Parameterisation of Degenerate Solutions of the Determinantal Assignment Problem

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Abstract: The paper is concerned with defining and parametrising the families of all degenerate compensators (feedback, squaring down etc) in a variety of linear control problems. Such compensators indicate the boundaries of the control design, but they also provide the means for linearising the nonlinear nature of the Determinantal Assignment Problems, which provide the unifying description for all frequency assignment problems (pole, zero) under static and dynamic compensation schemes.

Keywords: Linear systems; algebraic systems theory; degenerate compensators; global linearization

EXTENDED ABSTRACT

Introduction

The Determinantal Assignment Problem (DAP) has emerged as the abstract problem to which the study of pole, zero assignment of linear systems may be reduced (Karcanias & Giannakopoulos, (1984), Leventides & Karcanias (1995)). This approach unifies the study of frequency assignment problems (pole, zero) of multivariable systems under constant, dynamic centralised, or decentralised control structure, has been developed. The *Determinantal Assignment Problem* (DAP) demonstrates the significance of exterior algebra and classical algebraic geometry for control problems. The importance of tools and techniques of algebraic geometry for control theory problems has been demonstrated by the work in (Brockett & Byrnes (1981) etc.). The multi-linear nature of DAP suggests that the natural framework for its study is that of exterior algebra [1]. The study of DAP [7] may be reduced to a linear problem of zero assignment of polynomial combinants and a standard problem of multi-linear algebra, the decomposability of multivectors (Marcus, (1973)). The solution of the linear sub-problem, whenever it exists, defines a linear space in a projective space $\mathcal{P}^t$, whereas decomposability is characterised by the set of Quadratic Plücker Relations (QPR), which define the Grassmann variety of $\mathcal{P}^t$ (Hodge & Pedoe, (1952)). Thus, solvability of DAP is reduced to a problem of finding real intersections between the linear variety and the Grassmann variety of $\mathcal{P}^t$. This novel Exterior Algebra-Algebraic Geometry method, has provided new invariants (Plucher Matrices and the Grassmann vectors) for the characterisation of rational vector spaces, solvability of control problems, ability to discuss both generic and nongeneric cases and it is flexible as far as handling dynamic schemes, as well as structurally constrained compensation schemes. The multi-linear nature of DAP has been recently handled by a "blow up" type methodology, using the notion of degenerate solution and known as "Global Linearisation" (Leventides & Karcanias (1995), (1996)). Under certain conditions, this methodology allows the computation of solutions of the DAP problem.

There are many challenging issues in the development of the DAP framework and amongst them are its ability to provide solutions even for non-generic cases, as well as providing approximate solutions to the cases where generically there is no solution of the exact problem. The development of such solutions requires development of the methodology "Global Linearisation" (Leventides & Karcanias (1995), (1996)) and an essential part of this is the parameterisation of the degenerate solutions, which have been defined in (Brockett & Byrnes (1981)) as the compensation solutions where the feedback configuration sieges to exist. Although such solutions are prohibited from the systems viewpoint, they have the significant property that linearise asymptotically the multi-linear nature of DAP and thus they become key instruments for developing Global Linearisations. The classification of degenerate controllers and parameterization of Global Linearization Solutions play an integral role to the application of the Global Linearization methodology to special structure problems such as decentralized control etc. This paper deals with the parameterization of the degenerate solutions of the general DAP using the algebraic theory of minimal bases of rational vector spaces (Forney, (1975)).

Families of Determinantal Assignment Problems

We consider linear systems described by $S(A,B,C,D)$ state space descriptions with n states, p inputs and m outputs, where (A,B) is controllable, (A,C) is observable, or by the transfer function matrix $G(s) = C(sI - A)^{-1} B + D$, where the rank is min{$m,p$}. In terms of left, right coprime matrix fraction descriptions $G(s)$ may be represented as $G(s)= =D_l(s)^{-1}N_l(s) = N_r(s)D_r(s)^{-1}$, where $N_l(s)$, $N_r(s) \in \Re^{m\times p}[s]$, $D_l(s) \in \Re^{m\times m}[s]$ and $D_r(s) \in \Re^{p\times p}[s]$. The following frequency assignment problems are defined (Karcanias & Giannakopoulos, (1984)):

(i) Pole assignment by state feedback: Consider $L\in\Re^{n\times p}$, where L is a state feedback. If $B(s)=[sI-A, -B]$ and $\tilde{L} = [I_n, L^t]^t$, the closed loop characteristic polynomial is then given by $p_L(s)=\det\{sI-A-BL\} = \det\{B(s)\tilde{L}\}$. A similar expression is derived for the observer design.

(ii) Pole assignment by constant output feedback: Under an output feedback $K\in\Re^{m\times p}$, the closed loop characteristic polynomial $p_K(s)$ is given by $p_K(s)= \det\{D_l(s)+N_l(s)K\} = \det\{D_r(s) + K N_r(s)\}$. By defining the matrices

$$T_l(s) = [D_l(s), N_l(s)] \in\Re^{m\times(m+p)}[s],$$

$$T_r(s) = \begin{bmatrix} D_r(s) \\ N_r(s) \end{bmatrix} \in\Re^{(m+p)\times p}[s] \qquad (1)$$

and by setting $\tilde{K}_l=[I_m,K^t]^t\in\Re^{(m+p)\times m}$, $\tilde{K}_r=[I_l, K] \in\Re^{p\times(m+p)}$, $p_K(s) = \det\{T_l(s)\tilde{K}_l\} = \det\{\tilde{K}_r, T_r(s)\}$.

(iii) Zero assignment by squaring down: For a system with $m>p$ the squaring down problem is defined as finding $H\in\Re^{p\times m}$, a *squaring down post-compensator*, such that $G'(s)=HG(s)$ is the *squared down transfer function matrix*. Using a right MFD for $G'(s)$, $G'(s)=H N_r(s) D_r(s)^{-1}$ where $G(s)=N_r(s)D_r(s)^{-1}$ the squaring down is reduced to finding H such that $G'(s)$ has assigned zeros in the zero polynomial of $S(A,B,HC,HD)$ given by $z_K(s)=\det\{HN_r(s)\}$.

(iv) Dynamic Compensation Problems: For the standard feedback configuration of Figure (1) and with $G(s)\in\Re_{pr}(s)^{m\times p}$, $C(s)\in \Re(s)^{p\times m}$, with coprime MFD's where $C(s) = A_l(s)^{-1}B_l(s) = B_r(s)A_r(s)^{-1}$ the closed loop characteristic polynomial may be expressed as

$$f(s)=\det\left\{[D_\ell(s), N_\ell(s)]\begin{bmatrix} A_r(s) \\ B_r(s) \end{bmatrix}\right\}$$

$$=\det\left\{[A_\ell(s), B_\ell(s)]\begin{bmatrix} D_r(s) \\ N_r(s) \end{bmatrix}\right\} \qquad (2)$$

if $p \le m$, then $C(s)$ may be interpreted as Feedback Compensator and if $p \ge m$, the $C(s)$ may be interpreted as Pre-compensator. The above general dynamic formulation covers a number of important families of C(s) compensators as: **(a)** Constant**, (b)** PI, **(c)** PD, **(d)** PID, **(e)** Bounded degree. In fact:

(a) Constant Controllers: If $p \le m$, $A_l = I_p$, $B_l = K\in\Re^{p\times m}$, then the above expresses the constant output feedback case, whereas if $p \ge m$, $A_r = I_m$, $B_r = K\in\Re^{p\times m}$ expresses the constant pre-compensation formulation of the problem.

(b) PID Controllers: These controllers are expressed as $C(s) = K_0 + 1/sK_1 + sK_2 = [sI_p]^{-1}[s^2K_2 + sK_0 + K_1]$ where K_0, K_1, $K_2 \in \Re^{p\times m}$ and the left MFD is coprime with the only exception possibly at s=0, s=∞ (coprimeness at s=0 is guaranteed by rank(K_1)=p and at s=∞ by rank(K_2)=p). This scheme clearly covers also the case of PI and PD controllers. The determinantal output PID feedback is expressed as:

$$f(s)=\det\left\{[sI_p, s^2K_2+sK_0+K_1]\begin{bmatrix} D_r(s) \\ N_r(s) \end{bmatrix}\right\}=$$

$$=\det\{[I_p, K_0, K_1, K_2]\begin{bmatrix} sD_r(s) \\ sN_r(s) \\ N_r(s) \\ s^2N_r(s) \end{bmatrix}\} \qquad (3)$$

(c) Observability Index Bounded Dynamics (OBD) Controllers: These are defined by the property that their McMillan degree is equal to pk, where k is the observability index of the controller. Such controllers are expressed as $[A_l(s), B_l(s)] = T_ks^k + ... + T_0$ where T_k, T_{k-1},...,$T_0 \in \Re^{p\times(p\times m)}$ and $T_k = [I_p, X]$. Note that the above representation is not always coprime, and coprimeness has to be guaranteed first for McMillan degree to be pk; otherwise, the McMillan degree is less than pk. The dynamic determinantal OBD output feedback problem is expressed as

$$f(s)=\det\{(T_ks^k + ... + T_0)M(s)\} =$$

$$=\det\{[T_k, T_{k-1}, ..., T_0]\begin{bmatrix} s^kM(s) \\ s^{k-1}M(s) \\ . \\ . \\ M(s) \end{bmatrix}\} \qquad \textbf{(4)}$$

The above formulation is based on the assumption that $p\le m$ and thus output feedback configuration is used. If $p\ge m$, we can similarly formulate the corresponding problems as determinantal dynamic pre-compensation problems and use right coprime MFDs for $C(s)$.

The General Determinantal Assignment Problem and Degeneracy

All the problems above belong to the same problem family i.e. the determinantal assignment problem (DAP) and the dynamic problems are reduced to equivalent constant DAP. This involves the study of solution with respect to constant matrix

H of the equation based on $M(s) \in \Re[s]^{p\times r}[s]$, $r \leq p$, such that rank(M(s)) = r

$$\det(H\,M(s)) = f(s) \qquad (5)$$

where *f(s)* is the polynomial of an appropriate degree. Clearly, *M (s)* is the matrix defined by the system and the particular control problems and its invariant structure is essential to the solution of the corresponding DAP. The *Frequency Assignment Map* associated with the problem is the map assigning H to the coefficient vector $\underline{f}$ of *f(s)* ie. $F: R^{r\times p} \to R^{d+1}$: $F(H) = \underline{f}$. The solution of DAP is to find H such that F(H)= $\underline{f}$ for a given $\underline{f}$. Clearly, a system has the arbitrary zero assignment property if F is onto. An important family of compensators, which is crucial for the problem, is the family of the so called *degenerate solution*. A solution $\overline{H}$ is degenerate, iff $F(\overline{H}) = 0$ or equivalently det(H $M(s)$)$\equiv 0$. This concept was introduced for the case of output feedback in (Brockett & Byrnes (1981) and has been used in (Leventides & Karcanias (1995)) to develop a linearization methodology for DAP. The following result shows the importance of degenerate compensators:

Lemma (1) [12]**:** If there exists a degenerate matrix $\overline{H}_0$ such that the differential $DF_{\overline{K_0}}$ is onto, then any polynomial of degree d can be assigned via some static compensator. □

The above result suggests a sufficient approach for studying zero assignment using special forms of squaring down determined by degenerate solutions; such an approach is known as Global Linearisation methodology (Leventides & Karcanias (1995), (1996)). The role of degenerate solutions in the solvability of DAP motivates the study of their parameterization. In the following, we denote by $\mathcal{M} = colsp_{R[s]}\{M(s)\}$ the R[s]-module generated by the columns of $M(s)$.

Theorem (1): For the system described by $M(s)$, a p-dimensional space $\mathcal{V}$ =row span [H] corresponds to a degenerate gain, iff either of the following equivalent conditions holds true:

(i) There exists an px1 polynomial vector $\underline{t}(s) \in \mathcal{M}$ such that $[H]\underline{t}(s) = 0\ \forall s$.

(ii) There exists an px1 polynomial vector $\underline{t}(s) \in \mathcal{M}$ with coefficient matrix $P(t)$ such that the rank $P(t) \leq p$.

□

Note that in the characterisation of H we consider all possible gains (bounded and unbounded) and we may classify them as *finite degenerate* if dim $\mathcal{V} =$ r and as *infinite degenerate* if dim $\mathcal{V} <$ r. Every degenerate gain *rowspan* [H] can be approached via a sequence of spaces:

$$\mathcal{H}_\varepsilon = rowspan\ ([H] + \varepsilon[H']) \text{ as } \varepsilon \to 0\,,$$

where $[H']$ is an rxp matrix and can be seen as the *direction* via which we approach the degenerate gain *rowspan* [H]. The degenerate points can be viewed as the points of the Grassmannian where the frequency map cannot be continuously extended. In fact, degenerate points possess a very important property, that is, they scatter the sequences of gains approaching them; this implies that we may have two sequences of gains converging to a degenerate point, as $\varepsilon \to 0$, and yet the corresponding sequences of polynomials to converge into two different limits.

Theorem (1) clearly suggests that the parameterization of all degenerate gains, finite and infinite is related to the properties of the module $\mathcal{M}$ and in particular to the properties of the minimal bases of $\mathcal{M}$ (Forney, (1975)) as these are defined by the corresponding minimal indices and the associated real invariant spaces of $\mathcal{M}$ (Karcanias, (1994) (1996))). The properties of such spaces are then used to characterise finite and infinite degenerate solutions. We consider first the general $M(s)$ and we then specialise to the specific cases as defined by the corresponding control problems formulations (1)-(4).

The current investigation is the algebraic part of the development of the theory of Global Linearization that aims to parameterize all degenerate compensators and then identify those with the desired properties as far as spectrum assignment. The results reveal new structural invariants linked to the invariant real vector spaces of the $\mathcal{M}$ module. This work is required as an essential step in the further development of the methodology by using Homotopic methods.

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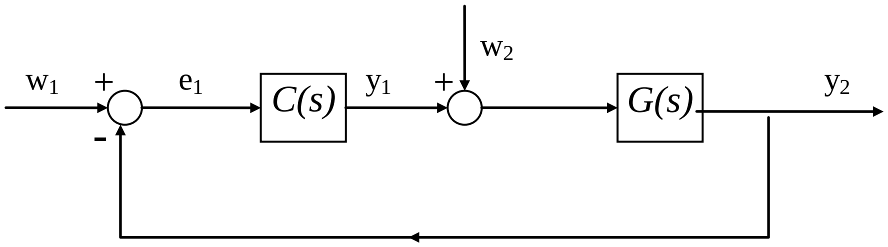

Fig.1. Standard feedback configuration