DIVERSIFICATION RETURNS, REBALANCING RETURNS AND VOLATILITY PUMPING

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Abstract

There is now a substantial literature on the effects of rebalancing on portfolio performance. It is widely argued in the theoretical literature that rebalanced strategies are inherently likely to generate greater terminal wealth than unrebalanced strategies, although empirical studies do not generally support this claim. We show that this claim is based on a misattribution between 'rebalancing returns' which are specific to the act of rebalancing, and 'diversification returns' which can be earned by both rebalanced and unrebalanced strategies. Confusion appears to have increased because in some situations these two distinct effects have the same magnitude. This issue has important implications for return attribution in diversified portfolios. Misleading claims about the benefits of rebalancing are likely to lead investors into strategies which involve insufficient diversification and excessive transactions costs.

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1. Introduction

Investment strategies are often classified as passive or active – although the distinction between these two approaches is a matter of degree. Passive strategies usually involve choosing portfolio weights according to some predetermined rule. A key practical element in a passive strategy is the frequency and hence cost of rebalancing. At one extreme, initial weights are chosen and no further rebalancing takes place, so the weights evolve over time according to the relative returns on the component assets. This is a buy and hold (B&H) strategy. A market-cap index is an example of B&H (although in practice, some rebalancing takes place when some low return stocks are replaced by "new" stocks). At the other extreme, some passive portfolio approaches assume continuous rebalancing to keep asset weights constant (this often produces tractable closed form results). Evaluation of alternative passive strategies may involve a "theoretical approach" using continuous time mathematics or Monte Carlo simulation or, most often, actual historical returns data. In all three approaches, incorporating transactions costs in sufficient detail to mimic "real world" outcomes is often problematic and simple fixed round-trip transactions costs are often used to determine the "optimal rebalancing frequency".

A branch of the portfolio choice literature which is complementary to the above methods is the optimal growth approach. Portfolio weights are chosen to maximise the expected geometric growth rate of the portfolio value – consistent with maximising the expected logarithm of final wealth (Kelly 1956, Thorpe 2010, Luenberger 1997). This approach often includes an examination of a portfolio consisting of one risky asset and one risk-free asset, as well as a portfolio of N risky assets. Another strand of the literature considers weights that are often not "optimal" but constitute an arbitrary fixed weight portfolio which is compared with a B&H strategy (e.g. Fernholz and Shay 1982, Luenberger 1997, Booth & Fama 1992, Mulvey et al. 2007, Qian 2012, Willenbrock, 2012). A theoretical result from this approach is that a portfolio of independently and identically distributed (IID) assets sees more rapid portfolio growth than the corresponding B&H strategy, and this "excess growth" increases with the volatility of the underlying portfolio assets. "Volatility pumping" is a strategy which seeks to take advantage of this by adding high volatility assets to a portfolio and rebalancing to fixed weights. This outperformance is attributed to the fact that a rebalanced strategy automatically "buys on the upticks and sells on the downticks" (Fernholz and Shay 1982).

In this paper we concentrate on this choice between constant portfolio weight rebalancing strategies and the corresponding B&H strategy. We examine under what circumstances and for what reasons one strategy outperforms the other. More precisely, we examine whether under similar stochastic conditions, a periodic rebalancing strategy gives a better outcome than a B&H strategy. We are particularly concerned with performance over a finite horizon, since this is of interest to most investors. The purpose of this paper is to correct misleading claims that are widely made about the benefits of rebalancing (and which are likely to lead to poor investment decisions); to investigate the implications of these claims on portfolio construction and to provide intuitive reasons for these results. We also examine some results from the continuous time literature and show how they can be interpreted in terms more familiar to investment practitioners.

Specifically, we reject the claim that rebalancing strategies automatically generate "excess growth" and hence greater terminal wealth than buy-and-hold strategies even when there is no predictable time structure to asset returns. We demonstrate instead that the different growth rates of these strategies are entirely explained by the different volatility levels of these portfolios (because a rebalanced portfolio tends to remain better diversified) with no evidence of the buy-low-and-sell-high effects that proponents claim. This has important implications for investors. The misleading claim that rebalancing generates excess growth encourages investors to hold volatile assets and to rebalance frequently in order to increase geometric returns. More efficient portfolios can be constructed simply by diversifying effectively and thus minimizing volatility drag.

The existence of a predictable element of time structure in relative asset returns can influence the performance of rebalanced portfolios. Mean reversion in relative asset returns tends to lead to higher terminal wealth for rebalanced strategies since, for example, assets which have recently underperformed are bought during rebalancing and subsequently tend to outperform. Conversely, momentum in returns tends to favour B&H. Thus in practice, the outcome of rebalancing versus a B&H strategy depends on the time series properties of the assets chosen as well as the weighting scheme and the performance metric used, the investment horizon, the frequency of rebalancing and transactions costs.

We demonstrate mathematically that unrebalanced portfolios also generate growth which is greater than the average growth of the underlying assets (this is the definition of "excess growth" generally used in this literature). Furthermore, we show that growth rates of unrebalanced and rebalanced portfolios are initially identical, and only diverge gradually as the composition of the unrebalanced portfolio gradually diverges from its initial weights. Specifically, we show that ─ contrary to the claims that are often made ─ for a portfolio of assets with IID returns which follow geometric Brownian motion, rebalancing only affects the expected portfolio growth rate to the extent that it alters portfolio volatility: the portfolio volatility of the strategy fully determines the expected portfolio growth.

Confusion on this issue appears to have arisen in part because of the difficulty in making meaningful comparisons between rebalanced and unrebalanced portfolios, since even when the portfolios are initially identical the composition of an unrebalanced portfolio tends to shift over time. We derive like-for-like comparisons between rebalanced and unrebalanced portfolios which show that in the absence of mean reversion in relative asset prices the difference is entirely explained by the different volatilities of the two portfolios.

The rest of this paper is structured as follows. In section 2 we provide a review of the relevant literature. Section 3 defines concepts such as the diversification return, excess growth, volatility drag and rebalancing return. In Section 4 we use continuous time math and simulation to analyze the outcome of B&H and rebalancing strategies on the expected geometric return of a portfolio of risky assets. Section 5 demonstrates that the widely-cited positive expected growth rate on a portfolio of one risk-free and one risky asset (each of which has zero expected growth rate) is due to the risky asset having a positive expected arithmetic return rather than – as is claimed – being due to the rebalancing trades themselves being profitable. It also explores the reasons for confusion in the existing literature by considering the distribution of final wealth from these two strategies and the probability that one dominates, over finite (and infinite) investment horizons. Finally, we examine the impact on investors of the misleading claims made in the previous literature. Section 6 concludes.

2. Literature Overview

Cheng and Deets (1971) compare the performance of B&H and rebalancing strategies assuming risky assets returns r_i are each IID but with different mean returns μ_i. They consider an equally weighted portfolio of *N* risky assets. The B&H strategy and the rebalancing strategy give the same outcome for expected terminal wealth when the mean returns of all risky assets are equal (furthermore, if returns are IID then expected terminal wealth is independent of the variance of individual asset returns). However, if at least one pair of assets have different mean returns $\mu_i \neq \mu_k$ then an initially equally weighted (*1/N*) B&H strategy always gives a higher expected terminal wealth than the corresponding rebalancing strategy: Intuitively, a B&H portfolio is likely over time to give increasing weights to the assets with higher μ_i. This relative superiority of B&H in terms of expected wealth is found to be larger the more frequently the portfolio is rebalanced to equal weights, the greater the dispersion in the μ_i and the longer the investment horizon and is smaller the greater the number of assets in the portfolio.

Evaluating the impact of rebalancing is important in a wide range of situations. Rebalancing back to constant weights is an important component of universal portfolios (Cover, 1991). Rebalancing is also inherent in any portfolio weighting strategy other than capitalisation-weighting, so it plays a part in the debate over fundamental and alternative forms of equity indexing (Kaplan, 2008; Hsu et al. 2011; Arnott, Hsu and Moore, 2005). Diversification returns have already been shown to be important in contexts other than equities and bonds. For example they account for a large part of the growth rate of portfolios of commodity futures (Gorton and Rouwenhorst 2006a and 2006b, Erb & Harvey 2006).

One important strand of the literature uses Black-Scholes continuous time framework to compare constant-weight rebalancing strategies (i.e. not necessarily equal weights) with corresponding static B&H strategy. This builds on the earlier work of Kelly (1956) and Thorpe (1967). A specific situation which is widely analysed (e.g. Gabay and Herlemont 2007, Luenberger 1997, Perold & Sharpe 1988 and Fernholz and Shay 1982) is a portfolio consisting of a proportion $1-\pi$ in the risk-free asset (which pays zero interest) and a proportion π in a single risky asset S which follows a lognormal diffusion process, with periodic drift μ. The risky asset follows $S_t = \exp(gt + \sigma W(t))$ where $g = \mu - \sigma^2 / 2$ and $W(t)$ is a Wiener process. Hence, if we set $g=0$ the long term growth rate of S_t approaches zero: $\lim\left(\frac{1}{t}\ln S_t - gt\right)=0$ as $t\to\infty$.

However, the terminal wealth for the rebalanced cash, risky-asset portfolio is:

$$V_t^{reb} = (S_t)^{\pi} e^{g^* t} \quad \text{where} \quad g^* = \pi(1-\pi)\sigma^2/2 \tag{1}$$

The term g^* is referred to as the "excess growth rate" of the portfolio (in excess of the weighted average of the growth rates of the component assets, which in this case is zero). The portfolio excess growth rate is positive for $0<\pi<1$, increases with volatility σ (hence the term "volatility pumping" given to this rebalancing strategy), and is maximized for $\pi=1/2$. The B&H portfolio has terminal wealth:

$$V_t^{B\&H} = (1-\pi)+\pi S_t \tag{2}$$

Hence for the risk-free plus risky asset portfolio:

$$\ln\frac{V_t^{reb}}{V_t^{B\&H}} = \frac{1}{2}\sigma^2\pi(1-\pi)t + \ln\frac{S_t^{\pi}}{1-\pi+\pi S_t} \tag{3}$$

The log of relative wealth for the two strategies depends positively on volatility and the time horizon and the distribution of relative wealth depends on the stochastic path of the risky asset. Dempster, Evstigneev and Schenk-Hoppe (2007) use a more general setting of stationary stochastic processes for returns to show that an equally-weighted rebalanced portfolio of *N* risky assets generates excess growth. They note that if individual assets have zero long-run growth, this "volatility pumping" seems to provide the rebalanced portfolio with "something for nothing".

It will already be apparent that one distinctive feature of the literature on rebalancing is that some studies use expected geometric growth rate (GM return) as their performance metric instead of more conventional metrics such as the mean portfolio return, variance, maximum drawdown, Sharpe ratio, information ratio, Fama-French 4-factor alpha and expected utility (or certainty equivalent return).[1] Our objective in this paper is to clear up the confusion which is apparent in the literature over the effects of rebalancing on these different metrics.

Granger et al. (2014) use a Black-Scholes framework to show that the terminal wealth for the cash plus risky asset portfolio leads to a fattened left tail for the rebalanced strategy in comparison to the B&H strategy, exacerbating drawdowns. As Perold and Sharpe (1988) note, a constant-weight rebalanced strategy which buys more risky assets after they have underperformed is doing the opposite of a portfolio insurance strategy, and could thus be seen as selling such portfolio insurance to other investors. The negative convexity of the rebalanced strategy is exacerbated if there are pronounced divergences in asset performance (e.g. between stocks and bonds over the period June 2008 to April 2009 when a B&H strategy outperformed a rebalanced strategy by 5% cumulative return). Granger et al. (2014) show conversely that a

[1] Using the GM as a target raises a number of questions. For example, maximising the GM will maximise the welfare of an investor whose utility is a logarithmic function of terminal wealth. It is less clear that it is an appropriate target for investors with other utility functions. This topic has previously been the subject of a long and rancorous debate, which we do not wish to revisit here. For our purposes it is sufficient to note that investors are encouraged to choose strategies such as volatility pumping or rebalanced strategies on the basis of their expected GM returns. This paper seeks to clarify how these apparently attractive GM returns come about.

momentum strategy (a moving average cross-over system) gives rise to positive convexity even when returns are not serially correlated. They therefore recommend augmenting a rebalanced stocks/bond portfolio with a momentum overlay (e.g. with 10% of funds allocated to the momentum strategy) which can offset the additional risk of large drawdowns that is inherent in rebalancing.

Rebalancing clearly has important implications for portfolio risk, but our key interest in this paper is in the claims that are made for the growth rates and terminal wealth achieved by rebalanced portfolios (either in expectation or asymptotically as $t \to \infty$). In particular, we examine the claims originally made in Fernholz and Shay (1982) and endorsed by an increasing number of subsequent papers (e.g. Luenberger 1997, Mulvey et al. 2007, Stein et al., 2009, Qian 2012, Willenbrock, 2012, Bouchey et al. 2012) that the process of rebalancing directly boosts long-term growth rates and terminal wealth even when returns have no predictable time structure (for example when asset prices evolve following geometric Brownian motion), and that rebalanced portfolios must be expected to outperform because they generate these "rebalancing returns" (also referred to as "excess growth").

A large empirical literature has attempted to assess the net effects of rebalancing in a simple two asset equity/bond portfolio. The results have varied, which is likely to be the result of the different performance metrics used, different markets and time horizons considered, the details of the rebalancing strategy (notably the frequency of rebalancing or the divergence levels which trigger rebalancing trades) and the extent to which transactions costs are taken into account. Some have found that a rebalancing strategy outperforms a B&H strategy (Arnott and Lovell, 1993, Tsai 2001, Donohue and Yip, 2003, Tokat and Wicas 2007, Harjoto and Jones, 2006, Bolognesi et al., 2013), while others have found the reverse. (Jaconetti, Kinniry and Zilbering 2010). Naturally, the metrics considered have an important impact on conclusions, for example Plaxco and Arnott (2002) conclude that a B&H strategy "may appear to outperform in a strongly trending market", but that rebalancing outperforms on a risk-adjusted basis. Another sizeable strand of the literature seeks to identify the optimal parameters for a rebalancing strategy (for example in terms of the frequency or divergence threshold used), such as Buetow et al. (2002), Masters (2003), Smith and Desormeau (2006), McLellan et al (2009). [There is little consensus from these studies on the optimal rebalancing frequency or no-trade interval around desired weights].

A common feature of many of these studies is that they test alternative rebalancing strategies in specific historic periods. Dichtl, Drobetz and Wambach (DDW, 2012) note that these studies consider at most a small number of different sets of realised equity and bond returns, and hence cannot attribute any statistical significance to their findings. DDW instead use a block bootstrap approach to generate a large number of alternative realisations whilst preserving any short-term time structure of the original historical data (for a portfolio initially 60% stock, 40% bond portfolio, using US, UK and German data 1981-2010). They find that constant proportions rebalancing strategies generate average returns which are "only marginally" and not consistently better: Rebalanced strategies generate higher average terminal wealth in the UK and Germany, but in the US buy and hold generates greater terminal wealth than rebalancing. They find instead that the key benefit of rebalancing strategies lies in volatility reduction, leading to outperformance on risk-adjusted measures such as the Sharpe ratio.

In a wide ranging study using 14 alternative constrained optimal weighting schemes and on various portfolios of US returns data (mostly over 1963-2004), DeMiguel et al (2009) find that an equally-weighted portfolio of *N* assets with monthly rebalancing does consistently better on three alternative performance metrics (i.e. Sharpe ratio, certainty equivalent return and turnover), than the optimal weighting schemes. They also report that the "B&H case in which the investor allocates *1/N* at the initial date and then holds this portfolio until the terminal date" gives results that are similar to those for the case with rebalancing". Hence it appears that a *1/N* B&H portfolio may perform as well as a monthly rebalanced *1/N* portfolio over a 40 year period – we investigate this possibility in our simulations below.

Plyakha et al. (2012) take a single period (1967-2009), but construct many alternative portfolios by randomly sampling 100 stocks from within the S&P500 index, allowing them to attribute statistical significance to their results. They find that rebalanced equally-weighted portfolios outperform in terms of mean return as well as risk-adjusted measures. Only some of this can be attributed to the fact that such equally-weighted portfolios give a higher weight to small and value stocks, since they also generate a significantly higher four-factor alpha. Reducing the frequency of rebalancing, and also evaluating the performance of rebalanced, but (approximately) capitalisation-weighted portfolios, helps to identify the source of this outperformance . The authors attribute it to the fact that rebalancing is "a contrarian strategy that exploits reversal in stock prices", consistent with the evidence of such reversals that has previously been identified (eg. Jegadeesh 1990, and Jegadeesh and Titman, 1993, 2002). However, the finding that rebalancing strategies may outperform because of such empirical regularities is very different from the claim made in the more theoretical literature that rebalancing strategies automatically outperform even when asset returns follow a geometric Brownian motion, and hence have no identifiable time structure.

The large body of empirical and theoretical research into the effects of rebalancing is testament to the practical importance of this issue to investors and the difficulty of deriving robust theoretical results. In this paper our focus is on the strand of research which claims that rebalancing directly generates "excess returns" even when returns exhibit no predictable time structure. This claim has important implications for investors, but we demonstrate that it is very misleading.

3. Terminology: Volatility Drag/Diversification Return vs. Rebalancing return/Excess Growth

This paper investigates the claimed outperformance of rebalanced strategies made by a number of prior papers which use expected growth rates as their performance metric. Use of the expected growth rate means that we must take account of "volatility drag". This effect will be familiar to many investors, and can be understood directly from the standard relationship between the arithmetic and geometric mean returns[2]:

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[2] Fama and Booth (1992) derive this relationship using a Taylor expansion for the expected continuously compounded return E[log(1+r)] around E[r]. This derivation makes no assumption about the nature of the

$$E[GM] \approx E[AM] - \frac{\sigma^2}{2} \quad (4)$$

This tells us that, all else equal, an asset or portfolio will generate a lower expected GM if it has a higher volatility. This is because the GM (and similarly, the continuous growth rate) is a concave function of terminal wealth[3]. If we compare two strategies with identical expected terminal wealth, the one with lower volatility will generate a higher E[GM] because this concave relationship effectively penalizes both exceptionally high and exceptionally low terminal wealth outturns (terminal wealth (TW) being a linear function of the AM). Figure 1 illustrates this volatility drag.

[Figure 1]

Volatility drag is inherent in the compounding relationship between periodic returns and terminal wealth and, as equation 4 shows, is merely a function of the volatility of returns. It affects both rebalanced and unrebalanced portfolios. A rebalanced portfolio will suffer lower volatility drag only to the extent that rebalancing keeps its variance lower than the corresponding

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asset or portfolio which generates these returns (except that the distribution of returns is differentiable with finite derivatives), yet it is an accurate approximation when compounding over small time periods over which E[r] is also small. The precise expression derived by Fama and Booth is $E[GM] = E[AM] - \sigma^2/2(1+E[r])^2$, but we use it here in the form in which it is most normally cited, which is of course still a good approximation for small E[r].

[3] GM=(Terminal wealth/Initial wealth)$^{1/n}$–1 in discrete time, or the equivalent continuous time growth rate $^1/_T$log(Terminal wealth/Initial wealth). Both are concave functions of terminal wealth. For simplicity we normalise initial wealth to 1.

unrebalanced portfolio. Another entirely different effect is that portfolio rebalancing could boost expected terminal wealth if there is negative autocorrelation in relative asset returns, since a fixed weight rebalancing strategy sells some of the assets that outperformed in the most recent period and buys assets that underperformed. Thus it will profit if these relative price movements tend to reverse in future.

These two effects are entirely distinct: one is an increase in the *expected growth rate* which results from any reduction in portfolio volatility, the other is an increase in *expected terminal wealth* which occurs only if there is (a) rebalancing, and (b) negative autocorrelation in relative asset returns. We argue in Section 5 below that proponents of rebalancing strategies tend to confuse these two effects by claiming that rebalancing generates "rebalancing returns" (excess growth) which are entirely absent from unrebalanced portfolios, but which are not dependent on any time structure in asset returns.

Equation (4) holds for any asset or portfolio. We can apply it to a diversified portfolio *p* and also to a portfolio containing a single asset *i*. Subtracting one of the resulting equations from the other gives us:

$$E[GM_p] - E[GM_i] \approx (E[AM_p] - \tfrac{1}{2}\sigma_p^2) - (E[AM_i] - \tfrac{1}{2}\sigma_i^2) \tag{5}$$

For simplicity we will assume that all these assets are IID[4]. Without this assumption, assets with larger expected returns are likely over time to comprise a larger proportion of an

[4] Fernholz and Shay (1982) [and…] similarly assume that all assets have identical expected growth rates.

unrebalanced portfolio, raising the expected return of the portfolio as a whole, as shown by Cheng and Deets (1971). By removing this effect, our assumption of IID asset returns simplifies the analysis − it is also generous to rebalanced portfolios. We show below that even with this assumption, unrebalanced portfolios give expected geometric returns equal to those of rebalanced portfolios with equal levels of volatility. Without it, unrebalanced portfolios would outperform. Rebalancing is worthwhile to the extent that it stops the portfolio variance from rising because the portfolio becomes more concentrated in the best-performing assets, but rebalancing does not have any direct effect on returns (the "rebalancing returns" that are widely claimed).

By definition, the expected portfolio AM is the weighted average of the expected AM of it component assets: $E[AM_p]=\Sigma w_i E[AM_i]$. With E[AM] assumed equal for every asset, $E[AM_p]=E[AM_i]$ regardless of the composition of the portfolio (we assume zero leverage, so portfolio weights always sum to unity). This gives us:

$$E[GM_p] - E[GM_i] \approx \tfrac{1}{2}(\sigma_i^2 - \sigma_p^2) \tag{6}$$

Booth and Fama (1992) define the "diversification return" as the degree to which the expected GM of a portfolio is greater than the weighted average of the expected GMs of its component assets. Our assumption of IID asset returns means that every asset has an identical E[GM] and so the weighted average of these component returns is the same for any unleveraged portfolio of these assets. Equation 6 thus represents the diversification return of shifting from a single asset to a portfolio of similar assets. It also tells us that this diversification return is entirely due to the associated reduction in portfolio volatility. This derivation makes it clear that the diversification return should be seen as simply the reduction in volatility drag caused by improved diversification. By contrast the "rebalancing return" is portrayed as resulting from a very different process, as rebalancing trades consistently buy on downticks and sell on upticks.

Multi-Asset Portfolios

We noted in section 2 above the widely-cited case in which a rebalanced portfolio which keeps a constant proportion Π of its value in a risky asset (and (1-Π) in cash). The risk-free and risky assets are both assumed to have zero expected growth rate, but a rebalanced portfolio nevertheless generates a positive expected growth of $g^* = \pi(1-\pi)\sigma^2 / 2$ per period. We return in Section 5 to consider the interpretation of this widely-quoted result, but in this section we first consider the more general case of portfolios containing more than one risky asset.

Fernholz and Shay (1982) claim that an equally-weighted portfolio of IID assets generates "excess growth" (a growth rate which is greater than the weighted average growth of its component assets) and so will outperform the corresponding unrebalanced portfolio, which they claim has the same expected growth rate as its component assets (i.e. zero excess growth). The higher growth achieved by the rebalanced portfolio they ascribe directly to the rebalancing process "buying on downticks and selling on upticks". This claim has been endorsed by subsequent papers (e.g. Qian, 2012, Luenberger, 1997, Bouchey et al., 2012, Willenbrock, 2012, Stein et al. 2012) and in the practitioner literature.

We can assess this claim directly and with complete generality by taking an unrebalanced portfolio and modelling it as two sub-portfolios which are initially of equal value. These sub-portfolios have growth paths described by the random variables x and y:

$$P_u = 0.5e^{x} + 0.5e^{y} \tag{7}$$

Using the Taylor expansion:

$$P_u \approx \frac{1}{2} + \frac{x}{2} + \frac{x^2}{4} + \frac{1}{2} + \frac{y}{2} + \frac{y^2}{4} \tag{8}$$

$$\Rightarrow P_u \approx 1 + \frac{x+y}{2} + \frac{1}{2}\left(\frac{x+y}{2}\right)^2 + \frac{1}{8}(x-y)^2 \approx e^{\frac{x+y}{2}} + \frac{1}{8}(x-y)^2 \tag{9}$$

Equation (9) shows that an unrebalanced portfolio has excess growth of $\frac{1}{8}(x-y)^2$. This is zero if P_u consists of only one asset, but if instead it can be divided into two non-identical sub-portfolios then this term is unambiguously positive.[5] Thus any diversified portfolio has "excess

[5] For clarity the analysis above expanded only to the quadratic terms, but the excess growth clearly remains positive if we add the third and fourth power terms of the expansion. Adding the extra terms to equation (5) gives us:

$$P_u \approx \frac{1}{2} + \frac{x}{2} + \frac{x^2}{4} + \frac{x^3}{12} + \frac{x^4}{48} + \frac{1}{2} + \frac{y}{2} + \frac{y^2}{4} + \frac{y^3}{12} + \frac{y^4}{48}$$

$$= 1 + \frac{x+y}{2} + \frac{1}{2}\left(\frac{x+y}{2}\right)^2 + \frac{1}{8}(x-y)^2 + \frac{1}{6}\left(\frac{x+y}{2}\right)^3 + \frac{3}{48}(x+y)(x-y)^2 + \frac{1}{24}\left(\frac{x+y}{2}\right)^4 + \frac{6}{384}(x^2-y^2)^2$$

$$+ \frac{1}{384}(x-y)^4$$

This gives us all the terms up to the fourth power in the Taylor expansion of $\exp\left(\frac{x+y}{2}\right)$ plus additional terms which represent "excess growth" by which growth in the portfolio exceeds the average (equally weighted in this case) of the growth rates of its components:

$$P_u \approx \exp\left(\frac{x+y}{2}\right) + \frac{1}{8}(x-y)^2 + \frac{3}{48}(x+y)(x-y)^2 + \frac{6}{384}(x^2-y^2)^2 + \frac{1}{384}(x-y)^4$$

growth" even if it is not rebalanced.[6] This directly contradicts the explicit claims made by proponents of rebalancing strategies. Labelling these "rebalancing returns" is extremely misleading – they are better regarded as the result of reduced volatility drag, since they clearly arise for any diversified portfolio.

We can demonstrate the same point using simulation. Such numerical techniques have been little used in this literature since without rebalancing portfolio asset weights tend to vary over time, making it difficult to derive like-for-like comparisons between rebalanced and unrebalanced strategies. Even when all asset returns are IID there are two effects which might be thought to cause E[GM] of rebalanced and unrebalanced portfolios to differ. First, unrebalanced

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If our portfolio consists of a single asset then x and y will be identical and the excess growth terms will all be equal to zero. If instead the portfolio is diversified then x and y are non-identical and these excess growth terms are:

$$Excess\ growth \approx \frac{1}{8}(x-y)^2 + \frac{3}{48}(x+y)(x-y)^2 + \frac{6}{384}(x^2-y^2)^2 + \frac{1}{384}(x-y)^4$$

$$= \frac{1}{64}(x-y)^2\left(4 + (x+y+2)^2 + \frac{1}{6}(x-y)^2\right)$$

This excess growth is clearly always positive for x≠y. We cannot rule out the possibility that over long horizons higher order terms will become significant, but at least for modest investment horizons (where x and y are well below 1) these can be safely ignored. Thus, contrary to what proponents of rebalancing explicitly claim, diversification unambiguously raises the expected growth rate of an unrebalanced portfolio, and this is true regardless of whether the portfolio is rebalanced.

[6] The growth rate g_u of a B&H portfolio over a horizon T can be expressed in terms of the growth rates g_i of its component assets as the following, since the terminal portfolio value is simply the sum of the terminal values of the component assets: $(1+g_u)^T = \sum_{i=1}^{n} w_i(1+g_i)^T$ where w_i are the initial portfolio weights of the assets. However, this complex non-linear relationship does not in general imply the simpler relationship $g_u = \sum_{i=1}^{n} w_i g_i$ that is sometimes alleged and which would if true imply that by definition a B&H portfolio generates no "excess growth".

strategies are likely to suffer greater "volatility drag" as they become less diversified over time, leading to higher portfolio volatility and a lower E[GM]. Second, it is widely claimed that there is an additional effect on E[GM] which is directly due to the act of rebalancing – a claim that we reject.

It is nevertheless possible to use simulations to disentangle these two possible effects. We simulate IID returns so that the mean portfolio return is the same in both strategies. We show that when the B&H and rebalanced portfolios have the same portfolio volatility, they have the same E[GM]. Hence, there is no additional impact on E[GM] from the act of periodic rebalancing to fixed weights. E[GM] is instead completely determined by portfolio volatility.

To simulate rebalanced and unrebalanced strategies with a wide range of different portfolio variances we use two approaches. First, we compare portfolios with different numbers of risky assets and second, we compare portfolios of two risky assets and alter the initial weights given to each assets.

We first consider portfolios containing N risky assets which we vary from N=2 to 100. For each value of N we simulate (i) an equally-weighted $1/N$ portfolio with rebalancing each month and (ii) an unrebalanced portfolio with initial weights $1/N$, but with the weights then evolve with relative asset returns (ie. B&H). As discussed above, monthly asset returns are assumed to be NIID, with annualized (arithmetic) mean 10% and standard deviation 20% per annum for each asset. For each portfolio we conduct 10,000 simulations, each over a horizon of 100 years.

[Figure 2 around here]

The results are shown in Figure 2. For both rebalanced and unrebalanced portfolios the portfolio variance falls as N rises. However for each level of N the unrebalanced portfolios have higher expected variances. Asset returns are all assumed identically distributed, so an equally weighted portfolio gives the minimum variance. Without rebalancing, the weights on each asset tend to diverge over time, leaving the portfolio less effectively diversified. The expected AM return remains constant for every portfolio (since the portfolio is unleveraged and all assets have identical expected AMs), but the geometric mean returns of these portfolios increase as their variances decrease, consistent with $E[GM] \approx E[AM] - \sigma^2/2$.

Figure 3 presents the same results, but with the average variance for each set of simulations plotted against the corresponding expected GM. The results for the rebalanced and unrebalanced portfolios now coincide. This shows that rebalancing affects the expected GM only to the extent that it affects the portfolio variance, and hence generates different levels of volatility drag. By contrast, if rebalancing generated returns by "buying low and selling high" as proponents suggest, we should expect different GMs for these portfolios even after correcting for their different variances.

[Figure 3 around here]

Next we examine the relationship between the expected GM and portfolio variance for portfolios of two risky assets with a range of different initial asset weights (Figure 4). A fixed

50:50 weighting is the minimum variance portfolio, with unrebalanced portfolios seeing higher variances as the portfolio weights subsequently drift over time. However if the initial portfolio weights are highly unequal (with one asset accounting for 86% or more of the portfolio) then the drift of these portfolio weights in the unrebalanced portfolios on average reduces variance because it can lead to weights becoming substantially more equal over time.

[Figure 4 around here]

Figure 5 plots the same results in terms of mean realised variance versus mean GM return for each set of simulations. The results coincide for rebalanced and unrebalanced portfolios just as they did for our earlier simulations. Furthermore, on this figure we have combined the results of both sets of simulations (i) varying the number of assets (with equal initial weights) and (ii) varying the initial weights in a two-asset portfolio. This shows that all these simulations describe the same linear relationship ($E[GM] \approx E[AM] - \sigma^2/2$), confirming that rebalancing only affects the average GM to the extent that it affects the average portfolio variance. The average GMs of our simulated portfolios differ by a maximum of only 0.8 basis points from those implied by this equation[7]. Thus the different E[GM]s of the rebalanced and unrebalanced portfolios can be

[7] Annex 1 shows that even though the portfolio variance shifts over time for an unrebalanced portfolio, when compounding over multiple short periods equation 1 is still a good approximation for the whole-horizon expected GM as a function of the average expected AM and average variance over this horizon. Using monthly (rather than continuous) compounding is inherently an approximation, but these results show that in these simulations it is a very good approximation.

entirely explained by the different degrees to which they suffer from volatility drag, and we have no evidence of "rebalancing returns" caused by the rebalancing trades themselves being profitable. For robustness we repeated the simulations using asset returns drawn from uniform and t distributions. These show the same result: that portfolios with the same variance generated the same expected GM regardless of whether they are rebalanced.

[Figure 5 around here]

Fernholz and Shay (1982) state that a fixed weights portfolio "buys on a downtick and sells on an uptick", and Luenberger (1997) that it will automatically "buy low and sell high". These authors claim that these effects boost returns even if asset returns follow a geometric Brownian motion, but Dempster et al. (2009) rightly note that any such profits are conditional on presume negative autocorrelation of returns. The rebalancing process will by construction sell some of an asset after a period in which it outperformed the rest of the portfolio, but this sale is only profitable if it takes place before a period (of whatever duration) of relative underperformance. Indeed, if rebalancing really did buy low and sell high, then it would increase the expected AM as well as the expected GM, but none of the proponents of rebalancing that we cite above claim that it does, and our simulations clearly show that it does not.

Similarly, Willenbrock (2011) argues that "the underlying source of the diversification return is the rebalancing", and Qian (2012) states that a "diversified portfolio, if left alone and not rebalanced, does not provide diversification return". These statements are misleading.

Rebalancing can be used to keep the portfolio at its minimum-variance weights and hence maximize the diversification return, but this does not imply that the diversification return will be zero in unrebalanced portfolios. Equation (9) and our simulations both clearly show that unrebalanced portfolios achieve higher expected GMs than their component assets as long as they retain some element of diversification.

Whether asset returns are autocorrelated in practice is an empirical question. Rebalancing will be profitable in markets which tend to mean-revert, and loss-making in markets which tend to trend (as assets which have underperformed in the past — and so are bought by the rebalancing strategy — tend to continue underperforming in the future, and vice versa). The misleading conclusion that rebalancing boosts expected GM returns even without any mean-reversion encourages investors to pursue strategies which may be inappropriate for the markets concerned.

4. Portfolios of One Risky Asset and One Risk-Free Asset

The previous section showed that a diversified portfolio generates an expected growth rate that is greater than the weighted average of its component assets. In the terminology used in this literature this shows that even unrebalanced portfolios generate "rebalancing returns". We demonstrated this algebraically in complete generality and numerically for portfolios of many IID assets (a scenario which should be least favourable to unrebalanced portfolios). In this section we return to a simpler situation: a portfolio consisting of risk-free deposits and a single risky asset with variance σ_a^2. This example is important, since it is widely used in the academic and practitioner literature and because the misleading conclusions drawn from it encourage investors to hold volatile assets and poorly diversified portfolios so as to maximise the scale of the rebalancing trades. It is important to understand how these papers have reached misleading conclusions.

These authors assume that the risk free and the risky asset have identical expected geometric mean returns. For simplicity we follow Dempster et al. (2007) and Qian (2012) in also normalising these returns to zero[8]. Under these assumptions these authors find that the expected geometric mean return of a portfolio which is 50% risky asset and 50% risk-free is $\sigma_a^2/8$. The fact that this return is achieved by combining two assets which each have zero expected GM makes this seem almost like achieving something out of nothing. Furthermore, maintaining the 50% asset weights requires the investor to rebalance by selling some of the risky asset after it has generated a positive return and buying some following a negative return. The positive GM return generated by this strategy is interpreted as resulting from these rebalancing trades. As Fernholz and Shay (1982) put it:

> "...a balanced cash-stock portfolio will buy on a downtick and sell on an uptick. The act of rebalancing the portfolio is like an infinitesimal version of buying at the lows and

[8] Without this normalisation, the AM and GM figures in Table 1 would all be increased by the risk-free rate. However the key result would remain unchanged: that the risky asset must by implication have E[AM] which is greater than the risk-free rate, and that this should be seen as the underlying source of the positive expected GM on the 50/50 portfolio, which has half the E[AM] of the risky asset but only one quarter of the volatility drag.

selling at the highs. The continuous sequence of fluctuations in the price of the stock produces a constant accrual of revenues to the portfolio."

The language used here suggests that these price movements are temporary and rapidly reversed but the authors explicitly assume a geometric Brownian motion in which the risky asset has an expected growth rate of zero, so after the asset price has initially diverged from its original value, the expected geometric return on any shares bought or sold at this new price is zero. This applies in every period, so any rebalancing trade shifts some wealth from one asset into another which is as likely to outperform as underperform the asset it replaces. This shift will raise the expected portfolio growth rate if it improves diversification and so reduces volatility drag, but the language used by proponents of rebalancing strategies suggests a very different effect is at work.

We can also demonstrate that the rebalancing trades are not the source of the increased expected growth rate by deriving the expected size of this increase without using any dynamic expressions, but instead merely using the standard arithmetic/geometric mean relationship ($E[GM] \approx E[AM] - \sigma^2/2$). This equation is of general applicability, and makes no presumption about rebalancing − it applies to the returns of both rebalanced and unrebalanced portfolios, and indeed to all positive numbers. In this case it tells us that the risk-free asset must have zero expected AM (since it is assumed to have zero GM and zero variance). The risky asset is assumed to have variance σ_a^2, but zero expected growth rate, so this equation tells us that it must have a positive expected AM of $\sigma_a^2/2$ (see Table 1). This positive arithmetic mean is generally not made explicit in discussions of this strategy, thus helping to maintain the impression that the expected geometric mean return on the 50:50 portfolio is caused by the rebalancing trades buying low and selling high.

Table 1: Expected AMs and GMs derived using E[GM]≈E[AM] - $\sigma^2/2$

	E[AM]	Variance	E[GM]
Risky asset	$\sigma_a^2/2$	σ_a^2	0
Risk-free asset	0	0	0
50:50 fixed weight portfolio	$\sigma_a^2/4$	$\sigma_a^2/4$	$\sigma_a^2/8$

The 50:50 portfolio has an expected AM equal to half that of the risky asset, but only one quarter of the variance. Thus it must have an expected growth rate of $\sigma_a^2/8$. Fernholz and Shay (1982) derive this result using stochastic calculus to model the dynamics of the portfolio, but it follows directly from the standard AM:GM relationship which applies for all assets and portfolios, regardless of whether they are rebalanced. This return is better interpreted as arising because the risky asset itself has a positive arithmetic mean return. For the 100% risky asset portfolio this positive expected AM is perfectly offset by volatility drag. By contrast, the 50/50 portfolio has an expected AM which is half as large, but it suffers only one quarter of the volatility drag, leaving it a positive expected GM. The positive E[GM] of the rebalanced portfolio is thus explained entirely by the reduced volatility drag, so there is no evidence of the buy-low/sell-high effects which are claimed.

Portfolio rebalancing played no part in deriving the size of the expected GM for this portfolio, but in practice if there is no rebalancing then the proportion *a* of the portfolio which is held in this single risky asset is likely to vary from one period to the next. The expected AM in any period increases in direct proportion to *a*, whilst the variance of the portfolio increases with a^2. Thus the expected GM in any period has a quadratic relationship[9] with *a*, with the maximum expected GM at $a = 0.5$ as shown in Figure 6.

[Figure 6 around here]

Rebalancing is required to maximise E[GM] by keeping the portfolio composition at 50/50. If *a* falls to zero then the portfolio is composed entirely of risk-free asset, with zero GM. If *a* rises to 1 then the portfolio is entirely risky asset and the volatility drag will completely offset the positive E[AM], leaving E[GM] zero. However, for short time horizons the proportion of the risky asset in the portfolio should not be expected to diverge substantially from its initial 50%, so

[9] $E[GM] = E[AM] – \sigma_p^2/2 = a\ \sigma_a^2/2 - a^2\sigma_a^2/2 = a(1- a)\ \sigma_a^2/2$. This result is derived by Qian (2012) and, in continuous time form, by Fernholz and Shay (1982). [If there is only a single risky asset, with an expected GM equal to the risk-free rate, then the 50:50 fixed weight portfolio will indeed be the most attractive option for an investor who wishes to maximise his expected portfolio GM. But in practice there are likely to be many alternative risky assets which are less than perfectly correlated. This allows clearly superior strategies to be constructed. If, for example, two or more assets have the same expected AM and variance then a portfolio of them (however weighted) will have the same expected AM, but a lower variance, and thus a higher expected GM. Combining this multi-asset portfolio with a fixed a% of cash will (for any a>0) generate an expected GM which is greater than that shown in Figure 6.]

the expected GM will be only slightly below the expected GM of the rebalanced portfolio. Annex 2 confirms this by showing that an unrebalanced portfolio which is initially 50% cash generates expected growth of approximately $\frac{\sigma^2 t}{8} - \frac{\sigma^4 t^2}{64}$, compared to $\frac{\sigma^2 t}{8}$ for the equivalent rebalanced portfolio. Thus expected growth for the unrebalanced portfolio is not the zero that is claimed. Indeed, for short time horizons the $\sigma^4 t^2$ term will be negligible, showing that the rebalanced and unrebalanced portfolios initially have identical expected growth. This gradual divergence of expected growth rates for these two portfolios is inconsistent with the claim that one always generates rebalancing returns and the other does not. By contrast, it is entirely consistent with the different growth rates being due to the gradual increase in volatility (and hence volatility drag) as the unrebalanced portfolio's composition gradually drifts away from its initial weights. Similarly, Annex 2 also shows that an unrebalanced portfolio containing multiple IID assets has expected growth which is non-zero, and which only gradually drifts away from the growth achieved by the corresponding rebalanced portfolio.

If instead the risky asset tends to revert to previous levels, then the rebalancing trades will generate additional profits that are not available to a B&H strategy. This can be illustrated using the simplest possible example of two consecutive periods of duration t over which the risky asset price evolves according to a geometric binomial distribution, either rising or falling by a factor $\sigma\sqrt{t}$. The portfolio is initially equally-weighted, with 0.5 in the risky asset and 0.5 in the (zero return) risk-free asset. If the risky asset rises in the first period, our portfolio then has $0.5(1+ \sigma\sqrt{t})$ in the risky and 0.5 in the risk-free asset. The rebalancing trade then sells half of the first period gains on the risky asset (an amount $0.25\sigma\sqrt{t}$) in order to return to equal weights. If the risky asset price falls in the second period (by $\sigma\sqrt{t}$) then this trade generates a profit of $\sigma^2 t/4$, and this is the amount by which the rebalanced portfolio outperforms the unrebalanced portfolio (which simply returns to its starting value, as do the assets held throughout both periods by the rebalanced strategy, so the difference between the terminal wealths generated by the rebalanced and unrebalanced portfolios is entirely due the profitability of the rebalancing trade). The rebalancing strategy makes an identical profit if the risky asset falls in the first period and then rebounds in the second. Conversely, the rebalanced portfolio underperforms in the other two scenarios, in which the risky asset either rises in both periods (in which case the rebalancing trade sold some of this asset at the end of the first period before it continued to outperform in the second) or falls in both periods (so the rebalancing trade buys more risky asset at the end of period 1 before its price falls in the second). Thus over this two period horizon a rebalancing strategy generates greater terminal wealth than the B&H strategy at a rate of $\sigma^2 t/8$ per period, but this is strictly contingent on the assumption that the risky asset price reverts to exactly its starting value at the end of the horizon.

The same result can be derived if the price of the risky asset follows a standard geometric Brownian motion: $S_t = S_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W(t)}$. Without loss of generality we normalise $S_0=1$, and we follow Fernholz and Shay (1982) in assuming that the risky asset has zero expected growth rate, implying that $\mu=\sigma^2/2$ and $S_t = e^{\sigma W(t)}$. A portfolio P_r which is constantly rebalanced to keep 50% in the risky asset and 50% in the risk-free asset will always have half the upward drift rate and half the standard deviation of the risky asset itself ($\mu=\sigma^2/4$ and standard deviation $\sigma/2$), giving us $P_r = e^{\frac{\sigma^2}{8}t+\frac{\sigma}{2}W(t)}$.

Fernholz and Shay (1982) note that whenever the unrebalanced portfolio returns to its initial weights (i.e. $W(t)=0$, implying $S_t=S_0$), the rebalanced portfolio will have outperformed by a factor $e^{\frac{\sigma^2}{8}t}$ (since $P_r = e^{\frac{\sigma^2}{8}t}$ whilst the unrebalanced portfolio $P_u = 0.5 + 0.5S_t$ will merely have returned to its initial value of 1). Thus the rebalanced portfolio has a growth rate (i.e. 'excess growth') of $\sigma^2 t/8$ compared to the zero growth rate of each of its component assets, but again this is contingent on the assumption that the risky asset returns at the end of the horizon to exactly its starting value.

One reason for the apparent confusion in the literature may be that this equally weighted risky/risk-free asset portfolio generates two very different forms of return which happen to be of equal magnitude. First, we have the extra terminal wealth generated by the rebalancing trades if the risky asset reverts to its starting value. Second, as we saw above, we have the diversification return which means that this rebalanced portfolio generates an expected growth rate of $\sigma^2 t/8$ whereas each component asset has a growth rate of zero (the assumption that $\mu=\sigma^2/2$ means that the risky asset has zero growth as the volatility drag perfectly offsets the positive drift μ, but the equally-weighted portfolio has half the drift ($\mu/2=\sigma^2/4$) and only one quarter of the volatility, implying volatility drag of only $\sigma^2/8$ and a portfolio growth rate of $\sigma^2/8$). These two effects have the same magnitude, but they use different metrics. The first is an increase in *terminal wealth* (a move along the horizontal axis in Figure 1) whilst the second is an increase in the *growth rate* (the vertical axis). Moreover the first is contingent on the risky asset price reverting to its starting value, whilst the second is not.

Fernholz/Shay (1982) is still very widely cited in support of rebalancing strategies. It supports its claim that such strategies benefit from “buying on downticks and selling on upticks” by noting, as above, that every time that the B&H portfolio returns to equal proportions the corresponding rebalanced portfolio will have outperformed, and also noting that as $t \to \infty$ “this will occur infinitely often with probability one”. This statement is formally correct, and it appears to imply that every rebalancing trade must eventually end up generating a profit, but it is very misleading to assume that this will be of any practical benefit to investors with finite horizons. Figure 7 shows that for typical parameter values it takes several millennia for the probability that the rebalanced portfolio outperforms the unrebalanced portfolio to get anywhere close to unity. Over horizons of up to 100 years P_r outperforms P_u in less than 70% of our 10,000 simulations[10].

[Figure 7 around here]

Furthermore, even though the proportion of paths which at some point return to $S_t=S_0$ tends to 1, each period the subset of paths which have not yet returned to S_0 will on average have diverged further from S_0 than in the previous period, so that a rebalancing trade made when it

[10] Gabay and Herlemont (2007) derive a closed form solution for the single risky asset case which shows similarly slow convergence to unity of the probability that the rebalanced portfolio outperforms.

first diverged from S_0 will on average be recording ever-larger losses.[11] It is only by taking both these subsets into account that we can make meaningful statements about the expected return on the rebalanced portfolio. Figure 8 shows that P_u outperforms at both tails of the distribution. Rebalancing trades are profitable on average on paths where S_t tends to mean revert, leaving only small positive or negative cumulative returns. Conversely, when S_t makes large cumulative moves in either direction (i.e. tends instead to trend) then P_r underperforms P_u.

[Figure 8 around here]

As time passes the left tail of the P_r distribution becomes vanishingly small, since the expected growth rate of this portfolio is positive, and fewer and fewer outturns see P_r below 0.5. But on the right tail, P_u shows more extreme outturns than P_r. Over time this tail represents a smaller and smaller probability space, but the average size of (P_u-P_r) in these cases keeps increasing. It is straightforward to demonstrate that the expected terminal wealth is in fact greater for the unrebalanced portfolio than the rebalanced[12]. This ever-more-extended, but ever-less-

[11] Similarly, we have the standard result that the expected length of time required for a geometric Brownian motion with zero expected growth rate ($\mu=\sigma^2/2$) to converge to any arbitrarily distant level is infinite. The time to convergence for many paths may be very short, and the proportion of paths which reach this level inevitably rises over time, but the remaining paths that have not yet converged will on average have moved in the opposite direction.

[12] As above, portfolio P_r is constantly rebalanced to keep 50% in the risky asset and 50% in a risk free asset evolves according to $P_r = e^{\frac{\sigma^2}{8}t+\frac{\sigma}{2}W(t)}$. Using a Taylor expansion and simplifying using the standard properties of the Wiener process (E[W(t)=0 and E[W^2(t)]=t) gives us: $E[P_r] \approx e^{\frac{\sigma^2 t}{4}}$. Similarly, $E[P_u] =$

likely right tail explains why this is so even though the probability that $P_r > P_u$ tends to 1 as time passes. It also tells us that over finite investment horizons it is very misleading to ignore this right tail by assuming (as Fernholz and Shay (1982) explicitly do) that P_r always outperforms P_u – a property that is only true for infinite investment horizons.

5. <u>Conclusion</u>

A sizeable literature has now developed concerning the effects of rebalancing strategies. Within this it is widely claimed that rebalancing strategies generate "rebalancing returns" by buying on downticks and selling on upticks. This paper demonstrates instead that the difference between the expected growth rates of rebalanced and unrebalanced portfolios of IID assets can be entirely explained by their different degrees of volatility drag, with no evidence of "rebalancing returns". This paper also shows that the arguments used by key proponents of rebalancing strategies are based on properties of returns at infinite horizons which are not applicable over practical investor lifetimes unless we assume that risky asset prices return to precisely their starting values.

$E[0.5 + 0.5S_t] \approx 0.5 + 0.5e^{\frac{\sigma^2 t}{2}}$ (the accuracy of these approximations depends on standard assumptions that higher moments are well behaved). By inspection, the ratio $E[P_r]/E[P_u]$ clearly tends to zero as the time horizon increases, and our simulations confirm that the average terminal value is higher for unrebalanced portfolios. Cheng and Deets (1971) demonstrate a similar result in discrete time and we noted its antecedents in continuous time in the introduction.

Specifically, we demonstrate that all diversified portfolios generate expected growth rates greater than the average growth of their component assets (the definition of "excess growth" used in this literature). This is in direct contradiction of the claims made by proponents of rebalancing strategies that such excess returns are the direct result of the process of rebalancing, and so are entirely absent from unrebalanced portfolios. By contrast, we show that unrebalanced portfolios initially generate the same expected growth as the corresponding rebalanced portfolios and these growth rates only diverge as the composition of the unrebalanced portfolio evolves.

The misleading arguments that are widespread in the literature have important implications, since they lead to a misinterpretation of the benefits of rebalancing. Specifically, they encourage investors to hold portfolios which are concentrated in volatile assets so as to increase the scale of the resulting rebalancing trades: "The pumping effect is obviously most dramatic when the original variance is high. After being convinced of this, you will likely begin to enjoy volatility, seeking it out for your investment rather than shunning it" (Luenberger, 1997). Investors would be better advised to seek to minimize volatility drag by diversifying effectively and to rebalance no more than is necessary to keep their portfolio compositions adequately close to their target allocations. Frequent rebalancing is likely to be costly due to transaction and market impact costs. Furthermore, the desire to maximise these transactions may push investors into sub-optimal asset allocations.

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Annex 1

Fama and Booth (1992) show that the continuously compounded holding period return is well approximated by the expected return expressed in continuously compounded terms minus a fraction of the variance of the simple returns.

$$E[\log(1+r)] \approx \log(1+E[r]) - \frac{\sigma^2}{2(1+E[r])^2} \tag{A1}$$

This equation holds in each period, so we can sum each side over periods 1 to *T*:

$$\sum_{t=1}^{T} E[\log(1+r_t)] \approx \sum_{t=1}^{T} \log(1+E[r_t]) - \sum_{t=1}^{T} \frac{\sigma_t^2}{2(1+E[r_t])^2} \tag{A2}$$

Rearranging and dividing through by *T*:

$$\frac{1}{T} E[\log\left(\prod_{t=1}^{T}(1+r_t)\right)] \approx \frac{1}{T}\sum_{t=1}^{T} \log(1+E[r_t]) - \frac{1}{T}\sum_{t=1}^{T} \frac{\sigma_t^2}{2(1+E[r_t])^2} \tag{A3}$$

$$\approx E[r_t] - \frac{1}{2T}\sum_{t=1}^{T} \sigma_t^2 \tag{A4}$$

If each period is short then $E[r_t]$ will be small and the continuously-compounded GM and AM above will be close to their more commonly-used discretely compounded equivalents. Thus we end up with a form of the standard relationship E[GM] ≈ E[AM] - $\sigma^2/2$ which applies even if the distribution of r_t varies over time: the expected GM over the whole multi-period horizon is approximately equal to the average expected return over these periods minus half of the average variance. The linear relationships shown in Figures 3 and 5 confirm that this relationship holds for the unrebalanced portfolio, whose E[AM] and variance shift over time.

Annex 2: Expected Growth Rates Of Rebalanced and Unrebalanced Portfolios

(a) Single Risky Asset

$\frac{dS}{S} = \mu dt + \sigma dW, \quad S_t = e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W(t)}$ where $\mu = \sigma^2/2$, $S_0 \equiv 1$

The rebalanced portfolio P_r follows $\frac{dPr}{Pr} = \frac{\mu}{2}dt + \frac{\sigma}{2}dW \Rightarrow P_r = e^{\frac{\sigma^2}{8}t + \frac{\sigma}{2}W(t)}$

$$\Rightarrow E[\log(P_r)] = \frac{\sigma^2 t}{8}$$

The unrebalanced portfolio $P_u = 0.5 + 0.5S_t = 0.5 + 0.5e^{\sigma W(t)}$

$$\log(P_u) = \log(0.5 + 0.5e^{\sigma W(t)})$$

A Taylor expansion of the exponent up to terms in $\sigma^4 W(t)^4$ gives:

$$\log(P_u) \approx \log\left(\frac{1}{2} + \frac{1}{2}\left(1 + \sigma W(t) + \frac{\sigma^2 W(t)^2}{2} + \frac{\sigma^3 W(t)^3}{6} + \frac{\sigma^4 W(t)^4}{24}\right)\right)$$

A Taylor expansion of the log up to terms in $\sigma^4 W(t)^4$ gives:

$$\begin{aligned}\log(P_u) \approx & \frac{\sigma W(t)}{2} + \frac{\sigma^2 W(t)^2}{4} + \frac{\sigma^3 W(t)^3}{12} + \frac{\sigma^4 W(t)^4}{48} \\ & - \frac{1}{2}\left(\frac{\sigma^2 W(t)^2}{4} + \frac{\sigma^3 W(t)^3}{4} + \frac{\sigma^4 W(t)^4}{16} + \frac{\sigma^4 W(t)^4}{12}\right) \quad + \frac{1}{3}\left(\frac{\sigma^3 W(t)^3}{8} + \frac{3\sigma^4 W(t)^4}{16}\right) \\ & - \frac{1}{4}\left(\frac{\sigma^4 W(t)^4}{16}\right)\end{aligned}$$

Substituting in standard assumptions for the cumulative Wiener term in geometric Brownian motion $E[W(t)] \equiv E[W(t)^3] \equiv 0$, $E[W(t)^2] = t$, $E[W(t)^4] = 3t^2$:

$$E[\log(P_u)] \approx E\left[\frac{\sigma^2 W(t)^2}{8} + \sigma^4 W(t)^4\left(\frac{1}{48} - \frac{1}{32} - \frac{1}{24} + \frac{1}{16} - \frac{1}{64}\right)\right] \quad \approx \frac{\sigma^2 t}{8} - \frac{\sigma^4 t^2}{64}$$

Thus for small horizons (where the above will be a very good approximation) the expected growth of P_u is identical to that of P_r. The expected growth rate of P_u gradually declines over time as the composition of the unrebalanced portfolio shifts. This is inconsistent with the key claim made by proponents of rebalancing that P_r generates expected excess growth at a constant rate of $\sigma^2/8$, whilst P_u has no rebalancing returns and hence zero excess growth. By contrast, the gradual decline of the expected growth rate of P_u is entirely consistent with our contention that this decline is due to increased volatility drag as the composition of the portfolio drifts away from the optimal 50:50 mix.

(b) Portfolio Of N Risky Assets

Each asset is assumed to follow a standard Wiener process $\frac{dS}{S} = \mu dt + \sigma dW$, implying

$S_t = e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W(t)}$ where S_0 is normalized to 1.

A rebalanced portfolio P_r which gives equal weight to each of n such assets follows:

$$\frac{dPr}{Pr} = \sum \frac{\mu_i}{n} dt + \sum \frac{\sigma_i}{n} dW_i$$

But the assets are IID:

$$\Rightarrow \frac{dPr}{Pr} = \mu_i dt + \frac{\sigma_i}{\sqrt{n}} \sum \frac{dW_i}{\sqrt{n}}$$

The constant composition of this rebalanced portfolio means that $\sum \frac{dW_i}{\sqrt{n}}$ has constant unit variance (like dW in the standard wiener process above). Thus application of Ito's lemma gives a similar result for P_r, which simply reflects the lower standard deviation $\sigma_i/\sqrt{n}$ of the rebalanced portfolio [and hence the reduced volatility drag].

$$\Rightarrow P_r = e^{(\mu - \frac{\sigma_i^2}{2n})t + \frac{\sigma_i}{\sqrt{n}} \sum W_i(t)}$$

$$\Rightarrow \frac{1}{t} E[\log(P_r)] = \mu - \frac{\sigma_i^2}{2n}$$

The unrebalanced portfolio $P_u = \frac{1}{n} \sum S_{it} = \frac{1}{n} \sum e^{(\mu - \frac{\sigma_i^2}{2})t + \sigma_i W_i(t)}$

These *IID* assets share a common trend term:

$$P_u = e^{(\mu - \frac{\sigma_i^2}{2})t} \frac{1}{n} \sum e^{\sigma_i W_i(t)}$$

$$\log(P_u) = \left(\mu - \frac{\sigma_i^2}{2}\right)t + \log\left(\frac{1}{n}\sum e^{\sigma_i W_i(t)}\right)$$

Taylor expansion of the exponent to the fourth power gives:

$$\log(P_u) = \left(\mu - \frac{\sigma_i^2}{2}\right)t + \log\left(\frac{1}{n}\sum\left(1 + \sigma_i W_i(t) + \frac{\sigma_i^2 W_i^2(t)}{2} + \frac{\sigma_i^3 W_i^3(t)}{6} + \frac{\sigma_i^4 W_i^4(t)}{24}\right)\right)$$

Taylor expansion of the log term to the fourth power gives:

$$\begin{aligned}\log(P_u) = {} & \left(\mu - \frac{\sigma_i^2}{2}\right)t + \frac{\Sigma\sigma_i W_i(t)}{n} + \frac{\Sigma\sigma_i^2 W_i^2(t)}{2n} + \frac{\Sigma\sigma_i^3 W_i^3(t)}{6n} + \frac{\Sigma\sigma_i^4 W_i^4(t)}{24n} \\ & - \frac{1}{2}\left(\frac{\left(\Sigma\sigma_i W_i(t)\right)^2}{n^2} + \frac{\left(\Sigma\sigma_i^2 W_i^2(t)\right)^2}{4n^2} + \frac{\left(\Sigma\sigma_i W_i(t)\right)\left(\Sigma\sigma_i^2 W_i^2(t)\right)}{n^2}\right. \\ & \left. + \frac{\left(\Sigma\sigma_i W_i(t)\right)\left(\Sigma\sigma_i^3 W_i^3(t)\right)}{3n^2}\right) + \frac{1}{3}\left(\frac{\left(\Sigma\sigma_i W_i(t)\right)^3}{n^3} + \frac{3\left(\Sigma\sigma_i W_i(t)\right)^2\left(\Sigma\sigma_i^2 W_i^2(t)\right)}{2n^3}\right) \\ & - \frac{1}{4}\left(\frac{\left(\Sigma\sigma_i W_i(t)\right)^4}{n^4}\right)\end{aligned}$$

Note that E[$W_i(t)$]≡E[$W_i^3(t)$]≡0, so E[off-diagonal terms]=0 so, for example, E[$(\Sigma W_i(t))^2$]= E[$\Sigma W_i^2(t)$]:

$$E[\log(P_u)] = \left(\mu - \frac{\sigma_i^2}{2}\right)t + \left(\frac{\sigma_i^2}{2n} - \frac{\sigma_i^2}{2n^2}\right)E\left[\Sigma W_i^2(t)\right] + \frac{\sigma_i^4}{24n}E\left[\Sigma W_i^4(t)\right]$$
$$- \frac{\sigma_i^4}{2}\, E\left[\frac{\left(\Sigma W_i^4(t) + \Sigma\Sigma W_i^2(t)W_j^2(t)\,\right)}{4n^2} + \frac{\left(\Sigma W_i^4(t) + 3\Sigma\Sigma W_i^2(t)W_j^2(t)\,\right)}{3n^2}\right]$$
$$+ \frac{\sigma_i^4}{3}\, E\left[\frac{3\Sigma W_i^4(t) + 6\Sigma\Sigma W_i^2(t)W_j^2(t)}{2n^3}\right] - \frac{\sigma_i^4}{4}\, E\left[\frac{\Sigma W_i^4(t) + 6\Sigma\Sigma W_i^2(t)W_j^2(t)}{n^4}\right]$$

$E\left[\Sigma W_i^2(t)\right] = nt,\; E\left[\Sigma W_i^4(t)\right] = 3nt^2$ (standard assumptions for the Wiener term) and E[$\Sigma\Sigma w_i^2 w_j^2$]=n(n-1)t^2 since the ΣΣ terms above are defined for i≠j.

$$\Rightarrow\; E[\log(P_u)] = \left(\mu - \frac{\sigma_i^2}{2n}\right)t + \sigma_i^4 E\left[\Sigma W_i^4(t)\right]\left(\frac{1}{24n} - \frac{1}{8n^2} - \frac{1}{6n^2} + \frac{1}{2n^3} - \frac{1}{4n^4}\right)$$
$$+ \sigma_i^4\, E\left[\Sigma\Sigma W_i^2(t)W_j^2(t)\,\right]\left(-\frac{1}{8n^2} - \frac{1}{2n^2} + \frac{1}{n^3} - \frac{6}{4n^4}\right)$$

$$= \left(\mu - \frac{\sigma_i^2}{2n}\right)t + \sigma_i^4 t^2\left[3\left(\frac{1}{24} - \frac{7}{24n} + \frac{1}{2n^2} - \frac{1}{4n^3}\right) - (n-1)\left(\frac{5}{8n} - \frac{1}{n^2} + \frac{3}{2n^3}\right)\right]$$

For small t the $\sigma_i^4 t^2$ term is negligible, so E[$\log(P_u)$] = E[$\log(P_r)$] = $\left(\mu - \frac{\sigma_i^2}{2n}\right)t$. Thereafter the $\sigma_i^4 t^2$ term becomes significant so E[$\log(P_u)$] < E[$\log(P_r)$]. The term in square brackets is zero for n=1 and negative for all n>1. An intuitive interpretation is that initially P_u=P_r, since the two portfolios start with identical composition. Thereafter the arithmetic mean of the two portfolios remains identical (since the underlying assets are assumed IID), but P_u becomes less well diversified over time, so it suffers from increasing volatility drag. But it is not true to claim (over finite horizons) that P_u never shows any excess growth.

FIGURE 1

Rebalanced and unrebalanced portfolios – varying number of assets

This chart shows how volatility in terminal wealth results in a lower expected growth rate, in accordance with equation (4). A zero-volatility terminal wealth outturn of (A) generates a growth rate (B), but although two equally likely outturns equidistant above and below (A) generate the same expected terminal wealth, they generate lower expected growth (C) because the curvature of the log function effectively penalises both exceptionally high and exceptionally low outturns.

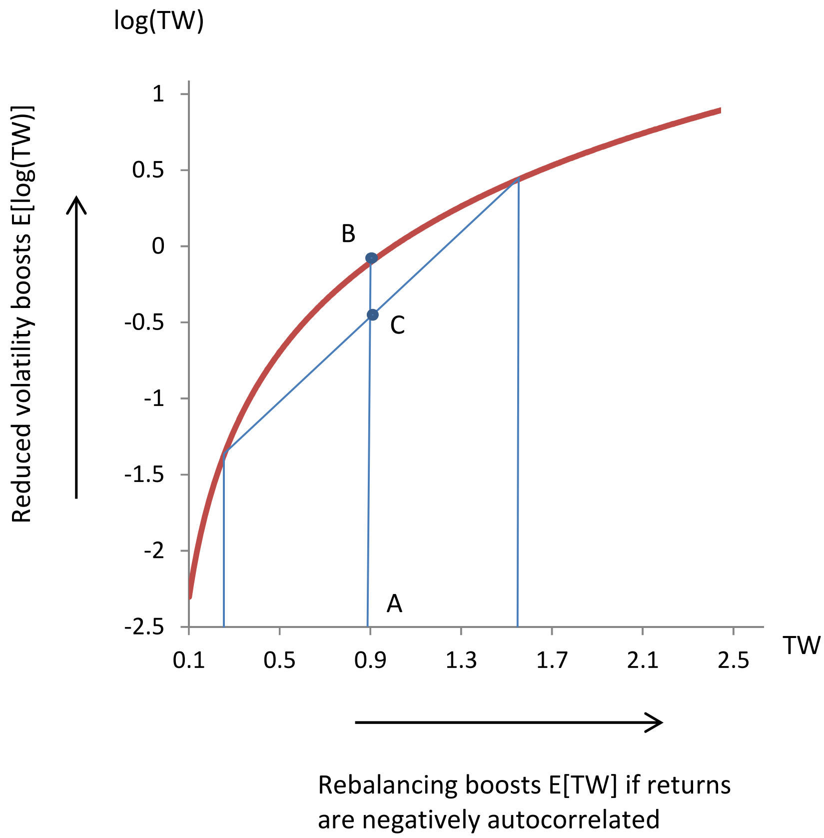

FIGURE 2

Rebalanced and unrebalanced portfolios – varying number of assets

This chart shows the annualized GM and variance of rebalanced and unrebalanced portfolios comprising different numbers (from 2 to 100) of component assets. This gives a total of 198 variants. Asset returns are assumed normal and IID, with annualized arithmetic mean 10% and variance 4%. Each path is calculated over 100 years, and the figure reports the average annualized GM and variance over 10,000 simulated paths.

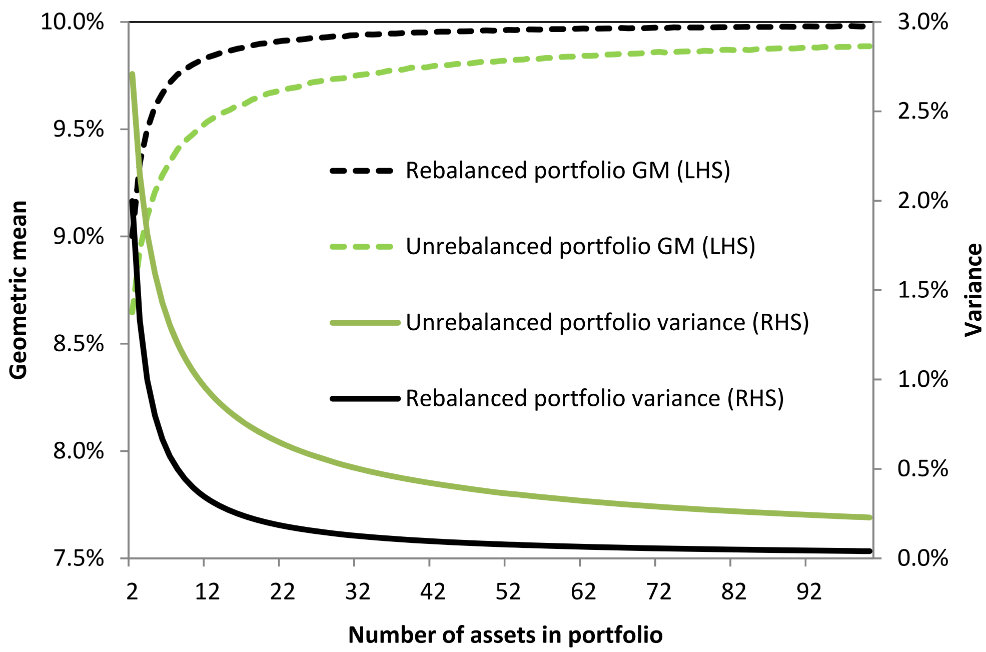

FIGURE 3

GM vs. Variance for Rebalanced and Unrebalanced Portfolios

(Varying Number of Assets)

This chart shows the same simulation results as in Figure 2, but plots the average annualized GM and variance for portfolios of each size N=2 to 100. The figure reports the average annualized GM and variance over 10,000 simulated paths for rebalanced and unrebalanced portfolios. The results show that the expected GM return depends on the average level of portfolio variance, but that for a given level of variance it makes no difference whether the portfolio is rebalanced or not.

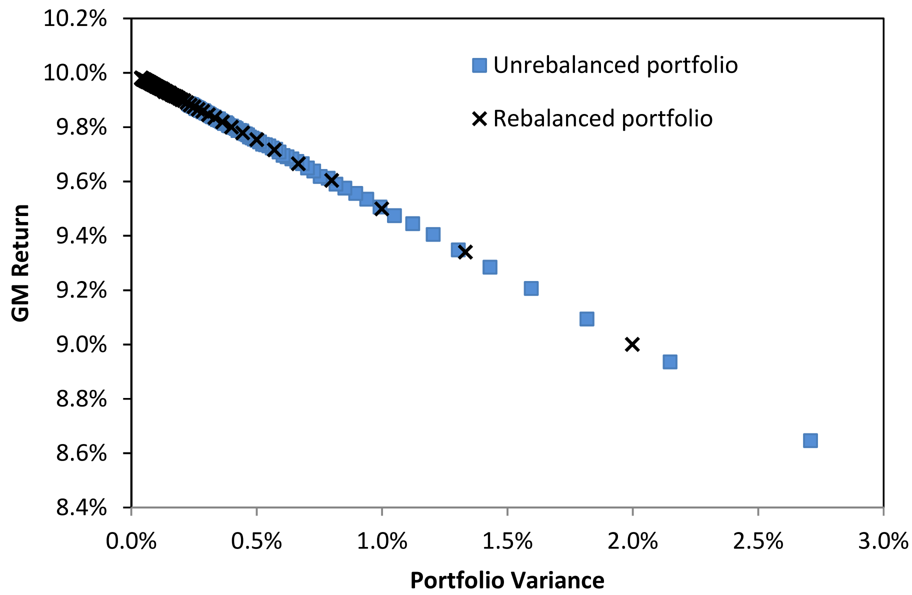

FIGURE 4

Rebalanced and Unrebalanced Portfolios – Varying Initial Portfolio Weights

This chart shows the annualized GM and variance of portfolios comprising two assets. The unrebalanced portfolio initially starts with the weight shown for asset A, but the weights are then allowed to evolve in line with relative asset returns. For the rebalanced portfolio the weight of asset A is returned at the end of each period to its initial value. Initial portfolio weights are given 101 different values (from 0% to 100% asset A) for both rebalanced and unrebalanced portfolios − a total of 202 variants. Asset returns are assumed normal and IID, with annualized arithmetic mean 10% and variance 4%. Each path is calculated over 100 years, and the figure reports the average GM and variance over 10,000 simulated paths.

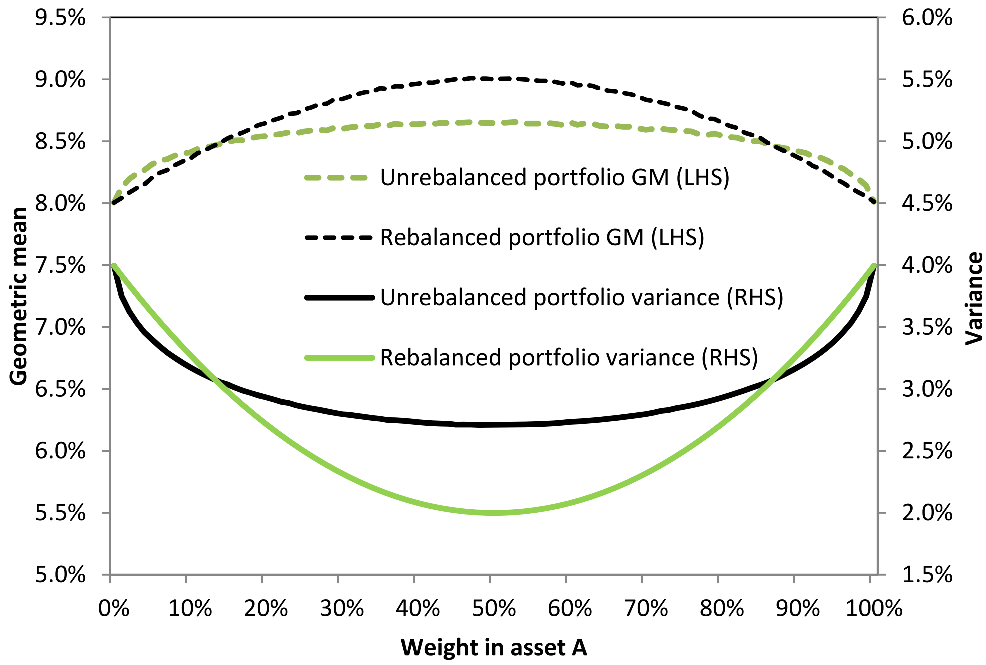

FIGURE 5

GM vs. Variance for Rebalanced and Unrebalanced Portfolios

This chart shows the same simulation results as in Figures 2 and 4, but plots the average annualized GM against the variance. The upper part of the figure shows the relationship between GM and variance when we vary the number of assets in the portfolio (which are initially equally weighted). The lower part of the figure shows the relationship between GM and variance for a two asset portfolio for different initial weights in asset A. The results show that rebalancing affects the expected GM return via its "volatility drag" effect on portfolio variance (following the linear relationship E[GM] ≈ E[AM] – $\sigma^2/2$), but it has no direct impact on the GM.

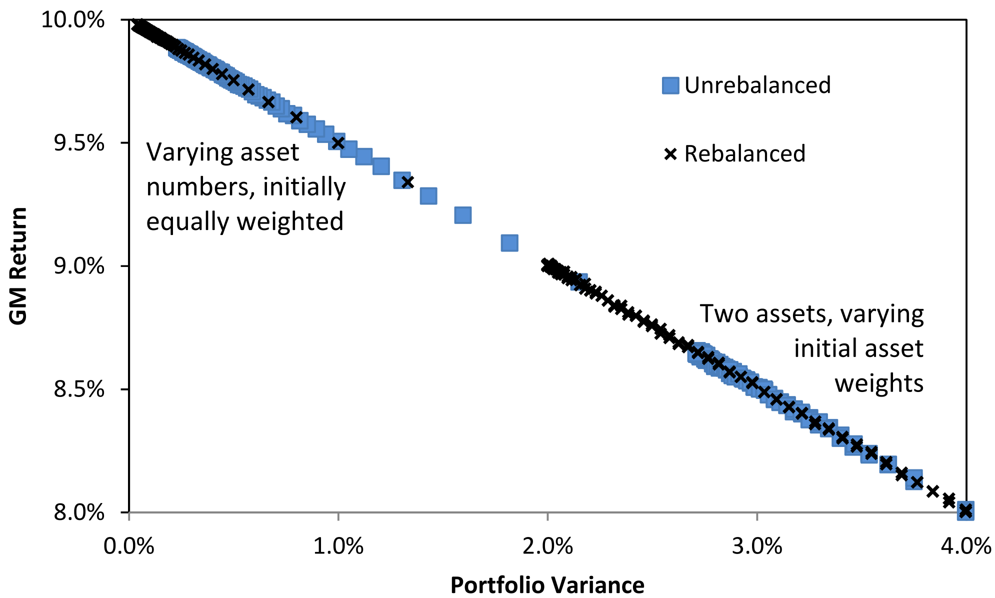

FIGURE 6

Volatility Pumping – Single Risky Asset

This chart shows the arithmetic and geometric means of portfolios comprising a risk-free asset (zero expected AM return and zero variance) and a risky asset (expected AM return 2%, and variance 4%). Equation (1) implies that the risky and risk-free assets each have zero expected GM, but for portfolios with a positive weight on each asset the expected GM must be positive.

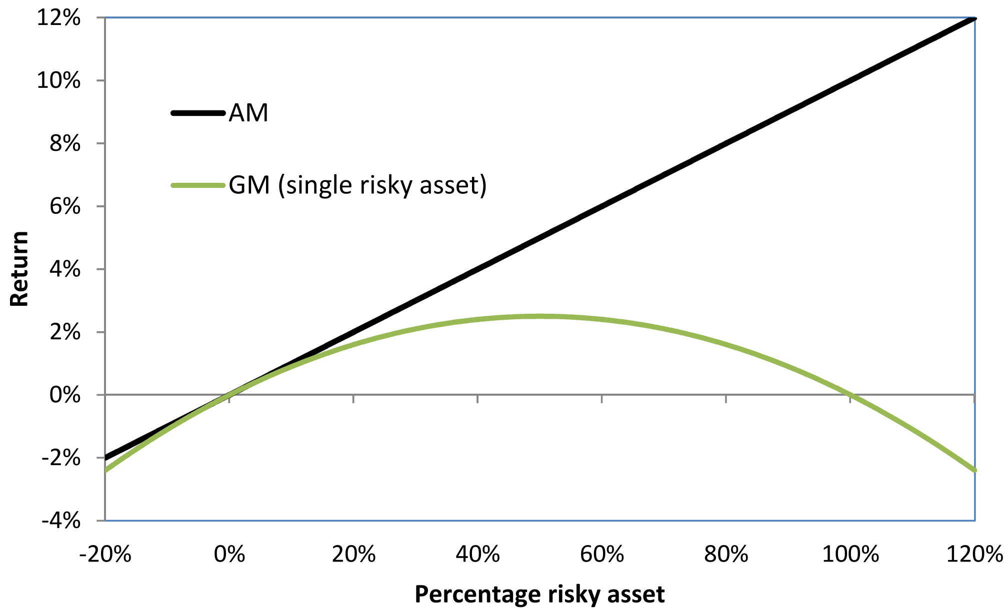

FIGURE 7

Proportion of Outcomes where $P_r > P_u$

This chart shows the proportion of the simulated paths for which the terminal wealth of the rebalanced portfolio P_r is greater than the unrebalanced portfolio P_u. This proportion is shown for simulations with a wide range of different time horizons. We follow Fernholz & Shay (1982) in assuming asset returns follow a geometric Brownian motion with zero expected geometric return. We consider two portfolios with starting value $1. Both invest $0.50 in a risk-free asset which has an interest rate of zero and $0.50 in the risky asset. The rebalanced portfolio rebalances back to 50/50 asset mix every month. We assume σ is 10% per annum for the risky asset (other simulations, not reported here, show our results are robust to alternative assumptions).

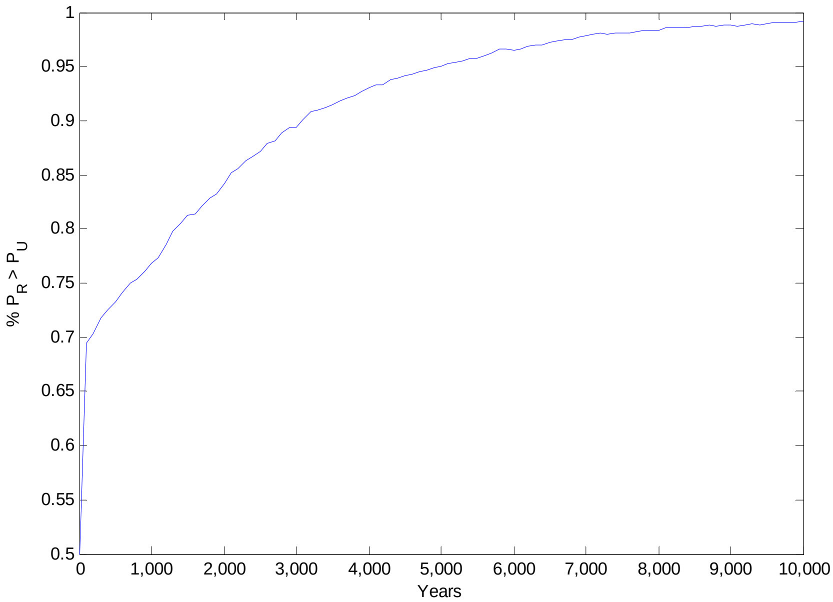

FIGURE 8

Probability Density of the Terminal Wealth of Rebalanced and Unrebalanced Portfolios after 100 Years

The chart shows the distribution of terminal wealth over 10,000 simulated paths of a 100 year time period. The parameters of the simulated values are the same as for Figure 7.