Bi-partite entanglement entropy in integrable models with backscattering

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In this paper we generalise the main result of a recent work by J. L. Cardy and the present authors concerning the bi-partite entanglement entropy between a connected region and its complement. There the expression of the leading order correction to saturation in the large distance regime was obtained for integrable quantum field theories possessing diagonal scattering matrices. It was observed to depend only on the mass spectrum of the model and not on the specific structure of the diagonal scattering matrix. Here we extend that result to integrable models with backscattering (i.e. with non-diagonal scattering matrices). We use again the replica method, which connects the entanglement entropy to partition functions on Riemann surfaces with two branch points. Our main conclusion is that the mentioned infrared correction takes exactly the same form for theories with and without backscattering. In order to give further support to this result, we provide a detailed analysis in the sine-Gordon model in the coupling regime in which no bound states (breathers) occur. As a consequence, we obtain the leading correction to the sine-Gordon partition function on a Riemann surface in the large distance regime. Observations are made concerning the limit of large number of sheets.

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1 Introduction

A quantity of current interest in quantum models with many local degrees of freedom is the bi-partite entanglement entropy [1]. It is a measure of quantum entanglement between the degrees of freedom of two regions, A and its complement, in the ground state $|\mathrm{gs}\rangle$ of the model. Other measures of entanglement also exist, see e.g. [1]-[5]. Such measures occur in the context of quantum computing, for instance. Since entanglement is a fundamental property of quantum systems, a measure of entanglement gives a good description of the quantum nature of a ground state, perhaps more so than correlation functions. For the formal definition of the entanglement entropy, consider the Hilbert space as a tensor product of local spaces associated to the sites of a quantum system. This can be written as a tensor product of the two quantum spaces associated to the regions A and its complement: $\mathcal{H} = \mathcal{A} \otimes \bar{\mathcal{A}}$. Then the entanglement entropy is the von Neumann entropy of the reduced density matrix ρ_A associated to A:

$$S_A = \mathrm{Tr}_{\mathcal{A}} \rho_A \log \rho_A \ , \quad \rho_A = \mathrm{Tr}_{\bar{\mathcal{A}}} |\mathrm{gs}\rangle\langle\mathrm{gs}| \ . \tag{1.1}$$

In this work we will be interested in 1-dimensional quantum systems. The entanglement of quantum spin chains has been extensively studied in the literature [6]-[14]. The scaling limit, describing the universal part of the quantum chain behaviour near a quantum critical point, is a quantum field theory (QFT) model (which we will assume throughout to possess Poincaré invariance). The scaling limit is obtained by approaching the critical point while letting the length of the region A go to infinity in a fixed proportion with the correlation length. It is known since some time [15]-[18] that the bi-partite entanglement entropy can be re-written in terms of more geometric quantities in this limit, using a "replica trick". It is related to the partition function $Z_n(x_1, x_2)$ of the (euclidean) QFT model on a Riemann surface $\mathcal{M}_{n,x_1,x_2}$ with two branch points, at the points x_1 and x_2, and n sheets cyclicly connected. The position of the branch points correspond to the end-points of the region A in the scaling limit. The relation is based on the simple re-writing $S_A = -\lim_{n\to 1} \frac{d}{dn} \mathrm{Tr}_{\mathcal{A}} \rho_A^n$, which gives:

$$S_A = -\lim_{n\to 1} \frac{d}{dn} \frac{Z_n(x_1,x_2)}{Z_1^n} \ . \tag{1.2}$$

Naturally, this expression implies that we must analytically continue the quantity $Z_n(x_1, x_2)$ from $n \in \mathbb{N}$, where it is naturally associated to Riemann surfaces, to $n \in [1, \infty)$. The object $\mathrm{Tr}_{\mathcal{A}} \rho_A^n$ certainly has a well-defined meaning for any n such that $\mathrm{Re}(n) > 0$. Indeed, ρ_A is hermitian (and has non-negative eigenvalues summing to 1), so that $\mathrm{Tr}_{\mathcal{A}} \rho_A^n$ is the sum of the n^{th} powers of its eigenvalues (with multiplicities). Note that this is an analytic continuation from positive integers n to complex n that satisfies the requirements of Carlson's theorem [19], hence the unique one that does. The scaling limit of this object is what defines the proper analytic continuation of $Z_n(x_1, x_2)$. It is natural to assume, as it has been done before [15], that the two branch points just become conical singularities with angle $2\pi n$, the rest of the space being flat. This is the point of view that we will take.

In [20], the ratio of partition functions $Z_n(x_1, x_2)/Z_1^n$ was studied at large distances $|x_1 - x_2| = r$ for certain 1+1-dimensional integrable QFTs. The main feature of these models is that there is no particle production in any scattering process and the scattering (S) matrix factorizes into products of 2-particle S-matrices which can be calculated exactly (for reviews see e.g. [21]-[25]). Taking the S-matrix as input it is then possible to compute the matrix elements of local operators (also called form factors). This is done by solving a set of consistency equations [26, 27]. This is known as the form factor bootstrap program for integrable QFTs. In [20], this program was used and generalised in order to compute $Z_n(x_1, x_2)/Z_1^n$ in the case of integrable models with diagonal scattering matrix (that is, without backscattering). It was deduced that for this class of models, the entanglement entropy behaves at large length r of the region A as

$$S_A = -\frac{c}{3}\log(\varepsilon m_1) - \frac{1}{8}\sum_{\alpha=1}^{\ell} K_0(2rm_\alpha) + O\left(e^{-3rm_1}\right). \tag{1.3}$$

Here, m_α are the masses of the ℓ particles in the QFT model, with $m_1 \leq m_\alpha$. The first term is the expected saturation of the entanglement entropy (with c the ultraviolet central charge), the precise value of which depends on the details of the microscopic theory included into the non-universal small distance ε. The interesting feature is the universal second term, where we see that the leading exponential corrections are independent of the scattering matrix, and only depend on the particle spectrum of the model. This is quite striking: for instance, a model of ℓ free particles of masses m_α will give the same leading exponential corrections as one with diagonally interacting particles with the same masses.

The purpose of this paper is to point out that this result still holds for *non-diagonal* integrable models, and to analyse the particular case of the sine-Gordon model. We obtain the first large-distance correction to $Z_n(x_1,x_2)/Z_1^n$ for that model and provide a detailed analysis of its large-n behaviour. This analysis gives a numerical confirmation of the main result (1.3). The large-n behaviour also shows interesting features that are in relation with the properties of the perturbing field: it is linear $\propto n$ in the super-renormalisable case, and has an extra logarithmic factor $\propto n\log n$ in the marginally renormalisable case (where the sine-Gordon model represents a sector of the $SU(2)$-Thirring model).

The paper is organised as follows: In section 2 we review the relationship, discussed in [20], between the partition function (hence the entanglement entropy) and the two-point function of branch-point twist fields in a theory consisting of n copies of a given integrable model. We generalise the procedure introduced in [20] for computing the entanglement entropy from analytic properties of form factors of branch-point twist fields to integrable models including backscattering. This is based on expressing the two-point function in terms of a form factor expansion. For all such models we conclude that, under certain assumptions, the leading order correction to the entropy in the infrared limit is of the same form as that obtained in [20] for theories with diagonal scattering. In section 3 we introduce the sine-Gordon model and obtain the two-particle form factors of the branch-point twist fields. In section 4 we check these form factors for consistency by extracting the underlying conformal dimension of the twist fields. In section 5 we check our general result from section 2 for the sine-Gordon model, and provide a large-n analysis of the two-particle contribution to the two-point function. We find good analytical and numerical agreement with the results of section 2. In section 6 we present our conclusions and outlook.

2 Form factor equations and the entanglement entropy

2.1 Form factor equations

This section follows closely [20]. Let us consider some model of 1+1-dimensional QFT. For clarity, we will characterise it by its lagrangian density $\mathcal{L}(\phi)$ depending on some "fundamental" field ϕ, although the results hold also when no lagrangian formulation is available. In [20], it was shown that the partition function $Z_n(x_1,x_2)$ on $\mathcal{M}_{n,x_1,x_2}$ is proportional to a two-point correlation function of certain twist fields in an extended model, composed of n independent copies of the initial theory. The lagrangian density of this extended model, for instance, is $\mathcal{L}^{(n)}(\phi_1,\ldots,\phi_n) = \sum_{j=1}^n \mathcal{L}(\phi_j)$. This model possesses a natural $\mathbb{Z}_n$ internal symmetry under cyclic exchange of the copies, $\sigma : \phi_j \mapsto \phi_{j+1}$ ($j = 1, \ldots, n-1$), $\phi_n \mapsto \phi_1$. By a standard procedure (see, for instance, the explanations in [20]), to this symmetry and to that of the opposite cyclic exchange σ^{-1}, we can associate, respectively, the twist fields $\mathcal{T}$ and $\tilde{\mathcal{T}}$, called *branch-point twist fields*. Their two-point function is simply related to the partition function of the original model on the Riemann surface:

$$\frac{Z_n(x_1,x_2)}{Z_1^n} = \mathcal{Z}_n \varepsilon^{2d_n} \langle \mathcal{T}(x_1)\tilde{\mathcal{T}}(x_2)\rangle \tag{2.1}$$

where here an below we use $\langle\cdots\rangle$ for denoting correlation functions in the extended model $\mathcal{L}^{(n)}$. Here ε is some short-distance scale, $\mathcal{Z}_n$ is a non-universal normalisation (with $\mathcal{Z}_1 = 1$), and d_n is the scaling dimension of the twist fields [17, 20]

$$d_n = \frac{c}{12}\left(n - \frac{1}{n}\right), \tag{2.2}$$

where c is the central charge of the ultra-violet conformal field theory associated to $\mathcal{L}$ (that describes the short-distance behaviours of correlation functions in the model $\mathcal{L}$). The fundamental property of the twist fields $\mathcal{T}$ and $\tilde{\mathcal{T}}$, as operators on the Hilbert space of $\mathcal{L}^{(n)}$, is the "semi-locality" property with respect to any local field $\mathcal{O}$:

$$\begin{aligned}
\mathcal{O}(y)\mathcal{T}(x) &= \mathcal{T}(x)(\sigma\mathcal{O})(y) \quad x^1 < y^1 \\
\mathcal{O}(y)\mathcal{T}(x) &= \mathcal{T}(x)\mathcal{O}(y) \quad x^1 > y^1 \\
\mathcal{O}(y)\tilde{\mathcal{T}}(x) &= \tilde{\mathcal{T}}(x)(\sigma^{-1}\mathcal{O})(y) \quad x^1 < y^1 \\
\mathcal{O}(y)\tilde{\mathcal{T}}(x) &= \tilde{\mathcal{T}}(x)\mathcal{O}(y) \quad x^1 > y^1 \; .
\end{aligned} \tag{2.3}$$

This implies that $\mathcal{O}$ and $\mathcal{T}$ (or $\tilde{\mathcal{T}}$) are mutually local* only when $\mathcal{O}$ is invariant under σ. Along with the fact that they have the lowest dimension, given by (2.2), and that they are invariant under all other symmetry transformations of the model $\mathcal{L}^{(n)}$ that are in agreement with σ, this fixes $\mathcal{T}$ and $\tilde{\mathcal{T}}$ uniquely up to normalisation (for definiteness, we will assume the CFT normalisation, $\langle \mathcal{T}(x)\tilde{\mathcal{T}}(0)\rangle \sim |x|^{-2d_n}$).

An important remark is that these twist fields are local fields: they commute with the energy density at space-like distances†. This is a consequence of the fact that they are associated to a symmetry. In particular, the resulting two-point function, which is proportional to the partition function of $\mathcal{L}^{(n)}$ with a defect line extending from x_1 to x_2 through which the fields are affected by the symmetry transformation, is independent of the shape of this defect line (this is sometimes called a topological defect). This is simply related to the fact that the partition function of $\mathcal{L}$ on the Riemann surface $\mathcal{M}_{n,x_1,x_2}$ is independent from the shape of the branch connections.

Let us now consider $\mathcal{L}$ to be a massive integrable QFT model. The Hilbert space of massive QFT is described by physical particles, and there are two bases, one corresponding to well-defined separated particles coming from far in the past, the other corresponding to those leaving far in the future. The overlap between these bases is the scattering matrix, and the main characteristic of integrable models is that their scattering preserves the number of particles involved and the set of momenta, and that it factorises into two-particle processes, as if particles were scattering by pairs at very distant space-time points from each other. Hence, only the two-particle to two-particle scattering matrix is relevant, and consistency under all possible pair-wise

*That is, they commute, hence are quantum mechanically independent, at space-like distances.

†The set of all local fields that constitute a QFT is the set of those that are mutually local with respect to the energy density.

separations gives the Yang-Baxter equation. Consider an integrable model with mass spectrum $\{m_\alpha,\ \alpha = 1,\ldots,\ell\}$ (some masses may be equal). Let us denote by $|\theta_1,\ldots,\theta_k\rangle_{\alpha_1\ldots\alpha_k}$ with $\theta_1 > \cdots > \theta_k$ the in-states with k particles of rapidities $\theta_1,\ldots,\theta_k$ and, respectively, of particle types (quantum numbers) $\alpha_1,\ldots,\alpha_k$. For $\theta_1 < \ldots < \theta_k$, this will represent the out-states. Then, the two-particle scattering matrix is defined by

$$|\theta_1,\theta_2\rangle_{\alpha_1\alpha_2} = \sum_{\beta_1,\beta_2} S_{\alpha_1\alpha_2}^{\beta_1\beta_2}(\theta_1-\theta_2)|\theta_2,\theta_1\rangle_{\beta_2\beta_1}$$

where we used relativistic invariance in order to write the scattering matrix as function of the rapidity differences. This equation holds for $\theta_1 > \theta_2$ as well as $\theta_1 < \theta_2$, thanks to unitary. One of the main achievements of massive integrable QFT is the exact calculation of scattering matrices in many models, from its expected physical properties along with the Yang-Baxter equation (or, in some cases, from Bethe ansatz solution of an underlying integrable microscopic model).

Another important result of massive integrable QFT is the exact calculation of some form factors in many models [26, 27] (see [28] for a recent review). Form factors are matrix elements of local fields between the vacuum and a many-particle state (say an in-state):

$$F_k^{\mathcal{O}|\alpha_1\ldots\alpha_k}(\theta_1,\ldots,\theta_k) = \langle \mathrm{vac}|\mathcal{O}(0)|\theta_1,\ldots,\theta_k\rangle_{\alpha_1\ldots\alpha_k}\ . \tag{2.4}$$

They are, more precisely, the analytic continuation of these matrix elements in the rapidity variables. The exact evaluation of these objects follows from solving a set of expected properties that involve the exact two-particle scattering matrix and form a Riemann-Hilbert problem, along with certain minimality assumptions on the analytic structure. Besides integrability, an important requirement for this Riemann-Hilbert problem to hold is locality of the field $\mathcal{O}$. Indeed, a strong indication of its validity is that it is possible to show that two fields whose form factors solve it commute at space-like distances. For the simplest local fields, this Riemann-Hilbert problem is well known and can be solved in many cases.

If $\mathcal{L}$ is integrable, then certainly $\mathcal{L}^{(n)}$ also is, with n times as many particles, which we will denote by the double index (α, i) for $\alpha = 1,\ldots,\ell$ and $i = 1,\ldots,n$. Its scattering matrix is simply given by

$$S_{(\alpha_1,i_1)\,(\alpha_2,i_2)}^{(\beta_1,j_1)\,(\beta_2,j_2)}(\theta) = \delta_{i_1,j_1}\delta_{i_2,j_2} \times \left\{ \begin{array}{ll} \delta_{\alpha_1,\beta_1}\delta_{\alpha_2,\beta_2} & i_1 \neq i_2 \\ S_{\alpha_1\alpha_2}^{\beta_1\beta_2}(\theta) & i_1 = i_2 \end{array}\right. . \tag{2.5}$$

Locality of the twist fields $\mathcal{T}$ and $\tilde{\mathcal{T}}$ along with the exchange relations (2.3) were used in [20] in order to justify a Riemann-Hilbert problem for their form factors in integrable models with diagonal scattering (that is, with $S_{\alpha_1\alpha_2}^{\beta_1\beta_2}(\theta) = S_{\alpha_1\alpha_2}(\theta)\delta_{\alpha_1,\beta_1}\delta_{\alpha_2,\beta_2}$). The generalisation to non-diagonal scattering is straightforward. We will consider only $\mathcal{T}$, since $\tilde{\mathcal{T}} = \mathcal{T}^\dagger$ on the Hilbert space. Let us denote by μ, ν and ω double indices of the form (α, j), and by $F_k^{\mathcal{T}|\mu_1\ldots\mu_k}(\theta_1,\ldots,\theta_k,n)$ the form factors of $\mathcal{T}$ in the n-copy model $\mathcal{L}^{(n)}$. They are analytic in the rapidity differences except for poles (that may accumulate at infinity), and satisfy the relations

$$\begin{aligned} F_k^{\mathcal{T}|\ldots\mu_i\mu_{i+1}\ldots}(\ldots,\theta_i,\theta_{i+1},\ldots,n) &= \sum_{\nu_i,\nu_{i+1}} S_{\mu_i\mu_{i+1}}^{\nu_i\nu_{i+1}}(\theta_{i\,i+1}) F_k^{\mathcal{T}|\ldots\nu_{i+1}\nu_i\ldots}(\ldots,\theta_{i+1},\theta_i,\ldots,n), \\ F_k^{\mathcal{T}|\mu_1\mu_2\ldots\mu_k}(\theta_1+2\pi i,\ldots,\theta_k,n) &= F_k^{\mathcal{T}|\mu_2\ldots\mu_k\hat{\mu}_1}(\theta_2,\ldots,\theta_k,\theta_1,n), \end{aligned} \tag{2.6}$$

and

$$
\begin{aligned}
-\,i\mathrm{Res}_{\bar{\theta}_0=\theta_0} F_{k+2}^{\mathcal{T}|\bar{\mu}\mu\mu_1\ldots\mu_k}(\bar{\theta}_0+i\pi,\theta_0,\theta_1\ldots,\theta_k,n) &= F_k^{\mathcal{T}|\mu_1\ldots\mu_k}(\theta_1,\ldots,\theta_k,n),\\
-i\mathrm{Res}_{\bar{\theta}_0=\theta_0} F_{k+2}^{\mathcal{T}|\bar{\mu}\hat{\mu}\mu_1\ldots\mu_k}(\bar{\theta}_0+i\pi,\theta_0,\theta_1\ldots,\theta_k,n) &= \sum_{\nu_1,\ldots,\nu_k} \mathcal{P}_{\hat{\mu}\mu_1\ldots\mu_k}^{\nu_1,\ldots,\nu_k}(\theta_0,\theta_1,\ldots,\theta_k,n)\\
&\quad \times F_k^{\mathcal{T}|\nu_1\ldots\nu_k}(\theta_1,\ldots,\theta_k,n) \qquad (2.7)
\end{aligned}
$$

with

$$
\mathcal{P}_{\hat{\mu}\mu_1\ldots\mu_k}^{\nu_1,\ldots,\nu_k}(\theta_0,\theta_1,\ldots,\theta_k,n) = \sum_{\omega_1,\ldots,\omega_{k-1}} S_{\hat{\mu}\mu_1}^{\omega_1\nu_1}(\theta_{01}) S_{\omega_1\mu_2}^{\omega_2\nu_2}(\theta_{02}) \cdots S_{\omega_{k-1}\mu_k}^{\hat{\mu}\nu_k}(\theta_{0k}). \qquad (2.8)
$$

Here $\theta_{ij}=\theta_i-\theta_j$. As function of θ_1 for real $\theta_2,\ldots,\theta_k$, there are no poles of $F_k^{\mathcal{T}|\mu_1\ldots\mu_k}(\theta_1,\ldots,\theta_k,n)$ in the strip $\mathrm{Im}(\theta_1)\in[0,\pi]$, except for those given by the last two equations, and for poles at purely imaginary values (and with purely imaginary residues) corresponding to bound states between the associated particles (note that we will not consider any bound states in the sine-Gordon example studied below). In the second equation, the crossing or locality relation, we introduced the symbol $\hat{\mu}=(\alpha,j+1)$. As compared to the usual form factor equations, it is altered by the nature of the exchange relation and it now relates form factors associated to different particle sets (belonging to different copies). Finally, the last two equations generalise the standard kinematic residue equation to branch-point twist fields, where we introduced the symbol $\bar{\mu}=(\bar{\alpha},j)$ with $\bar{\alpha}$ denoting the quantum number of the anti-particle of α in the theory $\mathcal{L}$. Once more, the exchange relations (2.3) are responsible for the splitting into two equations, and the shift in $\hat{\mu}$.

It is instructive to specialise to two particles; this is what gives the main result (1.3) for the entropy (one-particle form factors of spinless fields are θ-independent). In this case, the first two form factor equations specialise to

$$
F_2^{\mathcal{T}|(\alpha,j)\,(\beta,k)}(\theta,n) = \sum_{\delta,\gamma} F_2^{\mathcal{T}|(\gamma,k)\,(\delta,j)}(-\theta,n) S_{(\alpha,j)\,(\beta,k)}^{(\delta,j)\,(\gamma,k)}(\theta) = F_2^{\mathcal{T}|(\beta,k)\,(\alpha,j+1)}(2\pi i-\theta,n), \qquad (2.9)
$$

for all values of j, k, α and β, where θ is now the rapidity difference. From the equations above, from (2.5) and from application of the $\mathbb{Z}_n$ symmetry, it follows:

$$
\begin{aligned}
F_2^{\mathcal{T}|(\alpha,i)\,(\beta,i+k)}(\theta,n) &= F_2^{\mathcal{T}|(\alpha,j)\,(\beta,j+k)}(\theta,n) \qquad \forall \quad i,j,k,\alpha,\beta \qquad (2.10)\\
F_2^{\mathcal{T}|(\alpha,1)\,(\beta,j)}(\theta,n) &= F_2^{\mathcal{T}|(\beta,1)\,(\alpha,1)}(2\pi(j-1)i-\theta,n) \quad \forall \quad \alpha,\beta,j\in\{2,\ldots,n+1\}. \qquad (2.11)
\end{aligned}
$$

The last equation at $j=n+1$, and the first equation of (2.6), give

$$
F_2^{\mathcal{T}|(\alpha,1)\,(\beta,1)}(\theta,n) = \sum_{\delta,\gamma} F_2^{\mathcal{T}|(\gamma,1)\,(\delta,1)}(-\theta,n) S_{\alpha\beta}^{\delta\gamma}(\theta) = F_2^{\mathcal{T}|(\beta,1)\,(\alpha,1)}(2\pi ni-\theta,n), \qquad (2.12)
$$

so, as in [20], we can solve for the form factors associated to particles in the first copy and then use (2.11) to obtain all other solutions. From now on we will abbreviate

$$
F_2^{\mathcal{T}|(\alpha,1)(\beta,1)}(\theta,n) := F_2^{\mathcal{T}|\alpha\beta}(\theta,n). \qquad (2.13)
$$

The simple form of the scattering matrix for particles living in different sheets, and the fact that no bound state can occur between particles in different copies, simplifies drastically the pole structure of the form factors. A combination of the equations above along with the two-particle case of the last two equations of (2.7) gives that $F_2^{\mathcal{T}|\alpha\beta}(\theta,n)$ is analytic for $\mathrm{Im}(\theta)\in[0,2\pi n]$ except if $\bar{\alpha}=\beta$ for poles at $\theta=i\pi$, with residue $i\langle\mathcal{T}\rangle$, and at $\theta=2i\pi n-i\pi$, with residue $-i\langle\mathcal{T}\rangle$, and for possible poles in $\mathrm{Im}(\theta)\in(0,\pi)$ and in $\mathrm{Im}(\theta)\in(2\pi n-\pi,2\pi n)$ corresponding to bound states.

2.2 Conical singularities and the entanglement entropy

The analytic structure described above is sufficient to establish the result (1.3), following arguments of [20]. The most delicate point of these arguments is the analytic continuation in n, which still needs further justification. Below we attempt to support the arguments from the geometrical picture of conical singularities. Another point is that we must assume that form factors at real rapidities vanish faster than $(n-1)^{\frac{1}{2}}$ as $n \to 1$. They certainly do vanish as $n \to 1$, since then the branch-point twist field becomes the identity field. They were observed to vanish proportionally to $n-1$ in [20]. We first write the two-point function in the two-particle approximation:

$$\langle \mathcal{T}(r)\tilde{\mathcal{T}}(0)\rangle \approx \langle \mathcal{T}\rangle^2 \left(1 + \begin{array}{c}\text{1-part.}\\ \text{terms}\end{array} + \frac{n}{8\pi^2}\sum_{\alpha,\beta=1}^{\ell}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\theta_1 d\theta_2 f_{\alpha,\beta}(\theta_{12},n)\, e^{-r(m_\alpha\cosh\theta_1 + m_\beta\cosh\theta_2)}\right) \tag{2.14}$$

where

$$\begin{aligned} \langle \mathcal{T}\rangle^2 f_{\alpha,\beta}(\theta,n) &= \sum_{j=1}^{n}\left|F_2^{\mathcal{T}|(\alpha,1)(\beta,j)}(\theta,n)\right|^2 \\ &= \left|F_2^{\mathcal{T}|\alpha\beta}(\theta,n)\right|^2 + \sum_{j=1}^{n-1}\left|F_2^{\mathcal{T}|\alpha\beta}(-\theta+2\pi i j,n)\right|^2 . \end{aligned} \tag{2.15}$$

There is no contribution from the one-particle terms when we take the derivative with respect to n and evaluate it at $n=1$, since they are squares of (analytically continued) one-particle form factors, which are θ-independent and vanish faster than $n-1$ as $n \to 1$ by assumption.

As for the function $f_{\alpha,\beta}(\theta,n)$, coming from two-particle form factors, let us denote by $\tilde{f}_{\alpha,\beta}(\theta,n)$ its analytic continuation from $n = 1,2,\ldots$ to $n \in [1,\infty)$. The analytic continuation in n of the form factors themselves, for fixed rapidity, is natural from the geometrical picture of conical singularities with angle $2\pi n$. For any n real and positive, form factors should have kinematic poles at $\theta = i\pi$ and $\theta = 2i\pi n - i\pi$ representing particles going in a straight line on either sides of the conical singularity. They also possibly have bound state poles as described above corresponding to bound states forming on either side, and they satisfy all other properties stated above, now with n real and positive. For technical reasons, we must assume that the residues at bound state poles are not diverging as $n \to 1$. It is natural to expect that for any fixed rapidity, the resulting form factors do not "oscillate" as functions of n. More precisely, that they have definite convexity, at least for large enough n. As for the sum over j in (2.15), it should be understood, when multiplied by n like in (2.14), as the total two-particle contribution with particles allowed to cover the region all around the conical singularities. Intuitively, it should be smooth and should not present any oscillations either in n, since the space around the conical singularities just increases linearly with n. Hence it should have definite convexity, at least for n large enough (where parallel incoming straight trajectories just to the right and to the left of the conical singularity are far apart past it). In fact, a natural and more constraining requirement is that no oscillatory terms are present in the large-n expansion. These properties may fix uniquely the analytic continuation in n.

A smooth function of $n > 1$ can be obtained easily from the sum over j. Methods used in [20] lead to the fact that in the limit $n \to 1$, the only possible contribution to the derivative w.r.t. n is from the collision of the kinematic poles in $F_2^{\mathcal{T}|\alpha\beta}(\theta,n)$ when $\alpha = \bar{\beta}$. This uses the assumption that form factors vanish faster than $(n-1)^{\frac{1}{2}}$, and the fact that these are the only singularities that can collide as $n \to 1$. Hence the contribution can be evaluated exactly by extracting these kinematic poles. For completeness and clarity, let us provide here the explicit analytic continuation of the sum over j, following [20] and emphasising the general principles used. We write

$$\sum_{j=1}^{n-1} \left| F_2^{\mathcal{T}|\alpha\beta}(-\theta+2\pi i j,n)\right|^2 = \frac{1}{2i}\oint ds\, g(s)\cot \pi s - Q \tag{2.16}$$

where

$$g(s) = F_2^{\mathcal{T}|\alpha\beta}(-\theta+2\pi i s,n)(F_2^{\mathcal{T}|\alpha\beta})^*(-\theta-2\pi i s,n)\ . \tag{2.17}$$

The contour ranges from $\mathrm{Re}(s)=\epsilon$ to $\mathrm{Re}(s)=n-\epsilon$ encircling counter-clockwise the segment $s\in[\epsilon,n-\epsilon]$ for some $0<\epsilon<1/2$ near enough to $1/2$ so that no bound state singularity can be encircled. If the form factors at infinite rapidities vanish sufficiently fast (in all examples calculated, they vanish exponentially, which is sufficient), then the contour can be widened up to $\mathrm{Im}(s)=\pm\infty$, since the contribution at $\mathrm{Im}(s)=\pm\infty$ vanishes. The quantity Q is the sum of the residues from the kinematic poles in the product of form factors themselves:

$$\begin{aligned} Q &= \sum_{\text{kinematic poles } \tilde{s}} \mathrm{Res}_{s=\tilde{s}}\left(F_2^{\mathcal{T}|\alpha\beta}(-\theta+2\pi i s,n)(F_2^{\mathcal{T}|\alpha\beta})^*(-\theta-2\pi i s,n)\right)\pi\cot\pi\tilde{s} \\ &= \delta_{\bar{\alpha},\beta}\tan\left(\frac{\theta}{2}\right)\mathrm{Im}\left(F_2^{\mathcal{T}|\alpha\bar{\alpha}}(-2\theta+i\pi)-F_2^{\mathcal{T}|\alpha\bar{\alpha}}(-2\theta+2i\pi n-i\pi)\right)\ . \end{aligned} \tag{2.18}$$

Using the n-periodicity of $\cot\pi s$, the part of the integration at $\mathrm{Re}(s)=n-\epsilon$ can be re-written, and the contour integral in (2.16) becomes

$$\frac{1}{2}\int_{-\infty}^{\infty} dy\,\left(g(n+iy-\epsilon)\cot\pi(iy-\epsilon)-g(iy+\epsilon)\cot\pi(iy+\epsilon)\right) \tag{2.19}$$

This expression is analytic in n; as n is varied continuously down to $n=1$, no pole crosses the y integration line. The expression for Q is analytic in n for $n>1$ as well.

Note that we used the function $\pi\cot\pi s$, which is a function that remains unchanged under $s\mapsto s+n$ for any integer n and that has simple poles with equal residues, in the band $\mathrm{Re}(s)\in(0,n)$, only on the integers $1,2,\ldots,n-1$. The poles were used to reproduce the correct sum, and periodicity was used to cure the part of the integration contour at $\mathrm{Re}(s)=n-\epsilon$, which otherwise would have crossed poles as n is varied continuously. However, we wish to emphasise that no periodicity property of the function $g(s)$ (related to the form factors) itself is used, contrary to the derivation presented in [20]. The function $g(s)$ does not cause problems since as n is varied, its poles move accordingly. Could we replace $\pi\cot\pi s$ by another function $u(s)$? If we require that $u(s)$ do not depend explicitly on n in addition to the properties above, then it is uniquely fixed to $\pi\cot\pi s$. Relaxing periodicity to, for instance, $u(s+1)=v(s)u(s)$ would impose $v(s)=1$ for s integer (in order to have the correct residues), and conditions on convergence at infinity would require $v(s)=1$. Note also that the expression (2.19) does not depend on ϵ even for n non-integer.

The integral (2.19) have definite convexity as function of n, at least for large enough n. This follows from expected convexity properties of the form factors themselves. The same holds for the residues Q. Integration over θ should preserve these properties for large enough n, and it seems natural to expect that the full large-n expansion does not contain oscillatory terms. Of course, a more complete analysis of the n-dependence of Q and of the integral (2.19) would be useful.

It is clear that the integral in (2.19) vanishes when $n\to 1$ faster then $n-1$ by our assumption, and that Q vanishes at least like $(n-1)^{3/2}$ for any $\theta\neq 0$. The only possible obstruction in Q is when $\theta \to 0$, due to the kinematic pole of the first term in (2.18) at argument $(2n-1)i\pi$, and the kinematic pole of the second term at argument $i\pi$. Extracting these poles gives, for $n \to 1$ and $\theta \to 0$,

$$\tilde{f}_{\alpha,\beta}(\theta, n) = \delta_{\bar{\alpha},\beta}\tilde{f}_{\alpha,\bar{\alpha}}(0,1)\left(\frac{i\pi(n-1)}{2(\theta+i\pi(n-1))} - \frac{i\pi(n-1)}{2(\theta-i\pi(n-1))}\right) \tag{2.20}$$

with

$$\tilde{f}_{\alpha,\bar{\alpha}}(0,1) = \frac{1}{2} \ . \tag{2.21}$$

Here and below we adopt the convention that $\tilde{f}_{\alpha,\bar{\alpha}}(0,1) = \lim_{n\to 1} \tilde{f}_{\alpha,\bar{\alpha}}(0,n)$. This immediately gives

$$\left(\frac{\partial}{\partial n}\tilde{f}_{\alpha,\beta}(\theta,n)\right)_{n=1} = \frac{\pi^2}{2}\delta(\theta)\delta_{\bar{\alpha},\beta} \ . \tag{2.22}$$

This has a non-zero measure at $\theta = 0$, and proves (1.3).

As we mentioned above, there are two main requirements for the argument to be valid: first, an appropriate analytic continuation has to be taken, in agreement with the intuition from conical singularities, and second, the analytically continued one- and two-particle form factors have to vanish, for any real rapidity, faster than $(n-1)^{\frac{1}{2}}$ as $n \to 1$. As we said above, for the latter requirement, in the cases constructed in [20], it was observed that form factors vanish like $n-1$, and this is what is expected in general. Below we provide further evidence for both requirements in the case of the two-particle form factors by constructing them explicitly in the sine-Gordon model. In particular, we verify numerically the striking fact that $\tilde{f}_{\alpha,\bar{\alpha}}(0,1) = \frac{1}{2}$ is in agreement with $n\tilde{f}_{\alpha,\bar{\alpha}}(0,n)$ having no oscillatory terms in its large-n expansion. We also find that it diverges like n as $n \to \infty$ in the super-renormalisable case, and like $n \log n$ in the marginally renormalisable case.

3 The sine-Gordon case

Let us now consider the specific sine-Gordon case. This model can be defined by the lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_0\varphi)^2 - \frac{1}{2}(\partial_1\varphi)^2 + \mu\cos(\beta\varphi) \ , \tag{3.1}$$

where μ has scaling dimension $2\beta^2 - 2$ and β does not renormalise. We will use the variable

$$\nu = \frac{\beta^2}{8\pi - \beta^2} \ . \tag{3.2}$$

For $\nu \geq 1$, the spectrum of the model is known to be composed of two particles with equal masses, which we will label by $+$ and $-$, representing the quantised version of soliton and anti-soliton in the classical theory. For simplicity we will consider this region only. For $\nu < \infty$, the model is super-renormalisable. At $\nu = 1$, the particles are free with fermionic statistics, and the model is equivalent to a massive Dirac theory. The model with $\nu > 1$ can be seen as the massive Thirring model, a perturbation of the massive Dirac theory that preserves the $U(1)$ symmetry. From this viewpoint, the asymptotic particles are the positively and negatively charged version of the same particle. As $\nu \to \infty$, the model specialises to (a sector of) the $SU(2)$-Thirring model, which is marginally renormalisable.

The scattering of soliton with anti-soliton is non-diagonal. The scattering matrix is given by [29]

$$S_{+-}^{+-}(\theta)=\frac{\sinh\left(\frac{\theta}{\nu}\right)}{\sinh\left(\frac{i\pi-\theta}{\nu}\right)}S_{++}^{++}(\theta),\qquad S_{+-}^{-+}(\theta)=\frac{\sinh\left(\frac{i\pi}{\nu}\right)}{\sinh\left(\frac{i\pi-\theta}{\nu}\right)}S_{++}^{++}(\theta),\tag{3.3}$$

with

$$S_{++}^{++}(\theta)=S_{--}^{--}(\theta)=-\exp\left[\int_0^\infty\frac{dt}{t}\frac{\sinh\left(\frac{t(1-\nu)}{2}\right)\sinh\left(\frac{t\theta}{i\pi}\right)}{\sinh(t\nu/2)\cosh(t/2)}\right].\tag{3.4}$$

Since branch-point twist fields have zero $U(1)$ charge, their one-particle form factors vanish. On the other hand, the relations (2.12) give the following system of coupled equations for their two-particle form factors:

$$\begin{aligned}F_2^{\mathcal{T}|+-}(\theta,n) &= \sum_{\gamma,\delta=\pm}F_2^{\mathcal{T}|\gamma\delta}(-\theta,n)S_{-+}^{\delta\gamma}(\theta)=F_2^{\mathcal{T}|-+}(2\pi in-\theta,n), &(3.5)\\ F_2^{\mathcal{T}|-+}(\theta,n) &= \sum_{\gamma,\delta=\pm}F_2^{\mathcal{T}|\gamma\delta}(-\theta,n)S_{+-}^{\delta\gamma}(\theta)=F_2^{\mathcal{T}|+-}(2\pi in-\theta,n). &(3.6)\end{aligned}$$

These equations can be diagonalised by introducing the linear combinations $F_2^{\mathcal{T}|+-}(\theta,n)\pm F_2^{\mathcal{T}|-+}(\theta,n)$. However, the branch-point twist fields are also invariant under charge conjugation $+\leftrightarrow-$. Indeed, this is a symmetry of the action, which keeps unchanged partition functions on Riemann surfaces, and branch-point twist fields are associated to such partition functions. Hence $F_2^{\mathcal{T}|+-}(\theta,n)=F_2^{\mathcal{T}|-+}(\theta,n)$ and we are left with

$$F_2^{\mathcal{T}|+-}(\theta,n)=\left(S_{+-}^{+-}(\theta)+S_{+-}^{-+}(\theta)\right)F_2^{\mathcal{T}|+-}(-\theta,n)=F_2^{\mathcal{T}|+-}(2\pi in-\theta,n)\ .\tag{3.7}$$

It is possible to find integral representations for the combination of S-matrix elements above,

$$\begin{aligned}S_{+-}^{+-}(\theta)+S_{+-}^{-+}(\theta) &= \frac{\sin\left(\frac{\pi-i\theta}{2\nu}\right)}{\sin\left(\frac{\pi+i\theta}{2\nu}\right)}S_{++}^{++}(\theta) &(3.8)\\ &= -\exp\left[\int_0^\infty\frac{dt}{t}\frac{2\sinh\left(\frac{t(\nu-1)}{2}\right)\cosh\left(\frac{t(\nu-2)}{2}\right)\sinh\left(\frac{t\theta}{i\pi}\right)}{\sinh(\nu t)\cosh(t/2)}\right], &(3.9)\end{aligned}$$

where we used

$$\frac{\sin\frac{\pi}{2}(a+x)}{\sin\frac{\pi}{2}(a-x)}=\exp\left[\int_0^\infty\frac{dt}{t}\frac{2\sinh t(1-a)\sinh(tx)}{\sinh t}\right],\quad\text{for}\quad 0<a<1.\tag{3.10}$$

Following [20], it is now straightforward to find a minimal solution to (3.7) (that is analytic for $\mathrm{Im}(\theta)\in[0,2\pi n]$) up to normalisation:

$$F_2^{\min}(\theta,n)=-i\sinh\left(\frac{\theta}{2n}\right)\exp\left[\int_0^\infty\frac{dt}{t}\frac{2\sinh\left(\frac{t(\nu-1)}{2}\right)\cosh\left(\frac{t(\nu-2)}{2}\right)\sin^2\frac{it}{2}\left(n-\frac{\theta}{i\pi}\right)}{\sinh(nt)\sinh(t\nu)\cosh(t/2)}\right].\tag{3.11}$$

The two particle form factor $F_2^{\mathcal{T}|+-}(\theta,n)$ can be fixed by including the right pole structure as in [20]:

$$F_2^{\mathcal{T}|+-}(\theta,n)=\frac{\langle\mathcal{T}\rangle\sin\left(\frac{\pi}{n}\right)}{2n\sinh\left(\frac{i\pi-\theta}{2n}\right)\sinh\left(\frac{i\pi+\theta}{2n}\right)}\frac{F_2^{\min}(\theta,n)}{F_2^{\min}(i\pi,n)},\tag{3.12}$$

where the normalization has been chosen so that the kinematic residue equation gives

$$F_0^{\mathcal{T}}=\langle\mathcal{T}\rangle.\tag{3.13}$$

4 Identifying the ultraviolet conformal dimension

It is interesting to check the solutions above for consistency. A possible consistency check consists of analyzing the short-distance behaviour of two-point functions involving the twist field and compare that behaviour to the one expected from conformal field theory predictions. In particular, one can look at the two-point function of the twist field with the trace of the stress-energy tensor, Θ. It was found in [30] that

$$\Delta^{\mathcal{T}} = \Delta^{\tilde{\mathcal{T}}} = -\frac{1}{2\langle \mathcal{T} \rangle} \int_0^\infty r \, \langle \Theta(r) \mathcal{T}(0) \rangle \, dr \tag{4.1}$$

(where the integration is on a space-like ray). This formula is known as the Δ-sum rule. The first equality, expected from CFT, holds from the Δ-sum rule thanks to the fact that Θ commute with $\mathcal{T}$ and that $\Theta^\dagger = \Theta$. Following the derivations in [17, 20] the conformal dimension is expected to be $\Delta^{\mathcal{T}} = d_n/2$, where d_n was given in (2.2).

As explained in detail in [20] we can now employ the expansion of the two-point function in terms of two particle form factors and obtain

$$\begin{aligned} \Delta^{\mathcal{T}} &\approx -\frac{n}{\langle \mathcal{T} \rangle} \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} \frac{d\theta_1 d\theta_2 F_2^{\Theta|+-}(\theta_{12}) F_2^{\mathcal{T}|+-}(\theta_{12}, n)^*}{2(2\pi)^2 m^2 \left(\cosh\theta_1 + \cosh\theta_2\right)^2} \\ &= -\frac{n}{8\pi^2 m^2 \langle \mathcal{T} \rangle} \int\limits_{0}^{\infty} \frac{d\theta F_2^{\Theta|+-}(\theta) F_2^{\mathcal{T}|+-}(\theta, n)^*}{\cosh^2(\theta/2)}, \end{aligned} \tag{4.2}$$

where we have used the fact that $F_2^{\Theta|+-}(\theta_{12}) = F_2^{\Theta|-+}(\theta_{12})$ and $F_2^{\Theta|\pm\pm}(\theta_{12}) = 0$.

The two particle form factors of Θ can be obtained from results already known in the literature. In particular, by employing the results obtained in [31, 32] for the form factors of exponential fields and the relation between the sine-Gordon coupling constant and the soliton mass we find:

$$F_2^{\Theta|+-}(\theta) = \frac{2\pi i m^2}{\nu} \frac{G(\theta)\cosh\frac{\theta}{2}}{\sinh\left(\frac{i\pi-\theta}{2\nu}\right)}, \tag{4.3}$$

where

$$G(\theta) = -i \sinh\frac{\theta}{2} \exp\left[\int_0^\infty \frac{dt}{t} \frac{\sinh t(1-\nu)\sin^2 it\left(1-\frac{\theta}{i\pi}\right)}{\sinh(2t)\cosh t \sinh(t\nu)}\right]. \tag{4.4}$$

A table of values of the integral (4.2) for different values of n and ν is presented below. The values given in the top row in brackets are the exact values of the dimension, as predicted by CFT. Recall that we assumed the non-existence of bound states, which is why we only consider the case $\nu > 1$.

	$n=2$ (0.0625)	$n=3$ (0.1111)	$n=4$ (0.1563)	$n=5$ (0.2)
$\nu = 1.1$	0.0627	0.1115	0.1568	0.2008
$\nu = 1.6$	0.0628	0.1118	0.1573	0.2013
$\nu = 2.1$	0.0627	0.1115	0.1570	0.2010
$\nu = 2.6$	0.0626	0.1113	0.1566	0.2003
$\nu = 3.1$	0.0625	0.1111	0.1562	0.1999
$\nu = 3.6$	0.0624	0.1111	0.1560	0.1995

	$n=6$ (0.2431)	$n=7$ (0.2857)	$n=8$ (0.3281)	$n=9$ (0.3704)	$n=10$ (0.4125)
$\nu=1.1$	0.2440	0.2868	0.3293	0.3717	0.4139
$\nu=1.6$	0.2447	0.2876	0.3303	0.3728	0.4152
$\nu=2.1$	0.2443	0.2871	0.3297	0.3722	0.4145
$\nu=2.6$	0.2434	0.2861	0.3287	0.3707	0.4129
$\nu=3.1$	0.2428	0.2854	0.3277	0.3698	0.4119
$\nu=3.6$	0.2424	0.2848	0.3270	0.3690	0.4109

As we can see, all values are extremely close to the exact ones, even in the two particle approximation. This provides a consistency check for our form factor (3.12). A further check concerns the correct normalization of the form factor (4.3). Indeed the normalization employed here corresponds to $F_2^{\Theta|+-}(i\pi) = 2\pi m^2$ and differs from that in other places in the literature [31]. A consistency check for this normalization consists of extracting the central charge $c=1$ of the underlying CFT by employing Zamolodchikov's c-theorem [33]:

$$c = \frac{3}{2}\int_{-\infty}^{\infty} drr^3 \left\langle \Theta(r)\Theta(0)\right\rangle . \tag{4.5}$$

Employing again the expansion of the two point function in terms of form factors, changing variables and performing one integral as in (4.2) we obtain

$$c \approx \frac{3}{8\pi^2}\int_0^{\infty} \frac{\left|F_2^{\Theta|+-}(\theta)\right|^2}{\cosh^4(\theta/2)} d\theta. \tag{4.6}$$

This integral can be easily evaluated numerically for different values of ν and gives

	$\nu=1.1$	$\nu=2.1$	$\nu=3.1$	$\nu=4.1$	$\nu=5.1$
c	0.9999	0.9918	0.9888	0.9877	0.9874

5 Computation of $\tilde{f}_{\pm\mp}(0,n)$: Numeric and analytic results

In this section we wish to study the behaviour of the function $\tilde{f}_{\pm\mp}(0,n)$ (see (2.15)) in detail and, most importantly, show that $\tilde{f}_{\pm\mp}(0,1) = 1/2$ as expected. Let us introduce the notation $\tilde{f}(n) \equiv \tilde{f}_{+-}(0,n) = \tilde{f}_{-+}(0,n)$. This function, by definition, is the natural analytic continuation of $f_{+-}(0,n)$ from $n = 2,3,\ldots$ to $n \in [1,\infty)$, that is with the prescription of smoothness and monotony as described in paragraph 2.2. We will obtain the form of its full large-n expansion, with explicit first few coefficients, showing that it grows proportionally to n for any finite ν, and to $n\log n$ for $\nu = \infty$. This large-n expansion is interesting in its own right, and turns out to differ quite dramatically from the $1/n$ expansion found for the sinh-Gordon model [20]. We will then perform a numerical study to confirm it, and most importantly, verify the value $\tilde{f}(1) = 1/2$.

Let us commence by our analytical considerations. Note that the first term of (2.15) is zero at $\theta = 0$. For the summand, the second identity of (3.7) implies that $F_2^{\mathcal{T}|+-}(2\pi ij, n) = F_2^{\mathcal{T}|+-}(2\pi i(n-j), n)$ for $j = 1,\ldots,n-1$. In addition, for any fixed value of n the value of the function is largest for smaller values of j and decreases quickly as j approaches $[n/2]$. This behaviour is particularly extreme for n very large. In that case, only for $j \ll n$ does the function above have non-negligible values. Therefore, the value of $\tilde{f}(n)$ for $n\to\infty$ can be obtained by replacing the sum $\sum_{j=1}^{n-1} |F_2^{\mathcal{T}|+-}(2\pi ij,n)|^2$ by

$$2\sum_{j=1}^{\infty} \lim_{n\to\infty} |F_2^{\mathcal{T}|+-}(2\pi ij,n)|^2 = \tilde{f}(\infty). \tag{5.1}$$

This can be written as

$$\tilde{f}(\infty) = \frac{2}{b(1/2)^2} \sum_{j=1}^{\infty} a(j)^2 b(j)^2, \tag{5.2}$$

where

$$a(j) = \lim_{n\to\infty} a(j,n) = \frac{2(2j)^{\frac{3}{2}-\frac{1}{2\nu}}}{\pi(1-4j^2)}, \tag{5.3}$$

with

$$a(j,n) = \frac{\sin(\frac{\pi}{n})\sin(\frac{j\pi}{n})^{\frac{3}{2}-\frac{1}{2\nu}}}{2n\sin(\frac{(1-2j)\pi}{2n})\sin(\frac{(1+2j)\pi}{2n})\sin(\frac{\pi}{2n})^{\frac{3}{2}-\frac{1}{2\nu}}}, \tag{5.4}$$

and

$$\begin{aligned} b(j) &= \lim_{n\to\infty} b(j,n) = \lim_{n\to\infty} F_2^{\min}(2\pi i j, n)\left[\sin\left(\frac{j\pi}{n}\right)\right]^{\frac{1}{2\nu}-\frac{3}{2}} \\ &= \exp\left[-\int_0^\infty \frac{dt}{2t}\left(\frac{2\sinh\left(\frac{t(\nu-1)}{2}\right)\cosh\left(\frac{t(\nu-2)}{2}\right)}{\sinh(t\nu)\cosh(t/2)} - \frac{\nu-1}{\nu}\right)e^{-2tj}\right], \end{aligned} \tag{5.5}$$

with

$$b(j,n) = \exp\left[\int_0^\infty \frac{dt}{t}\left(\frac{2\sinh\left(\frac{t(\nu-1)}{2}\right)\cosh\left(\frac{t(\nu-2)}{2}\right)}{\sinh(t\nu)\cosh(t/2)} - \frac{\nu-1}{\nu}\right)\frac{\sin^2\frac{it}{2}(n-2j)}{\sinh(nt)}\right]. \tag{5.6}$$

Notice that the factor $\sin\left(\frac{j\pi}{n}\right)^{\frac{1}{2\nu}-\frac{3}{2}}$ in $a(j)$ is cancelled by a similar factor in $b(j)$. They have been introduced in order to guarantee that the integral in $b(j)$ is convergent as $t \to 0$. Formula (5.2) holds only for ν finite; we will come back to the case $\nu = \infty$ below. Evaluating (5.2) numerically we obtain the values listed in the table at the end of this section.

However, contrarily to the sinh-Gordon example studied in [20], the large-n corrections to this linear behaviour are not just of $1/n$ type but depend on the particular value of ν. The form of these corrections can be determined by some analysis. Notice that

$$\tilde{f}(n) = \begin{cases} \dfrac{2}{b(1/2,n)^2}\displaystyle\sum_{j=1}^{n/2} a(j,n)^2 b(j,n)^2 - \dfrac{1}{b(1/2,n)^2} a(n/2,n)^2 b(n/2,n)^2 & n \text{ even} \\ \dfrac{2}{b(1/2,n)^2}\displaystyle\sum_{j=1}^{(n-1)/2} a(j,n)^2 b(j,n)^2 & n \text{ odd} \end{cases} \tag{5.7}$$

Let us first analyse $b(j,n)$ and $a(j,n)$ independently. It is trivial to see that $\lim_{j\to\infty} b(j)^2 = 1$. More precisely, it has a large-j asymptotic expansion of the form

$$b(j)^2 = \sum_{k=0}^{\infty} b_k\,(2j)^{-2k} + O(j^{-\infty}), \qquad \text{with} \qquad b_0 = 1. \tag{5.8}$$

where we use powers of $2j$ instead of j for later convenience. This formula is obtained from a small-t expansion of the coefficient of e^{-2tj} in the integrand in the last line of (5.5). The corrections for finite but large n can be obtained by looking at $b(j,n)/b(j)$. Expanding the resulting integrand in t while keeping intact the factor $\sin^2 \frac{it}{2}(n-2j)/\sinh(nt)$ and the factor e^{-2tj}, then by integrating this expansion term by term, we find

$$\frac{b(j,n)}{b(j)} = 1 + \frac{(\nu-1)(7\nu-5)}{96\nu n^2}\left(\zeta\left(2,1-\frac{j}{n}\right)+\zeta\left(2,1+\frac{j}{n}\right)-\pi^2\right)+O\left(\frac{1}{n^4}\right) \tag{5.9}$$

where $\zeta(z,a)=\sum_{k=0}^{\infty}(k+a)^{-z}$ is the generalized Riemann zeta function. The full series is just the exponential of a series that has terms in n^{-2k} for $k=1,2,\ldots$, with coefficients that are functions of j/n, linear in the functions $\zeta\left(2k,1-\frac{j}{n}\right)+\zeta\left(2k,1+\frac{j}{n}\right)$‡. This series expansion is convergent for all $j=1,\ldots,n/2$.

As for the function $a(j)$, its square has a large-j series expansion

$$a(j)^2 = \frac{4}{\pi^2(2j)^{1+\frac{1}{\nu}}}\sum_{k=0}^{\infty} a_k j^{-2k}, \qquad \text{with} \qquad a_0 = 1. \tag{5.10}$$

The square of its corrected form $a(j,n)$ can be expanded at large n as follows:

$$a(j,n)^2 = a(j)^2 + \frac{(2j)^{3-\frac{1}{\nu}}}{(4j^2-1)n^2}\sum_{k=0}^{\infty} p_k(j^2)n^{-2k} \tag{5.11}$$

where $p_k(j^2)$ is a polynomial of order k in j^2. Again, this series expansion is convergent for all $j=1,\ldots,n/2$. In fact, it converges for all values of n such that $|n|>|j\pm 1/2|$ and $|n|>1/2$.

We now consider the large-n expansion of (5.7). We will do explicitly the case where n is even, and comment about the agreement with the case where n is odd afterwards. First, the subtraction $-\frac{1}{b(1/2,n)^2}a(n/2,n)^2 b(n/2,n)^2$ for n even can be seen, from the analysis above, to be

$$-\frac{1}{b(1/2)^2}\left(\frac{2}{\pi}\right)^{1-\frac{1}{\nu}} n^{-1-\frac{1}{\nu}}(1+n^{-2}[[n^{-2}]]) \tag{5.12}$$

where here and below we use the notation

$$[[x]] = \text{some series in non-negative integer powers of } x. \tag{5.13}$$

Let us then consider only the term which is a sum, and subtract for convenience the value $b(1/2)^2/b(1/2,n)^2\tilde{f}(\infty)$, which is just $[[n^{-2}]]$. We are left with

$$\frac{2}{b(1/2,n)^2}\left(\sum_{j=1}^{n/2} a(j,n)^2 b(j,n)^2 - \sum_{j=1}^{\infty} a(j)^2 b(j)^2\right). \tag{5.14}$$

The pre-factor $2/b(1/2,n)^2$ is also $[[n^{-2}]]$, so we will consider the sum without pre-factor. We will consider three parts:

$$\text{I.} \qquad \sum_{j=1}^{n/2} a(j)^2 b(j)^2 - \sum_{j=1}^{\infty} a(j)^2 b(j)^2 = -\sum_{j=n/2+1}^{\infty} a(j)^2 b(j)^2, \tag{5.15}$$

$$\text{II.} \qquad \sum_{j=1}^{n/2}\{\text{corrections to } a(j)^2\}\cdot b(j)^2, \tag{5.16}$$

$$\text{III.} \qquad \sum_{j=1}^{n/2} a(j,n)^2\cdot\{\text{corrections to } b(j)^2\}, \tag{5.17}$$

‡Note that all such functions can be written in terms of trigonometric functions, for instance $\zeta(2,1+z)+\zeta(2,1-z)=\pi^2/\sin^2\pi z - 1/z^2$, but the analytic structure is clearer using the generalised zeta functions.

where the correction terms are those from (5.11) and (5.9). In the part I, we can use the large-j expansions of $a(j)^2$ and $b(j)^2$. Consider the identity

$$\sum_{j=n/2+1}^{\infty} (2j)^{-z} = 2^{-z}\zeta\left(z, 1+\frac{n}{2}\right) = -n^{1-z}\left(\frac{1}{2(1-z)} + \frac{1}{2n} + n^{-2}[[n^{-2}]]\right) \tag{5.18}$$

(here, the coefficients of the series $[[n^{-2}]]$ are polynomials in z), where $\zeta(z) = \zeta(z,1)$ is the Riemann zeta function. This implies that

$$\mathrm{I}: -\sum_{j=n/2+1}^{\infty} a(j)^2 b(j)^2 = \frac{2n^{-\frac{1}{\nu}}}{\pi^2}\left(-\nu + \frac{1}{n} + n^{-2}[[n^{-1}]]\right) . \tag{5.19}$$

The contributions II can be analysed using the formula

$$\begin{aligned}\sum_{j=1}^{n/2} \frac{(2j)^z}{4j^2-1} &= \sum_{k=0}^{\infty} 2^{z-2-2k} H_{n/2}^{(2+2k-z)} \\ &= \sum_{k=0}^{\infty} 2^{z-2-2k}\zeta(2+2k-z) + n^{z-1}\left(\frac{1}{2(z-1)} + \frac{1}{2n} + n^{-2}[[n^{-1}]]\right) .\end{aligned} \tag{5.20}$$

Here, $H_a^{(b)}$ is the Harmonic number of order b

$$H_a^{(b)} = \sum_{j=1}^{a} \frac{1}{j^b} = \zeta(b) - \zeta(b, a+1) . \tag{5.21}$$

Note that the infinite sum on the right-hand side of (5.20) is convergent for any non-integer z. Also, the coefficients in $[[n^{-1}]]$ are rational functions of z. Let us write $b(j)^2 = \sum_{k'=0}^{k} b_{k'}(2j)^{-2k'} + r(j)$ where $r(j)$ is the rest. The function $r(j)$ behaves proportionally to j^{-2k-2} as $j \to \infty$, so that the sum from 1 to $n/2$ of the product of $r(j)$ times the k^{th} correction to $a(j)^2$ in (5.11) can be extended to a sum from 1 to ∞ without problems. Hence for the k^{th} contribution to II we can write:

$$\begin{aligned}&n^{-2k-2}\sum_{k'=0}^{k}\sum_{j=1}^{n/2} b_{k'}\frac{(2j)^{3-\frac{1}{\nu}-2k'}}{4j^2-1} p_k(j^2) + n^{-2k-2}\sum_{j=1}^{\infty} r(j)\frac{(2j)^{3-\frac{1}{\nu}}}{(4j^2-1)} p_k(j^2) \\ &-n^{-2k-2}\sum_{k'=k+1}^{\infty}\sum_{j=n/2+1}^{\infty} b_{k'}\frac{(2j)^{3-\frac{1}{\nu}-2k'}}{(4j^2-1)} p_k(j^2) .\end{aligned} \tag{5.22}$$

The first term can be evaluated using (5.20). The second term is a contribution of the order n^{-2k-2} only, and the last term can be evaluated using the negative of (5.20) without the infinite sum of zeta functions and is of order $n^{-2k-2-1/\nu}$. Hence, the result is a term proportional to n^{-2k-2} plus an infinite series of the form $n^{-\frac{1}{\nu}}[[n^{-1}]]$. For instance, the leading contributions to this series are

$$2^{-2k} p_{k,k} n^{-\frac{1}{\nu}}\left(\frac{1}{2(2+2k-1/\nu)} + \frac{1}{2n} + O(n^{-2})\right) \tag{5.23}$$

where $p_{k,k}$ is the coefficient of j^{2k} in $p_k(j^2)$. When we consider all corrections, for all $k = 0, 1, 2, \ldots$, we obtain, for the coefficient of any given order $n^{-1/\nu+k'}$ with fixed $k' = 0, 1, 2, \ldots$, an infinite sum over k. for instance, these are the sums over all k of (5.23) in the cases $k' = 0$

and $k' = 1$. We must make sure that all these infinite sums give finite results. By putting $n = aj$ for $a > 1$ in (5.11), a value where the series in (5.11) converges for all $j \geq 1$, we see that $\sum_{k=0}^{\infty} p_k(j^2)(aj)^{-2k} < \infty$. This implies that $\sum_{k=0}^{\infty} p_{k,k-l} a^{-2k} < \infty$, where $p_{k,k-l}$ is the coefficient of j^{2k-2l} in $p_k(j^2)$. Since the coefficients in $[[n^{-1}]]$ in (5.20) are rational functions of z, a little analysis of (5.22) shows that this is sufficient to prove that the infinite sums that are the coefficients of $n^{-1/\nu+k'}$ for any $k' = 0, 1, 2, \ldots$ give finite results. For instance, the contributions to $n^{-1/\nu}$ and $n^{-1/\nu-1}$ can be found from

$$\begin{aligned} h(s) \equiv \sum_{k=0}^{\infty} s^{2k} p_{k,k} &= \lim_{n\to\infty} \left((a(j,n)^2 - a(j)^2) \frac{(4j^2-1)n^2}{(2j)^{3-\frac{1}{\nu}}} \right)_{j=ns} \\ &= \left(\left(\frac{\pi s}{\sin \pi s} \right)^{1+\frac{1}{\nu}} - 1 \right) \frac{1}{\pi^2 s^2} \end{aligned} \tag{5.24}$$

by

$$\sum_{k=0}^{\infty} 2^{-2k} p_{k,k} = h(1/2) \ , \quad \sum_{k=0}^{\infty} \frac{2^{-2k} p_{k,k}}{2+2k-1/\nu} = 2^{2-\frac{1}{\nu}} \int_0^{\frac{1}{2}} ds\, s^{1-\frac{1}{\nu}} h(s) \ . \tag{5.25}$$

The result is that the leading contributions for the case II are

$$\mathrm{II}: n^{-2}[[n^{-2}]] + \frac{2}{\pi^2 n^{\frac{1}{\nu}}} \left(\nu - \nu \left(\frac{\pi}{2}\right)^{\frac{1}{\nu}} \sqrt{\pi} \frac{\Gamma\left(1-\frac{1}{2\nu}\right)}{\Gamma\left(\frac{1}{2}-\frac{1}{2\nu}\right)} + \frac{1}{n} \left(\left(\frac{\pi}{2}\right)^{1+\frac{1}{\nu}} - 1 \right) + n^{-2}[[n^{-1}]] \right) \ . \tag{5.26}$$

Finally, a similar analysis can be done with the part III, involving the correction terms of $b(j)^2$ in (5.9), by expanding the generalised zeta functions in powers of j/n. Considerations similar to those above, with the fact that the expansion of the generalised zeta functions is valid for $|n| > |j|$, lead to the same structure for the large-n expansion as that of parts I and II, with an extra factor n^{-2k} for each correction term with $k = 1, 2, 3, \ldots$. This gives:

$$\mathrm{III}: n^{-4}[[n^{-2}]] + n^{-\frac{1}{\nu}-2}[[n^{-1}]] \ . \tag{5.27}$$

Putting everything together, along with the subtraction (5.12) specific to the case n even, we find

$$n\tilde{f}(n) = c_0 n + \frac{c_2}{n} + \frac{c_4}{n^3} + \ldots + n^{-\frac{1}{\nu}} \left(d_0 n + d_1 + \frac{d_2}{n} + \ldots \right) \tag{5.28}$$

where

$$\begin{aligned} c_0 &= \tilde{f}(\infty) \\ d_0 &= -\frac{\nu}{b(1/2)^2} \left(\frac{2}{\pi}\right)^{2-\frac{1}{\nu}} \sqrt{\pi} \frac{\Gamma\left(1-\frac{1}{2\nu}\right)}{\Gamma\left(\frac{1}{2}-\frac{1}{2\nu}\right)} \\ d_1 &= 0 \ . \end{aligned} \tag{5.29}$$

It is striking that although the intermediate steps of the analysis give terms $n^{k-\frac{1}{\nu}}$ for $k = 1, 0, -1, -2, \ldots$, the term with $k = 0$ identically vanishes for all ν. In fact, one can see that the same analysis for the case n odd, starting from the second form of (5.7), directly gives vanishing coefficients for $k = 0, -2, -4, \ldots$. For instance, changing $n \mapsto n-1$ into (5.18) erases the term $1/(2n)$ on the right-hand side. The other terms computed above are also unchanged in the odd case. Hence we conjecture that the large-n expansion valid from both n even and n odd is

$$n\tilde{f}(n) = c_0 n + \frac{c_2}{n} + \frac{c_4}{n^3} + \ldots + n^{-\frac{1}{\nu}} \left(d_0 n + \frac{d_2}{n} + \frac{d_4}{n^3} + \ldots \right) \tag{5.30}$$

Indeed a very precise numerical fit of $nf_{+-}(0,n)$ for several values of ν, is given by the function above. Let us commence by evaluating $n\tilde{f}(n)$ for integers $n > 1$ and several values of ν (see Fig. 1). Notice that $\nu = 1$ corresponds to the free Fermion point and, indeed for this value

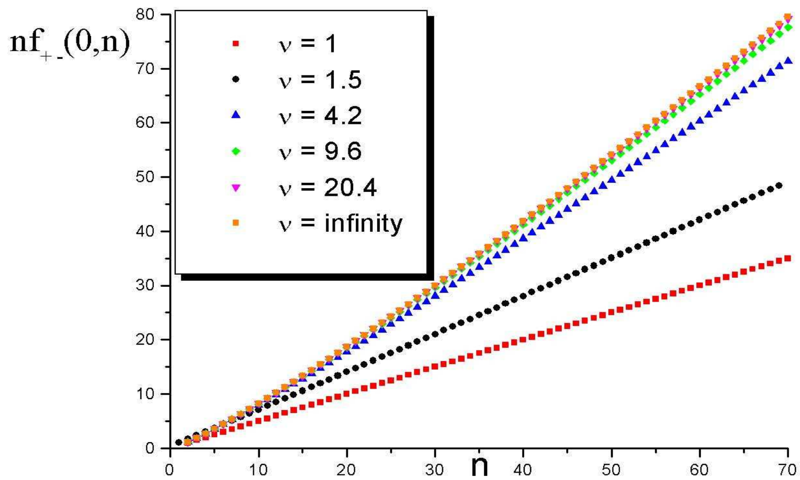

Figure 1: The function $nf_{+-}(0,n)$ computed numerically for several values of ν and $2 \leq n \leq 70$.

of ν we recover the result obtained analytically in [20] for the Ising model, namely $\tilde{f}(n) = 1/2$ for all values of n (in particular, also for $n = 1$). For other values of ν the large-n linear behaviour seems also apparent from Fig. 1 and is clearer the smaller the values of ν. In fact, for $\nu = 9.6$ and $\nu = 20.4$, the correct linear behaviour can only be seen at much larger values of n, because of the importance of the term in $n^{1-1/\nu}$.

Our numerical analysis has revealed that the numerical values of $nf_{+-}(0,n)$ (especially for small n) are best fitted by a function of the form:

$$F(\nu,n) = (1+n)c_0 + \tilde{c}_1 + \frac{\tilde{c}_2}{1+n} + d_0(1+n)^{1-\frac{1}{\nu}} + \frac{\tilde{d}_1}{(1+n)^{\frac{1}{\nu}}} + \frac{\tilde{d}_2}{(1+n)^{1+\frac{1}{\nu}}}, \tag{5.31}$$

This function has the same large-n behaviour as (5.28) but since the expansion (5.28) is only asymptotic its working is not guaranteed for small values of n (in particular at $n = 1$). Shifting $n \to n+1$ constitutes a re-summation that allows us to consider small values of n. The tables below show the values of all constants involved in (5.31) as well as the values of c_0 and d_0 as obtained by numerically evaluating the sum (5.2) and the function (5.29), respectively. The latter are in remarkably good agreement with the coefficients obtained from the fit. In addition, we give the value $F(\nu,1)$ which, for all ν considered, is compatible with the expected value $\tilde{f}(1) = 1/2$.

ν	$\tilde{f}(\infty)$	c_0 (fit)	d_0 (exact)	d_0 (fit)	$F(\nu,1)$
1.10	0.548110	0.548100	-0.089218	-0.078176	0.496148
1.50	0.711079	0.711023	-0.326980	-0.319507	0.488700
4.20	1.400983	1.400000	-1.049451	-1.043071	0.483927
9.60	2.418856	2.413256	-2.042131	-2.028276	0.485305
20.4	4.266412	4.264102	-3.866293	-3.858392	0.489379

ν	$\tilde{c}_1$	$\tilde{c}_2$	$\tilde{d}_1$	$\tilde{d}_2$
1.10	-0.564827	-0.224208	0.267706	0.066029
1.50	-0.745179	-0.189914	0.375405	0.231296
4.20	-1.613371	-0.470833	1.251032	0.567956
9.60	-3.397554	-3.514617	3.215461	3.431349
20.4	-5.850448	-9.119535	5.615018	9.110232

Concerning the values on the last table we see that, for example, the sum $c_0+\tilde{c}_1$ which we expect to be vanishing from the large-n expansion, indeed gives relatively small numbers compared to c_0 and $\tilde{c}_1$, especially for small ν. This is consistent with our working precision which is considerably reduced for sub-leading terms. A plot of the function $F(n, 1.5)$ as well as of the numerical values of $nf_{+-}(0,n)$ for $n = 2, 3, ..., 70$ and $\nu = 1.5$ from (5.7) can be seen in Fig. 2. The difference $|nf_{+-}(0,n) - F(n, 1.5)| < 10^{-4}$ for all $n = 2, 3, \ldots, 70$. In fact, this holds for all finite values of ν considered here.

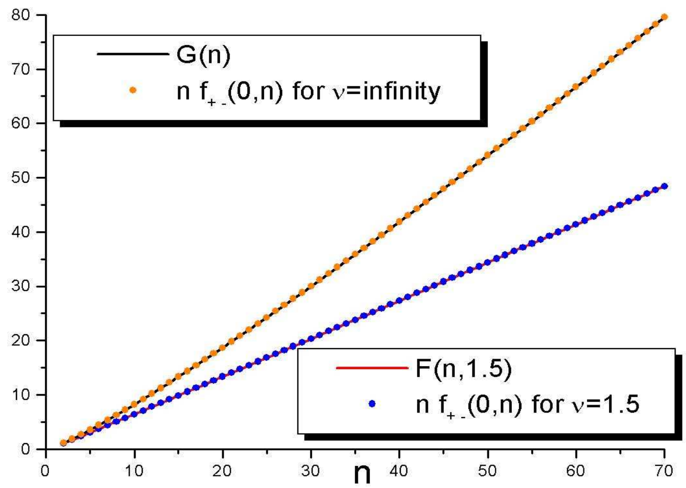

Figure 2: The functions $F(n, 1.5)$ and $G(n)$ and the corresponding values of $nf_{+-}(0,n)$ for $2 \leq n \leq 70$.

As we can see from the expansion, the term $n^{1-1/\nu}$ will give a very important contribution for ν large, so that $n\tilde{f}(n)$ should become linear in n very quickly for ν close to its minimum value 1 and very slowly for large values of ν. In fact, in the limit $\nu \to \infty$, a similar analysis as carried out above can be performed (although some complications appear). Both $a(j,n)$ and $b(j,n)$ have well-defined $\nu = \infty$ limit which commutes with the $n = \infty$ limit, giving

$$a(j)^2_{\nu=\infty} = \frac{4}{\pi^2}\frac{8j^3}{(1-4j^2)^2} \ , \quad b(j)^2_{\nu=\infty} = \frac{\Gamma\left(\frac{3}{2}+j\right)^2}{j^3\Gamma(j)^2} \ . \tag{5.32}$$

The leading term at large n, then, can be obtained from

$$\sum_{j=1}^{n/2} a(j)^2_{\nu=\infty} b(j)^2_{\nu=\infty} \sim \sum_{j=1}^{n/2} \frac{2}{\pi^2 j} = \frac{2}{\pi^2}\left(\Psi\left(1+\frac{n}{2}\right)+\gamma\right) \tag{5.33}$$

where γ is Euler's number and $\Psi(z)$ is the logarithmic derivative of Euler's gamma function $\Psi(z) = \Gamma(z)'/\Gamma(z)$. For n large the function $\Psi\left(1 + n/2\right)$ behaves as

$$\Psi\left(1+\frac{n}{2}\right) = \log\left(\frac{n}{2}\right) + O\left(\frac{1}{n}\right) \ , \tag{5.34}$$

so that the large-n expansion at $\nu = \infty$ starts with

$$\lim_{\nu\to\infty} n\tilde{f}(n) \sim \frac{1}{2\pi} n \log n \tag{5.35}$$

(where we used $b(1/2)^2_{\nu=\infty} = 8/\pi$). It is quite interesting to note that this logarithmic behaviour can also be obtained from the expansion (5.30) by simply taking the limit $\nu \to \infty$. Both c_0 and d_0 are divergent proportionally to ν, and these divergencies cancel out. The expansion in $1/\nu$ of $n^{1-1/\nu}$ then gives the correct logarithmic term. A more precise analysis of c_0 follows from

$$\sum_{j=1}^{\infty} a(j)^2 b(j)^2 \overset{\nu\to\infty}{=} \sum_{j=1}^{\infty}\left(a(j)^2 b(j)^2 - \frac{2}{\pi^2 j}\right) + \frac{2^{1-\frac{1}{\nu}}}{\pi^2}\zeta\left(1+\frac{1}{\nu}\right) + o(1) \tag{5.36}$$

where the zeta function contains the linear divergence at large ν. The first term can be evaluated exactly:

$$\begin{aligned}
\sum_{j=1}^{\infty}\left(a(j)^2 b(j)^2 - \frac{2}{\pi^2 j}\right) &= \frac{2}{\pi^2}\sum_{j=1}^{\infty}\left(\frac{\Gamma\left(j-\frac{1}{2}\right)^2}{\Gamma(j)^2} - \frac{1}{j}\right) \\
&= \frac{2}{\pi^2}\lim_{x\to 1}\left(2xK(x) + \log(1-x)\right) \\
&= -\frac{4}{\pi^2}\left(\Psi\left(\frac{1}{2}\right) + \gamma\right)
\end{aligned} \tag{5.37}$$

(where $K(x)$ is the complete elliptic integral of the first kind). From this, the $\nu = \infty$ limit of (5.30) gives the conjecture

$$\lim_{\nu\to\infty} 2\pi n\tilde{f}(n) = \left(n + \frac{e_2}{n} + \frac{e_4}{n^3} + \ldots\right)\log n + (\gamma + \log 32 - \log\pi)n + \frac{f_2}{n} + \frac{f_4}{n^3} + \ldots \tag{5.38}$$

We will now fit the values of $nf_{+-}(0,n)$ obtained numerically to a function of the type:

$$G(n) = e_0(1+n)\log(n+1) + \tilde{e}_1\log(n+1) + \frac{\tilde{e}_2\log(n+1)}{1+n} + f_0(n+1) + \tilde{f}_1 + \frac{\tilde{f}_2}{n+1}. \tag{5.39}$$

Similarly as $F(\nu,n)$, the function $G(n)$ has a large n expansion of the type (5.38). The coefficients e_0 and f_0 from the fit, as well as their exact values from (5.38) are given in the following table

e_0 (exact)	e_0 (fit)	f_0 (exact)	f_0 (fit)	$\tilde{e}_1$	$\tilde{e}_2$	$\tilde{f}_1$	$\tilde{f}_2$
0.159155	0.158971	0.461266	0.462474	-0.183370	-0.124718	-0.615249	0.248293

so that $G(1) = 0.4838988$. Fig. 2 shows the function $G(n)$ as well as the numerical values of $nf_{+-}(0,n)$ from (5.7). The agreement between the two sets of values is very good. More precisely $|nf_{+-}(0,n) - G(n)| < 10^{-4}$ for all $n = 2,\ldots,70$.

5.1 Discussion of the large-n results

We notice from the graphs and from the fitting functions that the value of $1/2$ at $n = 1$ is perfectly in agreement with smooth and *convex* analytic continuations agreeing with the given values at $n = 2, 3, \ldots$, in all cases analysed. Also, the large-n expansion makes it clear that no oscillatory terms appear. We also remark that the leading large-n behaviour is linear in all super-renormalisable cases $\nu < \infty$, and is proportional to $n \log n$ in the marginally renormalisable case $\nu = \infty$ (this is still a non-conformal case, where a scale appears by dimensional transmutation). This seems to suggest that the large-n behaviour is related to the type of perturbing field, and we recall that logarithmic functions (of the cutoff) usually appear with naively marginal perturbations. Moreover, the full large-n expansion in the finite-ν case seems to encode some characteristics of the perturbing field, as it also has a part with non-integer powers of n (although these powers are not in linear relation with the dimension of the field). Also striking and pointing towards a perturbation-theory origin of the large-n behaviour is the fact that the limit $\nu \to \infty$ can be taken directly in this large-n expansion, although it is expected to be only an asymptotic expansion, and the limit $\nu \to \infty$ *cannot* be taken in the various intermediate steps separately. We recall that a good regularisation scheme of the sine-Gordon model at $\nu = \infty$ is a "non-perturbative" dimensional regularisation, whereby ν is taken finite (thus changing the dimension of the perturbing field) and sent to infinity at the end of the calculation (see for instance [34]). This leads to logarithmic functions in a similar way to that by which $\log n$ appeared here.

6 Conclusions and outlook

This paper provides an extension and generalisation of the work [20] to integrable QFTs with backscattering. The main conclusion following from [20] and the present work is that the form of the leading correction to the bi-partite entropy at large distances of any integrable QFT is "universal" in the sense that it does not depend on the precise form of the scattering matrix (in particular, whether or not it is diagonal) but only on the mass spectrum of the model.

The argument leading to this result, developed in [20], uses the well-known replica trick, which involves the partition function on Riemann surfaces with two branch points and n sheets, related to the correlation function of twist fields. We obtained the first large-distance (distance between the branch points) correction to this partition function in the sine-Gordon model. This correction presents an interesting large-n asymptotic, which seems to be in relation with the type of perturbing field giving the massive theory. It would be very interesting to find a satisfying explanation for this large-n behaviour, in particular in the super-renormalisable case, where the leading behaviour is just linear.

From the replica trick, the main subtlety in obtaining the entanglement entropy is the analytic continuation in n, the number of Riemann sheets. In the present paper we attempted to provide support for the results using the geometrical picture of conical singularities with angle $2\pi n$ instead of branch points, and appealing to some convexity properties as function of n. However, a more precise statement about the uniqueness of the analytic continuation we obtained is still missing.

It would be interesting to investigate higher order corrections to the entropy, that is, higher particle form factor contributions to the correlation function (2.14) and, in particular, determine whether or not they are of a similar "universal" nature as the leading correction already obtained. This would involve the computation of higher particle form factors of the twist fields. Two-particle form factors were obtained here by solving a set of form factor consistency equations, but they could also be obtained using Lukyanov's method of angular quantisation [35], as was done in the sinh-Gordon model in [20]. This may be quite helpful for higher particle numbers.

Also, investigation in other models may help decipher the properties of form factors of twist fields, for low-particle numbers as well. The geometrical meaning of the field may provide simplifications. For instance, can there be non-zero one-particle form factors?

Another outstanding point is the generalisation of the approach employed here to the computation of the entanglement entropy of quantum systems consisting of multiply disconnected regions. Such an extension could provide an alternative way to proving certain natural general properties of the entanglement entropy shown to be valid in conformal field theory [17] and recently discussed in a more general framework in [36].

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