency of those two views in their assessment of the aggregate risk. k_{ij}^* is the average loss of X_{ij} under the scenarios specified by the i^{th} line manager, plus two adjustment terms, reflecting inconsistencies in the assessment of the risk X_i and of the aggregate risk X. Note that $\mathbb{E}[\zeta X_i]$ represents the part of risk $\mathbb{E}[\zeta X]$ allocated to X_i, while $\mathbb{E}[\zeta_i X_{ij}]$ represents the part of risk $\mathbb{E}[\zeta_i X_i]$ allocated to X_{ij}, in the sense of the weighted capital allocation principle of Furman and Zitikis [7].

To make matters more concrete, assume that at both the aggregate and line-of-business level, risk is measured with Tail-Value-at-Risk (TVaR) measure at respected security levels $p,\ q$. This implies the use of weights

$$\zeta = \frac{1}{1-p}\mathbf{1}\{X > Q_p(X)\}, \quad \zeta_i = \frac{1}{1-q}\mathbf{1}\{X > Q_q(X_i)\}. \tag{26}$$

Hence, only adverse scenarios for X and X_i respectively are considered for capital allocation purposes. The total capital is given by

$$K = \mathbb{E}[X|X > Q_p(X)], \tag{27}$$

which is the TVaR measure of X at security level p.

The allocated capital to X_i is

$$K_i^* = (1-\lambda)\mathbb{E}[X_i|X > Q_p(X)] + \lambda\mathbb{E}[X_i|X_i > Q_q(X_i)] \tag{28}$$

$$+ \lambda\frac{\nu_i}{\nu}\left(\mathbb{E}[X|X > Q_p(X) - \sum_{r=1}^{n}\mathbb{E}[X_r|X_r > Q_q(X_r)]\right). \tag{29}$$

Here, the first term involves a marginal TVaR-based allocation of the total risk to X_i (see e.g. Overbeck [10]), while the second term involves the TVaR measure of X_i at level q. The two terms give different types of information. $\mathbb{E}[X_i|X > Q_p(X)]$ quantifies the marginal contribution of X_i to the risk of the portfolio X, taking into account the diversification of X_i against other lines of business, whereas $\mathbb{E}[X_i|X_i > Q_q(X_i)]$ captures the stand-alone risk of X_i, without explicitly considering diversification effects. The former view is important from a portfolio management perspective, but the latter view is useful in order to give clear incentives to line managers.

Note that the third term in (28) can be made to be zero, by judicious choice of p, q. If $p = 0.99$, consistently with regulatory practice (e.g. in the context of the Swiss Solvency Test [11]) then some choice of $q^* < p$ is possible that leads to

$$\mathbb{E}[X|X > Q_p(X)] = \sum_{r=1}^{n}\mathbb{E}[X_r|X_r > Q_{q^*}(X_r)], \tag{30}$$

in a logical extension of the *haircut* allocation principle discussed in Dhaene et al. [5].

Assuming that $q = q^*$ is chosen as above, the allocation to X_{ij} is

$$k_{ij}^* = \mathbb{E}[X_{ij}|X_i > Q_{q^*}(X_i)] + (1-\lambda)\frac{\nu_{ij}}{\nu_i}\left(\mathbb{E}[X_i|X > Q_p(X)] - \mathbb{E}[X_i|X_i > Q_{q^*}(X_i)]\right). \tag{31}$$

Here, the first term involves the marginal contribution of X_{ij} to the TVaR of X_i, while the second term reflects the inconsistency between the stand-alone and marginal risk assessments of X_i. Note that this adjustment term could also be made to vanish if a different choice of security level q_i was allowed for each line of business. However this would create obvious problems of equity between lines of business and is therefore not further considered here. $\square$

Example 3. In this example we consider the case of an insurance company, where the risk preferences of the company board and of line managers are consistent, but there are differences in their *beliefs* about loss distributions.

Assume for simplicity that all random variables considered have a continuous distribution and denote by f_i, F_i the density and distribution function of X_i respectively, as formulated by the model used by the company's central risk function, which is taken to also represent the beliefs of the board. It is agreed across the company that scenario weights should be determined for each line of business reflecting the loss of that line of business only. In particular the scenario weights are defined as

$$\xi_i = \phi\left(F_i(X_i)\right), \ i = 1, \dots, n, \tag{32}$$

where ϕ is a non-negative, non-decreasing function with $\int_0^1 \phi(t) = 1$. Such scenarios generate a *distortion* or *spectral risk measure* (see Wang [15], Acerbi [1]), by

$$\rho_\phi(X_i) = \mathbb{E}[\xi_i X_i] = \int_{x \in \mathcal{S}_i} \phi(F_i(x)) x f_i(x) dx, \tag{33}$$

where $\mathcal{S}_i$ is the support of f_i. The TVaR measure considered earlier is well known to arise as a special case (e.g Acerbi [1]).

Line managers agree to use the same principle of risk measurement for all their policies. However, given their local expertise, they potentially disagree about the assessment of the distribution of the losses considered. In particular, the manager of the i^{th} portfolio believes that the density and distribution of X_i are $\tilde{f}_i, \tilde{F}_i$ respectively, again with support $\mathcal{S}_i$. Hence he wants to measure the risk of his portfolio by

$$\int_{x \in \mathcal{S}_i} \phi(\tilde{F}_i(x)) x \tilde{f}_i(x) dx = \int_{x \in \mathcal{S}_i} \phi(\tilde{F}_i(x)) x \frac{\tilde{f}_i(x)}{f_i(x)} f_i(x) dx = \mathbb{E}\left[\phi(\tilde{F}_i(X_i)) \frac{\tilde{f}_i(X_i)}{f_i(X_i)} X_i\right], \tag{34}$$

where the expectation is calculated using the model F_i favored by the central risk function. This motivates the definition of the following scenario weights by line managers

$$\xi_{ij} = \tilde{\xi}_i = \phi(\tilde{F}_i(X_i)) \frac{\tilde{f}_i(X_i)}{f_i(X_i)}, \ i = 1, \dots, n, \ j = 1, \dots, n_i. \tag{35}$$

Denote now

$$\zeta_i = (1-\lambda)\xi_i + \lambda\tilde{\xi}_i, \implies (1-\lambda)\mathbb{E}\left[\xi_i X_i\right] + \lambda \sum_{j=1}^{n_i} \mathbb{E}\left[\xi_{ij} X_{ij}\right] = \mathbb{E}[\zeta_i X_i], \tag{36}$$

where ζ_i are scenario weights produced as a compromise between the beliefs of the company board and the manager of the i^{th} line. Consequently, equations (12) and (13) become

$$K_i^* = \mathbb{E}[\zeta_i X_i] + \frac{\nu_i}{\nu}\left(K - \sum_{r=1}^{n} \mathbb{E}\left[\zeta_r X_r\right]\right) \tag{37}$$

$$k_{ij}^* = \mathbb{E}\left[\tilde{\xi}_i X_{ij}\right] + \frac{\nu_{ij}}{\nu_i}\left(K_i^* - \mathbb{E}\left[\tilde{\xi}_i X_i\right]\right). \tag{38}$$

Thus the capital allocation formulas take their simplest form, structurally similar to (2) appearing in Dhaene et al. [5]. However, the capital K_i^* is calculated under the compromise scenario weights, while K_i^* is allocated to k_{ij}^* according to weights reflecting the line managers' beliefs only.

$\square$

3.2 Proportional allocations

A somewhat different simplification to the results of Theorem 1 occurs if the business volumes are risk adjusted, in the sense that they are chosen to be equal to weighted expected losses, such that

$$\nu_i = \mathbb{E}[\xi_i X_i], \; i = 1, \ldots, n \tag{39}$$

$$\nu_{ij} = \mathbb{E}[\xi_{ij} X_{ij}], \; j = 1, \ldots, n_i. \tag{40}$$

Define the constants

$$\eta_i = \frac{\nu_i \sum_{j=1}^{n_i} \nu_{ij}}{(1-\lambda)\sum_{j=1}^{n_i} \nu_{ij} + \lambda \nu_i} = \frac{\mathbb{E}[\xi_i X_i] \sum_{j=1}^{n_i} \mathbb{E}[\xi_{ij} X_{ij}]}{(1-\lambda)\sum_{j=1}^{n_i} \mathbb{E}[\xi_{ij} X_{ij}] + \lambda \mathbb{E}[\xi_i X_i]}, \tag{41}$$

for $i = 1, \ldots, n$. Then the quantities appearing in Theorem 1 become:

$$w_i = \frac{\lambda \eta_i}{\sum_{j=1}^{n_i} \nu_{ij}}, \quad \tilde{w}_i = \frac{(1-\lambda)\eta_i}{\nu_i}, \quad \frac{\nu_i \tilde{w}_i}{\sum_{r=1}^{n} \nu_r \tilde{w}_r} = \frac{\eta_i}{\sum_{r=1}^{n} \eta_r}. \tag{42}$$

Substituting into equation (3) yields

$$K_i^* = \eta_i + \frac{\eta_i}{\sum_{r=1}^{n} \eta_r} \left(K - \sum_{r=1}^{n} \eta_r \right). \tag{43}$$

Consequently, simple algebra allows us to write equations (3) and (4) as

$$K_i^* = \frac{\eta_i}{\sum_{r=1}^{n} \eta_r} K \tag{44}$$

$$k_{ij}^* = \frac{\mathbb{E}[\xi_{ij} X_{ij}]}{\sum_{s=1}^{n_i} \mathbb{E}[\xi_{is} X_{is}]} K_i^*. \tag{45}$$

Hence the optimal capital allocation becomes a proportional sharing of available funds. In particular, k_{ij}^* is derived by simply allocating the optimal capital K_i^* according to the weighted expected losses $\mathbb{E}[\xi_{ij} X_{ij}]$. On the other hand, K_i^* derives from a proportional allocation of the total capital K, where the proportions $\eta_i / \sum_{r=1}^{n} \eta_r$ reflect the weighted expected loss of X_i, at the different levels of the organization.

A simple example illustrates the situation of proportional capital allocations further.

Example 4. Similarly to Example 2, we consider a single company with n lines of business, each of which consists of n_i insurance policies. Again it is assumed that $X_i = \sum_{j=1}^{n_i} X_{ij}$ for all $j = 1, \ldots, n_i$ and $X = \sum_{i=1}^{n} X_i$. As in Example 2, the company management preferences are given by weights $\xi_i = \zeta$ for all $i = 1, \ldots, n$. These weights again taken to reflect the aggregate risk. In this example, rather than using scenario weights related to TVaR, we use instead the weights

$$\xi_i = \zeta = 1 + \gamma \frac{X - \mathbb{E}[X]}{\sigma[X]}, \; \gamma > 0, \tag{46}$$

implying

$$\nu_i = \mathbb{E}[\zeta X_i] = \mathbb{E}[X_i] + \gamma \frac{\mathrm{Cov}[X_i, X]}{\sigma[X]}. \tag{47}$$

In this example, line managers are interested in allocating capital to the level of individual policies. For that reason, a very simple weight and business volume measure is proposed, with $\xi_{ij} \equiv 1$ for all $i = 1, \ldots, n, \; j = 1, \ldots, n_i$, leading to

$$\nu_{ij} = \mathbb{E}[X_{ij}]. \tag{48}$$

Consequently the allocation of the optimal capital K_i^* to X_{ij} is simply given by

$$k_{ij}^* = \frac{\mathbb{E}[X_{ij}]}{\mathbb{E}[X_i]} K_i^*. \tag{49}$$

To determine K_i^* itself by (44), we need to consider the quantities

$$\eta_i = \mathbb{E}[X_i] \left(\mathbb{E}[X_i] + \gamma \frac{\mathrm{Cov}[X_i, X]}{\sigma[X]} \right) \left(\mathbb{E}[X_i] + \lambda\gamma \frac{\mathrm{Cov}[X_i, X]}{\sigma[X]} \right)^{-1} \tag{50}$$

For the sake of exposition we now consider the limiting cases for λ, as discussed in Section 2.3. For $\lambda \to 0$ we have

$$\begin{aligned} \eta_i &\to \mathbb{E}[X_i] + \gamma \frac{\mathrm{Cov}[X_i, X]}{\sigma[X]} \implies \\ K_i^* &\to \left(\mathbb{E}[X_i] + \gamma \frac{\mathrm{Cov}[X_i, X]}{\sigma[X]} \right) \frac{K}{\mathbb{E}[X] + \gamma\sigma[X]}, \end{aligned} \tag{51}$$

which is a form of the covariance-based allocation discussed by Dhaene et al. [5]. For $\lambda \to 1$ the allocation reduces to the simple case

$$\eta_i \to \mathbb{E}[X_i] \implies K_i^* \to \mathbb{E}[X_i] \frac{K}{\mathbb{E}[X]}. \tag{52}$$

□

4 Proof of Theorem 1

The following notations are used throughout the proof.

$$\begin{aligned}
a_{ij} &= \left(\frac{\lambda}{\nu_{ij}}\right)^{1/2} \quad \forall i = 1, ..., n \ , \ \forall j = 1, ..., n_i \\
\boldsymbol{A}_i &= \left(\frac{1}{a_{i1}}, ..., \frac{1}{a_{in_1}}\right) \\
\boldsymbol{A} &= \left(\frac{1}{a_{11}}, ..., \frac{1}{a_{1n_1}}, ..., \frac{1}{a_{n1}}, ..., \frac{1}{a_{nn_n}}\right) \\
b_{ij} &= \mathbb{E}\left[\xi_{ij} X_{ij}\right] a_{ij} \ \forall i = 1, ..., n \ , \ \forall j = 1, ..., n_i \\
\boldsymbol{b}_i &= \left(b_{i1}, ..., b_{in_i}\right) \\
\boldsymbol{b} &= \left(b_{11}, ..., b_{1n_i}, ..., b_{n1}, ..., b_{nn_n}\right)
\end{aligned}$$

Vectors $\boldsymbol{A}, \boldsymbol{b}$ are of length $\sum\limits_{i=1}^{n} n_i$ while the vectors $\boldsymbol{A}_i, \boldsymbol{b}_i$ are of length n_i. In addition denote

$$\begin{aligned}
\alpha_i &= \langle \boldsymbol{A}_i, \boldsymbol{A}_i \rangle = \sum_{j=1}^{n_i} a_{ij}^{-2} = \frac{1}{\lambda} \sum_{j=1}^{n_i} \nu_{ij} \\
\alpha &= \langle \boldsymbol{A}, \boldsymbol{A} \rangle = \frac{1}{\lambda} \sum_{i=1}^{n} \sum_{j=1}^{n_i} \nu_{ij} \\
\beta_i &= \langle \boldsymbol{b}_i, \boldsymbol{A}_i \rangle = \sum_{j=1}^{n_i} \mathbb{E}\left[\xi_{ij} X_{ij}\right] \\
\beta &= \langle \boldsymbol{b}, \boldsymbol{A} \rangle = \sum_{i=1}^{n} \sum_{j=1}^{n_i} \mathbb{E}\left[\xi_{ij} X_{ij}\right]
\end{aligned}$$

In the objective function of (1) the variables K_i, k_{ij} are multiplied by the random variables $X_i, X_{ij}, \xi_i, \xi_{ij}$. The main idea of the proof is to transform the objective function, such that only expectations involving those random variables are present, and proceed by geometric arguments. This is done by the relation

$$\frac{\lambda}{\nu_{ij}} \mathbb{E}\left[\xi_{ij} \left(k_{ij} - X_{ij}\right)^2\right] = \left(k_{ij} \left(\frac{\lambda}{\nu_{ij}}\right)^{1/2} - \mathbb{E}\left[\xi_{ij} X_{ij}\right] \left(\frac{\lambda}{\nu_{ij}}\right)^{1/2}\right)^2 - \frac{\lambda \left(\mathbb{E}\left[\xi_{ij} X_{ij}\right]\right)^2}{\nu_{ij}} + \frac{\lambda \mathbb{E}\left[\xi_{ij} X_{ij}^2\right]}{\nu_{ij}} \tag{53}$$

Note that the last two terms on the right hand side of (53) do not depend on k_{ij} and therefore they have not affect on the optimal solution. Thus, we may replace the term $\frac{\lambda}{\nu_{ij}} \mathbb{E}\left[\xi_{ij} \left(k_{ij} - X_{ij}\right)^2\right]$ with the term $\left(a_{ij} k_{ij} - b_{ij}\right)^2$. Similarly we can replace the first term in the objective function in (1). Hence, we rewrite (1) as

$$\begin{cases} \min\limits_{K_i,k_{ij}} & \left\{\sum\limits_{i=1}^{n}(r_iK_i-s_i)^2+\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{n_i}(a_{ij}k_{ij}-b_{ij})^2\right\} \\ s.t. & \sum\limits_{i=1}^{n}K_i=K \\ & \sum\limits_{j=1}^{n_i}k_{ij}=K_i \quad \forall i=1,\dots,n,\ j=1,\dots,n_i. \end{cases} \tag{54}$$

where $r_i=\left(\frac{1-\lambda}{\nu_i}\right)^{1/2}$, $s_i=\mathbb{E}\left[\xi_iX_i\right]r_i$.

Let $U_i=r_iK_i$ and $T_{ij}=a_{ij}k_{ij}$. Hereby we will describe the optimization problem in terms of vectors, where $\boldsymbol{U}=(U_1,...,U_n)$, $\boldsymbol{T}_i=(T_{i1},...,T_{in_i})$. Substituting $\boldsymbol{U},\boldsymbol{T}_i$ in (54) leads to:

$$\begin{cases} \underset{\boldsymbol{U},\boldsymbol{T}_i}{min} & \left\{\|\boldsymbol{U}-\boldsymbol{s}\|^2+\sum\limits_{i=1}^{n}\|\boldsymbol{T}_i-\boldsymbol{b}_i\|^2\right\} \\ s.t. & (i)\ \ \langle\boldsymbol{U},\boldsymbol{R}\rangle=K \\ & (ii)\ \ \langle\boldsymbol{T}_i,\boldsymbol{A}_i\rangle=\frac{U_i}{r_i} \quad \forall i=1,,,n \end{cases} \tag{55}$$

where $\boldsymbol{R}=\left(\frac{1}{r_1},...,\frac{1}{r_n}\right)$ and $\boldsymbol{s}=(s_1,...,s_n)$.

In order to analyze (55) we rewrite the terms $\|\boldsymbol{T}_i-\boldsymbol{b}_i\|^2=\langle\boldsymbol{T}_i-\boldsymbol{b}_i,\boldsymbol{T}_i-\boldsymbol{b}_i\rangle$ in terms of U_i. For every choice of $\boldsymbol{U}$, such that the first constraint holds, the optimal $\boldsymbol{T}_i$ is the Euclidean projection of $\boldsymbol{b}_i$ onto the hyperplane $\langle\boldsymbol{T}_i,\boldsymbol{A}_i\rangle=\frac{U_i}{r_i}$, thus:

$$\boldsymbol{T}_i^*=\boldsymbol{b}_i+\frac{\frac{U_i}{r_i}-\langle\boldsymbol{b}_i,\boldsymbol{A}_i\rangle}{\langle\boldsymbol{A}_i,\boldsymbol{A}_i\rangle}\boldsymbol{A}_i\ . \tag{56}$$

Therefore, $\|\boldsymbol{T}_i^*-\boldsymbol{b}_i\|^2=\frac{\left(\frac{U_i}{r_i}-\beta_i\right)^2}{\alpha_i}$. This implies that for every choice of $\boldsymbol{U}$ and the corresponding $\boldsymbol{T}_i^*$, the value of the objective function is $\|\boldsymbol{U}-\boldsymbol{s}\|^2+\sum\limits_{i=1}^{n}\frac{\left(\frac{U_i}{r_i}-\beta_i\right)^2}{\alpha_i}$. In order to write this expression in terms of a Euclidean norm we define the following notations:

$$\theta_i=\left(1+\frac{1}{\alpha_ir_i^2}\right)^{1/2} \quad , \quad \tau_i=\frac{1}{r_i\theta_i} \quad , \quad \delta_i=\frac{1}{\theta_i}\left(s_i+\frac{\beta_i}{\alpha_ir_i}\right) \quad , \quad D_i=\theta_iU_i$$

for every $i=1,...,n$.

By substituting these notations in (55) we get the following equivalent optimization problem:

$$\begin{cases} \underset{\boldsymbol{D}}{min} & \left\{\|\boldsymbol{D}-\boldsymbol{\delta}\|^2\right\} \\ s.t. & \langle\boldsymbol{D},\boldsymbol{\tau}\rangle=K \end{cases} \tag{57}$$

where $\boldsymbol{D}=(D_1,...,D_n)$, $\boldsymbol{\delta}=(\delta_1,...,\delta_n)$ and $\boldsymbol{\tau}=(\tau_1,...,\tau_n)$. Hence, the optimal solution is

$$\boldsymbol{D}^*=\frac{K-\langle\boldsymbol{\delta},\boldsymbol{\tau}\rangle}{\langle\boldsymbol{\tau},\boldsymbol{\tau}\rangle}\boldsymbol{\tau}+\boldsymbol{\delta} \tag{58}$$

Recall that $U_i = r_i K_i$ and $D_i = \theta_i U_i$, hence $K_i = \frac{D_i}{\theta_i r_i}$. We derive the optimal capital allocation at the top level from (58):

$$\boldsymbol{K}^* = \frac{1}{r_i \theta_i} \left[\frac{K - \langle \boldsymbol{\delta}, \boldsymbol{\tau} \rangle}{\langle \boldsymbol{\tau}, \boldsymbol{\tau} \rangle} \boldsymbol{\tau} + \boldsymbol{\delta} \right]_i \tag{59}$$

and the $i-th$ element of $\boldsymbol{K}^*$ is

$$K_i^* = \frac{1}{r_i \theta_i} \left(\frac{K - \langle \boldsymbol{\delta}, \boldsymbol{\tau} \rangle}{\langle \boldsymbol{\tau}, \boldsymbol{\tau} \rangle} \tau_i + \delta_i \right) \tag{60}$$

By simple algebraic calculation we obtain the following equalities:

$$r_i \theta_i = \left(\frac{(1-\lambda)\, \alpha_i + \nu_i}{\alpha_i \nu_i} \right)^{1/2}$$

$$\theta_i^2 = \frac{(1-\lambda)\, \alpha_i + \nu_i}{(1-\lambda)\, \alpha_i}$$

$$\tau_i = \left(\frac{\alpha_i \nu_i}{(1-\lambda)\, \alpha_i + \nu_i} \right)^{1/2}$$

$$\langle \boldsymbol{\tau}, \boldsymbol{\tau} \rangle = \sum_{r=1}^{n} \frac{\alpha_r \nu_r}{(1-\lambda)\, \alpha_r + \nu_r}$$

$$\delta_i \tau_i = \frac{(1-\lambda)\, \alpha_i \mathbb{E}\left[\xi_i X_i\right] + \beta_i \nu_i}{\alpha_i (1-\lambda) + \nu_i}$$

$$\langle \boldsymbol{\delta}, \boldsymbol{\tau} \rangle = \sum_{r=1}^{n} \frac{(1-\lambda)\, \alpha_r \mathbb{E}\left[\xi_r X_r\right] + \beta_r \nu_r}{\alpha_r (1-\lambda) + \nu_r}$$

By Substituting these notations in (60) we obtain,

$$K_i^* = \left(\frac{\alpha_i \nu_i}{(1-\lambda)\, \alpha_i + \nu_i} \right) \left(\frac{K - \sum\limits_{r=1}^{n} \frac{(1-\lambda)\alpha_r \mathbb{E}[\xi_r X_r] + \beta_r \nu_r}{(1-\lambda)\alpha_r + \nu_r}}{\sum\limits_{r=1}^{n} \frac{\alpha_r \nu_r}{(1-\lambda)\alpha_r + \nu_r}} + \frac{(1-\lambda)}{\nu_i} \mathbb{E}\left[\xi_i X_i\right] + \frac{\beta_i}{\alpha_i} \right) \tag{61}$$

from which (3) follows by simple algebraic manipulations.

Following (56) we obtain the optimal capital allocation at the bottom level

$$\boldsymbol{k}_i^* = \frac{1}{a_{ij}} \left[\boldsymbol{b}_i + \frac{K_i^* - \beta_i}{\alpha_i} \alpha_i \right]_j \tag{62}$$

where the allocated capital to the $j-th$ risk in the $i-th$ line is

$$k_{ij}^* = \frac{1}{a_{ij}} \left(b_{ij} + \frac{K_i^* - \beta_i}{\langle \boldsymbol{A}_i, \boldsymbol{A}_i \rangle\, a_{ij}} \right) = \mathbb{E}\left[\xi_{ij} X_{ij}\right] + \frac{K_i^* - \beta_i}{\lambda \alpha_i} \nu_{ij}, \tag{63}$$

from which (4) follows.

5 Conclusions

Conflicting risk preferences are a fact of life within any insurance or financial institution. Rather than trying to eliminate these inconsistencies, we propose to embrace the diversity of views of risk and synthesize them into an optimal capital allocation that reflects all stakeholders' preoccupations. While different combinations of risk preferences (scenario weights) lead to very different looking capital allocations, a unifying principle emerges. Capital allocation in a hierarchical structure becomes a two-step procedure. First, the total capital is allocated to portfolios, according to synthesized preferences, reflecting the compromises reached. Second, such portfolio capitals are in turn allocated to more granular sub-portfolios, using bottom-up preferences that are only relevant to those sub-portfolios. In that sense, the top-level preferences set constraints but do not lead to micro-management. Thus, the solution to this optimization problem may lead to an organizationally acceptable capital allocation method.

It may be reasonable that in a large financial conglomerate, more than two organizational levels are relevant for the purposes of synthesizing preferences and allocating capital. In that case, optimal capital allocations can still be obtained following similar steps as in the proof of Theorem 1 – for a hierarchy with d relevant levels, capital allocation would become a d-step exercise.

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