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On the Castelnuovo-Mumford regularity of the cohomology of fusion systems and of the Hochschild cohomology of block algebras

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Abstract

Symonds’ proof of Benson’s regularity conjecture implies that the regularity of the cohomology of a fusion system and that of the Hochschild cohomology of a $p$-block of a finite group is at most zero. Using results of Benson, Greenlees, and Symonds, we show that in both cases the regularity is equal to zero.

Let $p$ be a prime and $k$ an algebraically closed field of characteristic $p$. Given a finite group $G$, a block algebra of $kG$ is an indecomposable direct factor $B$ of $kG$ as a $k$-algebra. A defect group of $B$ is a minimal subgroup $P$ of $G$ such that $B$ is isomorphic to a direct summand of $B \otimes_k kP$ as a $B$-$B$-bimodule. The defect groups of $B$ form a $G$-conjugacy class of $p$-subgroups of $G$. The Hochschild cohomology of $B$ is the algebra $HH^*(B) = \text{Ext}_{B \otimes_k B^{op}}^*(B)$, where $B^{op}$ is the opposite algebra of $B$, and where $B$ is regarded as a $B \otimes_k B^{op}$-module via left and right multiplication. By a result of Gerstenhaber, the algebra $HH^*(B)$ is graded-commutative; that is, for homogeneous elements $\zeta \in HH^m(B)$ and $\eta \in HH^m(B)$ we have $\eta \zeta = (-1)^{mn} \zeta \eta$, where $m, n$ are nonnegative integers. In particular, if $p = 2$, then $HH^*(B)$ is commutative, and if $p$ is odd, then the even part $HH^{ev}(B) = \oplus_{n \geq 0} HH^{2n}(B)$ is commutative and all homogeneous elements in odd degrees square to zero. The extension of the Castelnuovo-Mumford regularity to graded-commutative rings with generators in arbitrary positive degrees is due to Benson [2, §4]. We follow the notational conventions in Symonds [18]. In particular, if $p$ is odd and $T = \oplus_{n \geq 0} T^n$ is a finitely generated graded-commutative $k$-algebra and $M$ a finitely generated graded $T$-module, we denote by $\text{reg}(T, M)$ the Castelnuovo-Mumford regularity of $M$ as a graded $T^{ev}$-module, where $T^{ev} = \oplus_{n \geq 0} T^{2n}$ is the even part of $T$. We set $\text{reg}(T) = \text{reg}(T, T)$; that is, $\text{reg}(T)$ is the Castelnuovo-Mumford regularity of $T$ as a graded $T^{ev}$-module. See also [3] and [8] for more background material and references. We note that Benson’s definition of regularity uses the ring $T$ instead of $T^{ev}$, but the two definitions are equivalent. This can be seen by noting that [18, Proposition 1.1] also holds for finitely generated graded commutative $k$-algebras.

**Theorem 0.1** Let $G$ be a finite group and $B$ a block algebra of $kG$. We have $\text{reg}(HH^*(B)) = 0$.

This will be shown as a consequence of a statement on Scott modules. Given a finite group $G$ and a $p$-subgroup $P$ of $G$, there is up to isomorphism a unique
Let $\text{Theorem 0.4}$ systems arise as fusion systems of finite groups. There are examples of fusion properties of the regularity from [18, §2], [4], and Symonds’ proof in [18] of Benson’s regularity conjecture. We use the spectral sequence [9, Theorem 2.1], results and techniques in work of Benson [1], [2], [4], and Symonds’ proof in [18] of Benson’s regularity conjecture. We use the properties of the regularity from [18, §1] and [19, §2].

indecomposable $kG$-module $\text{Sc}(G; P)$ with vertex $P$ and trivial source having a quotient (or equivalently, a submodule) isomorphic to the trivial $kG$-module $k$. The module $\text{Sc}(G; P)$ is called the Scott module of $kG$ with vertex $P$. It is constructed as follows: Frobenius reciprocity implies that $\text{Hom}_{kG}(\text{Ind}^G_P(k), k) \cong \text{Hom}_{kP}(k, k) \cong k$, and hence $\text{Ind}^G_P(k)$ has up to isomorphism a unique direct summand $\text{Sc}(G; P)$ having $k$ as a quotient. Since $\text{Ind}^G_P(k)$ is selfdual, the uniqueness of $\text{Sc}(G; P)$ implies that $\text{Sc}(G; P)$ is also selfdual, and hence $\text{Sc}(G; P)$ can also be characterised as the unique summand, up to isomorphism, of $\text{Ind}^G_P(k)$ having a nonzero trivial submodule. Moreover, it is not difficult to see that $\text{Sc}(G; P)$ has $P$ as a vertex. See [7] for more details on Scott modules, as well as [11] for connections between Scott modules and fusion systems. For a finitely generated graded module $X$ over $H^*(G; k)$ we denote by $H^*_m(X)$ the local cohomology with respect to the maximal ideal of $H^*(G; k)$ generated by all elements in positive degree. The first grading is here the local cohomological grading, and the second is induced by the grading of $X$.

**Theorem 0.2** Let $G$ be a finite group and $P$ a $p$-subgroup of $G$. We have

$$\text{reg}(H^*(G; k); H^*(G; \text{Sc}(G; P))) = 0.$$  

**Remark 0.3** Using Benson’s reinterpretation in [1, §4], of the ‘last survivor’ from [5, §7], applied to the Scott module instead of the trivial module, one can show more precisely that

$$H^r_m - r(H^*(G; \text{Sc}(G, P))) \neq \{0\},$$

where $r$ is the rank of $P$. It is not clear whether this property, or even the property of having cohomology with regularity zero, characterises Scott modules amongst trivial source modules.

For $\mathcal{F}$ a saturated fusion system on a finite $p$-group $P$, we denote by $H^*(P; k)^{\mathcal{F}}$ the graded subalgebra of $H^*(P; k)$ consisting of all elements $\zeta$ satisfying $\text{Res}_Q^P(\zeta) = \text{Res}_\varphi(\zeta)$ for any subgroup $Q$ of $P$ and any morphism $\varphi: Q \to P$ in $\mathcal{F}$. If $\mathcal{F}$ is the fusion system of a finite group $G$ on one of its Sylow-$p$-subgroups $P$, then $H^*(P; k)^{\mathcal{F}}$ is isomorphic to $H^*(G; k)$ through the restriction map $\text{Res}_G^P$, by the characterisation of $H^*(G; k)$ in terms of stable elements due to Cartan and Eilenberg. In that case we have $\text{reg}(H^*(P; k)^{\mathcal{F}}) = 0$ by [18, Corollary 0.2]. If $\mathcal{F}$ is the fusion system of a block algebra $B$ of $kG$ on a defect group $P$, then $H^*(P; k)^{\mathcal{F}}$ is the block cohomology $H^*(B)$ as defined in [14, Definition 5.1]. It is not known whether all block fusion systems arise as fusion systems of finite groups. There are examples of fusion systems which arise neither from finite groups nor from blocks; see [10], [13].

**Theorem 0.4** Let $\mathcal{F}$ be a saturated fusion system on a finite $p$-group $P$. We have

$$\text{reg}(H^*(P; k)^{\mathcal{F}}) = 0.$$  

The key ingredients for proving the above results are Greenlees’ local cohomology spectral sequence [9, Theorem 2.1], results and techniques in work of Benson [1], [2], [4], and Symonds’ proof in [18] of Benson’s regularity conjecture. We use the properties of the regularity from [18, §1] and [19, §2].
Lemma 0.5 Let $G$ be a finite group and $V$ an indecomposable trivial source $kG$-module. Then $\text{reg}(H^*(G; k); H^* (G; V)) \leq 0$.

Proof Since $V$ is a direct summand of $\text{Ind}_P^G(k)$, we have

$$\text{reg}(H^*(G; k); H^* (G; V)) \leq \text{reg}(H^*(G; k); H^* (G; \text{Ind}_P^G(k))).$$

By [12, Lemma 4], the right side is equal to $\text{reg}(H^*(P; k))$, hence zero by [18, Corollary 0.2].

Lemma 0.6 Let $G$ be a finite group and $V$ a finitely generated $kG$-module. If $H_0(G; V) \neq \{0\}$, then $\text{reg}(H^*(G; k); H^* (G; V)) \geq 0$.

Proof It follows from the assumption $H_0(G; V) \neq \{0\}$ and Greenlees’ spectral sequence [9, Theorem 2.1] that there is an integer $s$ such that $H_0^{n-s}(H^*(G; V)) \neq \{0\}$, which implies the result.

□

Proof of Theorem 0.2 Set $V = \text{Sc}(G; P)$. By Lemma 0.5 we have

$$\text{reg}(H^*(G; k); \text{Ext}^n_{kG}(k; V)) \leq 0.$$

Since $V$ has a nonzero trivial submodule, we have $H_0(G; V) \neq \{0\}$, and hence the other inequality follows from Lemma 0.6. □

Theorem 0.1 will be a consequence of Theorem 0.2 and the following well-known observation (for which we include a proof for the convenience of the reader; the block theoretic background material can be found in [20]).

Lemma 0.7 Let $G$ be a finite group, $B$ a block algebra of $kG$ and $P$ a defect group of $B$. As a module over $kG$ with respect to the conjugation action of $G$ on $B$, the $kG$-module $B$ has an indecomposable direct summand isomorphic to the Scott module $\text{Sc}(G; P)$.

Proof Since the conjugation action of $G$ on $B$ induces the trivial action on $Z(B)$ and since $Z(B) \neq \{0\}$, it follows that the $kG$-module $B$ has a nonzero trivial submodule. Moreover, $B$ is a direct summand of $kG$, hence $B$ is a $p$-permutation $kG$-module, and the vertices of the indecomposable direct summands of $B$ are conjugate to subgroups of $P$. Thus $B$ has a Scott module with a vertex contained in $P$ as a direct summand. Since $Z(B)$ is not contained in the kernel of the Brauer homomorphism $\text{Br}_P$, it follows that $B$ has a direct summand isomorphic to the Scott module $\text{Sc}(G; P)$. □

Proof of Theorem 0.1 By [12, Proposition 5] we have $\text{reg}(HH^*(B)) \leq 0$. Recall that $HH^*(kG)$ is an $H^*(G; k)$-module via the diagonal induction map, and we have a canonical graded isomorphism $HH^*(B) \cong H^*(G; B)$ as $H^*(G; B)$-modules where $G$ acts on $B$ by conjugation; see e. g. [17, (3.2)]. It follows from [12, Lemma 4] that

$$\text{reg}(HH^*(B)) = \text{reg}(H^*(G; k); H^*(G; B)).$$
By Lemma 0.7, the $kG$-module $B$ has a direct summand isomorphic to $V = Sc(G; P)$, where $P$ is a defect group of $B$. Thus as an $H^*(G; k)$-module, $H^*(G; B)$ has a direct summand isomorphic to $H^*(G; V)$. It follows that

$$\text{reg}(HH^*(B)) \geq \text{reg}(H^*(G; k); H^*(G; V)) = 0,$$

where the last equality is from Theorem 0.2. This completes the proof of Theorem 0.1. \hfill \Box

**Remark 0.8** The above proof can be adapted to show that the regularity of the stable quotient $\underline{HH}^*(B)$ of $HH^*(B)$ also equals zero. Recall that $\underline{HH}^*(B)$ is the quotient of $HH^*(B)$ by the ideal $Z^p(B) = \text{Tr}^G_1(B)$ of $Z(B) \cong HH^0(B)$. Note that $Z^p(B)$ is concentrated in degree 0. Alternatively, $\underline{HH}^*(B)$ may be defined as the non-negative part of the Tate Hochschild cohomology of $B$. Our interest in $\underline{HH}^*(B)$ comes from the fact that Tate Hochschild cohomology of symmetric algebras is an invariant of stable equivalence of Morita type. We briefly indicate how the regularity of $\underline{HH}^*(B)$ may be calculated. Let $B = \oplus_i M_i$ be a decomposition of $B$ into a direct sum of indecomposable $kG$-modules $M_i$, where $G$ acts by conjugation on $B$. The canonical graded $H^*(G; k)$-module isomorphism $HH^*(B) \cong H^*(G; B)$ induces an isomorphism

$$HH^0(B) \cong H^0(G; B) = \oplus_i H^0(G; M_i)$$

in degree zero. Composing this with the the canonical isomorphisms $Z(B) \cong HH^0(B)$ and $H^0(G; M_i) \cong M_i^G$, it is easy to check that the image of $Z^p(B)$ in $\oplus_i M_i^G$ is $\oplus_i \text{Tr}^G_1(M_i)$. Since $B$ is a $p$-permutation $kG$-module, $\text{Tr}^G_1(M_i)$ is non-zero precisely if $M_i$ is isomorphic to the Scott module $Sc(G; 1)$ (which is a projective cover of the trivial $kG$-module). Let $M'$ denote the sum of all $M_i$'s in the above decomposition which are isomorphic to $Sc(G; 1)$ and let $M''$ be the complement of $M'$ in $B$ with respect to the above decomposition. Since $Z^p(B)$ is concentrated in degree zero, we have a direct sum decomposition $HH^*(B) \cong \oplus H^*(G; M'') \oplus Z^p(B)$ as $H^*(G; k)$-modules. In particular,

$$\text{reg}(H^*(G; k); HH^*(B)) = \max\{\text{reg}(H^*(G; k); H^*(G; M'')), \text{reg}(H^*(G; k); Z^p(B))\}.$$

We may assume that a defect group $P$ of $B$ is non-trivial. By Lemma 0.7, $M''$ contains a direct summand isomorphic to $Sc(G; P)$. Hence by Theorem 0.2

$$\text{reg}(H^*(G; k); H^*(G; M'')) \geq 0.$$ 

It follows from Theorem 0.1 and the above displayed equation that $\underline{HH}^*(B) \cong H^*(G; M'')$ has regularity zero.

**Proof of Theorem 0.4** By [18, Proposition 6.1] we have $\text{reg}(H^*(P; k))^F \leq 0$. For the other inequality we follow the arguments in [1, §3, §4], applied to transfer maps using fusion stable bisets. For $Q$ a subgroup of $P$ and $\varphi : Q \to P$ an injective group homomorphism, we denote by $P \times_{(Q, \varphi)} P$ the $P$-$P$-biset of equivalence classes in $P \times P$ with respect to the relation $(uw, v) \sim (u, \varphi(w)v)$, where $u, v \in P$, and $w \in Q$. The $kP$-$kP$-bimodule having $P \times_{(Q, \varphi)} P$ as a $k$-basis is canonically isomorphic to $kP \otimes_k Q (\varphi kP)$. This biset gives rise to a transfer map $\text{tr}_{P \times (Q, \varphi) P}$ on $H^*(P; k)$.
obtained by composing the restriction map \( \text{res}_\varphi^P : H^*(P; k) \to H^*(\varphi(Q); k) \), the isomorphism \( H^*(\varphi(Q); k) \cong H^*(Q; k) \) induced by \( \varphi \), and the transfer map \( \text{tr}_Q^P : H^*(Q; k) \to H^*(P; k) \). Let \( X \) be an \( \mathcal{F} \)-stable \( P \)-biset satisfying the conclusions of [6, Proposition 5.5]. That is, every transitive subbiset of \( X \) is isomorphic to \( P \times_{(Q, \varphi)} P \) for some subgroup \( Q \) of \( P \) and some group homomorphism \( \varphi : Q \to P \) belonging to \( \mathcal{F} \), the integer \( |X|/|P| \) is prime to \( p \), and for any subgroup \( Q \) of \( P \) and any group homomorphism \( \varphi : Q \to P \) in \( \mathcal{F} \), the \( Q \)-\( P \)-biset \( X \) and \( X_\varphi \) are isomorphic. By taking the sum, over the transitive subbisets \( P \times_{(Q, \varphi)} P \) of the transfer maps \( \text{tr}_{P \times_{(Q, \varphi)} P} \), we obtain a transfer map \( \text{tr}_X \) on \( H^*(P; k) \). Following [15, Proposition 3.2], the map \( \text{tr}_X \) acts as multiplication by \( |X|/|P| \) on \( H^*(P; k)^{\mathcal{F}} \), hence \( \text{Im}(\text{tr}_X) = H^*(P; k)^{\mathcal{F}} \), and we have a direct sum decomposition

\[
H^*(P; k) = H^*(P; k)^{\mathcal{F}} \oplus \ker(\text{tr}_X)
\]
as \( H^*(P; k)^{\mathcal{F}} \)-modules. A similar decomposition holds for Tate cohomology, and for homology (using either the canonical duality \( H_n(P; k) \cong H^\mathcal{F}(P; k)^\vee \) or the isomorphism \( H_n(P; k) \cong H_{-n-1}(P; k) \) obtained from composing the previous duality with Tate duality). By [1, Equation (4.1)], the transfer map \( \text{tr}_Q^P \) induces a homomorphism of Greenlees’ local cohomology spectral sequences

\[
\begin{array}{ccc}
H^{i,j}_m H^*(Q; k) & \longrightarrow & H_{-i-j}(Q; k) \\
\text{th}_k^\varphi \downarrow & & \downarrow \text{res}_k^\varphi \ast \\
H^{i,j}_m H^*(P; k) & \longrightarrow & H_{-i-j}(P; k)
\end{array}
\]

where \( \text{th}_k^\varphi \) and \( \text{res}_k^\varphi \ast \) are the maps induced by \( \text{tr}_Q^P \) and the inclusion \( Q \to P \), respectively. The isomorphism \( \varphi : Q \to \varphi(Q) \) induces an obvious isomorphism of spectral sequences

\[
\begin{array}{ccc}
H^{i,j}_m H^*(\varphi(Q); k) & \longrightarrow & H_{-i-j}(\varphi(Q); k) \\
= & & = \\
H^{i,j}_m H^*(Q; k) & \longrightarrow & H_{-i-j}(Q; k)
\end{array}
\]

Restriction and transfer on Tate cohomology are dual to each other under Tate duality, and hence the dual version of [1, Equation (4.1)] implies that the restriction \( \text{res}_{\varphi(Q)}^P \) induces a homomorphism of spectral sequences

\[
\begin{array}{ccc}
H^{i,j}_m H^*(P; k) & \longrightarrow & H_{-i-j}(P; k) \\
\text{res}_{\varphi(Q)}^P \downarrow & & \downarrow \text{tr}_{\varphi(Q)}^P \ast \\
H^{i,j}_m H^*(\varphi(Q); k) & \longrightarrow & H_{-i-j}(\varphi(Q); k)
\end{array}
\]

Composing the three diagrams above yields a homomorphism induced by \( \text{tr}_{P \times_{(Q, \varphi)} P} \) on the spectral sequence for \( P \), and taking the sum over all transitive subbisets of
X yields a homomorphism of spectral sequences
\[
\begin{array}{ccc}
H^i_m H^*(P; k) & \longrightarrow & H_{-i-j}(P; k) \\
(\text{tr}_X)_* & \downarrow & (\text{tr}_X^\vee)_* \\
H^i_m H^*(P; k) & \longrightarrow & H_{-i-j}(P; k)
\end{array}
\]
where \(X^\vee\) is the \(P\)-\(P\)-biset \(X\) with the opposite action \(u \cdot x \cdot v = v^{-1}xu^{-1}\) for all \(u, v \in P\) and \(x \in X\). One easily checks that \(X^\vee\) is isomorphic to a dual basis of \(X\) in the dual bimodule \(\text{Hom}_k(kX, k)\). By [6, Proposition 5.2], \(H^*(P; k)\) is finitely generated as a module over \(H^*(P; k)^F\). Thus the local cohomology spaces \(H^i_m H^*(P; k)\) can be calculated using for \(m\) the maximal ideal of positive degree elements in \(H^*(P; k)^F\) instead of \(H^*(P; k)\). It follows that \(\text{tr}_X\) induces a homomorphism of spectral sequences
\[
\begin{array}{ccc}
H^i_m H^*(P; k) & \longrightarrow & H_{-i-j}(P; k) \\
(\text{tr}_X)_* & \downarrow & (\text{tr}_X^\vee)_* \\
H^i_m H^*(P; k)^F & \longrightarrow & H_{-i-j}(P; k)^F
\end{array}
\]
For \(i = -j = r\), where \(r\) is the rank of \(P\), the edge homomorphism yields a commutative diagram of the form
\[
\begin{array}{ccc}
H^r_m H^*(P; k) & \overset{\gamma_P}{\longrightarrow} & H_0(P; k) \\
(\text{tr}_X)_* & \downarrow & (\text{tr}_X^\vee)_* \\
H^r_m H^*(P; k)^F & \overset{\delta_F}{\longrightarrow} & H_0(P; k)^F
\end{array}
\]
where the right vertical map is multiplication on \(k\) by \(\frac{|X|}{|P|}\). By [1, Theorem 4.1], the map \(\gamma_P\) is surjective, and hence so is the map \(\delta_F\). In particular, \(H^r_m H^*(P; k)^F \neq \{0\}\), whence the result.

\textbf{Remark 0.9} The fact that transfer and restriction on Tate cohomology are dual to each other under Tate duality can be deduced from a more general duality for transfer maps on Tate-Hochschild cohomology of symmetric algebras induced by bimodules which are finitely generated projective as left and right modules (cf. [16]).

\textbf{References}