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Massive Symmetric Space sine-Gordon Soliton Theories and Perturbed Conformal Field Theory

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Abstract

The perturbed conformal field theories corresponding to the massive Symmetric Space sine-Gordon soliton theories are identified by calculating the central charge of the unperturbed conformal field theory and the conformal dimension of the perturbation. They are described by an action with a positive-definite kinetic term and a real potential term bounded from below, their equations of motion are non-abelian affine Toda equations and, moreover, they exhibit a mass gap. The unperturbed CFT corresponding to the compact symmetric space G/G_0 is either the WZNW action for G_0 or the gauged WZNW action for a coset of the form $G_0/U(1)^p$. The quantum integrability of the theories that describe perturbations of a WZNW action, named Split models, is established by showing that they have quantum conserved quantities of spin $+3$ and -3. Together with the already known results for the other massive theories associated with the non-abelian affine Toda equations, the Homogeneous sine-Gordon theories, this supports the conjecture that all the massive Symmetric Space sine-Gordon theories will be quantum integrable and, hence, will admit a factorizable S-matrix. The general features of the soliton spectrum are discussed, and some explicit soliton solutions for the Split models are constructed. In general, the solitons will carry both topological charges and abelian Noether charges. Moreover, the spectrum is expected to include stable and unstable particles.

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1 Introduction.

The non-abelian affine Toda (NAAT) equations [1, 2] are integrable (multi-component) generalizations of the sine-Gordon equation for a bosonic field that takes values in a non-abelian Lie group, in contrast with the usual Toda field theories where the field takes values in the (abelian) Cartan subgroup of a Lie group. In [3] (see also [4]), it was found the subset of NAAT equations that can be written as the classical equations of motion of an action with a positive-definite kinetic term, a real potential term bounded from below and, moreover, with a mass-gap in order to make possible an S-matrix description. The resulting theories were named Homogeneous sine-Gordon (HSG) and Symmetric Space sine-Gordon (SSSG) theories, which are associated with the different compact Lie groups and compact symmetric spaces, respectively. Actually, the HSG and SSSG theories of [3] are particular examples of the deformed coset models constructed by Park in [5] and the Symmetric Space sine-Gordon models constructed by Bakas, Park and Shin in [6], respectively, where the specific form of the potential makes them exhibit a mass gap.

There are some general features of these theories that have to be emphasized. The first one is that all of them have soliton solutions. Taking into account that both the HSG and SSSG theories are described by actions with sensible properties, this is in contrast with the usual affine Toda field theories where the condition of having soliton solutions (imaginary coupling constant) always leads to an ill defined action which makes the quantum theory problematic [7]. The second is that they are defined by an action of the form

$$S[h] \;=\; \frac{1}{\beta^2}\left\{S_{WZNW}[h] \;-\; \int d^2x\, V(h)\right\}, \tag{1.1}$$

where $h = h(x,t)$ is a field that takes values in a non-abelian compact group G, $V(h)$ is some potential function on the group manifold, and S_{WZNW} is either the WZNW action for the group G or the gauged WZNW action for a coset of the form G/H, where H is an abelian subgroup of G to be specified. Therefore, if the quantum theory is to be well defined then the coupling constant has to be quantized: $\beta^2 = 1/k$, for some positive integer k (see (3.9) for a more precise form of this quantization rule). Such a quantization does not occur in the sine-Gordon theory or the usual affine Toda theories because the field takes values in an abelian group in those cases. An important consequence of this is that, in the quantum theory, the β^2 will not be a continuous coupling constant. However, the quantum theory will have other continuous coupling constants that appear in the potential and, in particular, determine the mass spectrum. A third feature which follows from (1.1) is that these theories are naturally described as perturbations of conformal field theory (CFT) coset models. Therefore, they will provide a Lagrangian formulation for some already known integrable perturbations of CFT's and, furthermore, they will also lead us to discovering new ones. Finally, these theories are expected to exhibit realistic properties of quantum particles not captured by other integrable field theories; for instance, the presence of unstable particles.

The main quantum properties of the HSG theories are quite well understood. They are integrable perturbations of the G-parafermion theories, which are coset CFT's of the form G_k/H, where G is a compact simple Lie group of rank r_g, $H \simeq U(1)^{r_g}$ is a maximal torus, and the 'level' k is a positive integer. The perturbation is given by a spinless primary field of conformal dimension $\Delta = \bar{\Delta} = h_g^\vee/(k + h_g^\vee)$, with $h_g^\vee$ the dual Coxeter number of G. The simplest HSG theory is associated to $G = SU(2)$, whose equation of motion is the complex sine-Gordon equation [5]. This theory corresponds to the perturbation of the usual $\mathbb{Z}_k$-parafermions by the first thermal operator [8], whose exact factorizable scattering matrix is the minimal one associated to a_{k-1} [9, 10]. The quantum integrability of the HSG theories for arbitrary G was established in [11], its soliton spectrum was obtained in [12], and a proposal for the exact factorizable S-matrices of the theories related to simply laced Lie groups G has been made in [13]. The main feature of those S-matrices is that they possess resonance poles which can be associated directly to the presence of unstable particles in the spectrum via the classical Lagrangian. Actually, to the best of our knowledge, these are the only known integrable quantum field theories that describe unstable particles. The S-matrices of [13] have been probed in [14] using the thermodynamic Bethe ansatz. In particular, this analysis confirms the expected value of the central charge of the unperturbed CFT for any simply laced G, and supports the interpretation of the resonance poles as a trace of the existence of unstable particles in the theory. These scattering matrices have been recently generalized in a Lie algebraic sense by Fring and Korf in [15].

In contrast, the quantum properties of the generality of SSSG theories are not known, and the purpose of this paper is to partially fill this gap. Namely, we will find the class of perturbed CFT's corresponding to the SSSG theories, investigate their quantum integrability, and discuss the general features of their spectrum of solitons.

The SSSG theories are related to a compact symmetric space G/G_0, with a G_0-valued field, and describe perturbations of either the WZNW CFT corresponding to G_0 or a coset CFT of the form G_0/H, where $H \simeq U(1)^p$ is a torus of G_0, not necessarily maximal. The equations of motion of this kind of theories for more general choices of the normal subgroup H were originally considered in the context of the, so called, reduced two-dimensional σ-models [16], although their Lagrangian formulation was not known until much later [6]. The results of [3, 4] show that they fit quite naturally into the class of non-abelian affine Toda theories and, what is more important, that the condition of having a mass gap requires that H is either trivial or abelian. The simplest SSSG theories are the ubiquitous sine-Gordon field theory, which corresponds to $G/G_0 = SU(2)/SO(2)$, and the complex sine-Gordon theory, which is related this time to $Sp(2)/U(2)$ [4] (recall that it is also the HSG theory associated to $SU(2)$). Actually, these two theories serve as paradigms of what can be expected in more complex situations. Another theories already discussed in the literature that belong to the class of SSSG theories are the integrable perturbations of the $SU(2)_k$ WZNW model and its $\widetilde{so(2)}$ reduction constructed by Brazhnikov [17]. Both of them are related to the symmetric space $SU(3)/SO(3)$ and, moreover, the second is identified with the perturbation of the usual $\mathbb{Z}_k$-parafermions by the second thermal operator.

The classification of the SSSG theories as perturbed CFT's is achieved through the calculation of the central charge of the unperturbed CFT and the conformal dimension of the perturbation. Since the unperturbed CFT is always a coset CFT of the form G_0/H, the calculation of its central charge is straightforward. In contrast, the calculation of the conformal dimension of the perturbation requires the knowledge of the structure of the symmetric space. The symmetric space G/G_0 is associated with a Lie algebra decomposition $g = g_0 \oplus g_1$ that satisfies the commutation relations

$$[g_0 \, , \, g_0] \subset g_0 \, , \quad [g_0 \, , \, g_1] \subset g_1 \, , \quad [g_1 \, , \, g_1] \subset g_0 \, , \tag{1.2}$$

where g and g_0 are the Lie algebras of G and G_0, respectively. Then, the conformal properties of the perturbation depend on the structure of the representation of g_0 provided by $[g_0, g_1] \subset g_1$. First of all, if the perturbation is to be given by a single primary field, then this representation has to be irreducible. This amounts to restrict the choice of G/G_0 to the, so called, 'irreducible symmetric spaces' [18], which have been completely classified by Cartan and are labelled by type I and type II.

In Section 2, we summarize the main features of the SSSG theories. In Section 3, the conformal dimension of the perturbation corresponding to all the SSSG models related to type I symmetric spaces is calculated by making use of the relationship between the classification of type I symmetric spaces and the classification of the finite order automorphisms of complex Lie algebras. It is worth noticing that this analysis only depends on the structure of the representation of g_0 given by $[g_0, g_1] \subset g_1$. Therefore, our results apply to any SSSG related to a type I symmetric space irrespectively of the choice of the normal subgroup H that determines the coset G_0/H and specifies the underlying CFT. For example, they provide the conformal dimension of the perturbation in the SSSG models constructed by Bakas, Park and Shin in [6], which include the generalized sine-Gordon models related to the NAAT equations based on $sl(2)$ embeddings constructed by Hollowood, Miramontes and Park in [4].

The classical integrability of all these theories is a consequence of the relationship between their equations of motion and the NAAT equations, which ensure the existence of an infinite number of classically conserved quantities. Concerning quantum integrability, it can be established by invoking a well known result due to Parke, which affirms that the existence of two higher spin conserved quantities of different spin in a two-dimensional quantum field theory is enough to ensure that there is no particle production in scattering processes and that the S-matrix is factorizable [19]. This way, the quantum integrability of the HSG theories was demonstrated in [11] by checking that the classically conserved quantities of spin ± 2 remain conserved in the quantum theory after an appropriate renormalization. The majority of SSSG theories also have classically conserved quantities of spin ± 2, and we expect that the analysis in [11] can be generalized to those cases without much effort. However, there is a class of SSSG where the simplest higher spin classically conserved quantities are of spin ± 3: the theories related to symmetric spaces of maximal rank, *i.e.*, those which satisfy $\mathrm{rank}(G/G_0) = \mathrm{rank}(G)$, where the rank of the symmetric space is the dimension of the maximal abelian subspaces contained in g_1. In Section 4, we explicitly check that at least two of those classically conserved quantities of spin ± 3 remain conserved in the quantum theory for all the theories related to a symmetric space of maximal rank where H is trivial. We name these theories 'Split models', and they are the only SSSG theories which correspond to massive quantum integrable perturbations of WZNW theories. It is worth noticing that (marginal) perturbations of WZNW models have been recently considered in the context of AdS_3 black holes [20].

The results of Section 4, together with those of [11], support the conjecture that all the SSSG are quantum integrable. This implies that they should admit a factorizable S-matrix and the next stage of analysis consists in establishing its form. Since we expect that it should be possible to infer the form of the exact S-matrix through the semiclassical quantization of the solitons, in Section 5 we investigate the general features of the spectrum of solitons. In view of the different kinds of SSSG theories, we have restricted the analysis to the Split models, which illustrate the main properties to be expected. Like the sine-Gordon and complex sine-Gordon theories, the fundamental particles of the theory can be identified with some of the classical soliton solutions. Moreover, the solitons of the Split models are topological and do not carry Noether charges, again like the solitons of the sine-Gordon theory. The origin of the topological charge is both the existence of different vacua and the fact that G_0 can be non simply connected. This is in contrast with the solitons of the HSG theories which are not topological but carry a $U(1)^{r_g}$ abelian Noether charge [12]. In general, the solitons of a generic SSSG theory corresponding to a perturbation of a coset CFT of the form G/H are expected to carry both topological charges and abelian Noether charges related to a global symmetry of the classical action specified by H. In this sense, they are analogous to the dyons in four-dimensional non-abelian gauge theories [21]. Another relevant feature of the solitons of the Split models, which is expected to be shared with other SSSG theories, is that their mass spectrum suggests that some of them might describe unstable particles in the quantum theory, which is analogous to what happens in the HSG theories [12, 13]. Actually, the unstability of the heaviest solitons in the spectrum has been checked by Brazhnikov in the SSSG theories related to $SU(3)/SO(3)$.

Our conclusions are presented in Section 6, and we have collected the explicit expressions for the classically conserved densities of the Split models together with some useful algebraic notation in the Appendix.

2 The Symmetric Space sine-Gordon theories.

The different SSSG models of [3] are characterized by four algebraic data: $\{g, \sigma, \Lambda_\pm\}$. g is a compact semisimple finite Lie algebra, and σ is an involutive ($\sigma^2 = 1$) automorphism of g that induces the decomposition

$$g = g_0 \oplus g_1, \tag{2.1}$$

with g_0 the set of fixed points of σ and $g_1 = \{u \in g \mid \sigma(u) = -u\}$. The subspaces g_0 and g_1 satisfy the commutation relations (1.2), which exhibits that G/G_0 is a compact symmetric space, where G and G_0 are the Lie groups corresponding to the compact Lie algebras g and g_0, respectively. Actually, the different choices for $\{g, \sigma\}$ are in one-to-one relation with the different compact symmetric spaces G/G_0, which prompted the name chosen in [3] for this class of models.

Finally, Λ_+ and Λ_-, are two semisimple elements in g_1 whose choice is only restricted by the condition that

$$\mathrm{Ker}(\mathrm{ad}_{\Lambda_+}) \cap g_0 = \mathrm{Ker}(\mathrm{ad}_{\Lambda_-}) \cap g_0 = g_0^0 \tag{2.2}$$

is abelian, which is required to ensure the existence of a mass gap [3]. They play the role of continuous coupling constants.

The SSSG model associated to $\{g, \sigma, \Lambda_\pm\}$ or, equivalently, to $\Lambda_\pm$ and the symmetric space G/G_0 is specified by the action

$$S_{\mathrm{SSSG}} = \frac{1}{\beta^2} \left\{ S_{\mathrm{WZNW}}[h] + \frac{m^2}{\pi} \int d^2x \, \langle \, \Lambda_+, h^\dagger \Lambda_- h \, \rangle \right\}, \tag{2.3}$$

where h is a bosonic field taking values in G_0, S_{WZNW} is the gauged WZNW action corresponding to the coset G_0/H, with H the abelian Lie group corresponding to g_0^0, and

m is the only mass scale of the theory [3]. The SSSG action (2.3) is invariant with respect to the abelian gauge transformations

$$h \to e^{\alpha} h e^{-\tau(\alpha)} \tag{2.4}$$

for any $\alpha = \alpha(x,t) \in g_0^0$. Since the potential $V(h) = -m^2 \langle \Lambda_+, h^\dagger \Lambda_- h \rangle / \pi$ introduced in (2.3) has $H \times H$ left-right symmetry, this gauge symmetry is essential to make the SSSG theories exhibit a mass gap. The precise form of the group of gauge transformations is specified by τ, which is an automorphism of g_0^0 that will not play any role in the following sections; we refer the reader to [3] for further details about the conditions to be satisfied by τ. The gauge symmetry leaves a residual global $H = U(1)^p$-symmetry, with $p = \dim(g_0^0)$, which, in particular, makes the classical solitonic solutions carry conserved abelian Noether charges, a feature that is shared with the HSG theories. Moreover, as will be discussed in Section 5, the solitonic solutions of the SSSG theories may also carry topological charges. This possibility does not exist in the case of the HSG theories where all the possible vacuum configurations get identified modulo gauge transformations.

The action (2.3) can be obtained by Hamiltonian reduction of a gauged two loop WZNW model [2, 22] associated to the affine Kac-Moody algebra $\bar{g}^{(r)}$, where $\bar{g}$ is the complexification of g and $r = 1, 2$, or 3 is the least positive integer for which σ^r is an inner automorphism (see the comments below (4.3)). In fact, the Hamiltonian reduction approach has been recently followed by Gomes *et al.* to construct another class of (classical) affine non-abelian Toda models different to the SSSG theories [23].

The classical integrability of these theories is a consequence of the connection between their equations of motion and the non-abelian affine Toda equations [1, 4]. This is made explicit by considering a particular gauge fixing prescription where the classical equations of motion reduce to [3]

$$\partial_-(h^\dagger \partial_+ h) \;=\; -m^2 \,[\, \Lambda_+ \,,\, h^\dagger \Lambda_- h \,], \tag{2.5}$$

$$P(h^\dagger \partial_+ h) \;=\; P(\partial_- h h^\dagger) \;=\; 0, \tag{2.6}$$

where P is a projector onto the subalgebra g_0^0, and $x_\pm = t \pm x$ are the light-cone variables. The first equation in (2.6) is a non-abelian affine Toda equation, and the second provide a set of constraints which come from the variation of the action (2.3) with respect to the abelian gauge connections in the LS gauge [3]. Notice that the equations (2.6) are left invariant by the transformation

$$x \to -x\,, \quad h \to h^\dagger\,, \quad \Lambda_\pm \to \eta^{\pm 1} \Lambda_\mp\,, \tag{2.7}$$

for an arbitrary real number η, which shows that the SSSG theories are parity invariant only if $\Lambda_+ = \eta \Lambda_-$ for some real number η, *i.e.*, if Λ_+ and Λ_- are chosen to be parallel or anti-parallel [3, 4].

At the quantum level, the SSSG theories will be described as perturbed CFT's of the form

$$S = S_{\rm CFT} + \frac{m^2}{\pi \beta^2} \int d^2x \; \Phi(x,t). \tag{2.8}$$

If, according to (2.2), $g_0^0 = u(1)^p$ with $p \geq 0$, $S_{\rm CFT}$ will be the action of either the CFT associated to the coset $G_0/U(1)^p$ ($p \neq 0$) or the WZNW model corresponding to G_0 ($p = 0$). Moreover, the perturbation is given by $\Phi = \langle \Lambda_+, h^\dagger \Lambda_- h \rangle$, which will be understood as a matrix element of the WZNW field taken in the representation of G_0 provided by $[g_0, g_1] \subset g_1$. As shown originally by Bakas for the complex sine-Gordon theory [8], and used in [11] to define the HSG theories at the quantum level, these identifications constitute a non-perturbative definition of the SSSG theories.

For a given symmetric space G/G_0, it is important to emphasise that the form of the coset $G_0/U(1)^p$ is fixed by the choice of $\Lambda_\pm$. Since $\Lambda_\pm$ are semisimple elements of g, different choices will lead to different values of p in the range

$$0 \leq \mathrm{rank}(G) - \mathrm{rank}(G/G_0) \leq p \leq \min\Big[\mathrm{rank}(G_0)\ ,\ \mathrm{rank}(G) - \nu\Big], \tag{2.9}$$

where the rank of the symmetric space, $\mathrm{rank}(G/G_0)$, is the dimension of the maximal abelian subspaces contained in g_1, and $\nu = 2$ or 1 depending on whether Λ_+ and Λ_- are linearly independent or not, respectively. In particular, the lower bound is reached when $\Lambda_+, \Lambda_- \in g_1$ are regular and, hence, $\mathrm{Ker}(\mathrm{ad}_{\Lambda_\pm})$ is already a maximal abelian subspace of g. All this implies that the SSSG theories provide a rich variety of different integrable models that include, for $p = \mathrm{rank}(G_0)$, new massive perturbations of the theory of G_0-parafermions different than those provided by the Homogeneous sine-Gordon theories [11]. Notice that this case happens only if the symmetric space satisfies $\mathrm{rank}(G_0) \leq \mathrm{rank}(G) - \nu$. Another particularly interesting class of models occurs when $\mathrm{rank}(G) = \mathrm{rank}(G/G_0)$ and $p = 0$. In this case, the SSSG theory is just a massive perturbation of the WZNW model corresponding to G_0.

We will concentrate on the theories where the perturbation in (2.8) is given by a single spinless primary field of the CFT which, taking into account the properties of the WZNW field [24], amounts to restrict ourselves to the SSSG theories associated with symmetric spaces where the representation of g_0 provided by $[g_0, g_1] \subset g_1$ is irreducible. Otherwise, the perturbation will be the sum of more than one primary field. The symmetric spaces with that property are called 'irreducible', and have been completely classified by Cartan [18]. There are two types of compact irreducible symmetric spaces:

- *Type I,* where the compact Lie algebra g is simple.
- *Type II,* where the compact Lie algebra is of the form $g = g_1 \oplus g_2$ with $g_1 = g_2$ simple, and the involution σ interchanges g_1 and g_2.

In the following we will only consider the SSSG theories associated to the type I symmetric spaces, which admit a thorough classification.

3 The type I SSSG theories as perturbed CFT's.

In this section, we calculate the central charge of the unperturbed CFT and the conformal dimension of the perturbation corresponding to the SSSG theories associated with the symmetric spaces of type I. We will make use of the relationship between the Cartan classification of this type of symmetric spaces and the Kac classification of the automorphisms of finite order of complex Lie algebras, which provides a systematic and very convenient description of the involutive automorphism σ. This represents an important advantage with respect to previous works on integrable systems associated with symmetric spaces [6, 16] which generally make use of explicit parametrizations of the field h based on some matrix representation for the symmetric space.

3.1 Type I symmetric spaces and finite order automorphisms.

The basic result is due to Kac, who established the following correspondence between the involutions of a complex Lie algebra $\bar{g}$ and the involutions of its compact real form g.

Theorem 1 (Proposition 1.4 in [18], Ch. X) *Let $Aut(g)$ denote the set of automorphisms of g, $Inv(g)$ the subset containing the involutions, and $Inv(g)/Aut(g)$ the set of conjugacy classes in $Aut(g)$ of the elements in $Inv(g)$. We define $Inv(\bar{g})/Aut(\bar{g})$ similarly. Each automorphism $\sigma \in Inv(g)$ extends uniquely to $\bar{\sigma} \in Inv(\bar{g})$ and if σ_1, σ_2 are conjugate within $Aut(g)$, then $\bar{\sigma}_1$, $\bar{\sigma}_2$ are conjugate within $Aut(\bar{g})$. Taking into account all this, it can be proved that the mapping:*

$$\tau : Inv(g)/Aut(g) \;\rightarrow\; Inv(\bar{g})/Aut(\bar{g}), \tag{3.1}$$

induced by $\sigma \;\rightarrow\; \bar{\sigma}$ is a bijection.

Recall now that the different compact symmetric spaces of type I associated with a compact simple Lie algebra g are in one-to-one relation with the different involutive automorphisms of g, not distinguishing automorphisms which are conjugate by the group $Aut(g)$. Therefore, they are also in one-to-one relation with the involutive automorphisms of its complexification $\bar{g}$, modulo conjugations by $Aut(\bar{g})$.

In order to summarize the Kac classification of the automorphisms of finite order of complex Lie algebras, it is necessary to introduce the following notation (see [25], Ch. 8, for more details). Let $\bar{g}$ denote a complex simple Lie algebra and μ an automorphism of $\bar{g}$ induced by an automorphism of its Dynkin diagram of order $r = 1, 2$ or 3. μ induces a $\mathbb{Z}/r\mathbb{Z}$–gradation of $\bar{g}$, which means that $\bar{g}$ can be decomposed into the sum of a set of subspaces labelled by an integer $0 \leq k \leq r-1$ that satisfy

$$\begin{aligned} &\mu(u) \;=\; \mathrm{e}^{\frac{2\pi i}{r}\, j}\, u \quad \forall u \in \bar{g}_j(\mu)\,, \\ &\bar{g} = \bigoplus_{k=1}^{r-1} \bar{g}_k(\mu), \qquad [\bar{g}_j(\mu), \bar{g}_k(\mu)] \subset \bar{g}_{j+k \bmod r}\,(\mu). \end{aligned} \tag{3.2}$$

Then there is a particular set of generators of $\bar{g}$, $\{E_0, E_1, \ldots, E_l\}$, where l is the rank of the invariant subalgebra $\bar{g}_0(\mu)$, with the following properties:

a) $\{E_1, \ldots, E_l\} \in \bar{g}_0(\mu)$ for $(\bar{g}, r) \neq (A_{2l}, 2)$, and $\{E_0, E_1, \ldots, E_{l-1}\} \in \bar{g}_0(\mu)$ for $(\bar{g}, r) = (A_{2l}, 2)$.

b) If $(\bar{g}, r) \neq (A_{2l}, 2)$, then $E_0 \in \bar{g}_1(\mu)$ is the lowest–weight vector of the irreducible representation of $\bar{g}_0(\mu)$ given by $[\,\bar{g}_0(\mu), \bar{g}_1(\mu)\,] \subset \bar{g}_1(\mu)$. Conversely, when $(\bar{g}, r) = (A_{2l}, 2)$ this role is played by E_l.

c) $\{E_1, \ldots, E_l\}$ are positive Chevalley generators for $\bar{g}_0(\mu)$, except for $(\bar{g}, r) = (A_{2l}, 2)$ where the Chevalley generators are $\{E_0, E_1, \ldots, E_{l-1}\}$.

d) Results a), b), and c) correspond to the case $r > 1$. When $r = 1$, $l = \mathrm{rank}(\bar{g})$, $E_1 = E_{\alpha_1}, \ldots, E_l = E_{\alpha_l}$, and $E_0 = E_{-\Psi_g}$, where $\{\vec{\alpha}_1, \vec{\alpha}_2, \ldots, \vec{\alpha}_l\}$ is a set of simple roots of $\bar{g}$, and $\vec{\Psi}_g$ is the highest root.

The classification is given by the following theorem.

Theorem 2 (Theorem 8.6 in [25]) *Let $\vec{s} = (s_0, s_1, \ldots, s_l)$ be a sequence of non-negative relatively prime integers, and put*

$$m = r \sum_{i=0}^{l} a_i s_i \tag{3.3}$$

where $a_0, a_1, \ldots, a_l$ are the Kac labels corresponding to the Dynkin diagram of the (twisted if $r \neq 1$) affine Kac-Moody algebra $\bar{g}^{(r)}$. Then:

a) The relations
$$\sigma_{\vec{s};r}(E_j) = e^{2\pi i s_j/m} E_j, \quad j = 0, \ldots, l, \tag{3.4}$$
define uniquely an m-th order automorphism $(\vec{s}; r)$ of $\bar{g}$.

b) Up to conjugacy by an automorphism of $\bar{g}$, the automorphisms $\sigma_{\vec{s};r}$ exhaust all m-th order automorphisms of $\bar{g}$.

c) The elements $\sigma_{\vec{s};r}$ and $\sigma_{\vec{s}';r'}$ are conjugate by an automorphism of $\bar{g}$ if, and only if, $r = r'$ and the sequence $\vec{s}$ can be transformed into the sequence $\vec{s}'$ by an automorphism of the Dynkin diagram of $\bar{g}^{(r)}$.

Taking into account Theorems 1 and 2, the classification of the symmetric spaces of type I is equivalent to working out the equation

$$m \,=\, r \sum_{i=0}^{l} a_i s_i \,=\, 2, \tag{3.5}$$

which has only three possible types of solutions:

$$
\begin{array}{ll}
[\mathrm{A1}] & r = 1\,,\ a_{i_0} = 2\,,\ s_{i_0} = 1\,, \quad \text{and} \quad s_i = 0 \quad \text{for} \quad i \neq i_0\,, \\
[\mathrm{A2}] & r = 2\,,\ a_{i_0} = 1\,,\ s_{i_0} = 1\,, \quad \text{and} \quad s_i = 0 \quad \text{for} \quad i \neq i_0\,, \\
[B] & r = 1\,,\ a_{i_0} = a_{i_1} = 1\,,\ s_{i_0} = s_{i_1} = 1\,, \quad \text{and} \quad s_i = 0 \quad \text{for} \quad i \neq i_0\,,\ i_1.
\end{array} \tag{3.6}
$$

This classifies the type I (compact) symmetric spaces and, hence, the corresponding SSSG theories, into type A1, A2, and B [18].

Given an automorphism $\sigma_{\vec{s};r}$, the subspaces $\bar{g}_0$ and $\bar{g}_1$ can be easily characterized as follows.

Theorem 3 (Proposition 8.6 in [25])

a) Let $i_1, i_2, \ldots, i_p$ be all the indices for which $s_{i_1} = \cdots = s_{i_p} = 0$. Then the Lie algebra $\bar{g}_0$ is isomorphic to a direct sum of the (l-p)–dimensional centre and a semisimple Lie algebra whose Dynkin diagram is the subdiagram of the Dynkin diagram of $\bar{g}^{(r)}$ consisting of the vertices $i_1, \ldots, i_p$.

b) Let $j_1, \ldots, j_n$ be all the indices for which $s_{j_1} = \cdots = s_{j_n} = 1$. Then the representation of $\bar{g}_0$ provided by $[\bar{g}_0, \bar{g}_1] \subset \bar{g}_1$ is isomorphic to a direct sum of n irreducible modules with highest weights $-\vec{\alpha}_{j_1}, \ldots, -\vec{\alpha}_{j_n}$.

Eq. (3.6) and Theorem 3 imply that g_0 is of the form

$$g_0 \;=\; \begin{cases} \bigoplus_{i=1}^{q} g^{(i)} & \text{for type A1 and A2,} \\ \bigoplus_{i=1}^{q} g^{(i)} \oplus u(1) & \text{for type B,} \end{cases} \tag{3.7}$$

where q can be either 1 or 2 in both cases, and $g^{(i)}$ is always compact and simple.

Therefore, if the symmetric space is of type A1 or A2, $S_{\rm CFT}$ in (2.8) is the action of the CFT associated to a coset of the form

$$\bigotimes_{i=1}^{q} G^{(i)}_{k_i} \,/\, U(1)^p \;, \tag{3.8}$$

where $G^{(i)}$ is the compact simple Lie group corresponding to $g^{(i)}$, and p is some integer in the range (2.9). Since g_0 is non-abelian, the consistency of the quantum theory requires that the coupling constant in (2.3) is quantized. The precise form of the quantization rule is [24, 26]

$$\frac{1}{\hbar\beta^2} \;=\; \frac{\vec{\Psi}^2_{g^{(i)}}}{2}\, k_i \;, \tag{3.9}$$

where k_i is an integer for each simple factor in (3.7), and $\vec{\Psi}^2_{g^{(i)}}$ is the square length of the long roots of $g^{(i)}$ with respect to the bilinear form of g. In (3.9) we have shown explicitly the Plank constant to exhibit that, just as in the sine-Gordon theory, the semi-classical limit is the same as the weak coupling limit, and that both are recovered when $k_i \to \infty$. Since there is a unique coupling constant β^2, eq. (3.9) implies that the 'levels' in (3.8) are related by means of

$$\frac{k_i}{k_j} \;=\; \frac{\vec{\Psi}^2_{g^{(j)}}}{\vec{\Psi}^2_{g^{(i)}}} \;. \tag{3.10}$$

Nevertheless, our calculations show that $\vec{\Psi}^2_{g^{(1)}} = \vec{\Psi}^2_{g^{(2)}}$ for all the type I symmetric spaces, with the only exception of $G_2/SU(2)\times SU(2)$ where $\vec{\Psi}^2_{g^{(2)}} = 3\vec{\Psi}^2_{g^{(1)}}$ (see Section 3.2.2), and we will use the notation $k = k_1$ in the following.

Therefore, in this case, the central charge of the unperturbed CFT is

$$c_{\rm CFT} = \sum_{i=1}^{q} \frac{k_i \dim(g^{(i)})}{k_i + h_i^\vee} \;-\; p, \tag{3.11}$$

where $h_i^\vee$ is the dual Coxeter number of $g^{(i)}$. Since the perturbation Φ is just a matrix element of the WZNW field in the representation of G_0 provided by $[g_0, g_1] \subset g_1$, which is irreducible, Φ is a spinless primary field with conformal dimension [24, 26]

$$\Delta_\Phi = \overline{\Delta}_\Phi = \sum_{i=1}^{q} \frac{C_2(g^{(i)})/\vec{\Psi}^2_{g^{(i)}}}{k_i + h_i^\vee}, \tag{3.12}$$

where $C_2(g^{(i)})$ is the quadratic Casimir. Then, Theorem 3 shows that this representation is a highest weight representation and, hence, the quadratic Casimir is given by

$$C_2(g^{(i)}) \;=\; \langle\, \vec{\Lambda}\,,\, \vec{\Lambda} \,+\, 2\vec{\delta}^{(i)} \,\rangle \tag{3.13}$$

where $\vec{\Lambda} = -\vec{\alpha}_{i_0}$ is the highest weight, and $\vec{\delta}^{(i)}$ is half the sum of the positive roots of $\bar{g}^{(i)}$.

Let us consider now the SSSG theories associated with symmetric spaces of type B. The main difference with the SSSG models of type A1 or A2 is that g_0 includes now a one-dimensional centre: the $u(1)$ factor in (3.7). In this case, it will be convenient to choose the Cartan subalgebra of g such that it contains the Cartan subalgebras of $\bigoplus_{i=1}^{q} g^{(i)}$ in addition to the generator of the centre. Then, $u(1) = \mathbb{R}\, i\vec{u}\cdot\vec{h}$, where $\vec{u}$ is a vector that is orthogonal to all the roots of $\bigoplus_{i=1}^{q} g^{(i)}$, and the components of $i\vec{h}$ provide a basis for the Cartan subalgebra of g. According to 3, another important difference is that the representation of $\bar{g}_0$ given by $[\bar{g}_0, \bar{g}_1] \subset \bar{g}_1$ is the sum of two irreducible highest weight representations with highest weights $\vec{\Lambda}_1 = -\vec{\alpha}_{i_0}$ and $\vec{\Lambda}_2 = -\vec{\alpha}_{i_1}$, namely,

$$\bar{g}_1 \;=\; L(\vec{\Lambda}_1) \;\oplus\; L(\vec{\Lambda}_2)\,. \tag{3.14}$$

Let us consider the identity

$$\sum_{i=0}^{l} a_i\, \vec{\alpha}_i \;=\; 0 \tag{3.15}$$

satisfied by the Kac labels of the Dynkin diagram of $\bar{g}^{(1)}$, where $\{\vec{\alpha}_1, \vec{\alpha}_2, \ldots, \vec{\alpha}_l\}$ is a set of simple roots of $\bar{g}$, and $\vec{\Psi}_g$ is the highest root. Then, the two highest weights satisfy

$$\vec{\Lambda}_1 \;+\; \vec{\Lambda}_2 \;=\; \sum_{\substack{i=0\\ i\neq i_0, i_1}}^{l} a_i\, \vec{\alpha}_i\,, \tag{3.16}$$

which implies that $\vec{\Lambda}_1$ is minus the lowest weight of the representation with highest weight $\vec{\Lambda}_2$. This manifests that the two highest weight representations are conjugate, $L(\Lambda_2) = L^{\dagger}(\Lambda_1)$, which is consistent with the fact that the representation of g_0 given by $[g_0, g_1] \subset g_1$ is irreducible. Therefore, since $\Lambda_{\pm} \in g_1$, it can be decomposed as

$$\Lambda_{\pm} \;=\; \lambda_{\pm} \;+\; \lambda_{\pm}^{\dagger}\,, \quad \text{with} \quad \lambda_{\pm} \in L(\Lambda_1)\,. \tag{3.17}$$

Taking again into account Theorem 3, eq. (3.16) also implies that $\vec{\Lambda}_1 + \vec{\Lambda}_2$ is a linear combination of the roots of g_0, which are orthogonal to $\vec{u}$, and, hence,

$$\vec{u}\cdot\vec{\Lambda}_1 \;=\; -\,\vec{u}\cdot\vec{\Lambda}_2\,. \tag{3.18}$$

Our calculations show that, in all the symmetric spaces of type B, $\vec{u}\cdot\vec{\Lambda}_1 \neq 0$. Since $\Lambda_{\pm} \in g_1$, this implies that $[\vec{u}\cdot\vec{h}, \Lambda_{\pm}] \neq 0$ and, hence, the $u(1)$ is not in g_0^0 for any choice of $\Lambda_{\pm}$ (see (2.2)).

Consider a generic field configuration

$$h \;=\; \widetilde{h}\, \exp(i\varphi\, \vec{u}\cdot\vec{h})\,, \tag{3.19}$$

where $\widetilde{h}$ is a field taking values in the compact semisimple Lie group $\bigoplus_{i=1}^{q} G^{(i)}$, and $\varphi = \varphi(x,t)$ is a real scalar field; for convenience we will normalize $\vec{u}$ such that $\vec{u}\cdot\vec{u} = 4\pi$. Then, the action (2.3) becomes

$$S_{\rm SSSG} = \frac{1}{\beta^2}\Big\{ S_{\rm WZNW}[\widetilde{h}] \;+\; \frac{1}{2}\int d^2x\, \partial_\mu\varphi\, \partial^\mu\varphi$$
$$+\frac{m^2}{\pi}\int d^2x \left({\rm e}^{\,-i\varphi\,\vec{u}\cdot\vec{\Lambda}_1}\, \langle\, \lambda_+^{\dagger}, \widetilde{h}^{\dagger}\lambda_-\widetilde{h}\,\rangle \;+\; {\rm e}^{\,+i\varphi\,\vec{u}\cdot\vec{\Lambda}_1}\, \langle\, \lambda_+, \widetilde{h}^{\dagger}\lambda_-^{\dagger}\widetilde{h}\,\rangle \right) \Big\}\,. \tag{3.20}$$

Therefore, if the symmetric space is of type B, $S_{\rm CFT}$ in (2.8) is the action of the CFT associated to a coset of the form

$$\Big[\bigotimes_{i=1}^{q} G_{k_i}^{(i)} \,/\, U(1)^p\Big] \times U(1)\,, \tag{3.21}$$

i.e., a coset of the form (3.8) plus a massless boson, whose central charge is

$$c_{\rm CFT} = \sum_{i=1}^{q} \frac{k_i\, \dim(g^{(i)})}{k_i + h_i^\vee} \;+\; 1 \;-\; p, \tag{3.22}$$

with the levels k_i defined by the quantization rule (3.9). In (3.21), we have already taken into account that the centre of g_0 is not in g_0^0 as a consequence of (3.17) and (3.18). Therefore, in this case, $\mathrm{rank}(G_0)$ has to be substituted for $\mathrm{rank}(G_0) - 1$ on the right-hand-side of eq. (2.9). Concerning the perturbation Φ in (3.20), it is a primary field of conformal dimension

$$\Delta_\Phi = \overline{\Delta}_\Phi = \sum_{i=1}^{q} \frac{C_2(g^{(i)})/\vec{\Psi}^2_{g^{(i)}}}{k_i + h_i^\vee} \;+\; \frac{(\vec{u}\cdot\vec{\Lambda}_1)^2}{k\, \vec{\Psi}^2_{g^{(1)}}}\,, \tag{3.23}$$

where the quadratic Casimir is given by (3.13) with $\vec{\Lambda} = \vec{\Lambda}_1$ or $\vec{\Lambda}_2$, both leading to identical results.

At this stage, we would like to correct a wrong statement in [3] concerning the fields associated to the centre of g_0. In that article, it was said that those fields can always be decoupled whilst preserving integrability. However, the SSSG of type B shows that this is not true in general. In our case, there is only one field associated to the centre of g_0, $\varphi = \varphi(x,t)$, whose classical equation of motion is

$$\partial_+\partial_-\varphi \;=\; -\,i\, m^2\, \frac{\vec{u}\cdot\vec{\Lambda}_1}{4\pi}\, \Big({\rm e}^{\,-i\varphi\, \vec{u}\cdot\vec{\Lambda}_1}\, \langle\, \lambda_+^\dagger, \widetilde{h}^\dagger \lambda_- \widetilde{h}\,\rangle \;-\; {\rm e}^{\,+i\varphi\, \vec{u}\cdot\vec{\Lambda}_1}\, \langle\, \lambda_+, \widetilde{h}^\dagger \lambda_-^\dagger \widetilde{h}\,\rangle\Big)\,, \tag{3.24}$$

which clearly shows that φ cannot be decoupled simply by putting $\varphi(x,t) = 0$ unless $\langle \lambda_+^\dagger, \widetilde{h}^\dagger \lambda_- \widetilde{h}\rangle$ is real, which is equivalent to

$$[\Lambda_+, \widetilde{h}^\dagger \Lambda_- \widetilde{h}] \in \bigoplus_{i=1}^{q} g^{(i)}\,. \tag{3.25}$$

This condition was already noticed in [4].

There are two general features of the conformal dimensions given by (3.12) and (3.23) that is important to emphasize. The first one is that the conformal dimension of the perturbation is independent of the value of p in eqs. (3.8) and (3.21), which is a consequence of the fact that the potential in (2.3) and, hence, Φ are invariant with respect to the gauge transformations (2.4). The second is that Δ_Φ decreases with k, which means that the perturbation is always relevant for k above some minimal value characteristic of each SSSG theory. Actually, Δ_Φ vanishes when $k \to \infty$, which shows that the theory consists of $\dim(g_0) - p$ bosonic massive particles in the the semi–classical and/or weak coupling limit.

3.2 Explicit calculation of Δ_Φ.

In the following we will illustrate the general procedure to calculate the conformal dimension of the perturbation by considering three particular cases where g_0 is either simple ($SU(2n)/SO(2n)$), semisimple ($G_2/SU(2)\times SU(2)$), or the direct sum of a simple ideal and a one-dimensional centre ($Sp(n)/U(n)$). In all these examples, $\mathrm{rank}(G/G_0) = \mathrm{rank}(G_0)$, which, according to (2.9), means that $\Lambda_\pm$ can be chosen such that $p = 0$ in (3.8) and (3.21). The results for all the type I symmetric spaces are presented in table 2 and 3. Other useful features of the symmetric spaces of type I have been collected in tables 4–6. In these tables, we have already taken into account the following isomorphisms of Lie algebras:

$$su(2)\simeq so(3)\simeq sp(1)\,,\quad so(5)\simeq sp(2)\,,\quad so(4)\simeq su(2)\oplus su(2)\,,\quad su(4)\simeq so(6)\,.$$

In particular, this shows that $SO(4)$ is not simple and, therefore, no symmetric space with $G = SO(4)$ appears in the tables because it would not be of type I. Moreover, the symmetric space $SU(2)/SO(2)$ corresponds to the well known sine-Gordon theory where the field takes values in the abelian group $SO(2)\simeq U(1)$, and it has not been included in the tables. When G_0 is simple, it is worthwhile noticing that Δ_Φ admits the general expression

$$\Delta_\Phi = \frac{\vec{\Psi}_g^2}{\vec{\Psi}_{g_0}^2}\,\frac{h_g^\vee}{2\,(k+h_{g_0}^\vee)}\,. \tag{3.26}$$

3.2.1 Example I: $G/G_0 = SU(2n)/SO(2n)$, $n>2$.

In this case, $\bar{g} = A_{2n-1}$, $r = 2$, and $\vec{s} = (0,\ldots,0,1)$, which follows from Theorem 3, part a), and the observation that the Dynkin diagram of $\bar{g}_0 = D_n$ is a subdiagram of the Dynkin diagram of $\bar{g}^{(r)} = A_{2n-1}^{(2)}$, as can be seen in fig. 1. Therefore, and taking into account (3.6), this symmetric space is of type A2. Fig. 1 also shows that the roots of $\bar{g}_0 = D_n$ are short roots in the Dynkin diagram of $\bar{g}^{(r)} = A_{2n-1}^{(2)}$, which means that $\vec{\Psi}_{D_n}^2 = \vec{\Psi}^2_{A_{2n-1}^{(2)}}/2$.

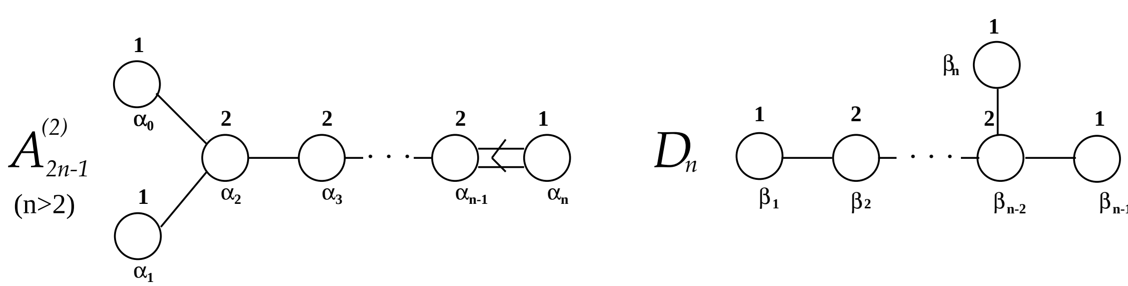

Figure 1: Dynkin diagrams of $A_{2n-1}^{(2)}$ and D_n. The numbers on top of the nodes are the Kac labels.

Theorem 3, part b), also implies that $[\bar{g}_0,\bar{g}_1]\subset\bar{g}_1$ is an irreducible representation of $\bar{g}_0$ with highest weight $\vec{\Lambda} = -\vec{\alpha}_n$. Then, we can use the identity (3.15) in order to write

$\vec{\Lambda}$ as a linear combination of the roots of $\bar{g}_0 = D_n$, namely

$$\vec{\Lambda} \,=\, -\vec{\alpha}_n \,=\, \vec{\alpha}_0 \,+\, \vec{\alpha}_1 \,+\, 2\sum_{i=2}^{n-1} \vec{\alpha}_i \,=\, \vec{\beta}_n \,+\, \vec{\beta}_{n-1} \,+\, 2\sum_{i=1}^{n-2} \vec{\beta}_i. \tag{3.27}$$

Moreover, taking into account that the highest root of D_n is

$$\vec{\Psi}_{D_n} \,=\, \vec{\beta}_1 \,+\, \vec{\beta}_n \,+\, \vec{\beta}_{n-1} \,+\, 2\sum_{i=2}^{n-2} \vec{\beta}_i, \tag{3.28}$$

eq. (3.27) simplifies to $\vec{\Lambda} = \vec{\Psi}_{D_n} + \vec{\beta}_1$. All this allows one to easily calculate the quadratic Casimir of this highest weight representation:

$$C_2(D_n) \,=\, \langle \vec{\Lambda}\,,\, \vec{\Lambda} \,+\, 2\vec{\delta}_{D_n}\rangle \,=\, 2n\, \vec{\Psi}^2_{D_n}, \tag{3.29}$$

where we have used the standard realization of the root system of D_n as a sublattice of the real euclidean space $\mathbb{R}^n$:

$$\Pi_{D_n} \,=\, \{\vec{\beta}_1 \,=\, \vec{v}_1 - \vec{v}_2\,, \ldots, \vec{\beta}_{n-1} \,=\, \vec{v}_{n-1} - \vec{v}_n\,,\; \vec{\beta}_n \,=\, \vec{v}_{n-1} + \vec{v}_n\}\,, \tag{3.30}$$

where

$$\vec{v}_i \cdot \vec{v}_j \,=\, \frac{\vec{\Psi}^2_{D_n}}{2}\, \delta_{ij}\,, \tag{3.31}$$

together with

$$\vec{\Psi}_{D_n} \,=\, \vec{v}_1 + \vec{v}_2\,, \quad \text{and} \quad 2\vec{\delta}_{D_n} \,=\, 2\sum_{i=1}^{n} (n-i)\, \vec{v}_i\,. \tag{3.32}$$

Therefore, taking into account (3.11) and (3.12), we conclude that the SSSG's associated with the symmetric space $SU(2n)/SO(2n)$ are integrable perturbations of either the WZNW model corresponding to $SO(2n)$ at level k ($p=0$) or a coset CFT of the form $SO(2n)_k/U(1)^p$ whose central charge is

$$c_{\rm CFT} = \frac{k\, n\, (2n-1)}{k + 2(n-1)} \,-\, p, \tag{3.33}$$

where $0 \leq p \leq n$, and the perturbation has conformal dimension

$$\Delta_\Phi \,=\, \frac{2n}{k \,+\, 2(n\,-\,1)}\,. \tag{3.34}$$

Notice that the perturbation is relevant for $k > 2$.

3.2.2 Example II: $G/G_0 = G_2/SU(2)\times SU(2)$.

In this case, $\bar{g} = G_2$, $r=1$, and $\vec{s} = (0,1,0)$, as can be seen in fig. 2. Therefore, this symmetric space is of type A1. $\bar{g}_0$ is of the form $\bar{g}_0 = \bar{g}^{(1)} \oplus \bar{g}^{(2)}$ with $\bar{g}^{(1)} = \bar{g}^{(2)} = A_1$. $\bar{g}^{(1)}$ and $\bar{g}^{(2)}$ are associated with the roots $\vec{\alpha}_2$ and $\vec{\alpha}_0$ of the Dynkin diagram of the affine algebra $G_2^{(1)}$, respectively, whose length is different and, therefore, $\vec{\Psi}^2_{g^{(2)}} = 3\vec{\Psi}^2_{g^{(1)}} = \vec{\Psi}^2_{G_2^{(1)}}$.

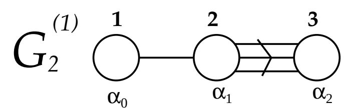

Figure 2: Dynkin diagram and Kac labels of $G_2^{(1)}$.

Theorem 3 implies that $[\bar{g}_0, \bar{g}_1] \subset \bar{g}_1$ gives an irreducible representation of $\bar{g}_0$ with highest weight $\vec{\Lambda} = -\vec{\alpha}_1$. Then, since $\vec{\alpha}_0 + 2\vec{\alpha}_1 + 3\vec{\alpha}_2 = 0$, one can write

$$\vec{\Lambda} = \frac{3}{2}\,\vec{\Psi}_{g^{(1)}} + \frac{1}{2}\,\vec{\Psi}_{g^{(2)}}\,, \tag{3.35}$$

which leads to

$$\begin{aligned} C_2(g^{(1)}) &= \langle\, \vec{\Lambda}\,,\, \vec{\Lambda} + \vec{\Psi}_{g^{(1)}}\,\rangle = \frac{15}{4}\,\vec{\Psi}^2_{g^{(1)}}\,, \\ C_2(g^{(2)}) &= \langle\, \vec{\Lambda}\,,\, \vec{\Lambda} + \vec{\Psi}_{g^{(2)}}\,\rangle = \frac{3}{4}\,\vec{\Psi}^2_{g^{(2)}}\,. \end{aligned} \tag{3.36}$$

Therefore, we conclude that the SSSG's associated with this symmetric space are integrable perturbations of either the WZNW model corresponding to $SU(2)_k \times SU(2)_{3k}$ ($p = 0$) or a coset CFT of the form $SU(2)_k \times SU(2)_{3k}/U(1)^p$ whose central charge is

$$c_{\rm CFT} = \frac{3k}{k+2} + \frac{9k}{3k+2} - p\,. \tag{3.37}$$

Notice the relationship between the levels of the two $SU(2)$ factors, which is a consequence of (3.10). The perturbation has conformal dimension

$$\Delta_\Phi = \frac{3}{4(k+2)} + \frac{15}{4(3k+2)}\,. \tag{3.38}$$

Moreover, since in this case $\mathrm{rank}(G_0) = \mathrm{rank}(G) = 2$, according to (2.9) there is a different SSSG theory for each integer p in the range $0 \leq p \leq 2 - \nu$. The perturbation is relevant for $k > 2$.

3.2.3 Example III: $G/G_0 = Sp(n)/U(n) = Sp(n)/U(1) \times SU(n)$, $n > 1$.

In this case, $\bar{g} = C_n$, $r = 1$, and $\vec{s} = (1, 0\ldots, 0, 1)$, as can be seen in fig. 3, and the symmetric space is of type B. Then, $\bar{g}_0 = \bar{g}^{(1)} \oplus u(1)$ with $\bar{g}^{(1)} = A_{n-1}$, and $\vec{\Psi}^2_{A_{n-1}} = \vec{\Psi}^2_{C_n^{(1)}}/2$.

Recall the standard realization of the root systems of C_n and A_{n-1} as sublattices of the real euclidean space $\mathbb{R}^n$:

$$\begin{aligned} &\Pi_{C_n} = \{\vec{\alpha}_1 = \vec{v}_1 - \vec{v}_2\,, \ldots, \vec{\alpha}_{n-1} = \vec{v}_{n-1} - \vec{v}_n\,,\, \vec{\alpha}_n = 2\vec{v}_n\}\,, \quad \vec{\Psi}_{C_n} = -\vec{\alpha}_0 = 2\vec{v}_1\,, \\ &\Pi_{A_{n-1}} = \{\vec{\alpha}_1 = \vec{v}_1 - \vec{v}_2, \ldots, \vec{\alpha}_{n-1} = \vec{v}_{n-1} - \vec{v}_n\} \subset \Pi_{C_n}\,, \quad \vec{\Psi}_{A_{n-1}} = \vec{v}_1 - \vec{v}_n\,, \end{aligned} \tag{3.39}$$

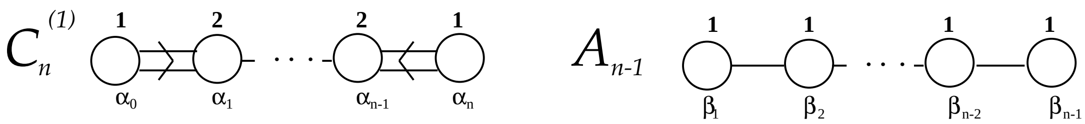

Figure 3: Dynkin diagram and Kac labels of $C_n^{(1)}$ and A_{n-1}.

where

$$\vec{v}_i \cdot \vec{v}_j \;=\; \frac{\vec{\Psi}^2_{C_n}}{4}\, \delta_{ij}\;. \tag{3.40}$$

Using (3.39), $\vec{\Lambda}_1 = 2\vec{v}_1$, which has to be split in two components. One, in the weight lattice of A_{n-1}, and another, corresponding to the $u(1)$ subalgebra of $\bar{g}_0$, orthogonal to it. The decomposition is as follows

$$\vec{\Lambda}_1 \;=\; \vec{\lambda}_{A_{n-1}} \;+\; \vec{\lambda}_{u(1)} \tag{3.41}$$

with

$$\vec{\lambda}_{A_{n-1}} \;=\; \sum_{i=1}^{n-1} \frac{2(n-i)}{n}\,(\vec{v}_i - \vec{v}_{i+1})\,, \qquad \vec{\lambda}_{u(1)} \;=\; \frac{2}{n}\sum_{i=1}^{n}\vec{v}_i \;=\; \frac{1}{\sqrt{\pi n}}\,\vec{u}\,. \tag{3.42}$$

This allows one to calculate the required quadratic Casimir

$$C_2(A_{n-1}) \;=\; \langle\, \vec{\lambda}_{A_{n-1}}\,,\, \vec{\lambda}_{A_{n-1}} \;+\; 2\vec{\delta}_{A_{n-1}}\,\rangle \;=\; \frac{(n-1)(n+2)}{n}\,\vec{\Psi}^2_{A_{n-1}}, \tag{3.43}$$

where we have used

$$2\vec{\delta}_{A_{n-1}} \;=\; \sum_{i=1}^{[n/2]} (\vec{v}_i \;-\; \vec{v}_{n+1-i})(n+1-2i), \tag{3.44}$$

with $[n/2]$ the integer part of $n/2$. The same result can be obtained by considering the other representation with highest weight $\vec{\Lambda}_2$.

Therefore, the SSSG theories associated with the symmetric space $Sp(n)/U(n)$ are integral perturbations of either the WZNW model corresponding to $SU(n)$ at level k plus a massless boson ($p = 0$) or a coset CFT of the form $[SU(n)_k/U(1)^p] \times U(1)$, whose central charge is given by (3.22):

$$c_{\rm CFT} \;=\; \frac{k\,(n^2-1)}{k+n} \;+\; 1 \;-\; p\,. \tag{3.45}$$

The conformal dimension of the perturbation is given by (3.23):

$$\Delta_\Phi \;=\; \frac{(n-1)(n+2)}{n(k+n)} + \frac{2}{kn}, \tag{3.46}$$

and there is a different SSSG theory for $0 \le p \le n-1$. The perturbation is relevant for $k > 2$.

4 Integrability of the SSSG theories.

The classical integrability of the Homogeneous (HSG) and Symmetric Space (SSSG) sine-Gordon theories is a consequence of the relationship between their equations of motion and the non-abelian affine Toda equations, which admit a zero-curvature representation. This implies the existence of an infinite number of conserved quantities, whose construction by means of the Drinfel'd-Sokolov procedure will be summarized in the first part of this section.

Concerning the HSG theories, their quantum integrability was established in [11] by explicitly checking that the conserved quantities of scale dimension ± 2 remain conserved in the quantum theory after an appropriate renormalization, and invoking a well known argument due to Parke [19].

For the SSSG theories, we expect that something similar happens and, hence, that they are also quantum integrable. However, the variety of different types of SSSG theories makes difficult to thoroughly check this conjecture. In general, and similarly to the HSG models, most of the SSSG theories exhibit classically conserved quantities of scale dimension $\pm 1, \pm 2, \ldots$, and it should be possible to generalize the proof in [11] to show that the conserved quantities of scale dimension ± 2 also give rise to quantum conserved quantities. Nevertheless, there is an exception to this pattern: the SSSG theories associated with symmetric spaces of maximal rank, *i.e.*, with symmetric spaces G/G_0 whose rank is $\mathrm{rank}(G/G_0) = \mathrm{rank}(G)$, which only have conserved quantities of odd scale dimension $\pm 1, \pm 3, \ldots$. According to (2.9), they are relevant perturbations of either the WZNW model corresponding to G_0 ($p = 0$) or a coset CFT of the form $G_0/U(1)^p$ with $0 \leq p \leq \mathrm{rank}(G_0)$. The type I symmetric spaces of maximal rank are the following [18]:

$$\begin{aligned}
& SO(2n)/SO(n) \times SO(n)\,, \quad SO(2n+1)/SO(n) \times SO(n+1)\,, \\
& SU(n)/SO(n)\,, \quad Sp(n)/U(n)\,, \quad E_6/Sp(4)\,, \quad E_7/SU(8)\,, \\
& E_8/SO(16)\,, \quad F_4/Sp(3) \times SU(2)\,, \quad G_2/SU(2) \times SU(2)\,. \qquad (4.1)
\end{aligned}$$

Notice that there is one for each simple compact Lie group G, which is related to its (unique) maximally non compact real form, also known as 'split form'. Besides $SU(2)/SO(2)$, which corresponds to the sine-Gordon theory, the simplest symmetric space of maximal rank is $SU(3)/SO(3)$. Then, eq. (2.9) reads $0 \leq p \leq 1$, which means that it gives rise to two different SSSG theories, depending on the choice of $\Lambda_{\pm}$. They are just the integrable perturbations of the $SU(2)_k$ WZNW model ($p = 0$) and its $\widetilde{so(2)}$ reduction ($p = 1$) constructed by Brazhnikov [17], which is identified with the perturbation of the usual $\mathbb{Z}_k$-parafermions by the second thermal operator.

In this section we explicitly prove the quantum integrability of the SSSG theories associated with a maximal rank symmetric space and $p = 0$, which will be called 'Split models'. Actually, they can be distinguished as the only ones that provide relevant perturbations of WZNW models. Namely, we will check that their simplest higher spin classically conserved densities, which have spin ± 3 instead of ± 2, remain conserved in the quantum theory after an appropriate renormalization. According to [19], this is enough to establish their quantum integrability, which, together with the results of [11] for the HSG theories, supports the conjecture that all the SSSG theories are quantum integrable.

4.1 Classical integrability.

The zero-curvature form of the equations of motion (2.6) is

$$[\partial_+ + m\, h\Lambda_+ h^\dagger\, ,\, \partial_- + m\, \Lambda_- - \partial_- h h^\dagger] = 0\,. \tag{4.2}$$

while the constraints arise naturally in the group theoretical description of the non-abelian affine Toda equations [1, 2]. In order to use the generalized Drinfel'd-Sokolov construction of [27] to get the infinite number of conserved densities of (4.2), we have to associate the zero-curvature equation with a loop algebra. In our case, the relevant loop algebra is $\mathcal{L}(g,\sigma) = \bigoplus_{j\in\mathbb{Z}} \mathcal{L}(g,\sigma)_j$ with

$$\mathcal{L}(g,\sigma)_{2j} = \lambda^j \otimes g_0\,, \quad \text{and} \quad \mathcal{L}(g,\sigma)_{2j+1} = \lambda^j \otimes g_1\,, \tag{4.3}$$

where λ is a spectral parameter; if $\sigma = \sigma_{(\vec{s};r)}$, then $\mathcal{L}(g,\sigma)$ is related to the (twisted if $r \neq 1$) affine Kac-Moody algebra $\bar{g}^{(r)}$ without central extension. Moreover, since

$$[\mathcal{L}(g,\sigma)_j\, ,\, \mathcal{L}(g,\sigma)_k] \subset \mathcal{L}(g,\sigma)_{j+k}\,, \tag{4.4}$$

the subspaces (4.3) define an integer gradation of $\mathcal{L}(g,\sigma)$. The equations of motion (4.2) remain unchanged under the transformation $\Lambda_\pm \mapsto \lambda^{\mp 1}\otimes\Lambda_\pm$ and, hence, the zero-curvature equation can actually be associated with $\mathcal{L}(g,\sigma)$ and the Lax operator $L = \partial_- + \Lambda + q$, where

$$\Lambda = m\,\lambda\otimes\Lambda_- \in \mathcal{L}(g,\sigma)_1 \quad \text{and} \quad q = -\partial_- h h^\dagger \in \mathcal{L}(g,\sigma)_0. \tag{4.5}$$

The generalized Drinfel'd-Sokolov construction goes as follows [27]. First, there is some local function y of the 'potential' q of the form [1]

$$y = \sum_{n>0} y^a(n)\lambda^{-k}\otimes t^a \in Im(ad_\Lambda)_{<0}, \tag{4.6}$$

that 'abelianizes' the Lax operator in the following sense:

$$e^y\, L\, e^{-y} = e^y(\partial_- + \Lambda + q)e^{-y} = \partial_- + \Lambda + H\,, \tag{4.7}$$

where $H \in \mathrm{Ker}(\mathrm{ad}_\Lambda)_{\leq 0}$ is another local function of q, and $\{t^a\}$ is a basis for g whose standard realization is presented in the Appendix. Then, eqs. (4.2) and (4.7) imply

$$e^y\Big(\partial_+ + mh(\lambda^{-1}\otimes\Lambda_+)h^\dagger\Big)e^{-y} = \partial_+ + \overline{H}. \tag{4.8}$$

where $\overline{H}$ also takes values in $\mathrm{Ker}(\mathrm{ad}_\Lambda)_{\leq 0}$ and, therefore, the zero-curvature equation becomes

$$\partial_-\overline{H} - \partial_+ H = [\,\overline{H}, H\,]\,. \tag{4.9}$$

The components of this equation along the centre of $\mathrm{Ker}(\mathrm{ad}_\Lambda)$ provide an infinite number of local conservation laws. To be precise, let $b \in \mathrm{Cent}(\mathrm{Ker}(\mathrm{ad}_\Lambda))_j$ with $j > 0$, and define

$$\mathcal{I}_j^{(0)}[b] = \langle b\, ,\, H\rangle\,, \qquad \overline{\mathcal{I}}_j^{(0)}[b] = \langle b\, ,\, \overline{H}\rangle\,. \tag{4.10}$$

[1] By a local function of q we mean a ∂_--differential polynomial in the components of q, *i.e.*, a polynomial in the components of q, $\partial_- q$, $\partial_-^2 q$, ...

The corresponding local conservation law is

$$\partial_+ \mathcal{I}_j^{(0)}[b] \;=\; \partial_- \overline{\mathcal{I}}_j^{(0)}[b]\,. \tag{4.11}$$

Moreover, one can check that the conserved densities given by (4.10) have scale dimension $j+1>1$ with respect to the scale transformations $x_\pm \to x_\pm/\rho$, which means that they give rise to conserved quantities of scale dimension $j>0$. The same procedure can be repeated by changing the Lax operator L by

$$\overline{L} \;=\; \partial_+ \;+\; m\lambda^{-1}\otimes \Lambda_+ \;+\; h^\dagger \partial_+ h \tag{4.12}$$

in eq. (4.7), and the result is the construction of another infinite number of local conserved quantities with scale dimension $j<0$, *i.e.*, negative scale dimension. Notice that L and $\overline{L}$ are conjugate by the transformation (2.7) and, moreover, that the dimensions of $\mathcal{L}(g,\sigma)_j$ and $\mathcal{L}(g,\sigma)_{-j}$ are equal. Therefore, for each conserved quantity of positive scale dimension j there will be another conserved quantity of scale dimension $-j$, and both are conjugate by (2.7). In particular, if the theory is parity invariant both conserved quantities are parity conjugate. This allows one to restrict the analysis to the conserved quantities with positive scale dimension.

Taking into account all this, the resulting number of classically conserved quantities with scale dimension $\pm j$ is given by the dimension of $\mathrm{Cent}(\mathrm{Ker}(\mathrm{ad}_\Lambda))_{\pm j}$. These dimensions can be easily calculated when $\Lambda_\pm$ are chosen to be regular, which means that $\mathrm{Ker}(\mathrm{ad}_{\Lambda_\pm})$ is a Cartan subalgebra of g and $p = \mathrm{rank}(G) - \mathrm{rank}(G/G_0)$ in (2.9). Then, the Drinfel'd-Sokolov construction produces exactly $\mathrm{rank}(G/G_0)$ local conserved quantities for each odd scale dimension $\pm 1, \pm 3, \ldots$, and $\mathrm{rank}(G) - \mathrm{rank}(G/G_0)$ for each even scale dimension $\pm 2, \pm 4, \ldots$.

Consider now a SSSG theory related to a symmetric space of maximal rank with p in the range (2.9) ($p=0$ corresponds to the case when $\Lambda_\pm$ are regular). Then, there are $\mathrm{rank}(G/G_0) - p$ local conserved quantities for each odd scale dimension, $\pm 1, \pm 3, \ldots$, and no conserved quantities with even scale dimension. This can be easily proved by choosing the Cartan subalgebra such that it contains the abelian subalgebra $g_0^0 = u(1)^p$ given by (2.2).

These results agree with what has been anticipated at the beginning of this section: in general, the simplest higher spin conserved quantities of a SSSG theory have spin ± 2. However, if $\mathrm{rank}(G/G_0) = \mathrm{rank}(G)$, the simplest conserved quantities will have scale dimension ± 3.

4.2 Quantum integrability of the Split models.

In the following, we will check that, after a suitable renormalization, the conserved densities of spin ± 1 and ± 3 remain conserved in the quantum version of the Split models. The explicit expressions for the relevant classical conserved densities are given in the Appendix. The proof will be similar to the one presented in [11] for the HSG theories, which uses conformal perturbation theory. However, since the simplest higher spin conserved quantity is of spin 3 instead of 2, this case will be even more involved, and in order to avoid unnecessary complications we will restrict ourselves to the Split models with G_0 simple. We will also fix the normalization of the invariant bilinear form of g, $\langle\ ,\ \rangle$, such that $\vec{\Psi}_{g_0}^2 = 2$.

Since the quantum SSSG theories can be described as perturbed conformal field theories, the existence of quantum conserved quantities can be investigated by using the methods of [28]. In the presence of the perturbation (2.8) any chiral field $\mathcal{I}(z)$, which in the unperturbed CFT satisfies $\bar{\partial}\mathcal{I}(z) = 0$, acquires a $\bar{z}$ dependence given by

$$\bar{\partial}\mathcal{I}(z,\bar{z}) \;=\; -km^2 \oint_z \frac{dw}{2\pi i}\Phi(w,\bar{z})\mathcal{I}(z)\;, \tag{4.13}$$

where we have introduced the notation $z = x_-$, $\bar{z} = x_+$, $\partial = \partial_-$, and $\bar{\partial} = \partial_+$ reminiscent of euclidean space. This contribution actually corresponds to the lowest order in perturbation theory; however, if the condition of super-renormalizability at first order $2\Delta_\Phi \leq 1$ is satisfied, no counterterms are needed to renormalize (2.8), and the previous equation is expected to be exact [28, 29]. Actually, for any SSSG theory, this condition is always fulfilled for k above some minimal value characteristic of the theory (see the comments at the end of Section 3.1). Therefore, in the perturbed CFT the chiral field $\mathcal{I}(z)$ will become a conserved quantity if the right-hand-side of (4.13) is a total ∂ derivative, i.e. if (4.13) can be written as $\bar{\partial}\mathcal{I} = \partial\bar{\mathcal{I}}$ where $\bar{\mathcal{I}}$ is another field of the original CFT. For this condition to be satisfied, the residue of the simple pole in the OPE between $\mathcal{I}(z)$ and $\Phi(w,\bar{z})$ has to be a total ∂ derivative; namely,

$$\Phi(w,\bar{z})\mathcal{I}(z) \;=\; \sum_{n>1} \frac{\{\Phi\mathcal{I}\}_n(z,\bar{z})}{(w-z)^n} \;+\; \frac{\partial\bar{\mathcal{I}}(z,\bar{z})}{(w-z)} \;+\cdots \tag{4.14}$$

Notice that the residues of the simple poles in the OPE's $\Phi(w,\bar{z})\mathcal{I}(z)$ and $\mathcal{I}(z)\Phi(w,\bar{w})$ differ only in a total ∂ derivative and, in practice, we will always consider the latter whose expression is usually simpler.

For the Split models, the unperturbed CFT is just the WZNW model corresponding to G_0 at level k. Using the conventions of the Appendix, the subset of generators of g given by t^α for each positive root $\vec{\alpha}$ of g provides a suitable basis of generators for the Lie subalgebra g_0. This means that the operator algebra of the G_0–WZNW model can be realized as a subset of the operator algebra of the WZNW model associated to G that, in particular, includes the chiral currents $J^\alpha(z)$ and $\bar{J}^\alpha(\bar{z})$ which satisfy the OPE

$$J^\alpha(w)J^\beta(z) \;=\; \frac{\hbar^2 k\,\delta^{\alpha\beta}}{(w-z)^2} \;+\; \frac{\hbar\, f^{\alpha\beta\gamma}\, J^\gamma(z)}{(w-z)} \;+\;\cdots \tag{4.15}$$

Moreover, it will be useful to recall the following well known identities [24, 26]. The first one is

$$J^\alpha(z)\,h(w,\bar{w}) \;=\; -\,\hbar\,\frac{t^\alpha\, h(w,\bar{w})}{z-w} \;+\cdots\;, \tag{4.16}$$

which is satisfied in an arbitrary representation of G_0 and exhibits that the WZNW field h is a primary field. The second is the relation between the WZNW field and the chiral currents

$$\hbar\,(k\;+\;h^\vee_{g_0})\,\partial h \;=\; \sum_\alpha (J^\alpha\, t^\alpha\, h)\;, \tag{4.17}$$

where $(AB)(z)$ is the normal ordered product of two operators $A(z)$ and $B(z)$, which will be defined by adopting the conventions of [30]. At this point, it is also convenient to recall that the classical expressions are recovered from the quantum ones by means of (see (4.5) and (4.17))

$$J^\alpha t^\alpha \;=\; -\,(\hbar k)\, q\,, \qquad (\hbar k) \to \frac{1}{\beta^2}\,, \quad \text{and} \quad k \to \infty\,, \tag{4.18}$$

which, according to (3.9), amounts to take the classical ($\hbar \to 0$) or weak coupling ($\beta^2 \to 0$) limits. Therefore, the differential polynomials in the components of q that appear in the classical local conserved densities have to be changed into differential polynomials in the chiral currents J^α. In the rest of this section we will show explicitly the Plank constant to make the distinction between classical and quantum contributions simpler.

4.2.1 Quantum spin-2 conserved densities (spin-1 conserved quantities).

Consider a generic normal ordered local spin-2 operator

$$\mathcal{I}_2(z) \;=\; D_{\alpha\beta} (J^\alpha J^\beta)(z), \tag{4.19}$$

where the numerical coefficients satisfy $D_{\alpha\beta} = D_{\beta\alpha}$. Taking into account that the perturbing operator $\Phi = \langle \Lambda_-, P\rangle$, with $P = (h\Lambda_+ h^\dagger)$, is just the WZNW field taken in the representation of G_0 provided by $[g_0, g_1] \subset g_1$, eqs. (4.16) and (4.17) become

$$J^\alpha(z)\, \langle v\,,\; P(w,\bar{w})\rangle \;=\; -\,\hbar\, \frac{\langle [v\,,\; t^\alpha]\,,\; P(w,\bar{w})\rangle}{(z-w)} \;+\; \cdots \tag{4.20}$$

$$\hbar\,(k + h^\vee_{g_0})\,\partial \langle v\,,\; P\rangle \;=\; (J^\alpha\, \langle [v\,,\; t^\alpha]\,,\; P\rangle)\,, \tag{4.21}$$

where v is an arbitrary element in g_1. Then, it is easy to show that the residue of the simple pole in the OPE $\mathcal{I}_2(z)\Phi(w,\bar{w})$ is given by

$$\mathrm{Res}\Big(\mathcal{I}_2(z)\;\Phi(w,\bar{w})\Big) \;=\; -2\,\hbar\,(J^\beta\, \langle D_{\alpha\beta}\,[\,\Lambda_-, t^\alpha\,], P(w,\bar{w})\,\rangle)\,. \tag{4.22}$$

Taking into account (4.21), this residue is a total ∂ derivative if the tensor $D_{\alpha\beta}$ satisfies [2]

$$(\hbar k)^2 D_{\alpha\beta}\,[\,\Lambda_-, t^\beta\,] \;=\; \frac{1}{2}\,[\vec{\mu}\cdot\vec{t}\,,\; t^\alpha\,], \tag{4.23}$$

for any rank(g)-component vector $\vec{\mu}$. This leads to the following solutions labelled by $\vec{\mu}$ ($\vec{\lambda}$ is defined in (A.4))

$$(\hbar k)^2 D_{\alpha\beta}(\vec{\mu}) \;=\; \frac{1}{2}\,\frac{\vec{\mu}\cdot\vec{\alpha}}{\vec{\lambda}\cdot\vec{\alpha}}\,\delta_{\alpha\beta} \;=\; m\, D^{(0)}_{\alpha\beta}(\vec{\mu})\,, \tag{4.24}$$

which satisfy

$$\mathrm{Res}\Big(\mathcal{I}_2(z)\;\Phi(w,\bar{w})\Big) \;=\; -\,\frac{(k + h^\vee_{g_0})}{k^2}\,\partial \langle\, \vec{\mu}\cdot\vec{t}\,,\; P\rangle(w,\bar{w})\,. \tag{4.25}$$

Therefore, the quantities

$$(\hbar k)^2 \mathcal{I}_2(\vec{\mu}) \;=\; \frac{1}{2}\sum_{\alpha>0} \frac{\vec{\mu}\cdot\vec{\alpha}}{\vec{\lambda}\cdot\vec{\alpha}}\,(J^\alpha J^\alpha) \tag{4.26}$$

[2] Explicit factors of $\hbar k$ are included to take account of the relation $J^\alpha t^\alpha = -(\hbar k)\, q$ in the $k \to \infty$ limit and recover the classical expressions given in the Appendix (see eq. (4.18)).

provide rank(g) linearly independent conserved densities of spin-2. The field $\bar{\mathcal{I}}_2(\vec{\mu})$ that satisfies the conservation law $\bar{\partial}\mathcal{I}_2 = \partial\bar{\mathcal{I}}_2$ is obtained through the explicit evaluation of the integral (4.13); the result is

$$(\hbar k)^2 \bar{\mathcal{I}}_2(\vec{\mu}) \;=\; -m^2\,(\hbar k)^2 \left(1 + \frac{4h_{g_0}^\vee - \Psi_g^2 h_g^\vee}{4k}\right) \langle \vec{\mu}\cdot\vec{t}\,,\; P\rangle\,. \tag{4.27}$$

It has already been pointed out that these quantum conserved quantities can be understood as the renormalization of the classical ones calculated in the Appendix. In this case, the relationship is particularly simple: the spin-2 quantum conserved densities equal the normal ordered classical conserved ones of scale dimension 2, up to the multiplicative renormalization in (4.27). In more general cases, one has to add quantum $O(1/k) \simeq O(\hbar\beta^2)$ corrections; however, the number of them must be finite as a reflection of the fact that the perturbation is relevant and, therefore, the quantum theory is super-renormalizable.

As expected, the stress-energy tensor is a particular example of a spin-2 conserved quantity. It is recovered from (4.26) for the particular choice $\vec{\mu} = \vec{\lambda}$:

$$(\hbar k)^2 \mathcal{I}_2(\vec{\lambda}) \;=\; \frac{1}{2}\,(J^\alpha J^\alpha) \;=\; -\,(k + h_{g_0}^\vee)\left\{\frac{-1}{2(k + h_{g_0}^\vee)}\,(J^\alpha J^\alpha)\right\} \;=\; -\,(k + h_{g_0}^\vee)\,T_{z,z}\,, \tag{4.28}$$

in agreement with the Sugawara construction. Consequently,

$$(\hbar k)^2 \bar{\mathcal{I}}_2(\vec{\lambda}) \;=\; -m^2\,(\hbar k)^2 \left(1 + \frac{4h_{g_0}^\vee - \Psi_g^2 h_g^\vee}{4k}\right) \langle \Lambda_-\,,\; P\rangle \;=\; -\,(k + h_{g_0}^\vee)\,T_{z,\bar{z}}\,. \tag{4.29}$$

4.2.2 Quantum spin-4 conserved densities (spin-3 conserved quantities).

Up to total ∂-derivatives, the most general form of a spin-4 normal ordered operator constructed with the WZNW currents J^α is

$$\mathcal{I}_4(z) = R_{\alpha\beta\gamma\rho}(J^\alpha(J^\beta(J^\gamma J^\rho)))\,(z) + P_{\alpha\beta\gamma}(J^\alpha(J^\beta \partial J^\gamma))\,(z) + Q_{\alpha\beta}(J^\alpha \partial^2 J^\beta)\,(z)\,, \tag{4.30}$$

with the following constraints:

$$\begin{aligned} R_{\alpha\beta\gamma\rho} \;&=\; R_{(\alpha\beta\gamma\rho)}, \qquad & Q_{\alpha\beta} \;&=\; Q_{\beta\alpha}\,, \\ P_{\alpha\beta\gamma} \;&=\; P_{\beta\alpha\gamma}, \qquad & P_{\alpha\beta\gamma} \,+\, P_{\gamma\beta\alpha} \,+\, P_{\alpha\gamma\beta} \;&=\; 0\,, \end{aligned} \tag{4.31}$$

where the first one indicates that $R_{\alpha\beta\gamma\rho}$ is a totally symmetric tensor. Then, the residue of the simple pole in the OPE $\mathcal{I}_4(z)\Phi(w,\bar{w})$ is

$$\begin{aligned} \mathrm{Res}\Big(\mathcal{I}_4(z)\Phi(w,\bar{w})\Big) \;=\;& \hbar\,\Big\{(J^\alpha(J^\beta(J^\gamma\langle\,\Omega_{\alpha\beta\gamma}\,,\,P\,\rangle))) + (\partial J^\alpha(J^\beta\langle\,M_{\alpha\beta}\,,\,P\,\rangle)) + \\ & + (\partial^2 J^\alpha\langle\,T_\alpha\,,\,P\,\rangle)\Big\}(w,\bar{w}), \end{aligned} \tag{4.32}$$

where

$$\Omega_{\alpha\beta\gamma} \;=\; 4\,R_{\alpha\beta\gamma\rho}\,[\Lambda_-, t^\rho], \tag{4.33}$$

$$M_{\alpha\beta} \;=\; -12\,\hbar\,R_{\alpha\beta\gamma\rho}\,[\,[\Lambda_-, t^\gamma], t^\rho] \,-\, 2\,(P_{\alpha\beta\gamma} - P_{\beta\gamma\alpha})\,[\Lambda_-, t^\gamma], \tag{4.34}$$

$$\begin{aligned} T_\alpha \;=\;& 2\,\hbar^2\,R_{\alpha\beta\gamma\rho}\,[\,[\,[\Lambda_-, t^\beta] t^\gamma], t^\rho] \,+\, \hbar(P_{\alpha\beta\gamma} - P_{\gamma\beta\alpha})\,[\,[\Lambda_-, t^\beta], t^\gamma] \,+ \\ & +\,(2\,Q_{\alpha\rho} \,+\, \hbar f^{\beta\gamma\alpha} P_{\beta\rho\gamma})\,[\Lambda_-, t^\rho]. \end{aligned} \tag{4.35}$$

The condition that $\mathcal{I}_4(z)$ is a conserved quantity is equivalent to the existence of two tensors $F_{\alpha\beta}$ and R_α taking values in g_1 such that

$$Res\Big(I_4(z)\Phi(w,\bar{w})\Big) \;=\; \partial(J^\alpha(J^\beta\langle\, F_{\alpha\beta}\,,\, P\,\rangle))\,+\,\partial(\partial J^\alpha\langle\, R_\alpha\,,\,P\,\rangle)\,. \tag{4.36}$$

We will assume that $F_{\alpha\beta}$ is totally symmetric because, otherwise, its antisymmetric part $F_{[\gamma\beta]}$ can be absorbed in R_α through the transformation

$$R_\alpha\;\rightarrow\;R_\alpha\,+\,\frac{\hbar}{2}f^{\alpha\beta\gamma}F_{[\gamma\beta]}\,. \tag{4.37}$$

If we define

$$\widehat{F}_{\alpha\beta}\;=\;\frac{F_{\alpha\beta}}{\hbar(k+h^\vee_{g_o})}\quad\text{and}\quad\widehat{R}_\alpha\;=\;\frac{R_\alpha}{\hbar(k+h^\vee_{g_o})}\,, \tag{4.38}$$

eqs. (4.33)–(4.38) can be combined to obtain the following tensor relations

$$12\,R_{\alpha\beta\gamma\rho}\,[\Lambda_-,t^\rho]\;=\;[\,\widehat{F}_{\alpha\beta}\,,\,t^\gamma\,]\,+\,[\,\widehat{F}_{\alpha\gamma}\,,\,t^\beta\,]\,+\,[\,\widehat{F}_{\gamma\beta}\,,\,t^\alpha\,], \tag{4.39}$$

$$\begin{aligned}-2\,(P_{\alpha\beta\gamma}-P_{\beta\gamma\alpha})[\Lambda_-,t^\gamma\,]\;&=\;2\,F_{\alpha\beta}\,+\,[\widehat{R}_\alpha,t^\beta]\,+\,\hbar\Big\{\,[\widehat{F}_{\beta\rho},[t^\alpha,t^\rho]\,]\,+\\&+[\,[\widehat{F}_{\alpha\beta},t^\rho],t^\rho]\,+\,[\,[\widehat{F}_{\alpha\rho},t^\beta],t^\rho]+\,[\,[\widehat{F}_{\beta\rho},t^\alpha],t^\rho]\,\Big\},\end{aligned} \tag{4.40}$$

$$\begin{aligned}-2\,Q_{\alpha\beta}\,[\,\Lambda_-,t^\beta\,]=-R_\alpha\,&+\,\hbar\Big\{\,(P_{\alpha\beta\gamma}-P_{\gamma\beta\alpha})\,[\,[\Lambda_-,t^\beta\,],t^\gamma]\,+\,f^{\beta\gamma\alpha}\,P_{\beta\rho\gamma}\,[\Lambda_-,t^\rho\,]\Big\}+\\&+2\hbar^2\Big\{\,R_{\alpha\beta\gamma\rho}\,[\,[\,[\Lambda_-,t^\beta],t^\gamma],t^\rho]\,+\,\frac{1}{3}\,f^{\xi\gamma\rho}f^{\beta\rho\alpha}\,[\,\widehat{F}_{\beta\xi},t^\gamma\,]\Big\}\,,\end{aligned} \tag{4.41}$$

where (4.39) does not contain any explicit quantum correction at all, which means that the same relation will hold at the classical level. In fact, taking into account the explicit expressions for the classical densities given in the Appendix and (4.18), for each rank(g)-component vector $\vec{\mu}$ there is a classical solution of (4.39)–(4.41) given by

$$R_{\alpha\beta\gamma\rho}=\frac{(\hbar k)^4}{m^3}\,R^{(0)}_{\alpha\beta\gamma\rho}(\vec{\mu})\,,\quad P_{\alpha\beta\gamma}=-\frac{(\hbar k)^3}{m^3}\,P^{(0)}_{\alpha\beta\gamma}(\vec{\mu})\,,\quad Q_{\alpha\beta}=\frac{(\hbar k)^2}{m^3}\,Q^{(0)}_{\alpha\beta}(\vec{\mu})\,, \tag{4.42}$$

which satisfy the classical limit of those equations:

$$12\,m^3R^{(0)}_{\alpha\beta\gamma\rho}(\vec{\mu})\,[\Lambda_-,t^\rho]=(\hbar k)^4\Big\{[\widehat{F}^{(0)}_{\alpha\beta}(\vec{\mu}),t^\gamma]+[\widehat{F}^{(0)}_{\alpha\gamma}(\vec{\mu}),t^\beta]+[\widehat{F}^{(0)}_{\beta\gamma}(\vec{\mu}),t^\alpha]\Big\}, \tag{4.43}$$

$$2\,m^3\,(P^{(0)}_{\alpha\beta\gamma}(\vec{\mu})-P^{(0)}_{\beta\gamma\alpha}(\vec{\mu}))\,[\Lambda_-,t^\gamma]=(\hbar k)^3\,\Big\{2\,(\hbar k)\,\widehat{F}^{(0)}_{\alpha\beta}(\vec{\mu})\,+\,[\widehat{R}^{(0)}_\alpha(\vec{\mu}),t^\beta]\Big\}, \tag{4.44}$$

$$2\,m^3\,Q^{(0)}_{\alpha\beta}(\vec{\mu})\,[\Lambda_-,t^\beta]\;=-(\hbar k)^3\,\widehat{R}^{(0)}_\alpha(\vec{\mu}). \tag{4.45}$$

Using eqs. (A.8)–(A.10), one can obtain the following explicit expressions for the classical limit of the corresponding tensors $\widehat{F}_{\alpha\beta}(\vec{\mu})$ and $\widehat{R}_\alpha(\vec{\mu})$

$$(\hbar k)^3R^{(0)}_\alpha(\vec{\mu})\;=\;-\frac{\vec{\alpha}\cdot\vec{\mu}}{(\vec{\lambda}\cdot\vec{\alpha})^2}\;t^{\bar{\alpha}}\,, \tag{4.46}$$

$$
\begin{aligned}
(\hbar k)^4 F^{(0)}_{\alpha\beta}(\vec{\mu}) \;=\; & \frac{t^{\bar\gamma}}{4}\Bigg\{\frac{1}{4}\Bigg(\frac{\vec\alpha\cdot\vec\mu}{(\vec\lambda\cdot\vec\alpha)^2} f^{\bar\alpha\beta\bar\gamma} \;+\; \frac{\vec\beta\cdot\vec\mu}{(\vec\lambda\cdot\vec\beta)^2} f^{\bar\beta\alpha\bar\gamma}\Bigg) + \\
& + \Bigg(\frac{f^{\gamma\alpha\beta}}{\vec\lambda\cdot\vec\gamma} \;+\; \frac{3}{4}\frac{\vec\lambda\cdot\vec\gamma\, f^{\bar\alpha\gamma\bar\beta}}{(\vec\lambda\cdot\vec\alpha)(\vec\lambda\cdot\vec\beta)}\Bigg)\Bigg(\frac{\vec\beta\cdot\vec\mu}{\vec\lambda\cdot\vec\beta} \;-\; \frac{\vec\alpha\cdot\vec\mu}{\vec\lambda\cdot\vec\alpha}\Bigg) - \\
& - \frac{\vec\gamma\cdot\vec\mu}{4\,\vec\lambda\cdot\vec\gamma}\Bigg(\frac{f^{\bar\beta\alpha\bar\gamma}}{\vec\lambda\cdot\vec\beta} \;+\; \frac{f^{\bar\alpha\beta\bar\gamma}}{\vec\lambda\cdot\vec\alpha}\Bigg)\Bigg\} - \frac{(\vec\alpha\cdot\vec\mu)\,\delta_{\alpha\beta}}{2\,(\vec\lambda\cdot\vec\alpha)^2}\,\vec\alpha\cdot\vec t\,,
\end{aligned}
\tag{4.47}
$$

where $\vec\alpha\cdot\vec t = \alpha^A t^A$ is in the Cartan subalgebra of g. In particular, when $\vec\mu = \vec\lambda$ the previous equations reduce to

$$
(\hbar k)^3 \widehat{R}^{(0)}_{\alpha}(\vec\lambda) = -\frac{1}{\vec\lambda\cdot\vec\alpha}\, t^{\bar\alpha}, \tag{4.48}
$$

$$
(\hbar k)^4 \widehat{F}^{(0)}_{\alpha\beta}(\vec\lambda) = -\frac{\delta_{\alpha\beta}}{2\,\vec\lambda\cdot\vec\alpha}\,\vec\alpha\cdot\vec t\,. \tag{4.49}
$$

Once we have solved the eqs. (4.39)–(4.41) in the classical limit, we will try to find the solutions for the full equations. Since they become quite complicated, we will only obtain a single one, which is enough to establish the quantum integrability of these models. Projecting eqs. (4.39)–(4.41) on the CSA we get

$$
\widehat{F}^{\bar\gamma}_{\alpha\beta}\gamma^A \;+\; \widehat{F}^{\bar\alpha}_{\beta\gamma}\alpha^A \;+\; \widehat{F}^{\bar\beta}_{\alpha\gamma}\beta^A \;=0, \tag{4.50}
$$

$$
\begin{aligned}
2\,\hbar\,(k + h^\vee_{g_o} - \frac{\Psi_g^2 h^\vee_g}{4})\,\widehat{F}^A_{\alpha\beta} \;-\; 2\,\hbar\,(\widehat{F}^B_{\alpha\beta}\alpha^B)\alpha^A \;=\; & \widehat{R}^{\bar\beta}_{\alpha}\beta^A \;+ \\
& + \hbar\,\{f^{\alpha\rho\delta}\,\widehat{F}^{\bar\delta}_{\beta\rho} \;+\; f^{\bar\gamma\alpha\bar\delta}\,\widehat{F}^{\bar\gamma}_{\beta\delta} \;+\; f^{\bar\gamma\beta\bar\delta}\,\widehat{F}^{\bar\gamma}_{\alpha\delta}\}\delta^A,
\end{aligned}
\tag{4.51}
$$

$$
\begin{aligned}
\hbar(k + h^\vee_{g_0})\widehat{R}^A_{\alpha} = & \frac{\hbar}{2}\widehat{R}^{\bar\xi}_{\beta} f^{\bar\xi\alpha\bar\beta}\beta^A + \frac{\hbar^2}{2}\Big\{ f^{\alpha\rho\delta}\Big\{\widehat{F}^{\bar\xi}_{\beta\rho} f^{\bar\xi\delta\bar\beta} + (\widehat{F}^{B}_{\beta\rho}\delta^B)\delta_{\bar\delta\bar\beta}\Big\} + \\
& + f^{\beta\rho\delta} f^{\bar\xi\delta\bar\beta}\widehat{F}^{\bar\xi}_{\alpha\rho} + 2 f^{\bar\xi\rho\bar\delta} f^{\bar\delta\rho\bar\beta}\widehat{F}^{\bar\xi}_{\alpha\beta} \;+\; 2 f^{\bar\xi\beta\bar\delta} f^{\bar\delta\rho\bar\beta}\widehat{F}^{\bar\xi}_{\alpha\rho} \;+\; 2 f^{\bar\xi\alpha\bar\delta} f^{\bar\delta\rho\bar\beta}\widehat{F}^{\bar\xi}_{\beta\rho} + \\
& + 2 f^{\bar\alpha\rho\bar\beta}(\widehat{F}^B_{\beta\rho}\alpha^B)\Big\}\beta^A + 2\hbar^2 f^{\bar\rho\gamma\bar\beta}\Big\{(\widehat{F}^B_{\gamma\rho}\alpha^B)\delta_{\bar\alpha\bar\beta} \;+\; \widehat{F}^{\bar\xi}_{\alpha\gamma} f^{\bar\xi\rho\bar\beta} \;+ \\
& + \widehat{F}^{\bar\xi}_{\rho\gamma} f^{\bar\xi\alpha\bar\beta} \;+\; \widehat{F}^{\bar\xi}_{\rho\alpha} f^{\bar\xi\gamma\bar\beta}\Big\}\rho^A \;-\; \frac{2\hbar^2}{3} f^{\xi\gamma\rho} f^{\beta\rho\alpha}\widehat{F}^{\bar\gamma}_{\beta\xi}\gamma^A\,.
\end{aligned}
\tag{4.52}
$$

where we have used the following definitions:

$$
\widehat{F}^{\bar\gamma}_{\alpha\beta} = \langle\,\widehat{F}_{\alpha\beta}\,,\,t^{\bar\gamma}\,\rangle, \quad \widehat{F}^{A}_{\alpha\beta} = \langle\,\widehat{F}_{\alpha\beta}\,,\,t^{A}\,\rangle, \quad \widehat{R}^{\bar\beta}_{\alpha} = \langle\,\widehat{R}_{\alpha}\,,\,t^{\bar\beta}\,\rangle\,, \quad \widehat{R}^{A}_{\alpha} = \langle\,\widehat{R}_{\alpha}\,,\,t^{A}\,\rangle. \tag{4.53}
$$

Taking into account (4.50)–(4.52) together with the constraints (4.31) one can check that there is a particular solution for eq. (4.36) given by

$$
(\hbar k)^4 \widehat{F}_{\alpha\beta} \;=\; (\hbar k)^4 \widehat{F}^{(0)}_{\alpha\beta}(\vec\lambda) \;= -\frac{\delta_{\alpha\beta}}{2\,\vec\lambda\cdot\vec\alpha}\,\vec\alpha\cdot\vec t. \tag{4.54}
$$

$$
(\hbar k)^3\,\widehat{R}_{\alpha} \;=\; (\hbar k)^3\Big\{\widehat{R}^{(0)}_{\alpha}(\vec\lambda) \;+\; \frac{1}{k}\widehat{R}^{(1)}_{\alpha}(\vec\lambda)\Big\}, \tag{4.55}
$$

where $\widehat{R}_\alpha^{(0)}(\vec{\lambda})$ is the classical solution (4.48) and

$$\widehat{R}_\alpha^{(1)}(\vec{\lambda}) \;=\; \frac{\Psi_g^2 h_g^\vee \;-\; 4h_{g_0}^\vee \;+\; 4\vec{\alpha}^2}{4\,\vec{\lambda}\cdot\vec{\alpha}}\, t^{\bar{\alpha}}. \tag{4.56}$$

For this particular solution, eqs. (4.39)–(4.41) imply

$$(\hbar k)^4 R_{\alpha\beta\gamma\rho} \;=\; m^3 R_{\alpha\beta\gamma\rho}^{(0)}(\vec{\lambda}), \tag{4.57}$$

where $R_{\alpha\beta\gamma\rho}^{(0)}(\vec{\lambda})$ is just the classical solution (A.11), which is a consequence of the absence of quantum corrections in eq. (4.39),

$$(\hbar k)^3 P_{\alpha\beta\gamma} \;=\; -m^3\Big\{P_{\alpha\beta\gamma}^{(0)}(\vec{\lambda}) \;+\; \frac{1}{k}P_{\alpha\beta\gamma}^{(1)}(\vec{\lambda})\Big\}, \tag{4.58}$$

where $P_{\alpha\beta\gamma}^{(0)}(\vec{\lambda})$ is the classical solution given by (A.12) and the quantum correction $P_{\alpha\beta\gamma}^{(1)}(\vec{\lambda})$ is

$$\begin{aligned} -m^3 P_{\alpha\beta\gamma}^{(1)}(\vec{\lambda}) \;=\; & \frac{1}{6\,\vec{\lambda}\cdot\vec{\gamma}}\bigg\{\frac{\Psi_g^2 h_g^\vee - 4\,h_{g_0}^\vee}{4}\,\Big\{\frac{f^{\bar{\alpha}\bar{\beta}\bar{\gamma}}}{\vec{\lambda}\cdot\vec{\alpha}} \;+\; \frac{f^{\bar{\beta}\alpha\bar{\gamma}}}{\vec{\lambda}\cdot\vec{\beta}}\Big\} + \\ & +\frac{3f^{\alpha\beta\gamma}}{2}\Big\{\frac{\vec{\alpha}\cdot\vec{\gamma}}{\vec{\lambda}\cdot\vec{\alpha}} \;-\; \frac{\vec{\beta}\cdot\vec{\gamma}}{\vec{\lambda}\cdot\vec{\beta}}\Big\}\bigg\}, \end{aligned} \tag{4.59}$$

and finally

$$(\hbar k)^2 Q_{\alpha\rho} \;=\; m^3\Big\{Q_{\alpha\rho}^{(0)}(\vec{\lambda}) \;+\; \frac{1}{k}Q_{\alpha\rho}^{(1)}(\vec{\lambda}) \;+\; \frac{1}{k^2}Q_{\alpha\rho}^{(2)}(\vec{\lambda})\Big\}, \tag{4.60}$$

where the first contribution is, again, the classical one given by (A.13) and the contributions $Q_{\alpha\rho}^{(1)}(\vec{\lambda})$ and $Q_{\alpha\rho}^{(2)}(\vec{\lambda})$ denote respectively first and second order quantum corrections whose explicit expressions are

$$m^3 Q_{\alpha\rho}^{(1)}(\vec{\lambda}) \;=\; \frac{{\Psi_g}^2 h_g^\vee - 8h_{g_0}^\vee + 4\vec{\alpha}^2}{8(\vec{\lambda}\cdot\vec{\alpha})^2}\,\delta_{\alpha\rho} \;+\; \frac{f^{\bar{\beta}\gamma\bar{\rho}} f^{\bar{\beta}\gamma\bar{\alpha}}}{4(\vec{\lambda}\cdot\vec{\alpha})(\vec{\lambda}\cdot\vec{\rho})}\,, \tag{4.61}$$

$$\begin{aligned} m^3 Q_{\alpha\rho}^{(2)}(\vec{\lambda}) =\; & \frac{5(\vec{\gamma}\cdot\vec{\beta})\{f^{\bar{\beta}\gamma\bar{\rho}} f^{\bar{\gamma}\beta\bar{\alpha}} \;+\; f^{\bar{\beta}\gamma\bar{\alpha}} f^{\bar{\gamma}\beta\bar{\rho}}\}}{48(\vec{\lambda}\cdot\vec{\alpha})(\vec{\lambda}\cdot\vec{\rho})} \;+\; h_{g_0}^\vee\frac{{\Psi_g}^2 h_g^\vee - 4h_{g_0}^\vee + 4\vec{\alpha}^2}{8(\vec{\lambda}\cdot\vec{\alpha})^2}\,\delta_{\alpha\rho} + \\ & +\frac{f^{\bar{\beta}\gamma\bar{\rho}} f^{\bar{\beta}\gamma\bar{\alpha}}}{4(\vec{\lambda}\cdot\vec{\alpha})(\vec{\lambda}\cdot\vec{\rho})}\Big\{\frac{{\Psi_g}^2 h_g^\vee - 4h_{g_0}^\vee}{4} \;-\; \frac{\vec{\gamma}^2}{3} \;-\; \frac{(\vec{\gamma}\cdot\vec{\beta})(\vec{\lambda}\cdot\vec{\beta})}{2\,\vec{\lambda}\cdot\vec{\gamma}}\Big\} - \\ & -\frac{2\vec{\rho}^4 + (\vec{\beta}\cdot\vec{\rho})^2}{24(\vec{\lambda}\cdot\vec{\rho})^2}\delta_{\alpha\rho} \;-\; \frac{f^{\beta\gamma\rho} f^{\beta\gamma\alpha}}{8\,\vec{\lambda}\cdot\vec{\gamma}}\Big\{\frac{\vec{\gamma}\cdot\vec{\rho}}{\vec{\lambda}\cdot\vec{\rho}} + \frac{\vec{\gamma}\cdot\vec{\alpha}}{\vec{\lambda}\cdot\vec{\alpha}} - \frac{\vec{\gamma}\cdot\vec{\beta}}{\vec{\lambda}\cdot\vec{\beta}}\Big\}. \end{aligned} \tag{4.62}$$

The existence of this particular solution for eq. (4.36) means that, at least, there is a spin-4 quantum conserved density or, equivalently, a quantum conserved quantity of spin 3. Moreover, as explained below (4.12), there will be also a conserved quantity of spin -3. Taking into account Parke's results [19], this allows one to conclude that the Split models are quantum integrable and, hence, that they should admit a factorizable S-matrix.

5 The spectrum of the Split models.

An important property of both the SSSG and the HSG theories is that they admit soliton solutions [3]. Moreover, the semi-classical quantization of the solitons is expected to help in deducing the form of the exact S-matrix. The complete analysis of the soliton spectrum of the SSSG theories is beyond the scope of this paper and will be presented in subsequent publications. However, in this section we will discuss its main features and provide some explicit soliton solutions.

We will restrict ourselves to the Split models, whose quantum integrability has been explicitly established in the previous section. They are associated with a type I symmetric space G/G_0 of maximal rank, *i.e.*, $\mathrm{rank}(G/G_0) = \mathrm{rank}(G) = r_g$, and two regular elements $\Lambda_\pm \in g_1$. This means that $\mathrm{Ker}(\mathrm{ad}_{\Lambda_\pm})$ is a Cartan subalgebra of g contained in g_1, and we will use the realization of g_0 and g_1 given in the Appendix. Actually, the simplest Split model is the well known sine-Gordon model, which is recovered with the symmetric space $SU(2)/SO(2)$. In this case, G_0 is abelian and, hence, the coupling constant does not have to be quantized; this is the reason why this SSSG theory has not been mentioned in the tables 2–6.

First of all, we have to obtain the vacuum manifold $\mathcal{M}_0$, which consists of the constant field configurations h_0 that minimise the potential in (2.3). This condition amounts to

$$[\Lambda_+ \,,\, h_0^\dagger \Lambda_- h_0] \;=\; 0\,, \quad \text{and} \quad \vec{\alpha}(\Lambda_+)\, \vec{\alpha}(h_0^\dagger \Lambda_- h_0) > 0\,, \tag{5.1}$$

for all roots $\vec{\alpha}$ of g, which requires that Λ_+ and $h_0^\dagger \Lambda_- h_0$ belong to the same Weyl chamber of the Cartan subalgebra of g. Since they are regular, and taking into account that the conjugation $h_0^\dagger \cdot h_0$ permutes the Weyl chambers [18], we can assume that Λ_+ and Λ_- already belong to the same Weyl chamber without any loss of generality. Then (5.1) implies $h_0^\dagger \Lambda_- h_0 = \Lambda_-$ and, therefore, h_0 has to be of the form

$$h_0 \;=\; \mathrm{e}^{\,\pi \vec{\mu}\cdot\vec{t}}\,, \tag{5.2}$$

where $\vec{\mu}$ either vanishes or belongs to the co-root lattice of G, which is the root lattice of the dual group $G^\vee$, $\Lambda_R(G^\vee)$. Recall that $G^\vee$ is defined by requiring that its Lie algebra has roots which are the duals of the roots of g defined by $\vec{\alpha}^\vee = 2\,\vec{\alpha}/\vec{\alpha}^2$. However, any element of the form (5.2) satisfies $h_0^2 = 1$, which implies that the vacuum manifold is given by

$$\mathcal{M}_0 \;=\; \{1\,,\, \mathrm{e}^{\,\pi\vec{\mu}\cdot\vec{t}} \mid \vec{\mu} = \sum_{i=1}^{r_g} n_i\, \vec{\alpha}_i^\vee \,, n_i = 0,1\}\,, \tag{5.3}$$

where $\vec{\alpha}_1, \ldots, \vec{\alpha}_{r_g}$ are simple roots of g. Therefore, $\mathcal{M}_0$ is an abelian discrete group isomorphic to the coset $\Lambda_R(G^\vee)/2\Lambda_R(G^\vee)$. Moreover, using the normalization in the Appendix, one can check that

$$\mathrm{e}^{\,\pi\vec{\alpha}^\vee\cdot\vec{t}} \;=\; \exp\left(+\frac{2\pi\, t^\alpha}{\sqrt{\vec{\alpha}^2}}\right) \;=\; \exp\left(-\frac{2\pi\, t^\alpha}{\sqrt{\vec{\alpha}^2}}\right)\,, \tag{5.4}$$

for any root of g, which emphasises that $\mathcal{M}_0 \subset G_0$.

5.1 Fundamental particles.

They correspond to the fluctuations of the field h around a vacuum configuration $h_0 \in \mathcal{M}_0$. Let us take, $h = h_0 e^{\Phi}$, where $\Phi \in g_0$. The linearized eqs. (2.6) are

$$P(\Phi) = 0 \quad \text{and} \quad \partial_\mu \partial^\mu \Phi = -4m^2 [\Lambda_+, [\Lambda_- , \Phi]], \tag{5.5}$$

which show that the fundamental particles are associated with the non-vanishing eigenvalues of the mass-matrix $-4m^2[\Lambda_+, [\Lambda_-, \Phi]]$ on g_0. They correspond to the field configurations of the form $\Phi = \phi t^{\alpha}$, where $\vec{\alpha}$ is an arbitrary positive root of g, whose mass is

$$m_\alpha = 2m\sqrt{\vec{\alpha}(\Lambda_+)\, \vec{\alpha}(\Lambda_-)}, \tag{5.6}$$

which is real because of (5.1). Therefore, for each positive root $\vec{\alpha}$ of g, there is a fundamental particle described by a real field $\phi = \phi(x,t)$ whose mass is given by (5.6). It is worth noticing that (5.6) is the mass formula giving the spectrum of fundamental particles of the HSG theory associated with G; however, in that case the particles are described by complex fields.

A very peculiar property of this spectrum that is shared with the HSG and all the other SSSG theories is that the mass formula (5.6) satisfies the following kind of inequalities

$$m_{\alpha+\beta} \geq m_\alpha + m_\beta \,, \tag{5.7}$$

which suggests that some of the fundamental particles might be unstable. For the HSG theories, this has been checked using perturbation theory [13] and, consequently, their exact S-matrix exhibits resonance poles associated with the unstable particles [13, 14]. For the Split model related to $SU(3)/SO(3)$ it has also been checked that some of the fundamental particles actually decay [17]; however, more work is needed in order to establish this for the generality of the SSSG models and construct their exact S-matrix.

5.2 Soliton solutions.

In order for a solution to eqs. (2.6) to have finite energy the field h must tend to limits in $\mathcal{M}_0$ as $x \to \pm\infty$. So,

$$h(+\infty, t)\, h^\dagger(-\infty, t) \in \mathcal{M}_0 \,, \tag{5.8}$$

and its value is conserved as the system evolves in time. This means that, at fixed t, a solution $h = h(t,x)$ with finite energy is a path on the G_0 manifold connecting two elements in $\mathcal{M}_0$ and, since G_0 is not simply connected in general, there could be different solutions sharing the same value of $h(+\infty,t)h^\dagger(-\infty,t)$. Therefore, each solution will be characterized by two topological 'quantum numbers'. The first will be the value of (5.8), which is an element in $\mathcal{M}_0$ or, equivalently, in the discrete group $\Lambda_R(G^\vee)/2\Lambda_R(G^\vee)$, and the second will be an element in $\pi_1(G_0)$, the fundamental group of G_0, which can be found in table 1. In other words, the solitons of the Split models will be topological, like the solitons of the sine-Gordon equation or of the affine Toda equations with imaginary coupling constant. This is in contrast with the solitons of the HSG theories, which only carry Noether charges. On their side, the solitons of the Split models do not carry any Noether charge because $g_0^0 = \{0\}$ in (2.2) ($p = 0$). Nevertheless, for a generic SSSG theory

G_0	$SO(n)$ $SO(16)$	$SO(n)\times SO(n+1)$ $SO(n)\times SO(n)$	$U(n)$	$Sp(4)$ $SU(8)$ $Sp(3)\times SU(2)$ $SU(2)\times SU(2)$
$\pi_1(G_0)$	$\mathbb{Z}_2$	$\mathbb{Z}_2\oplus\mathbb{Z}_2$	$\mathbb{Z}$	0

Table 1: Fundamental groups $\pi_1(G_0)$ corresponding to the Split models.

$p\neq 0$ and the solitons will carry both topological and $U(1)^p$ Noether charges, which make them similar to the dyons in four-dimensional non-abelian gauge theories [21]

Taking into account that the sine-Gordon theory is the Split model corresponding to $SU(2)/SO(2)$, a number of explicit soliton solutions for the Split models can be obtained by embedding the sine-Gordon solitons in the regular $SU(2)$ subgroups of G. This method is widely used in the context of Yang-Mills theories based on arbitrary Lie groups to construct monopole or instanton solutions by embeddings of the $SU(2)$ spherically symmetric 't-Hooft-Polyakov monopole [31] or the self-dual $SU(2)$ Belavin-Polyakov-Schwartz-Tyupkin instanton [32]. It has also been used to construct the soliton solutions of the affine Toda theories with imaginary coupling constant [33] and, more recently, to construct the soliton solutions of the HSG theories starting with the Complex sine-Gordon solitons [12]. For each positive root $\vec{\alpha}$ of G, let us consider the field configuration $h=\exp(\phi t^{\alpha}/\sqrt{\vec{\alpha}^2})$, which trivially satisfies the constraints in (2.6). According to (2.3), its Lagrangian density is

$$\mathcal{L} \;=\; \frac{1}{4\pi\beta^2\vec{\alpha}^2}\Bigl(\frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi \;+\; m_\alpha^2(\cos\phi\;-\;1)\Bigr), \tag{5.9}$$

which is just the Lagrangian density of the sine-Gordon model.

This way, for each positive root $\vec{\alpha}$ of G, each soliton solution of the sine-Gordon equation provides a soliton solution for the Split model. Namely, the usual soliton and anti-soliton solutions allow one to construct two *a priori* different soliton solutions with mass

$$M_{s,\bar{s}}(\vec{\alpha}) = \frac{2}{\vec{\alpha}^2}\,\frac{1}{\pi\beta^2}\,m_\alpha, \tag{5.10}$$

while the masses of the solitons associated with the breathers of the sine-Gordon equation are

$$M_n(\vec{\alpha}) \;=\; 2M_s(\vec{\alpha})\,\sin\Bigl(\frac{\vec{\alpha}^2}{2}\frac{\pi\beta^2}{2}\,n\Bigr) \;\leq\; 2M_s(\vec{\alpha}), \tag{5.11}$$

and there is a different soliton for each integer n such that $n<2/(\beta^2\vec{\alpha}^2)$. As usual, in the weak coupling $\beta^2\rightarrow 0$ limit we obtain

$$M_n(\vec{\alpha}) \;\simeq\; nm_\alpha - \mathcal{O}\Bigl(\beta^2\Bigr) \tag{5.12}$$

and, hence, the fundamental particle associated with $\vec{\alpha}$ becomes identified with the lightest breather with mass $M_1(\vec{\alpha})$. It is important to notice that the mass formulae (5.10) and (5.11) satisfy inequalities similar to (5.7), which again suggests that some of these solitons might be unstable. Actually, some examples of unstable solitons are already known in the Split model related to $SU(3)/SO(3)$ [17].

The soliton (s) and antisoliton ($\bar{s}$) solutions of the sine-Gordon model satisfy

$$\phi_{s,\bar{s}}(+\infty,t) \,-\, \phi_{s,\bar{s}}(-\infty,t) \,=\, \pm 2\pi, \tag{5.13}$$

which implies the following asymptotic behaviour for the fields $h_s = \exp(\phi_s t^\alpha/\sqrt{\vec{\alpha}^2})$ and $h_{\bar{s}} = \exp(\phi_{\bar{s}} t^\alpha/\sqrt{\vec{\alpha}^2})$:

$$h_s(+\infty,t)\, h_s^\dagger(-\infty,t) \,=\, \mathrm{e}^{+\frac{2\pi t^\alpha}{\sqrt{\vec{\alpha}^2}}}, \qquad h_{\bar{s}}(+\infty,t)\, h_{\bar{s}}^\dagger(-\infty,t) \,=\, \mathrm{e}^{-\frac{2\pi t^\alpha}{\sqrt{\vec{\alpha}^2}}}, \tag{5.14}$$

both in $\mathcal{M}_0$. Therefore, taking into account (5.4), their asymptotic behaviour is the same and it will not be possible to distinguish these two soliton configurations unless G_0 is not simply connected and h_s and $h_{\bar{s}}$ are associated with different elements in $\pi_1(G_0)$.

6 Conclusions.

In this paper we have studied some of the quantum properties of the massive SSSG theories constructed in [3]. First of all, we have identified the perturbed conformal field theories corresponding to these theories when the symmetric space G/G_0 is of type I. This amounts to find which are the unperturbed CFT and the perturbing operator specified by the potential term in the classical action. Since the type I symmetric spaces are irreducible, the perturbation is given by a single spinless primary field whose conformal dimension has been calculated and can be found in tables 2 and 3. Actually, our calculation only depends on the algebraic structure of the symmetric space and, therefore, it provides the conformal dimension of the perturbations for the general class of SSSG theories constructed in [6]. They are obtained from ours by substituting Λ_+ and Λ_- for two arbitrary elements T and $\bar{T}$ in g_1, and g_0^0 for $\mathfrak{h}$, their simultaneous centralizer in g_0. The resulting SSSG theories are perturbations of the coset CFT related to G/H, where H is the Lie group corresponding to $\mathfrak{h}$. However, as shown in [3], these theories will not exhibit a mass gap unless H is either trivial or abelian.

Among others, the resulting class of perturbed CFT's include massive perturbations of WZNW models, which are related to symmetric spaces of maximal rank; we have named these theories 'Split models'. In addition, there are new massive perturbations of parafermion theories different to those provided by the HSG theories [11]. In particular, this class of theories includes the perturbations of the simplest $\mathbb{Z}_k$ parafermions by their first and second thermal operators [9], which are related to $G/G_0 = Sp(2)/U(2)$ [4] and $SU(3)/SO(3)$ [17], respectively.

Our second task was to investigate the quantum integrability of the SSSG theories. In view of the large variety of different types of perturbed CFT's corresponding to the SSSG theories, we have restricted ourselves to give a detailed proof of the quantum integrability only for the Split models. Classically, they exhibit rank(G) conserved quantities for each odd spin $\pm 1, \pm 3, \ldots$, and we have checked that there are at least two quantities of spin $+3$ and -3 that remain conserved in the quantum theory after an appropriate renormalization. This implies, via the usual folklore, the factorization of their scattering matrices [19] and, hence, their quantum integrability. This result, together with the integrability of the HSG theories [11] whose simplest higher spin conserved quantities have spin ± 2, lead us to conjecture that all the massive SSSG theories will be quantum integrable.

The quantum integrability of the SSSG theories implies that they should admit a factorizable S-matrix and the next stage of analysis consists in establishing its form. As a first step towards this aim, we have illustrated the general properties of the spectrum of the SSSG theories by discussing the spectrum of fundamental particles and solitons of the Split models. Its main features, which are expected to be shared with all the other SSSG theories, are, first, that the fundamental particles become identified with some of the solitons in the semiclassical limit, like in the sine-Gordon and complex sine-Gordon theories. This makes us expect that the spectrum will be entirely solitonic in the general case. Second, since there are different vacua, these solitons are topological, in contrast with the solitons of the HSG theories which are not. Moreover, in the general case, they are expected to carry conserved Noether charges as well. Third, some of these solitons are expected to correspond to unstable particles in the quantum theory. Therefore, like in the HSG theories, only the stable solitons should correspond to asymptotic states while the unstable ones will produce resonance poles in the S-matrix [13, 14]. Finally, both the fundamental particles and the solitons are labelled by the roots of the Lie algebra of G, which is reminiscent of the spectrum of the HSG theories.

Taking into account the form of the exact S-matrices of the HSG theories [13, 14], the last feature suggests that the S-matrices of the SSSG theories could be somehow related to the 'colour valued' scattering matrices constructed in [15]. These S-matrices are related to pairs $\{\tilde{g}|g\}$ of simply laced Lie algebras, where $\tilde{g}$ governs the mass spectrum and the fusing rules, while g provides the 'colour' quantum numbers. Using this construction, the S-matrix of the HSG theory corresponding to the Lie group G at level k is related to the pair $\{A_{k-1}|g\}$, where g is the Lie algebra of G. Then, it is worthwhile to notice that the conjectured [15] central charge of the ultraviolet CFT corresponding to the colour valued S-matrix specified by the pair $\{D_k|g\}$ coincides with the central charge of the unperturbed CFT of the G/G_0 Split model ($p = 0$) at level k, or level $2k$ if $G = SU(m)$, where G/G_0 is one of the symmetric spaces listed in (4.1) with simply laced G. However, additional work is needed in order to go beyond this numerical coincidence.

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A Classical conserved densities of the Split models.

In this appendix we give the explicit expressions for the classical conserved densities with scale dimension 2 and 4 corresponding to the Split models, which have been obtained by solving eqs. (2.6).

We will use the standard explicit realization of the basis t^a of g in terms of a Cartan basis of its complexification

$$t^A = iH^A , \qquad A = 1, \ldots, \mathrm{rank}(g) ,$$

0.2truecm

$$t^\alpha = \sqrt{\frac{1}{2\langle E_\alpha, E_{-\alpha} \rangle}} (E_\alpha - E_{-\alpha}) , \quad t^{\bar{\alpha}} = i \sqrt{\frac{1}{2\langle E_\alpha, E_{-\alpha} \rangle}} (E_\alpha + E_{-\alpha}), \quad (A.1)$$

where the H^A's provide an orthonormal basis for the Cartan subalgebra with respect to the invariant bilinear form $\langle \, , \, \rangle$, and the step operators E_α are normalized such that

$$[\, H^A, E_\alpha \,] = \vec{\alpha}(H^A)\, E_\alpha = \alpha^A \, E_\alpha , \qquad [\, E_\alpha, E_{-\alpha} \,] = \langle \, E_\alpha, E_{-\alpha} \rangle \, \alpha^A H^A, \qquad (A.2)$$

with $\vec{\alpha}$ a positive root of g.

The generators (A.1) are normalized in such a way that $\langle \, t^a, t^b \, \rangle = -\delta^{ab}$. Moreover, the corresponding structure functions are totally antisymmetric and satisfy:

$$\begin{aligned} & f^{\bar{\alpha}\bar{\beta}\bar{\gamma}} = f^{\alpha\beta\bar{\gamma}} = f^{\alpha\bar{\alpha}\beta} = f^{\alpha\bar{\alpha}\bar{\beta}} = 0, \quad \forall \, \vec{\alpha}, \vec{\beta}, \vec{\gamma} > 0, \\ & f^{ABC} = f^{AB\alpha} = f^{AB\bar{\alpha}} = 0, \quad \forall \, \vec{\alpha} > 0 \ \text{ and } \ \forall \ \ A, B = 1, \cdots, \mathrm{rank}(g), \\ & f^{\alpha\bar{\alpha}A} = \alpha^A, \quad \forall \, \vec{\alpha} > 0 \ \text{ and } \ \forall \ \ A = 1, \cdots, \mathrm{rank}(g). \end{aligned} \qquad (A.3)$$

Recall that the rank of a symmetric space G/G_0 is the dimension of any maximal abelian subspace contained in g_1. Therefore, if $\mathrm{rank}(G/G_0) = \mathrm{rank}(G)$, one can take $t^A \in g_1$ for all A and, hence, the subset of generators t^α provide a basis for the subalgebra g_0, while the $t^{\bar{\alpha}}$'s, together with the t^A's, generate g_1. We will use this realization of g_0 and g_1 in our calculations for the Split models. Therefore,

$$\Lambda_- = \lambda^A \, t^A = \vec{\lambda} \cdot \vec{t}, \qquad (A.4)$$

and the potential $q = q(x)$ in the Lax operator (4.5) is of the form $q = q^\alpha t^\alpha$.

Taking into account all this, the expressions for the conserved densities of scale-dimension 2 are

$$\mu^A \mathcal{I}_2^{(0)A} = \vec{\mu} \cdot \vec{\mathcal{I}}_2^{(0)} = D_{\alpha\beta} q^\alpha q^\beta, \quad \text{with} \quad D_{\alpha\beta} = \frac{1}{2m} \frac{\vec{\mu} \cdot \vec{\alpha}}{\vec{\lambda} \cdot \vec{\alpha}}, \qquad (A.5)$$

where $\vec{\mu}$ is an arbitrary rank(g)-component vector which allows one to write the rank(g) conserved quantities $\mathcal{I}_2^{(0)A}$ in a compact way. In particular, for $\vec{\mu} = \vec{\lambda}$, (A.5) reduces to

$$m \lambda^A \mathcal{I}_2^{(0)A} = \frac{1}{2} q^\alpha q^\alpha = 2\pi \beta^2 T_{--}, \qquad (A.6)$$

which is one of the components of the classical stress-energy tensor.

For scale dimension 4, the rank(g)-conserved densities can be written as:

$$\vec{\mu} \cdot \vec{\mathcal{I}}_4^{(0)} = R^{(0)}_{\alpha\beta\gamma\rho}(\vec{\mu}) q^\alpha q^\beta q^\gamma q^\rho + P^{(0)}_{\alpha\beta\gamma}(\vec{\mu}) q^\alpha q^\beta \partial_- q^\gamma + Q^{(0)}_{\alpha\beta}(\vec{\mu}) q^\alpha \partial_-^2 q^\beta, \qquad (A.7)$$

where

$$R^{(0)}_{\alpha\beta\gamma\rho}(\vec{\mu}) = \Big(- \frac{(\vec{\alpha}\cdot\vec{\gamma})(\vec{\gamma}\cdot\vec{\mu})}{8m^3(\vec{\lambda}\cdot\vec{\alpha})(\vec{\lambda}\cdot\vec{\gamma})^2}\,\delta_{\alpha\beta}\delta_{\gamma\rho} - \frac{f^{\bar{\beta}\gamma\bar{\xi}} f^{\xi\rho\alpha}\,(\vec{\alpha}\cdot\vec{\mu})}{6m^3(\vec{\lambda}\cdot\vec{\beta})(\vec{\lambda}\cdot\vec{\xi})(\vec{\lambda}\cdot\vec{\alpha})} + \\ + \frac{f^{\bar{\rho}\alpha\bar{\xi}} f^{\bar{\xi}\bar{\gamma}\beta}}{24m^3(\vec{\lambda}\cdot\vec{\gamma})(\vec{\lambda}\cdot\vec{\rho})}\Big\{ - \frac{\vec{\beta}\cdot\vec{\mu}}{\vec{\lambda}\cdot\vec{\beta}} + \frac{\vec{\xi}\cdot\vec{\mu}}{\vec{\lambda}\cdot\vec{\xi}}\Big\}\Big)_{\mathrm{sym}(\alpha\beta\gamma\rho)}, \tag{A.8}$$

$$P^{(0)}_{\alpha\beta\gamma}(\vec{\mu}) = \Big(\frac{f^{\gamma\alpha\beta}}{3m^3(\vec{\lambda}\cdot\vec{\gamma})^2}\,\frac{\vec{\beta}\cdot\vec{\mu}}{\vec{\lambda}\cdot\vec{\beta}} + \frac{f^{\bar{\alpha}\gamma\bar{\beta}}}{4m^3(\vec{\lambda}\cdot\vec{\alpha})(\vec{\lambda}\cdot\vec{\beta})}\,\frac{\vec{\beta}\cdot\vec{\mu}}{\vec{\lambda}\cdot\vec{\beta}} - \\ - \frac{f^{\bar{\beta}\alpha\bar{\gamma}}}{4m^3(\vec{\lambda}\cdot\vec{\gamma})(\vec{\lambda}\cdot\vec{\beta})}\Big\{ \frac{\vec{\gamma}\cdot\vec{\mu}}{3\,\vec{\lambda}\cdot\vec{\gamma}} + \frac{\vec{\beta}\cdot\vec{\mu}}{\vec{\lambda}\cdot\vec{\beta}}\Big\}\Big)_{\mathrm{sym}(\alpha\beta)}, \tag{A.9}$$

$$Q^{(0)}_{\alpha\beta}(\vec{\mu}) = -\frac{\vec{\alpha}\cdot\vec{\mu}}{2\,m^3(\vec{\lambda}\cdot\vec{\alpha})^3}\,\delta_{\alpha\beta}. \tag{A.10}$$

In these equations, sym$(\alpha\beta\gamma\rho)$ and sym$(\alpha\beta)$ means that the corresponding expressions have to be considered completely symmetrized in $(\alpha\beta\gamma\rho)$ or $(\alpha\beta)$, respectively. In particular, for $\vec{\mu} = \vec{\lambda}$ these expressions simplify to

$$R^{(0)}_{\alpha\beta\gamma\rho}(\vec{\lambda}) = -\frac{1}{24m^3\,\vec{\lambda}\cdot\vec{\alpha}}\left\{ \frac{\vec{\alpha}\cdot\vec{\gamma}}{\vec{\lambda}\cdot\vec{\gamma}}\,\{\delta_{\alpha\beta}\delta_{\gamma\rho} + \delta_{\alpha\rho}\delta_{\gamma\beta}\} + \frac{\vec{\alpha}\cdot\vec{\beta}}{\vec{\lambda}\cdot\vec{\beta}}\delta_{\alpha\gamma}\delta_{\beta\rho}\right\}, \tag{A.11}$$

$$P^{(0)}_{\alpha\beta\gamma}(\vec{\lambda}) = \frac{-1}{6m^3\,\vec{\lambda}\cdot\vec{\gamma}}\left\{ \frac{f^{\bar{\beta}\alpha\bar{\gamma}}}{\vec{\lambda}\cdot\vec{\beta}} + \frac{f^{\bar{\alpha}\beta\bar{\gamma}}}{\vec{\lambda}\cdot\vec{\alpha}}\right\}, \tag{A.12}$$

$$Q^{(0)}_{\alpha\beta}(\vec{\lambda}) = -\frac{\delta_{\alpha\beta}}{2\,m^3(\vec{\lambda}\cdot\vec{\alpha})^2}. \tag{A.13}$$

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G/G_0	$\mathrm{rank}(G/G_0)$	Δ_Φ
$SU(3)/SO(3)$	2	$\frac{6}{k+2}$
$SU(4)/SO(4)$	3	$\frac{4}{k+2}$
$SU(n)/SO(n), \quad (n \geq 5)$	$n-1$	$\frac{n}{2(k+n-2)}$
$SU(2n)/Sp(n), \quad (n \geq 1)$	$n-1$	$\frac{n}{k+n+1}$
$SO(n+2)/SO(n) \times U(1), \quad (n \geq 5)$	2	$\frac{n-1}{2(k+n-2)} + \frac{1}{2k}$
$SO(n+3)/SO(n) \times SO(3), \quad (n \geq 4)$	3	$\frac{n-1}{2(k+n-2)} + \frac{1}{k+2}$
$SO(n+m)/SO(n) \times SO(m), \quad (n, m \geq 4)$	$\min(n, m)$	$\frac{n-1}{2(k+n-2)} + \frac{m-1}{2(k+m-2)}$
$Sp(n+m)/Sp(n) \times Sp(m), \quad (n, m \geq 1)$	$\min(n, m)$	$\frac{1+2n}{4(k+n+1)} + \frac{1+2m}{4(k+m+1)}$
$SU(n+m)/SU(n) \times SU(m) \times U(1),$ $(n, m \geq 2)$	$\min(n, m)$	$\frac{(n-1)(n+1)}{2n(k+n)} + \frac{(m-1)(m+1)}{2m(k+m)} + \frac{n+m}{2nmk}$
$Sp(n)/SU(n) \times U(1), \quad (n \geq 2)$	n	$\frac{(n-1)(n+2)}{n(k+n)} + \frac{2}{nk}$
$SO(2n)/SU(n) \times U(1), \quad (n \geq 3)$	$[n/2]$	$\frac{n^2-n-4}{2n(k+n-1)} + \frac{2}{nk}$

Table 2: Conformal dimensions of the perturbations corresponding to the type I SSSG-models associated to the classical Lie groups G.

G/G_0	rank(G/G_0)	Δ_Φ
$E_6/Sp(4)$	6	$\frac{6}{k+5}$
E_6/F_4	2	$\frac{6}{k+9}$
$E_7/SU(8)$	7	$\frac{9}{k+8}$
$E_8/SO(16)$	8	$\frac{15}{k+14}$
$F_4/SO(9)$	1	$\frac{9}{2(k+7)}$
$E_6/SU(6) \times SU(2)$	4	$\frac{21}{4(k+6)} + \frac{3}{4(k+2)}$
$E_7/SO(12) \times SU(2)$	4	$\frac{33}{4(k+10)} + \frac{3}{4(k+2)}$
$E_8/E_7 \times SU(2)$	4	$\frac{57}{4(k+18)} + \frac{3}{4(k+2)}$
$F_4/Sp(3) \times SU(2)$	4	$\frac{15}{4(k+4)} + \frac{3}{4(k+2)}$
$G_2/SU(2) \times SU(2)$	2	$\frac{15}{4(3k+2)} + \frac{3}{4(k+2)}$
$E_6/SO(10) \times U(1)$	2	$\frac{767}{128(k+8)} + \frac{1}{128k}$
$E_7/E_6 \times U(1)$	3	$\frac{527}{72(k+12)} + \frac{1}{72k}$

Table 3: Conformal dimensions of the perturbations corresponding to the type I SSSG-models associated to the exceptional Lie groups G.

G/G_0	Type	$\vec{s}$	$\tilde{C}_2(g^{(i)}) = C_2(g^{(i)})/\vec{\Psi}^2_{g^{(i)}}$
$SU(3)/SO(3)$	A2	$(0,1)$	6
$SU(4)/SO(4)$	A2	$(0,1,0)$	$\tilde{C}_2(g^{(1)}) = \tilde{C}_2(g^{(2)}) = 2$
$SU(n)/SO(n)$, $(n \geq 5)$	A2	$(0,\ldots,0,1)$	n
$SU(2n)/Sp(n)$, $(n \geq 1)$	A2	$(1,\ldots,0)$	n
$SO(n+3)/SO(n) \times SO(3)$ $n = 2p,\ (p \geq 2)$	A1	$(0,\ldots,1,0)$	$\tilde{C}_2(D_p) = p - 1/2$ $\tilde{C}_2(B_1) = 1$
$SO(n+3)/SO(n) \times SO(3)$ $n = 2p+1,\ (p \geq 2)$	A2	$(0,\ldots,1,0)$	$\tilde{C}_2(B_p) = p$ $\tilde{C}_2(B_1) = 1$
$SO(n+m)/SO(n) \times SO(m)$ $n = 2p, m = 2q,\ (p,q \geq 2)$	A1	$(0,\ldots,1,\ldots,0)$, $s_p = 1$	$\tilde{C}_2(D_p) = p - 1/2$ $\tilde{C}_2(D_q) = q - 1/2$
$SO(n+m)/SO(n) \times SO(m)$ $n = 2p, m = 2q+1,\ (p,q \geq 2)$	A1	$(0,\ldots,1,\ldots,0)$, $s_p = 1$	$\tilde{C}_2(D_p) = p - 1/2$ $\tilde{C}_2(B_q) = q$
$SO(n+m)/SO(n) \times SO(m)$ $n = 2p+1, m = 2q+1,\ (p,q \geq 2)$	A2	$(0,\ldots,1,\ldots,0)$, $s_p = 1$	$\tilde{C}_2(B_p) = p$ $\tilde{C}_2(B_q) = q$
$Sp(n+m)/Sp(n) \times Sp(m),\ \ (n,m \geq 1)$	A1	$(0,\ldots,1,\ldots,0)$, $s_n = 1$	$\tilde{C}_2(C_n) = (2n+1)/4$ $\tilde{C}_2(C_m) = (2m+1)/4$

Table 4: Type, $\vec{s}$, and $C_2(g^{(i)})/\vec{\Psi}^2_{g^{(i)}}$ corresponding to the SSSG-models of types A1 and A2 associated to G classical.

G/G_0	Type	$\vec{s}$	$\tilde{C}_2(g^{(i)}) = C_2(g^{(i)})/\vec{\Psi}^2_{g^{(i)}}$
$SO(n+2)/SO(n) \times U(1)$ $n = 2p,\ (p \geq 3)$	B	$(1,1,0\ldots,0)$	$\tilde{C}_2(D_p) = p - 1/2$ $\vec{\Lambda} \cdot \vec{u} = \vec{\Psi}^2_{D_p}/2$
$SO(n+2)/SO(n) \times U(1)$ $n = 2p+1,\ (p \geq 2)$	B	$(1,1,0\ldots,0)$	$\tilde{C}_2(B_p) = p$ $\vec{\Lambda} \cdot \vec{u} = \vec{\Psi}^2_{B_p}/2$
$SU(n+m)/SU(n) \times SU(m) \times U(1)$ $(n,m \geq 2)$	B	$(1,0,\ldots,1,\ldots,0)$ $s_0 = s_n = 1$	$\tilde{C}_2(A_{n-1}) = \frac{(n-1)(n+1)}{2n}$ $\tilde{C}_2(A_{m-1}) = \frac{(m-1)(m+1)}{2m}$ $\vec{\Lambda} \cdot \vec{u} = \frac{n+m}{2nm}\vec{\Psi}^2_{A_{n-1}}$
$Sp(n)/SU(n) \times U(1)$ $(n \geq 2)$	B	$(1,0,\ldots,0,1)$	$\tilde{C}_2(A_{n-1}) = \frac{(n-1)(n+2)}{n}$ $\vec{\Lambda} \cdot \vec{u} = \frac{2}{n}\vec{\Psi}^2_{A_{n-1}}$
$SO(2n)/SU(n) \times U(1)$ $(n \geq 3)$	B	$(1,0,\ldots,0,1)$	$\tilde{C}_2(A_{n-1}) = \frac{(n^2-n-4)}{2n}$ $\vec{\Lambda} \cdot \vec{u} = \frac{2}{n}\vec{\Psi}^2_{A_{n-1}}$

Table 5: Type, $\vec{s}$, and $C_2(g^{(i)})/\vec{\Psi}^2_{g^{(i)}}$ corresponding to the SSSG models of type B associated to G classical.

G/G_0	Type	$\vec{s}$	$\tilde{C}_2(g^{(i)}) = C_2(g^{(i)})/\vec{\Psi}^2_{g^{(i)}}$
$E_6/Sp(4)$	A2	$(0,0,0,0,1)$	6
E_6/F_4	A2	$(1,0,0,0,0)$	6
$E_7/SU(8)$	A1	$(0,0,0,0,0,0,1)$	9
$E_8/SO(16)$	A1	$(0,0,0,0,0,0,1,0)$	15
$F_4/SO(9)$	A1	$(0,0,0,0,1)$	$9/2$
$E_6/SU(6)\times SU(2)$	A1	$(0,0,0,0,0,0,1)$	$\tilde{C}_2(A_5) = 21/4$, $\tilde{C}_2(A_1) = 3/4$
$E_7/SO(12)\times SU(2)$	A1	$(0,0,0,0,0,1,0,0)$	$\tilde{C}_2(D_6) = 33/4$, $\tilde{C}_2(A_1) = 3/4$
$E_8/E_7\times SU(2)$	A1	$(0,1,0,0,0,0,0,0,0)$	$\tilde{C}_2(E_7) = 57/4$, $\tilde{C}_2(A_1) = 3/4$
$F_4/Sp(3)\times SU(2)$	A1	$(0,1,0,0,0)$	$\tilde{C}_2(C_3) = 15/4$, $\tilde{C}_2(A_1) = 3/4$
$G_2/SU(2)\times SU(2)$	A1	$(0,1,0)$	$\tilde{C}_2(g^{(1)}) = 3/4$, $\tilde{C}_2(g^{(2)}) = 15/4$
$E_6/SO(10)\times U(1)$	B	$(1,0,0,0,0,1,0)$	$\tilde{C}_2(D_5) = 767/128$, $\vec{\Lambda}\cdot\vec{u} = \vec{\Psi}^2_{D_5}/128$
$E_7/E_6\times U(1)$	B	$(1,0,0,0,0,0,1,0)$	$\tilde{C}_2(E_6) = 647/72$, $\vec{\Lambda}\cdot\vec{u} = \vec{\Psi}^2_{E_6}/72$

Table 6: Type, $\vec{s}$ and $C_2(g^{(i)})/\vec{\Psi}^2_{g^{(i)}}$ corresponding to the SSSG models associated to G exceptional.