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The Euclidean Division as an Iterative 
ERES-based Process

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Abstract. Considering the Euclidean Division of two real polynomials, 
we present an iterative process based on the ERES method to compute
the remainder of the division and we represent it using a simple matrix
form.

Introduction

The representation of the Euclidean algorithm process is presented using the
matrix-based methodology of Extended-Row-Equivalence and Shifting opera-
tions (ERES) \cite{3, 4}. This allows the use of numerical methodologies for algebraic
computation problems with the additional advantage of being able to handle
uncertain coefficients and numerical errors.

We consider two real polynomials:

\[ P(x) = \sum_{i=0}^{m} p_i x^i, \; p_m \neq 0 \quad \text{and} \quad Q(x) = \sum_{i=0}^{n} q_i x^i, \; q_n \neq 0, \quad m, n \in \mathbb{N} \quad (0.1) \]

with degrees \( \deg\{P(x)\} = m, \deg\{Q(x)\} = n \) respectively, and \( m \geq n \).

Definition 1. We define the set

\[ \mathcal{D}_{m,n} = \{(P(x), Q(x)) : P(x), Q(x) \in \mathbb{R}[x], \; m = \deg\{P(x)\} \geq \deg\{Q(x)\} = n\} \]

For any pair \( \mathcal{D} = (P(x), Q(x)) \in \mathcal{D}_{m,n}, \) we define a vector representative \( \mathcal{D}(x) \)
and a basis matrix \( D_m \) represented as :

\[ \mathcal{D}(x) = [P(x), Q(x)]^t = [p, q]^t \cdot \mathcal{L}_m(x) = D_m \cdot \mathcal{L}_m(x) \]

where \( D_m \in \mathbb{R}^{2 \times (m+1)}, \mathcal{L}_m(x) = [x^m, x^{m-1}, \ldots, x, 1]^t. \) The matrix \( D_m \) is formed
directly from the coefficients of the given polynomials \( P(x) \) and \( Q(x) \).

Definition 2. Given a pair \( \mathcal{D}_{m,n} \) of real polynomials with a basis matrix \( D_m \),
the following operations are defined \cite{3, 4}:
a) Elementary row operations with scalars from \( \mathbb{R} \) on \( D_m \).
b) Addition or elimination of zero rows on \( D_m \).
c) If \( a^t = [0, \ldots, 0, a_t, \ldots, a_k] \in \mathbb{R}^k, a_t \neq 0 \) then we define as the Shifting operation

\[
\text{shf} : \text{shf}(a^t) = [a_t, \ldots, a_k, 0, \ldots, 0] \in \mathbb{R}^k
\]

By \( \text{shf}(D_{m,n}) \equiv D^*_{m,n} \), we shall denote the pair obtained from \( D_{m,n} \) by applying shifting on the rows of \( D_m \). Type (a), (b) and (c) operations are referred to as Extended-Row-Equivalence and Shifting (ERES) operations.

The following theorem shows the relation between a matrix and its shifted form [1].

Theorem 1 (Matrix representation of Shifting). If \( D \in \mathbb{R}^{2 \times k}, k > 2 \), is an upper trapezoidal matrix with rank \( \rho(D) = 2 \) and \( D^* \in \mathbb{R}^{2 \times k} \) is the matrix obtained from \( D \) by applying shifting on its rows, then there exists a matrix \( S \in \mathbb{R}^{k \times k} \) such that:

\[
D^* = D \cdot S
\]

Corollary 1. If \( D_m \in \mathbb{R}^{2 \times (m+1)} \) is the basis matrix of a pair of real polynomials \( D = (P(x), Q(x)) \in D_{m,n} \), then \( D^*_m \in \mathbb{R}^{2 \times (m+1)} \) is the basis matrix of the pair \( D^* = (P(x), x^{m-n} Q(x)) \in D_{m,m} \) and there exists a matrix \( S_D \in \mathbb{R}^{(m+1) \times (m+1)} \) such that:

\[
D^*_m = D_m \cdot S_D
\]  

The ERES representation of the Euclidean Division

If we have a pair of polynomials \( D = (P(x), Q(x)) \in D_{m,n} \), then, according to Euclid’s division algorithm, it holds:

\[
P(x) = \frac{P_m}{q_n} x^{m-n} Q(x) + R_1(x)
\]  

This is the first and basic step of the Euclidean Division algorithm. The polynomial \( R_1(x) \in \mathbb{R}[x] \) is given by:

\[
R_1(x) = \sum_{i=m-n}^{m-1} \left( p_i - \frac{P_m}{q_n} q_{i-(m-n)} \right) x^i + \sum_{i=0}^{m-n-1} p_i x^i
\]

In the following, we will show that the remainder \( R_1(x) \) can be computed by applying ERES operations to the basis matrix \( D_m \) of the pair \( D \).

Proposition 1 (Matrix representation of the first remainder of the Euclidean Division). Applying the algorithm of the Euclidean Division to a pair \( D = (P(x), Q(x)) \in D_{m,n} \) of real polynomials, there exists a polynomial \( R_1(x) \in \mathbb{R}[x] \) with degree \( 0 \leq \deg(R_1(x)) < m \) such that:

\[
P(x) = \frac{P_m}{q_n} x^{m-n} Q(x) + R_1(x)
\]
Then, the remainder \( R_1(x) \) can be represented in matrix form as:

\[
R_1(x) = \mathbf{v}^t \cdot E_1 \cdot \mathbf{E}_m(x)
\]

where \( E_1 \in \mathbb{R}^{2 \times (m+1)} \) is the matrix, which occurs from the application of the ERES operations on the basis matrix \( D_m \) of the pair \( \mathcal{D} \) and \( \mathbf{v} = [0, 1]^t \).

Proof. If we consider the division \( P(x)/Q(x) \), then, according to Euclid’s algorithm, there is a polynomial \( R_1(x) \) with degree \( 0 \leq \text{deg}\{R_1(x)\} < m \) such that:

\[
R_1(x) = P(x) - \frac{p_m}{q_n} x^{m-n} Q(x) = [0, 1] \begin{bmatrix} 0 & 1 - \frac{p_m}{q_n} \end{bmatrix} \begin{bmatrix} P(x) \\ x^{m-n} Q(x) \end{bmatrix} \quad (0.5)
\]

If we take into account the result in corollary 1, we will have:

\[
R_1(x) = [0, 1] \begin{bmatrix} 0 & 1 - \frac{p_m}{q_n} \end{bmatrix} \cdot D_m \cdot S_D \cdot \mathbf{E}_m(x) = \mathbf{v}^t \cdot C \cdot D_m \cdot S_D \cdot \mathbf{E}_m(x) \quad (0.6)
\]

where \( \mathbf{v}^t = [0, 1], C = \begin{bmatrix} 0 & 1 - \frac{p_m}{q_n} \end{bmatrix}, D_m \) is the basis matrix of the polynomials \( P(x) \) and \( Q(x) \) and \( S_D \) the respective shifting matrix. Therefore, there exists a matrix \( E_1 \in \mathbb{R}^{2 \times (m+1)} \) such that:

\[
E_1 = C \cdot D_m \cdot S_D \quad \text{and} \quad R_1(x) = \mathbf{v}^t \cdot E_1 \cdot \mathbf{E}_m(x) \quad (0.7)
\]

We consider now the basis matrix \( D_m \) of the polynomials \( P(x) \) and \( Q(x) \):

\[
D_m = \begin{bmatrix} P(x) \\ Q(x) \end{bmatrix} = \begin{bmatrix} p_m & \ldots & p_{n+1} & p_n & p_{n-1} & \ldots & p_0 \\ 0 & \ldots & 0 & q_n & q_{n-1} & \ldots & q_0 \end{bmatrix} \cdot \mathbf{E}_m(x) \quad (0.8)
\]

and we will show that the above matrix \( E_1 \) is produced by applying the ERES operations to the basis matrix \( D_m \) of the polynomials \( P(x) \) and \( Q(x) \). We follow the next methodology:

1. We apply shifting on the rows of \( D_m \). Let \( S_D \in \mathbb{R}^{(m+1) \times (m+1)} \), be the proper shifting matrix: \( D_m^{(1)} = D_m \cdot S_D = \begin{bmatrix} p_m & \ldots & p_{m-n+1} & p_{m-n} & p_{m-n-1} & \ldots & p_0 \\ q_n & \ldots & q_1 & q_0 & 0 & \ldots & 0 \end{bmatrix} \).

2. We reorder the rows of the matrix \( D_m^{(1)} \). If \( J = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \) is the permutation matrix, then: \( D_m^{(2)} = J \cdot D_m^{(1)} = \begin{bmatrix} q_n & \ldots & q_1 & q_0 & 0 & \ldots & 0 \\ p_m & \ldots & p_{m-n+1} & p_{m-n} & p_{m-n-1} & \ldots & p_0 \end{bmatrix} \).

3. We apply stable row operations on \( D_m^{(2)} \) (LU factorization). If \( L = \begin{bmatrix} 1 & 0 \\ \frac{p_m}{q_n} & 1 \end{bmatrix} \) then \( L^{-1} = \begin{bmatrix} 1 & 0 \\ -\frac{p_m}{q_n} & 1 \end{bmatrix} \) and therefore:
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\[ D^{(3)}_m = L^{-1} \cdot D^{(2)}_m = \begin{bmatrix} 1 & 0 \\ -\frac{p_m}{q_n} & 1 \end{bmatrix} \cdot \begin{bmatrix} q_n & \cdots & q_1 & q_0 & 0 & \cdots & 0 \\ p_m & \cdots & p_{m-n+1} & p_{m-n} & p_{m-n-1} & \cdots & p_0 \end{bmatrix} \]

\[ = \begin{bmatrix} q_n & \cdots & q_1 & 0 & \cdots & 0 \\ 0 & \cdots & p_{m-n+1} & q_1 & p_{m-n} - q_0 \frac{p_m}{q_n} & p_{m-n-1} & \cdots & p_0 \end{bmatrix} \]

We notice that the term \( \frac{p_m}{q_n} \) emerges from the LU factorization.

The above process can be described by the following equation:

\[ D^{(3)}_m = L^{-1} \cdot J \cdot D_m \cdot S_D \] (0.9)

which represents the ERES methodology. Obviously \( L^{-1} \cdot J = C \) and therefore,

we conclude that \( D^{(3)}_m \equiv E_1 \). \( \square \)

The following theorem establishes the connection between the ERES method and the Euclidean Division of two real polynomials.

**Theorem 2 (Matrix representation of the remainder of the Euclidean Division).** Applying the algorithm of the Euclidean Division to a pair \( D = (P(x), Q(x)) \in D_{m,n} \) of real polynomials, there are polynomials \( G(x), R(x) \in \mathbb{R}[x] \) with degrees \( \deg{G(x)} = m - n \) and \( 0 \leq \deg{R(x)} < n \) respectively, such that:

\[ P(x) = G(x)Q(x) + R(x) \]

Then, the final remainder \( R(x) \) can be represented in matrix form as:

\[ R(x) = v^t \cdot E_N \cdot \xi_m (x) \]

where \( E_N \in \mathbb{R}^{2 \times (m+1)} \) is the matrix, which occurs from the successive application of the ERES operations on the basis matrix \( D_m \) of the pair \( D \) and \( v = [0, 1]^t \).

The proof of the previous theorem is based on the iterative application of the result from proposition 1 to the sequence \( \{(P(x), Q(x)), (R_i(x), Q(x))\} \), for \( 1 \leq i \leq (m - n) \). Therefore, we get a sequence of matrices \( E_i = L_i^{-1} \cdot E_{i-1} \cdot S_i \) for \( i = 1, 2, \ldots, N < m - n \), where the final matrix \( E_N \) gives the total remainder \( R(x) \) and every matrix \( L_i \) gives a specific coefficient of the quotient \( G(x) \).

**References**