Pricing Discretely Monitored Asian Options under Lévy Processes

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Abstract

We present methodologies to price discretely monitored Asian options when the underlying evolves according to a generic Lévy process. For geometric Asian options we provide closed-form solutions in terms of the Fourier transform and we study in particular these formulas in the Lévy-stable case. For arithmetic Asian options we solve the valuation problem by recursive integration and derive a recursive theoretical formula for the moments to check the accuracy of the results. We compare the implementation of our method to Monte Carlo simulation implemented with control variates and using different parametric Lévy processes. We also discuss model-risk issues.

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1 Introduction

We investigate the pricing problem for Asian options monitored at discrete times. The payoff of an arithmetic (geometric) Asian option depends on the arithmetic (geometric) average value of the underlying asset price over a given time period. Asian options have been very successful in the marketplace, because they reduce the possibility of market manipulations near the expiry and they offer better hedging possibilities to firms with a stream of exposures.

Several approaches have been attempted to obtain pricing formulas for the price of Asian options, assuming a continuous-time monitoring of the underlying under the geometric Brownian motion hypothesis, see Fusai and Roncoroni (2008) for a review and numerical comparisons. Among analytical approaches, we mention the Laplace transform approach in Geman and Yor (1993), the spectral expansion derived by Linetsky (2004), and the approximation of the average distribution by fitting integer moments in Turnbull and Wakeman (1991), Lévy (1992), Milevsky and Posner (1998) or logarithmic moments as in Fusai and Tagliani (2002). Another approach uses binomial trees, such as Gaudenzi et al. (2007). However, a large number of contracts specify discrete time monitoring, and the impact of the continuous-time assumption can be substantial for some path-dependent derivatives, see for instance the literature on lookback and barrier options, Kat (2001).

For the discrete case, Clewlow and Carverhill (1990), Andreasen (2002), Dempster et al. (1998), Zvan et al. (1999) focus their attention on the geometric Brownian motion. Benhamou (2002) enhances the algorithm of Clewlow and Carverhill (1990) based on a Fast Fourier technique and adapt it to some non-lognormal densities, like the Student t. Their approaches, although innovative, require computationally intensive numerical methods or approximations for which no clear error bound is available. Albrecher (2004) and Albrecher and Predota (2004) explore approximations based on the moments of the average, but in general it is difficult to evaluate the approximation error.

The contributions of this paper are three-fold. In the first place, we discuss geometric Asian options: we provide closed-form expressions for the price of geometric Asian options in terms of the Fourier transform, when the underlying asset evolves according to a generic Lévy process. This result extends the ones available in the literature relative to the Gaussian case. Moreover, our formula applies to both the discrete and continuous monitoring case. In particular, we analyze in depth our closed-form pricing formulas in the case of Lévy-stable process. Unlike arithmetic options, geometric options are not heavily traded. However, using the analytical formulas for geometric options it is possible to implement a very efficient Monte Carlo control-variate technique to price arithmetic Asian options. This idea has been introduced at first for Asian options by Kemna and Vorst (1990).

Secondly, we discuss the pricing of arithmetic Asian options when the underlying asset evolves according to a generic Lévy process. In addition to the above mentioned control variate approach, we present a new numerical procedure which combines a recursive numerical quadrature with a fast Fourier transform algorithm. Our procedure is also able to provide estimates of the Greeks, such as delta and gamma.

Finally, given the wealth of Lévy processes that potentially can be calibrated to a given set of prices, we discuss model-risk issues, along the lines of Schoutens et al. (2004): as intuition suggests, due to the smoothing effect of the averaging process, arithmetic Asian options are much less sensitive to model risk than, say, barrier options.

The paper is organized as follows. In Section 2 we model the underlying process, starting from the distribution of the log-increments. In Section 3 we discuss the closed-form pricing of geometric Asian options by means of the Fourier transform technique. In Section 4 we present the new recursive algorithm for the valuation of arithmetic Asian options. In Section 5 we discuss our numerical results with particular emphasis on the discrete monitoring feature and to model-risk issues. In Section 6 we conclude. An appendix, which can be skipped at first reading, contains some proofs.

2 The process for the underlying

We are interested in pricing discrete Asian options, for which the payoff depends on the geometric or on the arithmetic average of the prices observed at equally-spaced discrete times $t_0 \equiv 0, t_1 \equiv \Delta, \ldots, t_j \equiv j\Delta, \ldots$. We denote by S_t the underlying asset price at time t. Consider the demeaned log-increments of size Δ:

$$X_T^{\Delta} \equiv \ln\left(S_{T\Delta}\right) - \ln\left(S_{(T-1)\Delta}\right) - m_T^{\Delta}\Delta, \tag{1}$$

where m_T^Δ is the deterministic component under the risk-neutral measure of the log-increments X_T^Δ, whose value will be specified later on:

$$m_T^\Delta \equiv \frac{1}{\Delta}\mathbb{E}\left\{\ln\left(S_{T\Delta}\right) - \ln\left(S_{(T-1)\Delta}\right)\right\}. \tag{2}$$

We consider the demeaned log-increments instead of the simple log-increments for future notational convenience.

As far as the underlying process is concerned, we only assume that, under the risk-neutral measure, non-overlapping log-increments be independent and that increments of equal size be identically distributed. In other words, we assume that the logarithm of the prices be a Lévy process under the risk neutral measure. Then $m_T^\Delta \equiv m$ does not depend on either the time step T or the step size Δ and the underlying asset price reads:

$$S_{T\Delta} = S_0 e^{mT\Delta + X_1^\Delta + \cdots + X_T^\Delta}. \tag{3}$$

Lévy processes display a number of palatable features: they are the most direct generalization of the Brownian motion (BM); they are analytically tractable; Lévy processes are general enough to include a wealth of patterns and thus they account for smile and skew effects in option prices; the i.i.d. structure of the Lévy processes simplifies the estimation of the respective parameters under the real measure, see Meucci (2005). For a thorough introduction to Lévy processes with applications to finance see Geman (2002), Schoutens (2003), Cont and Tankov (2004a), Carr et al. (2003), Geman (2005).

A generic Lévy process is fully determined by the characteristic exponent of the log-increments, which is defined as the logarithm of the characteristic function:

$$\psi_\Delta(\omega) \equiv \ln\left(\mathbb{E}\left\{e^{i\omega X_T^\Delta}\right\}\right). \tag{4}$$

In Table 1 we list a few parametric Lévy processes and their associated characteristic exponent. The Gaussian model (g) is the benchmark assumption: the ensuing process is the purely diffusive Brownian motion, which gives rise to the geometric Brownian motion (GBM) process for the price of the underlying. The model (jd) introduced by Merton (1976) and the double exponential (de) model introduced by Kou (2002) are jump-diffusion processes that account for the presence of fat tails in the empirical distribution of the

Model	$\psi_\Delta(\omega)$
g	$-\frac{\sigma^2}{2}\omega^2\Delta$
nig	$-\delta\Delta\left(\sqrt{\alpha^2-(\beta+i\omega)^2}-\sqrt{\alpha^2-\beta^2}\right)$
cgmy	$C\Delta\Gamma(-Y)\left((M-i\omega)^Y-M^Y+(G+i\omega)^Y-G^Y\right)$
de	$-\frac{1}{2}\sigma^2\omega^2\Delta+\lambda\Delta\left(\frac{(1-p)*\eta_2}{\eta_2+i\omega}+\frac{p*\eta_1}{\eta_1+i\omega}-1\right)$
jd	$-\frac{1}{2}\sigma^2\omega^2\Delta+\lambda\Delta\left(e^{i\omega\alpha-\frac{1}{2}\omega^2\delta^2}-1\right)$
st	$-\kappa^\alpha\lvert\omega\rvert^\alpha\Delta\left(1-i\beta\,\mathrm{sign}(\omega)\tan\left(\frac{\alpha\pi}{2}\right)\right)$

Table 1: Characteristic exponents of some parametric Lévy processes

underlying[1]. The remaining models (nig and cgmy) are pure jump processes with finite variation that can display both finite and infinite activity[2]. They are subordinated Brownian motions: in other words, they can be interpreted as Brownian motions subject to a stochastic time change which is related to the level of activity in the market. In particular, stable processes (st) display the additional feature that their distribution does not depend on the monitoring interval, modulo a scale factor.

So far the drift parameter m in (3) has been left unspecified. Due to the incompleteness of the market, we have to choose a martingale measure for the risk-neutral pricing of derivatives. Except in the special case of the geometric Brownian motion, there are many equivalent measures under which the discounted price process is a martingale. Several different approaches have been suggested to select an appropriate martingale measure, but there is as yet no definitive way of pricing contingent claims in incomplete markets.

[1]In these models σ^2 represent the instantaneous variance of the diffusion part, whilst λ is the jump-intensity. In the jd model, α and δ^2 refer respectively to the mean and the variance of the jump size. In the de model, p is the probability of a up jump, whilst η_1 and η_2 govern the decay of the tails of the up and down jump sizes, that are exponentially distributed.

[2]In particular, the nig model has stochastic time change given by an Inverse Gaussian Process I_t with parameters 1 and $\delta\sqrt{\alpha^2-\beta^2}$, so that $nig_t = \beta\delta^2 I_t + \delta W_{I_t}$, where W_t is a Wiener process. The path behaviour of the CGMY process is determined by the Y parameter. If $Y < 0$, the paths have finite jumps in any finite interval; if not, the paths have infinitely many jumps in any finite time interval, i.e. the process has infinite activity. Moreover, if the Y parameter lie in the interval [1, 2), the process is of infinite variation. See Schoutens (2003).

A discussion on the different choices of an equivalent martingale measure with reference to Lévy Processes can be found for example in Chan (1999) and in Hubalek and Sgarra (2006). A mathematically tractable choice consists in choosing the value of m in such a way that the price S_t discounted by the money-market account B_t be a martingale, i.e. $\mathbb{E}\left[S_T/B_T\right] = S_0/B_0$, $\forall T \geq 0$, see Schoutens (2003). A simple algebraic manipulation shows that m must solve

$$m = (r - q) - \frac{\psi_\Delta(-i)}{\Delta}, \tag{5}$$

where r denotes the constant risk-free rate and q denotes the constant dividend payout rate. Another possible choice is represented by the Esscher transform, as advocated at first in Gerber and Shiu (1994). In the following we will also specify the process for the underlying price when the underlying itself represents the numeraire, see El Karoui et al. (1995). In other words, we need to set m equal to a parameter $\bar{m}$ such that, under the new measure, the equality $\overline{\mathbb{E}}\left[B_T/S_T\right] = B_0/S_0$ is satisfied for all $T > 0$. Another simple algebraic manipulation yields the appropriate value $\bar{m} = (r - q) + \psi_\Delta(i)/\Delta$.

3 Geometric Asian options

The payoff of a geometric Asian option depends on the following path-dependent random variable:

$$G_T^\Delta \equiv \left(\prod_{k=0}^{T} S_{\Delta k}\right)^{\frac{1}{T+1}}. \tag{6}$$

In this section we obtain new formulas that price both fixed-strike and floating-strike geometric Asian options when the underlying asset evolves according to a Lévy process. In particular, we detail our formula for the case of Lévy stable processes, for which considerable simplifications are possible. Furthermore, we consider the limit of our formula as the frequency of the monitoring dates increases, thereby obtaining new results for non-Gaussian processes for the continuous-time monitoring case.

3.1 Fixed-strike options

The price of a geometric Asian call option with fixed strike K reads:

$$C_{fx}^g(T, K) \equiv e^{-rT}\mathbb{E}\left\{\max\left\{G_T^\Delta - K, 0\right\}\right\}. \tag{7}$$

In order to price this option, we adopt a Fourier transform approach. If we derive the characteristic function of the variable $Y_T^{\Delta} \equiv \ln\left(G_T^{\Delta}\right)$, then, as we show in Appendix A, its characteristic function reads:

$$\begin{aligned}\phi_{Y_T^{\Delta}}(\omega) &\equiv \mathbb{E}\left\{e^{i\omega Y_T^{\Delta}}\right\} \\ &= \exp\left(i\omega\left(\ln S_0 + m\frac{\Delta T}{2}\right) + \sum_{k=1}^{T}\psi_{\Delta}\left(\omega\frac{T-k+1}{T+1}\right)\right).\end{aligned} \tag{8}$$

Then we follow Carr and Madan (1999) and we introduce a damping parameter ϱ so that we obtain an expression for the Fourier transform with respect to the logarithm of the strike of a modified call option:

$$\mathcal{F}\left[C_{fx}^{g}\right](\omega) \equiv \int_{-\infty}^{+\infty} e^{i\omega k} e^{\varrho k} C_{fx}^{g}\left(T, e^{k}\right) dk = \frac{e^{-rT}\phi_{Y_T^{\Delta}}(\omega - \varrho i + i)}{\varrho^2 + \varrho - \omega^2 + i(2\varrho + 1)\omega}. \tag{9}$$

In this expression ϱ is a positive constant such that the $(\varrho + 1)$-th moment of G_T^{Δ} exists (in our numerical experiments below we set $\varrho \equiv 1.5$). If we take ϱ to be a negative constant, the expression above returns the Fourier transform of a modified put option. A Fourier inversion yields the option price:

$$C_{fx}^{g}(T, K) = \frac{e^{-\varrho k}}{\pi}\int_{0}^{+\infty} e^{-i\omega k}\mathcal{F}\left[C_{fx}^{g}\right](\omega)\, d\omega. \tag{10}$$

To perform this inversion we can apply numerical quadrature (e.g. NIntegrate in Mathematica) or the Fast Fourier Transform (FFT) algorithm, see Press et al. (1997), which outputs N option prices at equally spaced values for the log-strike.

3.2 Floating-strike options

The price of a floating-strike geometric Asian call option reads:

$$C_{fl}^{g}(T, K) \equiv e^{-rT}\mathbb{E}\left\{\max\left\{G_T^{\Delta} - KS_{\Delta T}, 0\right\}\right\}, \tag{11}$$

where in general K is set equal to 1. It can be proved, see Fusai and Meucci (2007), that the the Fourier transform of the price reads:

$$\mathcal{F}\left[C_{fl}\right](\omega) \equiv \int_{-\infty}^{+\infty} e^{i\omega k} e^{\varrho k} C_{fl}^{g}\left(T, e^{k}\right) dk = S_0\frac{e^{-rT}\phi_{Z_T^{\Delta}}\left(\left(\omega - (\varrho + 1)\, i\right)\right)}{\varrho^2 + \varrho - \omega^2 + i(2\varrho + 1)\omega}, \tag{12}$$

where $\phi_{Z_T^\Delta}(\omega)$ is the characteristic function of $Z_T^\Delta \equiv \ln\left(G_T^\Delta / S_{\Delta T}\right)$ and is given by:

$$\phi_{Z_T^\Delta}(\omega) = e^{-i\omega\bar{m}\Delta\frac{T}{2}+\sum_{k=1}^{T}\psi_\Delta\left(-\omega\frac{k}{T+1}\right)}. \tag{13}$$

Then, the FFT algorithm yields the desired call prices.

3.3 Stable processes

A distribution is stable if the sum of its i.i.d. copies is an i.i.d. copy of the same distribution, modulo an affine transformation. Stability comes at a price, as stable distributions display infinite variance, except in the normal case. A Lévy process is stable if the distribution of its increments is stable. Stable distributions and stable Lévy processes have been the object of intense study in the past decades, see Samorodnitsky and Taqqu (1994) and Sato (1999) for a thorough introduction and Carr and Wu (2003) and McCulloch (1996) for applications to finance. For stable processes the characteristic exponent is of the form:

$$\psi_\Delta(\omega) = -\Delta\kappa^\alpha |\omega|^\alpha \left(1 - i\beta\,\mathrm{sign}(\omega)\tan\left(\frac{\alpha\pi}{2}\right)\right), \tag{14}$$

where $\kappa > 0$ is a scale parameter, $\beta \in [-1,1]$ is a skewness parameter and $\alpha \in (1,2]$ is known as the stability index.

For the fixed-strike case, the characteristic function (8) simplifies as follows, see Fusai and Meucci (2007):

$$\mathbb{E}\left\{e^{i\omega Y_T^\Delta}\right\} = e^{i\omega\left(\ln S_0 + m\frac{\Delta T}{2}\right) - \Delta c\kappa^\alpha|\omega|^\alpha\left(1-i\beta\,\mathrm{sign}(\omega)\tan\left(\frac{\alpha\pi}{2}\right)\right)}. \tag{15}$$

In this expression, c can be represented in terms of the Riemann zeta and the generalized Riemann zeta functions respectively:

$$\mathcal{Z}(s) = \sum_{k=1}^{\infty} k^{-s}, \quad \mathcal{Z}(s,p) = \sum_{k=1}^{\infty}(k+p)^{-s}, \tag{16}$$

and reads:

$$c \equiv \left(\frac{1}{T+1}\right)^\alpha \left(\mathcal{Z}(-\alpha) - \mathcal{Z}(-\alpha, T+1)\right). \tag{17}$$

When $\kappa \equiv \sigma/\sqrt{2}$, $\beta \equiv 0$ and $\alpha \equiv 2$ the stable process (14) becomes the standard Brownian motion, and c simplifies into $c = (2+T)(3+2T)/(6(1+T))$

and (15) becomes the characteristic function of a normal distribution with mean $\ln S_0 + m\Delta T/2$ and variance $\Delta c\frac{\sigma^2}{2}$. In other words, the geometric average (6) is lognormally distributed and we obtain the result of Kemna and Vorst (1990).

For floating-strike geometric Asian options, the characteristic function (13) becomes

$$\overline{\mathbb{E}}\left\{e^{i\omega Z_T^\Delta}\right\} = e^{i\omega\bar{m}\Delta\frac{T}{2} - \Delta d\kappa^\alpha|\omega|^\alpha\left(1 - i\beta\,\mathrm{sign}(\omega)\tan\left(\frac{\alpha\pi}{2}\right)\right)}. \tag{18}$$

In this expression, d can be represented in terms of the n-th harmonic number of order r, defined as $H_n^{(r)} = \sum_{k=1}^n 1/k^r$, and reads $d \equiv H_T^{(-\alpha)}/(T+1)^\alpha$. Again, when $\kappa \equiv \sigma/\sqrt{2}$, $\beta \equiv 0$ and $\alpha \equiv 2$ the stable process (14) becomes the standard Brownian motion, and (18) becomes the characteristic function of a normal distribution.

3.4 Continuous-time monitoring

The results of the continuous-time literature can be obtained by letting the number of monitoring dates T approach infinity while the observation frequency Δ approaches zero, in such a way that $\tau \equiv T\Delta$ remains constant. For fixed-strike geometric Asian options[3], the quantity of interest is the continuously monitored log-geometric average:

$$\widetilde{Y}_\tau \equiv \frac{1}{\tau}\int_0^\tau \ln(S_t)\,dt. \tag{19}$$

The characteristic function of $\widetilde{Y}_\tau$ reads:

$$\begin{aligned}\phi_{Y_T^\Delta}(\omega) &\equiv \mathbb{E}\left\{e^{i\omega\widetilde{Y}_\tau}\right\} \qquad (20)\\ &= \exp\left(i\omega\left(\ln S_0 + m\frac{\Delta T}{2}\right) + \lim_{x\to\infty}\int_0^x \psi_{\tau/x}\left(\omega\frac{x-u}{x}\right)du\right).\end{aligned}$$

The integral and the limit can be computed analytically for several parametric processes, see Table (3).

[INSERT TABLE (3) HERE]

[3] A similar result can be obtained for floating-strike geometric Asian options, see Fusai and Meucci (2007).

In particular, in the case of stable processes, the characteristic function of (19) admits an analytic expression:

$$\mathbb{E}\left\{e^{i\omega\widetilde{Y}_\tau}\right\} = e^{i\omega\left(\ln S_0+m\frac{\tau}{2}\right)-\frac{\tau}{1+\alpha}\kappa^\alpha|\omega|^\alpha\left(1-i\beta\,\mathrm{sign}(\omega)\tan\left(\frac{\alpha\pi}{2}\right)\right)}. \tag{21}$$

This also follows from applying to (15) and (17) the Euler-McLaurin formula $\lim_{T\to\infty} c\tau/T = \tau/(1+\alpha)$, see Abramowitz and Stegun (1974). In particular, in the Gaussian case $\alpha \equiv 2$ we obtain that, under the appropriate measures, $\widetilde{Y}_\tau$ is Normal with mean $\ln S_0 + m\tau/2$ and variance $\sigma^2\tau/3$, which is the result that appears in Hull (2005).

4 Arithmetic Asian options

The payoff of an arithmetic Asian option depends on the following path-dependent random variable:

$$A_T^\Delta \equiv \frac{1}{T+1}\sum_{k=0}^{T} S_{\Delta k}. \tag{22}$$

Notice that we use the convention that the average starts at period $k \equiv 0$. In the following, we exploit a recursive formulation to price the fixed strike arithmetic Asian option. For the floating strike version see Fusai and Meucci (2007). The payoff of an arithmetic Asian call option with fixed strike K reads:

$$C_{fx}^a(K,T) \equiv \max\left\{A_T^\Delta - K, 0\right\}. \tag{23}$$

As realized in Clewlow and Carverhill (1990), the distribution of A_T^Δ can be obtained recursively. If we define $Z_k^\Delta \equiv m\Delta + X_k^\Delta$, from (22) we are interested in the distribution of the following quantity $\sum_{k=1}^{T} S_{\Delta k} = e^{Z_1^\Delta}\left(1+e^{Z_2^\Delta}\left(\cdots\left(1+e^{Z_T^\Delta}\right)\right)\right)$. Starting from $L_T^\Delta \equiv e^{Z_1^\Delta}$ and introducing recursively the quantities

$$L_k^\Delta \equiv e^{Z_k^\Delta}\left(1+L_{k+1}^\Delta\right), \quad k = T-1, ..., 1, \tag{24}$$

we obtain $A_T^\Delta \equiv S_0\left(1+L_1^\Delta\right)/(T+1)$. Therefore, the key ingredient for the computation of fixed-strike arithmetic Asian options is the density of L_1^Δ or equivalently, the density of $B_1^\Delta \equiv \ln\left(L_1^\Delta\right)$. We discuss this computation in

Section 4.1. Once we obtain the density f_{B_1} we can price call options[4] with an additional numerical integration:

$$\mathbb{E}\left\{C_{fx}^{a}\left(K,T\right)\right\} = e^{-rT}\int_{\gamma}^{+\infty}\left(\frac{S_0}{T+1}\left(1+e^{x}\right)-K\right)f_{B_1}\left(x\right)dx, \tag{25}$$

where $\gamma \equiv \ln\left(K\left(T+1\right)/S_0-1\right)$. Furthermore, once we have the density f_{B_1} we can compute option prices for different strikes K and spot prices S_0. From (25) we can also easily compute the Greeks. For instance, for the delta we obtain:

$$\Delta = \frac{e^{-rT}\partial\mathbb{E}\left\{C_{fx}^{a}\left(K,T\right)\right\}}{\partial S_0} = \frac{e^{-rT}}{T+1}\int_{\gamma}^{+\infty}\left(1+e^{x}\right)f_{B_1}\left(x\right)dx. \tag{26}$$

Similarly, for the gamma we obtain:

$$\Gamma = \frac{e^{-rT}\partial^2\mathbb{E}\left\{C_{fx}^{a}(K,T)\right\}}{\partial S_0^2} = e^{-rT}\left(\frac{K}{S_0}\right)^2\frac{T+1}{K(T+1)-S_0}f_{B_1}\left(\gamma\right). \tag{27}$$

Finally, notice that the recursion (24) translates into a formula for the moments of the arithmetic average. Indeed, from the independence of Z_k^{Δ} and L_{k+1}^{Δ} as well as from the definition of Z_k^{Δ}, we obtain:

$$\begin{aligned}\mathbb{E}\left\{\left(L_k^{\Delta}\right)^n\right\} &= \mathbb{E}\left\{\left(e^{Z_k^{\Delta}}\left(1+L_{k+1}^{\Delta}\right)\right)^n\right\} \\ &= e^{nm\Delta}\phi_{X^{\Delta}}\left(-in\right)\sum_{q=0}^{n}\binom{n}{q}\mathbb{E}\left\{\left(L_{k+1}^{\Delta}\right)^q\right\},\end{aligned} \tag{28}$$

where the recursion starts with the following initial condition:

$$\mathbb{E}\left\{\left(L_T^{\Delta}\right)^n\right\} \equiv \mathbb{E}\left\{e^{nZ_1^{\Delta}}\right\} = \phi_{X^{\Delta}}\left(-in\right). \tag{29}$$

The moments of the arithmetic average then can be computed as follows:

$$\mathbb{E}\left\{\left(A_T^{\Delta}\right)^n\right\} \equiv \left(\frac{S_0}{T+1}\right)^n\sum_{j=0}^{n}\binom{n}{j}\mathbb{E}\left\{\left(L_T^{\Delta}\right)^j\right\}. \tag{30}$$

We will use this result to verify the accuracy of our numerical method. An expression similar to (30) was obtained also in Albrecher (2004).

[4] If we are interested in pricing put options, we have to integrate over the relevant domain or to exploit the put-call parity for Asian options.

4.1 Computation of the pricing density

Since Z_k^{Δ} and L_{k+1}^{Δ} are independent, the density of $B_k \equiv \ln\left(L_k^{\Delta}\right) = Z_k^{\Delta} + \ln\left(1 + L_{k+1}^{\Delta}\right)$ is the convolution of the density $f_{Z_k^{\Delta}}$ and that of $\ln\left(1 + e^{B_{k+1}^{\Delta}}\right)$. Furthermore, since the Z_k^{Δ} are i.i.d. the density $f_{Z_k^{\Delta}}$ does not depend on the monitoring time index k, which we drop from the notation. With a change of variable we obtain that the density of f_{B_j}satisfies the recursion:

$$f_{B_k}(x) = \int_{-\infty}^{+\infty} f_{Z^{\Delta}}\left(x - \ln\left(e^y + 1\right)\right) f_{B_{k+1}}(y)\, dy, \quad k = T-1, ..., 1, \tag{31}$$

where the initial condition is set as $f_{B_T} \equiv f_{Z^{\Delta}}$. For some specifications of the underlying Lévy process such as Gaussian, NIG, Double Exponential and Jump-Diffusion this density is known analytically; for other specifications, such as CGMY, it can be obtained by inverting the characteristic function of the log returns with the FFT. We remark that a recursion similar to (31) appears in Clewlow and Carverhill (1990) and then in Benhamou (2002). These authors exploit the convolution structure of the recursion to obtain the density of B_k^{Δ} by applying an FFT and an inverse FFT at each monitoring date. Instead, we use the FFT once to generate the density of Z_k^{Δ} given its characteristic function and then we implement a series of recursive quadratures.

We proceed by approximating the integral (31) using an M-point quadrature formula, see Press et al. (1997):

$$\begin{aligned} f_{B_k}(x) &\approx \int_{l}^{u} \varphi_{Z^{\Delta}}\left(x - \ln\left(e^y + 1\right)\right) f_{B_{k+1}}(y)\, dy \\ &\approx \sum_{j=1}^{M} w_j \phi\left(x - \ln\left(e^{y_j} + 1\right)\right) f_{B_{k+1}}(y_j), \end{aligned} \tag{32}$$

where y_j are the abscissas and w_j the corresponding weights in the quadrature formula. An issue in the implementation of the above procedure is the choice of the domain $[l, u]$. We use the results in Philips and Nelson (1995) that, for a given random variable X, yield a bound to $\Pr(X > c)$ and to $\Pr(X < -c)$ in terms of the integer moments of X. In our implementation we focus on the first ten integer moments of B_T^{Δ} to determine l such that $\Pr\left(B_T^{\Delta} < l\right) \leq 10^{-8}$ and similarly we focus on the first ten integer moments of $L_T^{\Delta} \equiv \exp\left(B_T^{\Delta}\right)$ to determine u such that $\Pr\left(L_T^{\Delta} > e^u\right) \leq 10^{-8}$ (the latter moments are readily provided by (28)). These choices have proved sufficient to achieve accurate results. Another issue in the implementation of (32) is the choice of the quadrature rule: in our numerical implementation we adopt a Gaussian quadrature rule. To motivate this choice with respect to alternative procedures we observe that the quantity to be estimated, i.e. the option price, can be represented as a multiple integral

$$I = \int_{\ln\left(\frac{KT}{S_0}\right)}^{+\infty} dx \int_{-\infty}^{+\infty} dy_n \cdots \int_{-\infty}^{+\infty} dy_1 \zeta\left(x, \mathbf{y}\right), \tag{33}$$

where

$$\zeta\left(x, \mathbf{y}\right) \equiv \left(\frac{S_0}{T+1}\left(1+e^x\right) - K\right) \varphi_{Z^\Delta}\left(x - \ln\left(e^{y_n}+1\right)\right) \cdots \varphi_{Z^\Delta}\left(y_1\right). \tag{34}$$

An M-point numerical quadrature approximates (33) with the following expression:

$$\hat{I} \equiv \sum_{k_{n+1}=1}^{M} \sum_{k_n=1}^{M} \cdots \sum_{k_1=1}^{M} w_{k_1} \cdots w_{k_n} w_{k_{n+1}} \zeta\left(x_{k_{n+1}}, y_{k_1}, \cdots, y_{k_n}\right). \tag{35}$$

The convergence rate using the trapezoid rule is $O\left(J^{-2/n}\right)$, where J is the number of evaluations of $\zeta\left(x, \mathbf{y}\right)$, see Haselgrove (1961). Using the Simpson rule we have an improvement to $O\left(J^{-4/n}\right)$. A crude Monte Carlo simulation samples points uniformly and averages the function at these points, providing an approximation to (33) which is characterized by a standard error which is independent of n and of order $O\left(J^{-1/2}\right)$: for sufficiently large values of n, this convergence rate is better than either the trapezoid or the Simpson rule. On the other hand, using an M-point Gaussian quadrature, the error is $O\left(J^{-(2M+1)/2n}\right)$, and for M sufficiently large we obtain a faster convergence than with any of the above methods, including Monte Carlo simulations. We mention that recursive quadrature has received attention in the literature on barrier options, see Aitsahalia and Lai (1997), Sullivan (2000), Andricopoulos et al. (2003), Fusai and Recchioni (2005).

A third issue in the implementation of the recursion (32) is the computational cost. We can write that recursion in matrix form as follows:

$$\mathbf{f}_k = \mathbf{KD}\mathbf{f}_{k+1}, \tag{36}$$

where $\mathbf{f}_k$ is a vector with elements $f_{B_k}(x_j)$; $\mathbf{K}$ is an $M \times M$ kernel whose (k, j)-th element reads $\phi(x_k - \ln(e^{y_j} + 1))$; and $\mathbf{D}$ is a diagonal matrix with elements $d_{jj} = w_j$. The density at the n-th monitoring date is then given by iterating (36) starting from $\mathbf{f}_n$. Therefore, the solution at the n-th monitoring date for each value of x requires $O(nM^2)$ function evaluations (matrices $\mathbf{K}$), plus $O(nM^2)$ elementary operations. Therefore the total cost is of the order of $O(nM^2)$ elementary operations.

To summarize, the algorithm, which was implemented in C, proceeds as follows:

- Define the parametric model for log-returns as in Table 1.
- Using the FFT algorithm compute the density function of the log-returns, and assign it as initial condition $\mathbf{f}_T$ to the recursion (36).
- Using the weights and abscissas of the Gaussian quadrature, construct the product matrix $\mathbf{KD}$ and then iteratively compute $\mathbf{f}_n, \mathbf{f}_{n-1}, \ldots, \mathbf{f}_1$.
- Compute the option price by numerical integration of the payoff function with the density $\mathbf{f}_1$ as in (25).
- Check the accuracy of the results by comparing the numerical moments of the arithmetic average with the theoretical expression provided by (30).

5 Numerical results

In this section we perform numerical tests to examine the accuracy of our procedure[5]. More precisely, we compare our recursive pricing procedure with the results of a standard Monte Carlo-based pricing with one million scenarios.

The first issue is the impact of the monitoring frequency. It is well known that for barrier options the discrepancy between option prices under continuous and discrete monitoring can be significant. Indeed, the convergence of the discrete monitored barrier option prices to the continuous case is extremely slow, of the order of $n^{-1/2}$, where n is the number of monitoring dates. In the sequel, we investigate if this is also the case for Asian options when the underlying follows a Lévy process.

The second issue is model risk: in its general formulation, model risk arises when different competing models properly account for the empirical

[5]Detailed results are available in the pre-print version of the present paper, see Fusai and Meucci (2007).

evidence, but decisions based on different models give rise to different results, see Meucci (2005) for a discussion in the general context of trading and portfolio management. In this specific context, the competing models are different parametric specifications for the underlying Lévy process; the empirical evidence is represented by a set of option prices; and the decision can be, represented for instance, by the choice of the most suitable hedge.

5.1 Calibration of the underlying processes

To perform a comparison between different Lévy models, we proceed as follows. We consider the calibration results reported in Schoutens (2003, p. 82) relative to the NIG and the CGMY models[6]. The calibrated parameters for the NIG process are:

$$\hat{\alpha} = 6.1882, \quad \hat{\beta} = -3.8941, \quad \hat{\delta} = 0.1622. \tag{37}$$

The calibrated parameters for the CGMY process are:

$$\hat{C} = 0.0244, \quad \hat{G} = 0.0765, \quad \hat{M} = 7.5515, \quad \hat{Y} = 1.2945. \tag{38}$$

Using the estimated CGMY model, we calibrate the Gaussian, the double-exponential and the Merton models. The calibration problem for the Merton model is ill-posed, as many different combinations of the mean jump size and its volatility reproduce option prices with about the same overall least square error, see He at al. (2006) and Cont and Tankov (2004b) where some regularization techniques based on splines and relative entropy are proposed. The problem stems from the low sensitivity of liquid vanilla option prices to the jump parameters. Since our aim is not the calibration to market prices, but to prices generated by the CGMY model, we calibrate the two jump-diffusion models in such a way to minimize the square integrated difference between the real part of the characteristic functions of the CGMY and the jump-diffusion models:

$$\min_{\theta=\{\sigma,\lambda,\alpha,\delta\}} \int_0^{\omega_{\max}} \sum_{j=1}^{3} \left(\mathrm{Re}\left(e^{\psi^{cgmy}_{\Delta_j}(\omega)}\right) - \mathrm{Re}\left(e^{\psi^{jd}_{\Delta_j}(\omega;\theta)}\right)\right)^2 d\omega, \tag{39}$$

[6] These models have been calibrated to a set of 77 mid-prices of European call options on the S&P 500 Index at the close of the market on 18 April 2002. On this day the S&P 500 closed at 1124.47. The considered options expire in May 2002, June 2002, September 2002, December 2002, March 2003, June 2003 and finally in December 2003. The strikes range mainly in the interval 1100-1300. See Schoutens (2003, p. 6).

where $\Delta_1 \equiv 0.25$, $\Delta_2 \equiv 0.5$, $\Delta_3 \equiv 1$ and $\omega_{\max}$ has been set equal to 250. The same procedure has been pursued to calibrate the pure diffusion (GBM) and the double-exponential models. A similar calibration procedure has been proposed by Belomestny and Reiß (2006) and by Schmidt (1982). The calibrated volatility parameter for the GBM is $\hat{\sigma} = 0.17801$. The calibrated parameters for the Merton model are:

$$\hat{\sigma} = 0.126349, \ \hat{\alpha} = -0.390078, \ \hat{\lambda} = 0.174814, \ \hat{\delta} = 0.338796.$$

The calibrated parameters for the double-exponential model are:

$$\hat{\sigma} = 0.120381, \quad \hat{\lambda} = 0.330966, \quad \hat{p} = 0.20761, \quad \hat{\eta}_1 = 9.65997, \quad \hat{\eta}_2 = 3.13868. \tag{40}$$

Figure 1 shows the density of the log-returns for the calibrated models. The Merton jump-diffusion model, the Kou double exponential model and the CGMY densities appear very similar and remarkably different from the Gaussian case. In particular, the skewness and kurtosis parameters are respectively equal to -19.813 and 986.936 for the CGMY model, to -2.16016 and 7.46374 for the Merton model, to -2.77006 and 13.5805 for the double exponential model and to -2.13745 and 9.93736 for the NIG model.

[Insert Figure 1 here]

5.2 Monte Carlo simulations

To simulate paths from the above Levy processes we follow the algorithms presented in Cont and Tankov (2004a). In addition, for the CGMY model we use the approximate simulation method based on a combination of Brownian subordination and rejection recently proposed by Madan and Yor (2005)[7]. However, their algorithm introduces a bias because small jumps are replaced by their expectations. Unfortunately, this bias is difficult to quantify but, to the best of our knowledge, no exact simulation method for the increments of this process is known, although a possible alternative is proposed by Poirot and Tankov (2007). To increase the accuracy of the Monte Carlo simulations we use the Geometric Asian option prices as control variate. In particular, the Control variate estimate has been obtained using

$$\hat{c}_{MC}^{CV} = \hat{c}_{MC}^{A} + \hat{\beta} \left(c^G - \hat{c}_{MC}^{G} \right),$$

[7] P. Tankov kindly provided the C code to implement the Yor-Madan algorithm.

where c^G is the analytical geometric Asian option obtained in previous section, $\hat{c}^A_{MC}$ and $\hat{c}^G_{MC}$ are the crude MC estimates of the arithmetic and geometric options and the coefficient $\hat{\beta}$ has been pre-computed from regressing 100000 simulations of the arithmetic payoff against the respective geometric payoff. Then 1000000 new simulations are performed and the control variate estimate is obtained by using the above formula.

5.3 Results for geometric options

We consider now the pricing of Asian options. Tables 4 to 10 report the main results of this analysis. In particular, Table 4 reports the prices for fixed-strike geometric-average Asian options. Different columns correspond to different models: this way we can assess the impact of model risk on the pricing of these derivatives. The first column selects the model, the second selects the number of monitoring dates (12, 50, 250, 1000, 10000, ∞), the third the CPU time and the remaining the Geometric Asian option price for different strikes (90, 100 and 110). We keep the maturity date fixed and equal to 1 year. The spot price has been set equal to 100. Option prices have been computed using the Carr-Madan formula and FFT inversion (10) with $N \equiv 2^{17}$, $\lambda \equiv 0.0004$ and setting the damping parameter $\varrho \equiv 1.5$.

Comparing the prices of the geometric options as the monitoring frequency varies, we notice that the differences between continuous and discrete monitoring are relevant only for high strike prices and a very low number of monitoring dates. Otherwise the percentage difference between the discrete and the continuous version is below 5%. Considering the non-exact estimation of the parameters due to calibration, this difference is acceptable: discrete monitoring provides a viable approximation for the continuous-time limit. This result is useful because the discrete-monitoring computation is less computationally intensive than its continuous counterpart.

The second remark is that the choice between Gaussian and non-Gaussian model appears very important. This is also consistent with Figure 2, where we report the density of the log-geometric mean. Comparing different Lévy models, we notice that the pure jump process (jd), the double exponential (de) and the CGMY models yield very similar option prices, whereas NIG prices tend to be undervalued. This result does not seem due to the calibration procedure: once we re-calibrate the NIG model using the same procedure followed for the jump-diffusion and double-exponential process, the new prices are still different from the ones obtained in the remaining models.

		Monte Carlo				Number of Quadrature Points					
						7000		5000		1000	
Modello	Dates	MC	CPU	MC+CV	CPU	MSE	CPU	MSE	CPU	MSE	CPU
g	12	0.08439	63	0.03525	68	0.00000	119	0.00000	65	0.00000	6
g	50	0.97323	125	0.02433	130	0.00640	135	0.00640	52	0.00640	5
g	250	0.56911	454	0.01786	459	0.02621	250	0.02621	130	0.02537	8
nig	12	0.93767	88	0.01573	93	0.01106	67	0.01106	35	1.91168	5
nig	50	0.85467	225	0.03892	230	0.02555	97	0.06330	52	n.a.	6
nig	250	1.69247	952	0.01453	957	n.a.	252	n.a.	131	n.a.	8
cgmy	12	25.37854	105	2.84911	111	0.52170	64	0.52170	37	0.52115	8
cgmy	50	7.25580	293	0.77980	299	1.07702	95	1.07702	53	4.10542	10
cgmy	250	4.00879	997	0.43916	1005	1.18273	218	4.74037	126	n.a.	12
jd	12	0.82418	70	0.82418	75	0.03609	65	0.03609	34	0.03609	5
jd	50	1.50117	156	1.50117	161	0.04138	95	0.04138	51	0.04138	6
jd	250	1.81340	602	1.81340	607	0.04307	246	0.04307	131	6.36434	9
de	12	0.38332	73	0.38332	78	0.10211	73	0.10211	40	0.10211	6
de	50	0.90982	156	0.90982	161	0.14144	107	0.14144	57	0.14145	6
de	250	0.99756	601	0.99756	606	0.11956	261	0.11956	144	n.a.	11

Table 2: In this table we give, for different models, the 1000*Squared Root of Sum of Squared (column ERR), the CPU time in Seconds (CPU column) for the Monte Carlo simulation (Crude and with Control Variate) and for the recursive quadrature method (with different number of grid points: from 1000 to 7000). As benchmark we have used the first five moments computed according to formula (30)

[Insert Table 4 here]
[Insert Figure 2 here]

5.4 Results for arithmetic options

We consider now arithmetic-average Asian options. In Table 5 we report the CPU time and the square root of the sum of squared errors, where the errors are the differences between the analytical moments and the numerical moments of order 0 to 5. The analytical moments are computed according to formula (30), whilst the numerical are computed in two ways: using one million Monte Carlo simulations (with and without the Geometric Average as control variate; for these results we report the standard error) and using the density obtained by recursion (36).

The most interesting results are that, except for the NIG process, the proposed numerical integration procedure provides much more accurate estimates in lesser computational time than Monte Carlo simulation also if implemented using control variates.

We stress that the Monte Carlo approach is a viable alternative only if enhanced by the control variates. In order to compute the latter, our expressions for the geometric-average options are essential. Indeed, the control variate substantially reduces the error with respect to the crude Monte Carlo simulation. An exception is represented by the CGMY results, which appear quite inaccurate. This can be due to the simulation algorithm we used, whose approximation error is very hard to quantify. In particular, more detailed results, given in Fusai and Meucci (2007), show up how the percentage error between analytical and numerical moments is larger than 1% only in exceptional cases and is very small in the Gaussian and Jump-Diffusion case. We have also examined the performance of our approach using different parameter configurations, but the greater accuracy respect to Monte Carlo simulation is in general preserved.

The computational cost of our technique is linear in the number of monitoring dates and quadratic in the number of quadrature points. Extremely accurate result can be obtained for the Gaussian, jump-diffusion and double exponential processes, even with a very large number of monitoring dates ($\approx$ 250) and a low number of nodes ($\approx$ 3000). Slightly less accurate results are obtained for the CGMY process, but in this case the Monte Carlo simulations do not appear to provide a reliable alternative at all. Instead, for the NIG model, our approach does not perform well, because the density peaks as we increase the monitoring frequency. In this case the Monte Carlo appears to be the only viable alternative.

[Insert Table 5 here]

In Tables 6-10 we consider prices of arithmetic Asian options. The numerical results previously presented seem to justify that model risk does not seem to be an issue (although Gaussian and NIG models produce somewhat different option prices), as important as for barrier options. Intuitively, the averaging process tapers the thickness of the tails, whilst for barrier options the model sensitivity is much higher, as different path properties are emphasized by the knock-out/in effect of the barrier, see Schoutens et al. (2004). The tapering effect for Asian options is confirmed by Figures 2 and 3, which display the densities of the geometric and arithmetic average according to different models. As we can see, with the exception of the Gaussian and NIG cases, these densities look very similar.

Furthermore, the number of monitoring dates does not seem as crucial as for barriers. Therefore, in order to approximate the continuously monitored solution, we can use the discrete solution with a low number of monitoring dates, for which our algorithm is reliable and fast.

[Insert Table 6 here]
[Insert Table 7 here]

[Insert Table 8 here]
[Insert Table 9 here]
[Insert Table 10 here]
[Insert Figure 2 here]
[Insert Figure 3 here]

Finally, in Figures 4, 5 and 6 we report the differences in prices, deltas and gammas computed according to the different models taking as benchmark the geometric Brownian motion. The differences among the Levy models are small whilst appear remarkable if compared to the GBM case: even from a hedging perspective the effect of model risk is limited, unlike in the case of barrier options, see Schoutens et al. (2004). This is welcome news from a risk-management perspective. Clearly, it remains to be investigated the relevance of the i.i.d. assumption, that underlies all Lévy processes. This will be possibly the topic of future work, although in the non i.i.d. setting our formulae for the geometric case and the numerical approximation for the arithmetic case do not apply.

[Insert Figure 4 here]
[Insert Figure 5 here]
[Insert Figure 6 here]

6 Conclusions

We introduce analytical closed formulas to price geometric Asian options, and a recursive algorithm as well as a control-variate technique to price arithmetic Asian options under the general assumption that the underlying evolves according to a Lévy process. In both cases, we consider discretely monitored options. However, differently from other path-dependent options like barrier and lookback, Asian option prices do not seem to be affected by the monitoring frequency. We also evaluate the impact of model risk. As it turns out, model risk is significant in the case of the Gaussian and NIG models. It remains to be investigatedthe effect of stochastic volatility. Unfortunately, both the closed-form analytical formulas for the geometric Asian options and the numerical algorithm for the arithmetic Asian options rely on the i.i.d. assumption for the log-increments of the underlying: this assumption is not satisfied by non-Lévy stochastic volatility models.

A The characteristic function of the geometric Average for a Generic Lévy processes

Let $Y_T^\Delta \equiv \ln\left(G_T^\Delta\right)$ where G_T^Δ is given in (6). With some algebraic manipulation, we can express Y_T^Δ in terms of the Lévy process increments X_k^Δ:

$$\begin{aligned}
Y_T^\Delta &= \frac{1}{T+1}\sum_{k=0}^{T}\ln S_{\Delta k} \qquad (41)\\
&= \ln S_0 + \frac{1}{T+1}\left(m\Delta\frac{(T+1)\,T}{2} + \sum_{k=1}^{T}\sum_{i=1}^{k}X_i^\Delta\right)\\
&= \ln S_0 + m\Delta\frac{T}{2} + \frac{1}{T+1}\sum_{k=1}^{T}X_k^\Delta\sum_{i=k}^{T}1\\
&= \ln S_0 + m\Delta\frac{T}{2} + \frac{1}{T+1}\sum_{k=1}^{T}(T-k+1)\,X_k^\Delta
\end{aligned}$$

Using the independence of each X_k^Δ and (4), we obtain the characteristic function of Y_T^Δ under the risk-neutral measure:

$$\begin{aligned}
\mathbb{E}\left\{e^{i\omega Y_T^\Delta}\right\} &= e^{i\omega\left(\ln S_0 + m\Delta\frac{T}{2}\right)}\prod_{k=1}^{T}\tilde{\mathbb{E}}\left\{e^{i\omega\frac{T-k+1}{T+1}X_k^\Delta}\right\} \qquad (42)\\
&= \exp\left(i\omega\left(\ln S_0 + m\frac{\Delta T}{2}\right) + \sum_{k=1}^{T}\psi_\Delta\left(\omega\frac{T-k+1}{T+1}\right)\right).
\end{aligned}$$

To price floating-strike geometric Asian options, we define $Z_T^\Delta \equiv \ln\left(G_T^\Delta / S_{T\Delta}\right)$. With some algebra, we have $Z_T^\Delta = -\bar{m}\Delta\frac{T}{2} - \frac{1}{T+1}\sum_{k=1}^{T}kX_k^\Delta$,and exploiting the independence of the increments X_k^Δ, we obtain

$$\overline{\mathbb{E}}\left\{e^{i\omega Z_T^\Delta}\right\} = e^{-i\omega\bar{m}\Delta\frac{T}{2} + \sum_{k=1}^{T}\psi_\Delta\left(-\omega\frac{k}{T+1}\right)}. \qquad (43)$$

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Model	$\lim_{x\to\infty}\int_0^x \psi_{\tau/x}\left(\omega\frac{x-u}{x}\right)du$
g	$-\frac{1}{6}\omega^2\sigma^2\tau$
nig	$\delta\tau\frac{\left(-i\tan^{-1}\left(\frac{\beta}{\sqrt{\alpha^2-\beta^2}}\right)\alpha^2+i\tan^{-1}\left(\frac{\beta+i\omega}{\sqrt{\alpha^2-(\beta+i\omega)^2}}\right)\alpha^2+i\sqrt{\alpha^2-(\beta+i\omega)^2}(\beta+i\omega)+\sqrt{\alpha^2-\beta^2}(2\omega-i\beta)\right)}{2\omega}$
cgmy	$C\tau\frac{\left(\omega\left((Y+1)G^Y-(i\omega+G)^Y-(M-i\omega)^Y+M^Y(Y+1)\right)-i\left(-G(i\omega+G)^Y+G^{Y+1}+M\left((M-i\omega)^Y-M^Y\right)\right)\right)\Gamma(-Y-1)}{\omega}$
de	$-\frac{\frac{\tau}{6}\left(\sigma^2\omega^3+6\lambda\omega+p\eta_1\lambda\left(-3i\log\left(\eta_1^2+\omega^2\right)+6i\log(\eta_1)-3\sqrt{\frac{1}{\omega^2}}\lambda\pi\omega+6\tan^{-1}\left(\frac{\eta_1}{\omega}\right)\right)+(1-p)\eta_2\lambda\left(3i\log\left(\eta_2^2+\omega^2\right)-6i\log(\eta_2)-3\sqrt{\frac{1}{\omega^2}}\pi\omega+6\tan^{-1}\left(\frac{\eta_2}{\omega}\right)\right)\right)}{\omega}$
jd	$-\frac{1}{6}\tau\left(\omega^2\sigma^2+6\lambda-\frac{3ie^{-\frac{\alpha^2}{2\delta^2}}\lambda\sqrt{2\pi}\left(\mathrm{Erfi}\left(\frac{\alpha}{\sqrt{2}\delta}\right)-\mathrm{Erfi}\left(\frac{i\omega\delta^2+\alpha}{\sqrt{2}\delta}\right)\right)}{\delta\omega}\right),$
st	$-\tau\frac{\lvert\omega\rvert^\alpha k^\alpha\left(1-i\beta\,\mathrm{sign}(\omega)\tan\left(\frac{\alpha\pi}{2}\right)\right)}{1+\alpha}$
$\mathrm{Erfi}\,(x)$ gives the imaginary error function $\mathrm{Erf}\,(iz)\,/i$.)	

Table 3: The expression of the limit appearing in the characteristic function of the Geometric Average

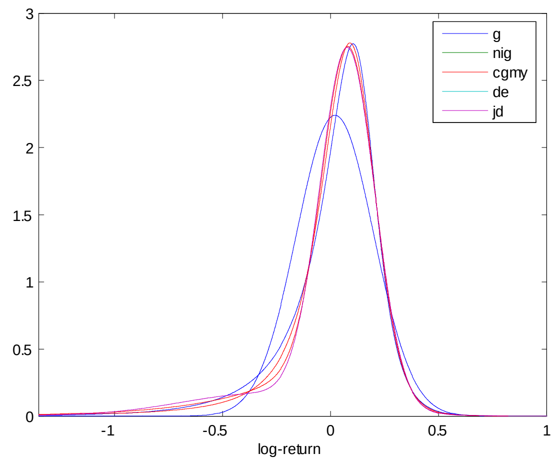

Figure 1: Densities of the log-returns at the 1 yr horizon.

Model	Dates	CPU	K=90	K=100	K=110
g	12	0.016	11.66744	4.703634	1.251243
g	50	0.047	11.7076	4.769273	1.299134
g	250	0.125	11.71856	4.787024	1.312165
g	1000	0.625	11.72065	4.790409	1.314654
g	10000	3.656	11.72128	4.791427	1.315403
g	cts	0.031	11.72135	4.791541	1.315486
nig	12	0.063	12.40399	4.903628	0.9217
nig	50	0.235	12.45446	4.956617	0.952823
nig	250	1.515	12.46819	4.971162	0.961418
nig	1000	10.703	12.47081	4.973945	0.963065
nig	10000	158.891	12.4716	4.974784	0.963562
nig	cts	0.047	12.47168	4.974877	0.963617
cgmy	12	0.078	12.49339	4.879854	0.930755
cgmy	25	0.14	12.52291	4.913607	0.952526
cgmy	50	0.329	12.53757	4.930348	0.963302
cgmy	100	0.703	12.54517	4.939021	0.968879
cgmy	250	1.906	12.54981	4.944327	0.97229
cgmy	1000	11.593	12.55216	4.947009	0.974013
cgmy	10000	177.657	12.55286	4.947818	0.974533
cgmy	cts	0.062	12.55294	4.947908	0.974591
de	12	0.141	12.49912	4.860329	0.950346
de	50	0.875	12.54201	4.91101	0.984821
de	250	5.094	12.55377	4.924996	0.994327
de	1000	28.218	12.55602	4.927676	0.996148
de	10000	304.688	12.5567	4.928483	0.996697
de	cts	0.078	12.55677	4.928573	0.996758
jd	12	0.078	12.49709	4.853707	0.959979
jd	50	0.296	12.53882	4.9046	0.995103
jd	250	1.968	12.55026	4.918632	1.00478
jd	1000	11.969	12.55245	4.92132	1.006633
jd	10000	192.5	12.55311	4.92213	1.007192
jd	cts	0.547	12.55319	4.92222	1.007254

Table 4: Prices of geometric Asian options for different Levý processes and different number of monitoring dates. Legend: Dates is the number of monitoring dates. Remaining parameters: $S_0 = 100$, $T = 1year$, $r = 0.0367$.

		Monte Carlo				Number of Quadrature Points					
						7000		5000		1000	
Modello	Dates	MC	CPU	MC+CV	CPU	MSE	CPU	MSE	CPU	MSE	CPU
g	12	0.08439	63	0.03525	68	0.00000	119	0.00000	65	0.00000	6
g	50	0.97323	125	0.02433	130	0.00640	135	0.00640	52	0.00640	5
g	250	0.56911	454	0.01786	459	0.02621	250	0.02621	130	0.02537	8
nig	12	0.93767	88	0.01573	93	0.01106	67	0.01106	35	1.91168	5
nig	50	0.85467	225	0.03892	230	0.02555	97	0.06330	52	n.a.	6
nig	250	1.69247	952	0.01453	957	n.a.	252	n.a.	131	n.a.	8
cgmy	12	25.37854	105	2.84911	111	0.52170	64	0.52170	37	0.52115	8
cgmy	50	7.25580	293	0.77980	299	1.07702	95	1.07702	53	4.10542	10
cgmy	250	4.00879	997	0.43916	1005	1.18273	218	4.74037	126	n.a.	12
jd	12	0.82418	70	0.82418	75	0.03609	65	0.03609	34	0.03609	5
jd	50	1.50117	156	1.50117	161	0.04138	95	0.04138	51	0.04138	6
jd	250	1.81340	602	1.81340	607	0.04307	246	0.04307	131	6.36434	9
de	12	0.38332	73	0.38332	78	0.10211	73	0.10211	40	0.10211	6
de	50	0.90982	156	0.90982	161	0.14144	107	0.14144	57	0.14145	6
de	250	0.99756	601	0.99756	606	0.11956	261	0.11956	144	n.a.	11

Table 5: In this table we give, for different models, the 1000*Squared Root of Sum of Squared (column ERR), the CPU time in Seconds (CPU column) for the Monte Carlo simulation (Crude and with Control Variate) and for the recursive quadrature method (with different number of grid points: from 1000 to 7000). As benchmark we have used the first five moments computed according to formula (30)

		MC simulation		Numerical Quadrature		
n	K	MC+CV	se	10000	5000	1000
12	90	11.90505	0.213	11.90497	11.90498	11.90428
12	100	4.88207	0.173	4.88210	4.88212	4.88199
12	110	1.36329	0.145	1.36314	1.36314	1.36371
50	90	11.93302	0.203	11.93301	11.93299	11.93339
50	100	4.93735	0.169	4.93736	4.93738	4.93711
50	110	1.40267	0.150	1.40264	1.40262	1.40199
250	90	11.94096	0.201	11.94068	11.94069	11.94137
250	100	4.95244	0.169	4.95233	4.95239	4.94942
250	110	1.41359	0.151	1.41351	1.41350	1.41290

Table 6: Prices of arithmetic Asian options for Gaussian process. Parameters: $S_0 = 100$, $r = 0.0367$, $\sigma = 0.17801$.

		MC simulation		Numerical Quadrature		
n	K	MC+CV	se	11000	5000	1000
12	90	12.62293	0.341	12.62243	12.62243	12.61315
12	100	5.06077	0.223	5.06060	5.06057	5.05723
12	110	1.01374	0.176	1.01355	1.01355	1.01379
50	90	12.66112	0.333	12.66118	12.66160	14.11925
50	100	5.10359	0.221	5.10367	5.10383	4.88735
50	110	1.03770	0.177	1.03770	1.03774	0.75191
250	90	12.67186	0.329	n.a.	n.a.	n.a.
250	100	5.11558	0.218	n.a.	n.a.	n.a.
250	110	1.04446	0.176	n.a.	n.a.	n.a.

Table 7: Prices of arithmetic Asian options for NIG process. Parameters: $S_0 = 100$, $r = 0.0367$, $\alpha = 6.1882$, $\beta = -3.8941$, $\delta = 0.1622$.

		MC simulation		Numerical Quadrature		
n	K	MC+CV	se	10000	5000	1000
12	90	12.69114	0.360	12.70625	12.70626	12.70557
12	100	5.02787	0.211	5.03492	5.03486	5.03649
12	110	1.01895	0.156	1.02115	1.02110	1.02214
50	90	12.73548	0.371	12.73854	12.73855	12.74033
50	100	5.07403	0.223	5.07570	5.07577	5.07225
50	110	1.04594	0.165	1.04674	1.04669	1.04948
250	90	12.74864	0.393	12.74737	12.72731	n.a
250	100	5.08744	0.233	5.08694	5.07912	n.a
250	110	1.05377	0.170	1.05389	1.05251	n.a

Table 8: Prices of arithmetic Asian options for CGMY process. Parameters: $S_0 = 100$, $r = 0.0367$, $C = 0.0244$, $G = 0.0765$, $M = 7.5515$, $Y = 1.2945$.

		MC simulation		Numerical Quadrature		
n	K	MC+CV	se	10000	5000	1000
12	90	12.71298	0.379	12.71236	12.71236	12.71252
12	100	5.01729	0.243	5.01712	5.01705	5.01772
12	110	1.04141	0.183	1.04142	1.04147	1.03957
50	90	12.74420	0.371	12.74369	12.74369	12.74386
50	100	5.05849	0.243	5.05809	5.05814	5.05781
50	110	1.06886	0.184	1.06878	1.06877	1.06983
250	90	12.75141	0.358	12.75241	12.75242	12.86165
250	100	5.06910	0.232	5.06949	5.06943	5.11830
250	110	1.07632	0.177	1.07646	1.07637	1.08915

Table 9: Prices of arithmetic Asian options for Double Exponential process. Parameters: $S_0 = 100$, $r = 0.0367$, $\sigma = 0.120381$, $\lambda = 0.330966$, $p = 0.2071$, $\eta_1 = 9.65997$, $\eta_2 = 3.13868$.

		MC simulation		Numerical Quadrature		
n	K	MC+CV	se	10000	5000	1000
12	90	12.71035	0.375	12.71066	12.71065	12.71038
12	100	5.01117	0.248	5.01127	5.01124	5.01008
12	110	1.05141	0.189	1.05162	1.05165	1.05140
50	90	12.74051	0.367	12.74093	12.74093	12.74112
50	100	5.05226	0.245	5.05246	5.05243	5.05337
50	110	1.07959	0.192	1.07959	1.07962	1.07966
250	90	12.74964	0.366	12.74917	12.74918	12.78492
250	100	5.06417	0.244	5.06381	5.06387	5.08129
250	110	1.08772	0.189	1.08740	1.08737	1.09043

Table 10: Prices of arithmetic Asian options for Merton Jump-Diffusion process. Parameters: $S_0 = 100$, $r = 0.0367$, $\sigma = 0.126349$, $\alpha = -0.390078$, $\lambda = 0.174814$, $\delta = 0.338796$.

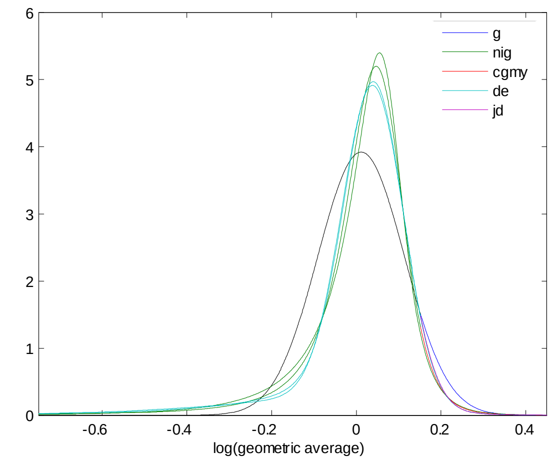

Figure 2: Density of the logarithm of the geometric mean for the different Lévy models (25 monitoring dates).

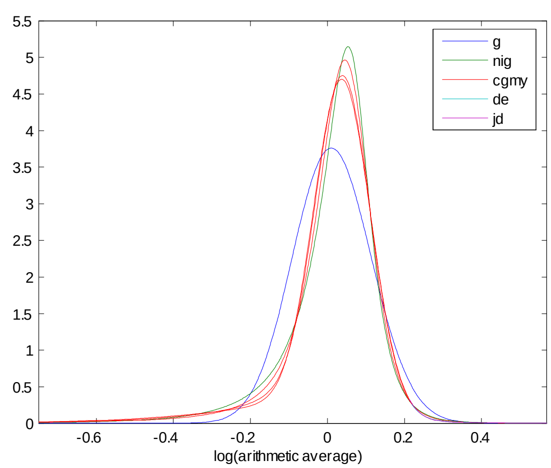

Figure 3: Density of the log(arithmetic mean) for the different Lévy models (25 monitoring dates).

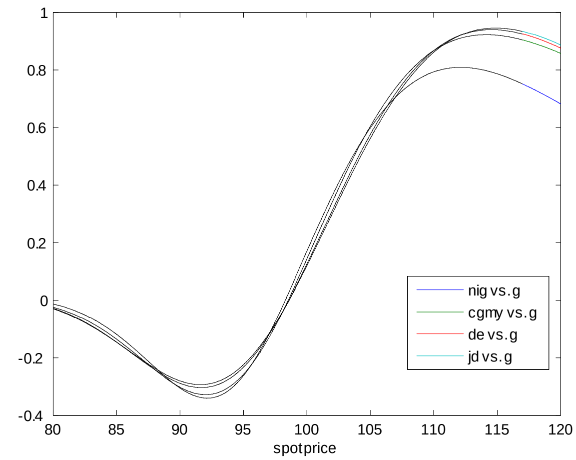

Figure 4: Price differences (Lévy model vs Gaussian) of the arithmetic Asian option under different Lévy models (25 monitoring dates).

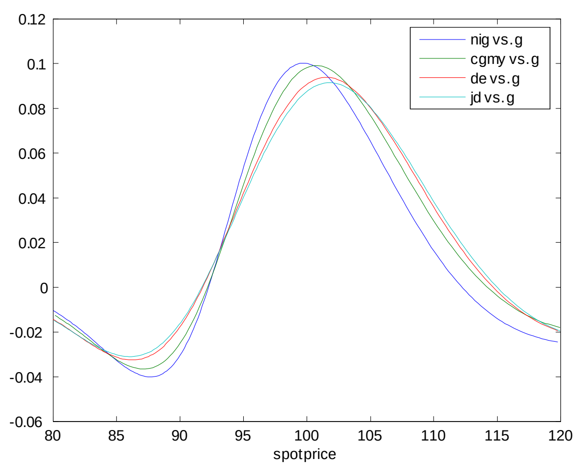

Figure 5: Delta differences (Lévy model vs Gaussian) of the arithmetic Asian option under different Lévy models (25 monitoring dates). The delta has been computed using formula (26).

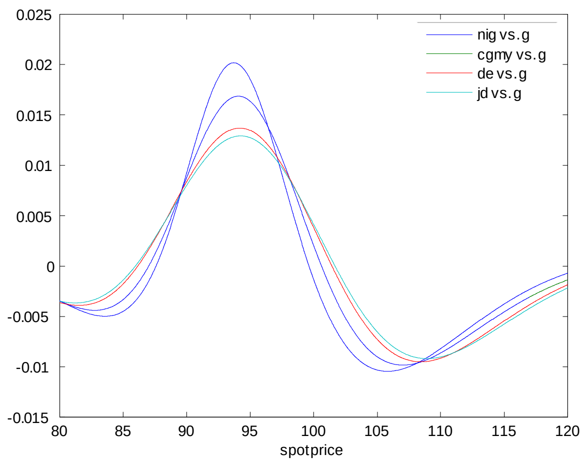

Figure 6: Gamma differences (Lévy model vs Gaussian) of the arithmetic Asian option under different Lévy models (25 monitoring dates). The gamma has been computed using formula (27).