MIT-CTP-2897

Z-D Brane Box Models and Non-Chiral Dihedral Quivers.

Bo Feng, Amihay Hanany and Yang-Hui He
fengb, hanany, yhe@ctp.mit.edu[a]

Center for Theoretical Physics,
Massachusetts Institute of Technology
Cambridge, Massachusetts 02139, U.S.A.

Generalising ideas of an earlier work [16], we address the problem of constructing Brane Box Models of what we call the Z-D Type from a new point of view, so as to establish the complete correspondence between these brane setups and orbifold singularities of the non-Abelian G generated by Z_k and D_d under certain group-theoretic constraints to which we refer as the BBM conditions. Moreover, we present a new class of $\mathcal{N}=1$ quiver theories of the ordinary dihedral group d_k as well as the ordinary exceptionals $E_{6,7,8}$ which have non-chiral matter content and discuss issues related to brane setups thereof.

1 Introduction

Configurations of branes [1] have been proven to be a very useful method to study the gauge field theory which emerges as the low energy limit of string theory (for a complete reference, see Giveon and Kutasov [2]). The advantage of such setups is that they provide an intuitive picture so that we can very easily deduce many properties of the gauge theory. For example, brane setups have been used to study mirror symmetry in 3 dimensions [1,3,4,5,6], Seiberg Duality in 4 dimensions [7], and exact solutions when lifting Type IIA setups to M-theory [8,9]. After proper T- or S-dualities, we can transform the above brane setups to D-brane as probes on some target space with orbifold singularities [10,11,12].

For example, the brane setup of streching Type IIA D4-branes between $n+1$ NS5-branes placed in a circular fashion (the "elliptic model" [8]) is precisely T-dual to D3-branes stacked upon ALE singularities of the type $\widehat{A_n}$, or in other words orbifold singularities of the form $\mathbb{C}^2/Z_{n+1}$, where Z_{n+1} is the cyclic group on $n+1$ elements and is a finite discrete subgroup of $SU(2)$. As another example, the Brane Box Model [13,14,15] is T-dual to D3-branes as probes on orbifold singularities of the type $\mathbb{C}^3/\Gamma$ with $\Gamma = Z_k$ or $Z_k \times Z_{k'}$ now being a finite discrete subgroup of $SU(3)$ [14]. A brief review of some of these contemporary techniques can be found in our recent work [16]. In fact, it is a very interesting problem to see how in general the two different methods, viz., brane setups and D3-branes as probes on geometrical singularities, are connected to each other by proper duality transformations [17].

The general construction and methodology for D3-branes as probes on orbifold singularities has been given [12]. However, the complete list of the corresponding brane setups is not yet fully known. For orbifolds $\mathbb{C}^2/\{\Gamma \in SU(2)\}$, we have the answer for the $\widehat{A_n}$ series (i.e., $\Gamma = Z_{n+1}$) and the $\widehat{D_n}$ series (i.e., $\Gamma = D_{n-2}$, the binary dihedral groups) [6], but not for the exceptional cases $\widehat{E_{6,7,8}}$. At higher dimensions, the situation is even more disappointing: for orbifolds of $\mathbb{C}^3/\{\Gamma \in SU(3)\}$, brane setups are until recently limited to only Abelian singularities, namely $\Gamma = Z_k$ or $Z_k \times Z_{k'}$ [14].

In a previous paper [16], we went beyond the Abelian restriction in three dimensions and gave a new result concerning the correspondence of the two methods. Indeed we showed that[b] for $\Gamma := G = Z_k * D_{k'}$ a finite

[a] Research supported in part by the CTP and the LNS of MIT and the U.S. Department of Energy under cooperative research agreement #DE-FC02-94ER40818; YHH is also supported by the NSF Graduate Fellowship.

[b] In that paper we used the notation $Z_k \times D_{k'}$ and pointed out that the symbol $\times$ was really an abuse. We shall here use

discrete subgroup of $SU(3)$, the corresponding brane setup (a Brane Box Model) T-dual to the orbifold discription can be obtained. More explicitly, the group $G \in SU(3)$ is defined as being generated by the following matrices that act on $\mathbb{C}^3$:

$$\alpha = \begin{pmatrix} \omega_k & 0 & 0 \\ 0 & \omega_k^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \beta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega_{2k'} & 0 \\ 0 & 0 & \omega_{2k'}^{-1} \end{pmatrix} \qquad \gamma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{pmatrix} \tag{1.1}$$

where $\omega_n := e^{\frac{2\pi i}{n}}$, the nth root of unity.

The abstract presentation of the groups is as follows:

$$\alpha\beta = \beta\alpha, \quad \beta\gamma = \gamma\beta^{-1}, \quad \alpha^m\gamma\alpha^n\gamma = \gamma\alpha^n\gamma\alpha^m \quad \forall m, n \in \mathbb{Z} \tag{1.2}$$

Because of the non-Abelian property of G, the preliminary attempts at the corresponding Brane Box Model by using the idea in a previous work [18] met great difficulty. However, via careful analysis, we found that the group G can be written as the semidirect product of Z_k and $D_{\frac{kk'}{\gcd(k,2k')}}$. Furthermore, when $\frac{2k'}{\gcd(k,2k')} =$ even, the character table of G as the semidirect product $Z_k \rtimes D_{\frac{kk'}{\gcd(k,2k')}}$ preserves the structure of that of $D_{\frac{kk'}{\gcd(k,2k')}}$, in the sense that it seems to be composed of k copies of the latter. Indeed, it was noted [16] that only under this parity condition of $\frac{2k'}{\gcd(k,2k')} =$ even, can we construct, with the two group factors Z_k and $D_{\frac{kk'}{\gcd(k,2k')}}$, a consistent Brane Box Model with the ideas in the abovementioned paper [18].

The success of the above construction, constrained by certain conditions, hints that something fundamental is acting as a key rôle in the construction of non-Abelian brane setups above two (complex) dimensions. By careful consideration, it seems that the following three conditions presented themselves to be crucial in the study of $Z_k * D_{k'}$ which we here summarize:

1. The whole group G can be written as a semidirect product: $Z_k \rtimes D_d$;

2. The semidirect product of G preserves the structure of the irreducible representations of D_d, i.e., it appears that the irreps of G consist of k copies of those of the subgroup D_d;

3. There exists a (possibly reducible) representation of G in 3 dimensions such that the representation matrices belong to $SU(3)$. Henceforth, we shall call such a representation, consistent with the $SU(3)$ requirement (see more discussions [16,19] on decompositions), as "*the chosen decomposition of* **3**".

We will show in this paper that these conditions are sufficient for constructing Brane Box Model of the Z-D type. Here we will call the Brane Box Model in our recent paper [16] as **Type** Z**-**D and similarly, that in earlier works [13,14] we shall call the Z-Z Type. We shall see this definition more clearly in subsection §2.3. It is amusing to notice that Brane Box Models of Type Z-Z also satisfy the above three conditions since they correspond to the group $Z_k \times Z_{k'}$, which is a direct product.

Furthermore, we shall answer a mysterious question posed at the end of our earlier work [16]. In that paper, we discussed the so-called *Inverse Problem*, i.e., given a consistent Brane Box Model, how may one determine, from the structure of the setup (the number and the positioning of the branes), the corresponding group Γ in the orbifold structure of $\mathbb{C}^3/\Gamma$. We found there that only when k is the divisor of d can we find the corresponding group defined in (1.1) with proper k, k'. This was very unsatisfying. However, the structure of the Brane Box Model of Type Z-D was highly suggestive of the solution for general k, d. We shall here mend that short-coming and for arbitrary k, d we shall construct the corresponding group Γ which satisfies

the symbol $*$ and throughout the paper reserve $\times$ to mean strict direct product of groups and $\rtimes$, the semi-direct product.

above three conditions. With this result, we establish the complete correspondence between the Brane Box Model of Type Z-D and D3-branes as probes on orbifold singularities of $\mathbb{C}^3/\Gamma$ with properly determined Γ.

The three conditions which are used for solving the inverse problem can be divided into two conceptual levels. The first two are at the level of pure mathematics, i.e., we can consider it from the point of view of abstract group theory without reference to representations or to finite discrete subgroup of $SU(n)$. The third condition is at the level of physical applications. From the general structure [12] we see that for constructing $\mathcal{N} = 2$ or $\mathcal{N} = 1$ theories we respectively need the group Γ to be a finite subgroup of $SU(2)$ or $SU(3)$. This requirement subsequently means that we can find a faithful (but possibly reducible) 2-dimensional or 3-dimensional representation with the matrices satisfying the determinant 1 and unitarity conditions. In other words, what supersymmetry ($\mathcal{N} = 2$ or 1) we will have in the orbifold theory by the standard procedure [12] depends only on the chosen representation (i.e., the decomposition of **2** or **3**). Such distinctions were already shown before [14,19]. The group Z_3 had been considered [14]. If we choose its action on $\mathbb{C}^3$ as $(z_1, z_2, z_3) \longrightarrow (e^{\frac{2\pi i}{3}} z_1, e^{\frac{-2\pi i}{3}} z_2, z_3)$ we will have $\mathcal{N} = 2$ supersymmetry, but if we choose the action to be $(z_1, z_2, z_3) \longrightarrow (e^{\frac{2\pi i}{3}} z_1, e^{\frac{2\pi i}{3}} z_2, e^{\frac{2\pi i}{3}} z_3)$ we have only $\mathcal{N} = 1$. This phenomenon mathematically corresponds to what are called sets of transitivity of collineation group actions [20].

Moreover, we notice that the ordinary dihedral group d_k which is excluded from the classification of finite subgroup of $SU(2)$ can be imbedded[c] into $SU(3)$. Therefore we expect that d_k should be useful in constructing some $\mathcal{N} = 1$ gauge field theories by the standard procedures [12,19]. We show in this paper that this is so. With the proper decompositions, we can obtain new types of gauge theories by choosing $\mathbb{C}^3$ orbifolds to be of the type d_k. For completeness, we also give the quiver diagrams of ordinary tetrahedral, octahedral and icosahedral groups ($E_{6,7,8}$), which by a similar token, can be imbedded into $SU(3)$.

The organisation of the paper is as follows. In §2 we give a simple and illustrative example of constructing a Brane Box Model for the direct product $Z_k \times D_{k'}$, whereby initiating the study of brane setups of what we call Type Z-D. In §3 we deal with the twisted case which we encountered earlier [16]. We show that we can imbed the latter into the direct product (untwisted) case of §2 and arrive at another member of Brane Box Models of the Z-D type. In §4 we give a new class of $SU(3)$ quiver which are connected to the ordinary dihedral group d_k. Also, we give an interesting brane configuration that will give matter matter content as the $d_{k=\text{even}}$ quiver but a different superpotential on the gauge theory level. Finally in §5 we give concluding remarks and suggest future prospects.

Nomenclature

Unless otherwise specified, we shall throughout the paper adhere to the notation that the group binary operator $\times$ refers to the strict direct product, $\rtimes$, the semi-direct product, and $*$, a general (twisted) product by combining the generators of the two operands[d]. Furthermore, ω_n is defined to be $e^{\frac{2\pi i}{n}}$, the nth root of unity; $H \triangleleft G$ mean that H is a normal subgroup of G; and a group generated by the set $\{x_i\}$ under relations $f_i(\{x_j\}) = 1$ is denoted as $\langle x_i | f_j \rangle$.

2 A Simple Example: The Direct Product $Z_k \times D_{k'}$

We recall that in a preceeding work [16], we constructed the Brane Box Model (BBM) for the group $Z_k * D_{k'}$ as generated by (1.1), satifying the three conditions mentioned above, which we shall dub as the **BBM**

[c] Since it is in fact a subgroup of $SU(2)/\mathbb{Z}_2 \cong SO(3)$, the embedding is naturally induced from $SO(3) \hookrightarrow SU(3)$. In fact the 3-dimensional representation in $SU(3)$ is faithful.

[d] Therefore in the previous paper [16], the group $G := Z_k \times D_{k'}$ in this convention should be written as $Z_k * D_{k'}$, *q.v. Ibid.* for discussions on how these different group compositions affect brane constructions.

condition for groups. However, as we argued in the introduction, there may exist in general, groups not isomorphic to the one addressed [16] but still obey these conditions. As an illustrative example, we start with the simplest member of the family of $Z * D$ groups that satisfies the BBM condition, namely the direct product $G = Z_k \times D_{k'}$. We define α as the generator for the Z_k factor and γ, β, those for the $D_{k'}$. Of course by definition of the direct product α must commute with both β and γ. The presentation of the group is clearly as follows:

$$\begin{array}{lll} \alpha^k = 1; & & \text{The Cyclic Group } Z_k \\ \beta^{2k'} = 1, \quad \beta^{k'} = \gamma^2, & \beta\gamma = \gamma\beta^{-1}; & \text{The Binary Dihedral Group } D_{k'} \\ \alpha\beta = \beta\alpha, \quad \alpha\gamma = \gamma\alpha & & \text{Mutual commutation} \end{array}$$

We see that the first two of the BBM conditions are trivially satisfied. To satisfy the third, we need a 3-dimensional matrix represenation of the group. More explicitly, as discussed [16], to construct the BBM of the Z-D type, one needs the decomposition of **3** into one nontrivial 1-dimensional irrep and one 2-dimensional irrep. In light of this, we can write down the $SU(3)$ matrix generators of the group as

$$\alpha = \begin{pmatrix} \omega_k^2 & 0 & 0 \\ 0 & \omega_k^{-1} & 0 \\ 0 & 0 & \omega_k^{-1} \end{pmatrix} \qquad \beta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega_{2k'} & 0 \\ 0 & 0 & \omega_{2k'}^{-1} \end{pmatrix} \qquad \gamma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{pmatrix} \tag{2.3}$$

Here, we notice a subtle point. When $k =$ even, $\alpha^{\frac{k}{2}}$ and $\beta^{k'}$ give the same matrix form. In other words, (2.3) generates a *non-faithful* representation. We will come back to this problem later, but before diving into a detailed discussion on the whole group $Z_k \times D_{k'}$, let us first give the necessary properties of the factor $D_{k'}$.

2.1 The Group $D_{k'}$

One can easily check that all the elements of the binary dihedral $D_{k'} = \langle \beta, \gamma \rangle$ group can be written, because $\gamma^2 = \beta^{k'}$, as

$$\gamma^n \beta^p, \quad \text{with} \quad n = 0, 1 \;\; p = 0, 1, ..., 2k' - 1.$$

From this constraint and the conjugation relation

$$(\gamma^{n_1}\beta^{p_1})^{-1}(\gamma^n\beta^p)(\gamma^{n_1}\beta^{p_1}) = \gamma^n \beta^{p_1(1-(-1)^n)+(-1)^{n_1}p},$$

we can see that the group is of order $4k'$ and moreover affords 4 1-dimensional irreps and $(k'-1)$ 2-dimensional irreps. The classes of the group are:

	$C_{n=0}^{p=0}$	$C_{n=0}^{p=k'}$	$C_{n=0}^{\pm p}$	$C_{n=1}^{p \bmod 2}$
$\lvert C\rvert$	1	1	2	k'
$\#C$	1	1	$k'-1$	2

To study the character theory of $G := D_{k'}$, we recognise that $H := \{\beta^p\}$ for p even is a normal subgroup of G. Whence we can use the Frobenius-Clifford theory of induced characters to obtain the irreps of G from the factor group $\widetilde{G} := G/H = 1, \beta, \gamma, \gamma\beta$. For k' even, $\widetilde{G}$ is $Z_2 \times Z_2$ and for k' odd, it is simply Z_4. these then furnish the 1-dimensional irreps. We summarise the characters of these 4 one dimensionals as follows:

	$k' =$ even				$k' =$ odd			
	$\beta^{p=\text{even}}$	$\beta(\beta^{\text{odd}})$	$\gamma(\gamma\beta^{\text{even}})$	$\gamma\beta(\gamma\beta^{\text{odd}})$	β^{even}	$\beta(\beta^{\text{odd}})$	$\gamma(\gamma\beta^{\text{even}})$	$\gamma\beta(\gamma\beta^{\text{odd}})$
χ^1	1	1	1	1	1	1	1	1
χ^2	1	-1	1	-1	1	-1	ω_4	$-\omega_4$
χ^3	1	1	-1	-1	1	1	-1	-1
χ^4	1	-1	-1	1	1	-1	$-\omega_4$	ω_4

The 2-dimensional irreps can be directly obtained from the definitions; they are indexed by a single integer l:

$$\chi_2^l(C_{n=1}) = 0, \quad \chi_2^l(C_{n=0}^p) = (\omega_{2k'}^{lp} + \omega_{2k'}^{-lp}), \quad l = 1, .., k'-1. \tag{2.4}$$

The matrix representations of these 2-dimensionals are given below:

$$\beta^p = \begin{pmatrix} \omega_{2k'}^{lp} & 0 \\ 0 & \omega_{2k'}^{-lp} \end{pmatrix} \qquad \gamma\beta^p = \begin{pmatrix} 0 & i^l\omega_{2k'}^{-lp} \\ i^l\omega_{2k'}^{lp} & 0 \end{pmatrix}$$

From (2.4) we immediately see that $\chi_2^l = \chi_2^{-l} = \chi_2^{2k'-l}$ which we use to restrict the index l in χ_2^l into the region $[1, k'-1]$.

Now for the purposes of the construction of the BBM, we aboveall need to know the tensor decompositions of the group; these we summarise below.

$\mathbf{1}\otimes\mathbf{1'}$	$k'=$ even $\chi_1^2\chi_1^2=\chi_1^1 \quad \chi_1^3\chi_1^3=\chi_1^1 \quad \chi_1^4\chi_1^4=\chi_1^1$ $\chi_1^2\chi_1^3=\chi_1^4 \quad \chi_1^2\chi_1^4=\chi_1^3 \quad \chi_1^3\chi_1^4=\chi_1^2$	$k'=$ odd $\chi_1^2\chi_1^2=\chi_1^3 \quad \chi_1^3\chi_1^3=\chi_1^1 \quad \chi_1^4\chi_1^4=\chi_1^3$ $\chi_1^2\chi_1^3=\chi_1^4 \quad \chi_1^2\chi_1^4=\chi_1^1 \quad \chi_1^3\chi_1^4=\chi_1^2$
$\mathbf{1}\otimes\mathbf{2}$	$\chi_1^h\chi_2^l = \begin{cases} \chi_2^l & h=1,3 \\ \chi_2^{k'-l} & h=2,4 \end{cases}$	
$\mathbf{2}\otimes\mathbf{2'}$	$\chi_2^{l_1}\chi_2^{l_2} = \chi_2^{(l_1+l_2)} + \chi_2^{(l_1-l_2)}$ where $\chi_2^{(l_1+l_2)} = \begin{cases} \chi_2^{(l_1+l_2)} & \text{if} \quad l_1+l_2<k', \\ \chi_2^{2k'-(l_1+l_2)} & \text{if} \quad l_1+l_2>k', \\ \chi_1^2+\chi_1^4 & \text{if} \quad l_1+l_2=k'. \end{cases}$ $\chi_2^{(l_1-l_2)} = \begin{cases} \chi_2^{(l_1-l_2)} & \text{if} \quad l_1>l_2, \\ \chi_2^{(l_2-l_1)} & \text{if} \quad l_1<l_2, \\ \chi_1^1+\chi_1^3 & \text{if} \quad l_1=l_2. \end{cases}$	

2.2 The Quiver Diagram

The general method of constructing gauge field theories from orbifold singularities of $C^3/\Gamma \subset SU(3)$ has been given [12,19]. Let us first review briefly the essential results. Given a finite discrete subgroup $\Gamma \subset SU(3)$ with irreducible representations $\{r_i\}$, we obtain, under the orbifold projection, an $\mathcal{N}=1$ super Yang-Mills theory with gauge group

$$\bigotimes_i SU(N|r_i|), \qquad |r_i| = \dim(r_i), N \in \mathbb{Z}$$

To determine the matter content we need to choose the decomposition of $\mathbf{3}$ (i.e., the 3×3 matrix form) of Γ which describes how it acts upon $\mathbb{C}^3$. We use R to denote the representation of chosen $\mathbf{3}$ and calculate the tensor decomposition

$$R \otimes r_i = \bigoplus_j a_{ij}^R r_j \tag{2.5}$$

The matrix a_{ij}^R then tells us how many bifundamental chiral multiplets of $SU(N_i)\times SU(N_j)$ there are which transform under the representation $(N_i, \bar{N}_j)$, where $N_i := N|r_i|$. Furthermore, knowing this matter content we can also write down the superpotential whose explicit form is given in (2.7) and (2.8) of Lawrence, Nekrasov and Vafa [12]. We do not need the detailed form thereof but we emphasize that all terms in the superpotential are cubic and there are no quatic term. This condition is necessary for finiteness [15,12] and we will turn to this fact later.

We can encode the above information into a "**quiver diagram**". Every node i with index $\dim r_i$ in the quiver denotes the gauge group $SU(N_i)$. Then we connect a_{ij}^R arrows from node i to j in order to denote the correpsonding bifundamental chiral multiplet $(N_i, \bar{N}_j)$. When we say that a BBM construction is **consistent** we mean that it gives the same quiver diagram as one obtains from the geometrical probe methods [12].

Now going back to our example $Z_k \times D_{k'}$, its character table is easily written: it is simply the Kronecker product of the character tables of Z_k and $D_{k'}$ (as matrices). We promote (2.4) to a double index

$$(a, \chi_i^l)$$

to denote the charaters, where $a = 0, ..., k-1$ and are characters of Z_k (which are simply various kth roots of unity) and χ are the characters of $D_{k'}$ as presented in the previous subsection. We recall that $i = 1$ or 2 and for the former, there are 4 1-dimensional irreps indexed by $l = 1, .., 4$; and for the latter, there are $k'-1$ 2-dimensional irreps indexed by $l = 1, .., k'-1$. It is not difficult to see from (2.3) that the chosen decomposition should be:

$$\mathbf{3} \longrightarrow (2, \chi_1^1) \oplus (-1, \chi_2^1)$$

The relevant tensor decomposition which gives the quiver is then

$$[(2, \chi_1^1) \oplus (-1, \chi_2^1)] \otimes (a, \chi_i^l) = (a+2, \chi_i^l) \oplus (a-1, \chi_i^l \otimes \chi_2^1), \tag{2.6}$$

which is thus reduced to the decompositions as tabulated in the previous subsection.

2.3 The Brane Box Model of $Z_k \times D_{k'}$

Now we can use the standard methodology [14,16,18] to construct the BBM. The general idea is that for the BBM corresponding to the singularity $\mathbb{C}^3/\Gamma$, we put D-branes whose number is determined by the irreps of Γ into proper grids in Brane Boxes constructed out of NS5-branes. Then the genetal rule of the resulting BBM is that we have gauge group $SU(N_i)$ in every grid and bifundamental chiral multiplets going along the North, East and SouthWest directions. The superpotential can also be read by closing upper or lower triangles in the grids [14]. The quiver diagram is also readily readable from the structure of the BBM (the number and the positions of the branes).

Indeed, in comparison with geometrical methods, because the two quivers (the orbifold quiver and the BBM quiver) seem to arise from two vastly disparate contexts, they need not match a priori. However, by judicious choice of irreps in each grid we can make these two quiver exactly the same; this is what is meant by the **equivalence** between the BBM and orbifold methods. The consistency condition we impose on the BBM for such equivalence is

$$\mathbf{3} \otimes r_i = \bigoplus_{j \in \{\text{North,East,SouthWest}\}} r_j. \tag{2.7}$$

Of course we observe this to be precisely (2.5) in a different guise.

Now we return to our toy group $Z_k \times D_{k'}$. The grids are furnished by a parallel set of k' NS5-branes with 2 ON^0 planes intersected by k (or $\frac{k}{2}$ when k is even; see explanation below) NS5$'$-branes perpendicular thereto and periodically identified such that $k(\text{or } \frac{k}{2}) \equiv 0$ as before [16]. This is shown in Figure 1. The general brane setup of this form involving 2 sets of NS5-branes and 2 ON-planes we shall call, as mentioned in the introduction, the BBM of **the Z-D Type**.

The irreps are placed in the grids as follows. First we consider the leftmost column. We place a pair of irreps $\{(a, \chi_1^1), (a, \chi_1^3)\}$ at the bottom (here a is some constant initial index), then for each incremental grid going up we increase the index a by 2. Now we notice the fact that when k is odd, such an indexing makes

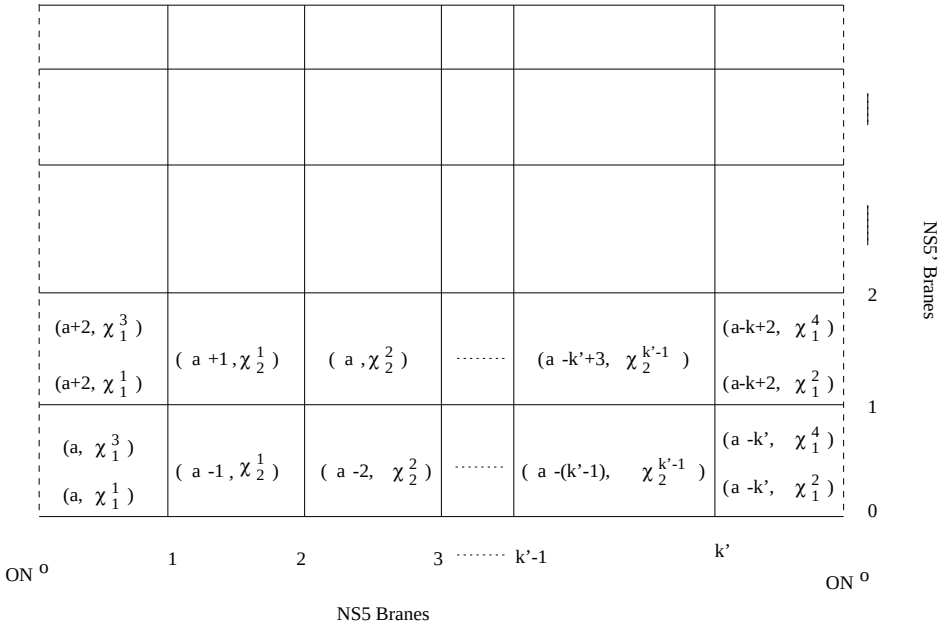

Figure 1: The Brane Box Model for $Z_k \times D_{k'}$. Notice that for every step along the vertical direction from the bottom to top, the first index has increment 2, while along the horizontal direction from left to right, the first index has decrement 1 and the second index, increment 1. The vertical direction is also periodically identified so that k(or $\frac{k}{2}$) $\equiv 0$.

one return to the bottom grid after k steps whereas if k is even, it suffices to only make $\frac{k}{2}$ steps before one returns. This means that when k is odd, the periodicity of a is precisely the same as that required by our circular identification of the NS5′-branes. However, when k is even it seems that we can not put all irreps into a single BBM. We can circumvent the problem by dividing the irreps (a, χ) into 2 classes depending on the parity of a, each of which gives a BBM consisting of $\frac{k}{2}$ NS5′-branes. We should not be surprised at this phenomenon. As we mentioned at the beginning of this section, the matrices (2.3) generate a non-faithful representation of the group when k is even (i.e., $\alpha^{\frac{k}{2}}$ gives the same matrix as $\beta^{k'}$). This non-faithful decomposition of **3** is what is responsible for breaking the BBM into 2 disjunct parts.

The same phenomenon appears in the $Z_k \times Z_{k'}$ BBM as well. For k even, if we choose the decomposition as $\mathbf{3} \longrightarrow (1,0) + (0,1) + (-1,-1)$ we can put all irreps into kk' grids, however if we choose $\mathbf{3} \longrightarrow (2,0) + (0,1) + (-2,-1)$ we can only construct two BBM's each with $\frac{kk'}{2}$ grids and consisting of one half of the total irreps. Indeed this a general phenomenon which we shall use later:

PROPOSITION 2.1 *Non-faithful matrix representations of* Γ *give rise to corresponding Quiver Graphs which are disconnected.*

Having clarified this subtlety we continue to construct the BBM. We have fixed the content for the leftmost column. Now we turn to the bottom row. Starting from the second column (counting from the left side) we place the irreps $(a-1, \chi_2^1), (a-2, \chi_2^2), ..., (a-(k'-1), \chi_2^{k'-1})$ until we reach the right side (i.e., $(a-j, \chi_2^j)$ with $j = 1, ...k'-1$) just prior to the rightmost column; there we place the pair $\{(a-k', \chi_1^2), (a-k', \chi_1^4)\}$. For the remaining rows we imitate what we did for the leftmost column and increment, for the i-th column, the first index by 2 each time we ascend one row, i.e., $(b, \chi_i^j) \rightarrow (b+2, \chi_i^j)$. The periodicity issues are as discussed above.

Our task now is to check the consistency of the BBM, namely (2.7). Let us do so case by case. First we check the grid at the first (leftmost) column at the i-th row; the content there is $\{(a+2i, \chi_1^1), (a+2i, \chi_1^3)\}$.

Then (2.7) dictates that

$$[(2,\chi_1^1)\oplus(-1,\chi_2^1)]\otimes(a+2i,\chi_1^1 \text{ or } \chi_1^3)$$
$$=(a+2(i+1),\chi_1^1 \text{ or } \chi_1^3)\oplus((a+2i)-1,\chi_2^1)$$

by using the table of tensor decompositions in subsection §2.1 and our chosen **3** from (2.6). Notice that the first term on the right is exactly the content of the box to the North and second term, the content of the East. Therefore consistency is satisfied. Next we check the grid in the second column at the i-th row where $((a+2i)-1,\chi_2^1)$ lives. As above we require

$$[(2,\chi_1^1)\oplus(-1,\chi_2^1)]\otimes((a+2i)-1,\chi_2^1)$$
$$=((a+2(i+1))-1,\chi_2^1)\oplus((a+2i)-2,\chi_2^2)\oplus(a+2(i-1),\chi_1^1)\oplus(a+2(i-1),\chi_1^3)$$

whence we see that the first term corresponds to the grid to the North, and second, East, and the last two, SouthWest. We proceed to check the grid in the $j+1$-th column ($2\leq j\leq k'-2$) at the i-th row where $((a+2i)-j,\chi_2^j)$ resides. Again (2.7) requires that

$$[(2,\chi_1^1)\oplus(-1,\chi_2^1)]\otimes((a+2i)-j,\chi_2^j)$$
$$=((a+2(i+1))-j,\chi_2^j)\oplus((a+2i)-(j+1),\chi_2^{j+1})\oplus((a+2(i-1))-(j-1),\chi_2^{j-1})$$

where again the first term gives the irrep the grid to the North, the second, East and the third, SouthWest. Next we check the grid in the k'-th column and i-th row, where the irrep is $((a+2i)-(k'-1),\chi_2^{k'-1})$. Likewise the requirement is

$$[(2,\chi_1^1)\oplus(-1,\chi_2^1)]\otimes((a+2i)-(k'-1),\chi_2^{k'-1})$$
$$=((a+2(i+1))-(k'-1),\chi_2^{k'-1})\oplus((a+2i)-k',\chi_1^2)$$
$$\oplus((a+2i)-k',\chi_1^4)\oplus((a+2(i-1))-(k'-2),\chi_2^{k'-2})$$

whence we see again the first term gives the grid to the North, the second and third, East and the fourth, SouthWest. Finally, for the last (rightmost) column, the grid in the i-th row has $((a+2i)-k',\chi_1^2)$ and $((a+2i)-k',\chi_1^4)$. We demand

$$[(2,\chi_1^1)\oplus(-1,\chi_2^1)]\otimes((a+2i)-k',\chi_1^2 \text{ or } \chi_1^4)$$
$$=((a+2(i+1))-k',\chi_1^2 \text{ or } \chi_1^4)\oplus((a+2(i-1))-(k'-1),\chi_2^{k'-1}))$$

where the first term gives the grid to the North and the second term, Southwest. So we have finished all checks and our BBM is consistent.

From the structure of this BBM it is very clear that each row gives a $D_{k'}$ quiver and the different rows simply copies it k times according to the Z_k. This repetition hints that there should be some kind of direct product, which is precisely what we have.

2.4 The Inverse Problem

Now we address the inverse problem: given a BBM of type Z-D, with k' vertical NS5-branes bounded by 2 ON^0-planes and k horizontal NS5′-branes, what is the corresponding orbifold, i.e., the group which acts on $\mathbb{C}^3$? The answer is now very clear: if k is odd we can choose the group $Z_k\times D_{k'}$ or $Z_{2k}\times D_{k'}$ with the action as defined in (2.3); if k is even, then we can choose the group to be $Z_{2k}\times D_{k'}$ with the same action.

In this above answer, we have two candidates when k is odd since we recall from discussions in §2.3 the vertical direction of the BBM for the group $Z_{2k}\times D_{k'}$ only has periodicity $\frac{k}{2}$ and the BBM separates into two pieces. We must ask ourselves, what is the relation between these two candidates? We notice that (2.3) gives an non-faithful representation of the group $Z_{2k}\times D_{k'}$. In fact, it defines a new group of which has the faithful representation given by above matrix form and is a factor group of $Z_{2k}\times D_{k'}$ given by

$$G := (Z_{2k} \times D_{k'})/H, \quad \text{with} \quad H = \langle 1, \alpha^k \beta^{k'} \rangle \tag{2.8}$$

In fact G is isomorphic to $Z_k \times D_{k'}$. We can see this by the following arguments. denote the generators of $Z_{2k} \times D_{k'}$ as α, β, γ and those of $Z_k \times D_{k'}$ as $\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}$. An element of G can be expressed as $[\alpha^a \beta^b \gamma^n] \equiv [\alpha^{a+k} \beta^{b+k'} \gamma^n]$. We then see the homomorphism from G to $Z_k \times D_{k'}$ defined by

$$[\alpha^a \beta^b \gamma^n] \longrightarrow \tilde{\alpha}^a \tilde{\beta}^{ak'+b} \tilde{\gamma}^n$$

is in fact an isomorphism (we see that $[\alpha^a \beta^b \gamma^n]$ and $[\alpha^{a+k} \beta^{b+k'} \gamma^n]$ are mapped to same element as required; in proving this the $k =$ odd condition is crucial).

We see therefore that given the data from the BBM, viz., k and k', we can uniquely find the $\mathbb{C}^3$ orbifold singularity and our inverse problem is well-defined.

3 The General Twisted Case

We have found [16] that the group $Z_k * D_{k'}$ (which in that paper we called $Z_k \times D_{k'}$) defined by (1.1) can be written in another form as $Z_k \rtimes D_{\frac{kk'}{\gcd(k,2k')}}$ where it becomes an (internal) semidirect product. We would like to know how the former, which is a special case of what we shall call a **twisted** group[e] is related to the direct product example, which we shall call the **untwisted** case, upon which we expounded in the previous section.

The key relation which describes the semidirect product structure was shown [16] to be $\alpha\gamma = \beta^{\frac{2k'}{\gcd(k,2k')}} \gamma\alpha$. This is highly suggestive and hints us to define a one-parameter family of groups[f] $G(a) := \{Z_k \rtimes D_d\}$ whose presentations are

$$\alpha\beta = \beta\alpha, \qquad \alpha\gamma = \beta^a \gamma\alpha. \tag{3.9}$$

When the parameter $a = 0$, we have $G(0) = Z_k \times D_{k'}$ as discussed extensively in the previous section. Also, when $a = \frac{kk'}{\gcd(k,2k')}$, $G(a)$ is the group $Z * D$ treated in the previous paper [16]. We are concerned with members of $\{G(a)\}$ that satisfy the BBM conditions and though indeed this family may not exhaust the list of all groups that satisfy those conditions they do provide an illustrative subclass.

3.1 Preserving the Irreps of D_d

We see that the first of the BBM conditions is trivially satisfied by our definition (3.9 of $G(a) := Z_k \rtimes D_d$. Therefore we now move onto the second condition. We propose that $G(a)$ preserves the structure of the irreps of the D_d factor if a is even. The analysis had been given in detail [16] so here we only review briefly. Deducing from (3.9) the relation, for $b \in \mathbb{Z}$,

$$\alpha(\beta^b \gamma)\alpha^{-1} = \beta^{b+a}\gamma,$$

we see that $\beta^b\gamma$ and $\beta^{b+a}\gamma$ belong to the same conjugacy class after promoting D_d to the semidirect product $Z_k \rtimes D_d$. Now we recall from subsection §2.1 that the conjugacy classes of D_d are β^0, β^d, $\beta^{\pm p} (p \neq 0, d)$, $\gamma\beta^{\text{even}}$ and $\gamma\beta^{\text{odd}}$. Therefore we see that when $a =$ even, the conjugacy structure of D_d is preserved since therein $\beta^b\gamma$ and $\beta^{b+a}\gamma$, which we saw above belong to same conjugate class in D_d, are also in the same conjugacy class in $G(a)$ and everything is fine. However, when $a =$ odd, they live in two different conjugacy

[e] As mentioned in the Nomenclature section, $*$ generically denotes twisted products of groups.

[f] We note that this is unambiguously the semi-direct product $\rtimes$: defining the two subgroups $D := \langle \beta, \gamma \rangle$ and $Z := \langle \alpha \rangle$, we see that $G(a) = DZ$ as cosets, that $D \triangleleft G(a)$ and $D \cap Z = 1$, whereby all the axioms of semi-directness are obeyed.

classes at the level of D_d but in the same conjugacy class in $G(a)$ whence violating the second condition. Therefore a has to be even.

3.2 The Three Dimensional Representation

Now we come to the most important part of finding the 3-dimensional representations for $G(a)$, i.e., condition 3. We start with the following form for the generators

$$\beta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega_{2d} & 0 \\ 0 & 0 & \omega_{2d}^{-1} \end{pmatrix} \qquad \gamma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{pmatrix} \tag{3.10}$$

and

$$\alpha = \begin{pmatrix} \omega_k^{-(x+y)} & 0 & 0 \\ 0 & \omega_k^x & 0 \\ 0 & 0 & \omega_k^y \end{pmatrix} \tag{3.11}$$

where $x, y \in \mathbb{Z}$ are yet undetermined integers (notice that the form (3.11) is fixed by the matrix (3.10) of β and the algebraic relation $\alpha\beta = \beta\alpha$). Using the defining relations (3.9) of $G(a)$, i.e relation $\alpha\gamma = \beta^a\gamma\alpha$, we immediately have the following constraint on x and y:

$$\omega_k^{x-y} = \omega_{2d}^a \tag{3.12}$$

which has integer solutions [g] only when

$$k = (\frac{2d}{\delta})l \qquad l \in \mathbb{Z} \quad \text{and} \quad \delta := \gcd(a, 2d) \tag{3.13}$$

with the actual solution being

$$x - y = \frac{a}{\delta}l.$$

Equation (3.13) is a nontrivial condition which signifiess that for arbitrary $k, 2d, a$, the third of the BBM conditions may be violated, and the solution, not found. This shows that even though $G(a = \text{even})$ satisfies the first two of the BBM conditions, they can not in general be applied to construct BBM's of Type *Z-D* unless (3.13) is also respected. However, before starting the general discussion of those cases of $Z * D$ where (3.13) is satisfied, let us first see how the group treated before [16] indeed satisfies this condition.

For $Z_k * D_{k'}$ [16] and defined by (1.1), let $\delta_1 := \gcd(k, 2k')$. We have $d = \frac{kk'}{\delta_1}$, $a = \frac{2k'}{\delta_1}$ from Proposition (3.1) in that paper. Therefore $\delta = \gcd(a, 2d) = a$ and $k = \frac{2d}{\delta}$ so that (3.13) is satisfied with $l = 1$ and we have the solution $x - y = 1$. Now if we choose $y = 0$, then we have

$$\alpha = \begin{pmatrix} \omega_k^{-1} & 0 & 0 \\ 0 & \omega_k^1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.14}$$

Combining with the matrices in (3.10), we see that they generate a faithful 3-dimensional representation of $Z_k * D_{k'}$. It is easy to see that what they generate is in fact isomorphic to a group with matrix generators, as given in (2.3),

[g] Since (3.12) implies $\frac{2\pi(x-y)}{k} - \frac{2\pi a}{2d} = 2\pi\mathbb{Z}$, we are concerned with Diophantine equations of the form $\frac{p}{q} - \frac{m}{n} \in \mathbb{Z}$. This in turn requires that $np = mq \Rightarrow q = \frac{nl}{\gcd(m,n)}$, $l \in \mathbb{Z}$ by diving through by the greatest common divisor of m and n. Upon back-substitution, we arrive at $p = \frac{ml}{\gcd(m,n)}$.

$$\alpha^{-1} = \begin{pmatrix} \omega_{2k}^{-2} & 0 & 0 \\ 0 & \omega_{2k}^{1} & 0 \\ 0 & 0 & \omega_{2k}^{1} \end{pmatrix} \qquad \beta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega_{2d} & 0 \\ 0 & 0 & \omega_{2d}^{-1} \end{pmatrix} \qquad \gamma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{pmatrix} \tag{3.15}$$

by noticing that $\alpha^{-1}\beta^{\frac{k'}{\delta}}$ in (3.15) is precisely (3.14). But this is simply a non-faithful representation of $Z_{2k} \times D_{d=\frac{kk'}{\gcd(k,2k')}}$, our direct product example! Furthermore, when $k = odd$, by recalling the results of §2.4 we conclude in fact that the group $Z_k * D_{k'}$ is isomorphic to $Z_k \times D_d$. However, for $k =$ even, although $Z_k * D_{k'}$ is still embeddable into $Z_{2k} \times D_{d=\frac{kk'}{\gcd(k,2k')}}$ with a non-faithful representation (2.3), it is not isomorphic to $Z_k \times D_d$ and the BBM thereof corresponds to an intrintically twisted case (and unlike when $k =$ odd where it is actually isomorphic to a direct product group). We emphasize here an obvious but crucial fact exemplified by (2.8): *non-faithful representations of a group A can be considered as the faithful representation of a new group B obtained by quotienting an appropriate normal subgroup of A.* This is what is happening above. This explains also why we have succeeded [16] in constructing the BBM only when we wrote $Z_k * D_{k'}$ in the form $Z_k \rtimes D_{d=\frac{kk'}{\gcd(k,2k')}}$.

Now let us discuss the general case. We recall from the previous subsection that a has to be even; we thus let $a := 2m$. With this definition, putting (3.12) into (3.11,) we obtain for the quantity $\alpha\beta^{-m}$:

$$\tilde{\alpha} = \alpha\beta^{-m} = \begin{pmatrix} \omega_k^{-2y}\omega_{2d}^{2m} & 0 & 0 \\ 0 & \omega_k^{y}\omega_{2d}^{-m} & 0 \\ 0 & 0 & \omega_k^{y}\omega_{2d}^{-m} \end{pmatrix} \tag{3.16}$$

This $\tilde{\alpha}$ generates a cyclic group $Z_{\tilde{k}}$ and combined with (3.10) gives the direct product group of $Z_{\tilde{k}} \times D_d$, but with a non-faithful representation as in (2.3). Therefore for the general twisted case, we can obtain the BBM of Z-D type of $G(a)$ by imbedding $G(a)$ into a larger group $Z_{\tilde{k}} \times D_d$ which is a direct product just like we did for $Z_k * D_{k'}$ embeding to $Z_k \rtimes D_{d=\frac{kk'}{\gcd(k,2k')}}$ two paragraphs before, and for which, by our etude in §2, a consistent BBM can always be established. However, we need to emphasize that in general such an embedding (3.16) gives non-faithful representations so that the quiver diagram of the twisted group will be a union of disconnected pieces, as demanded by Proposition 2.1, each of which corresponds to a Type Z-D BBM. We summarise these results by stating

PROPOSITION 3.2 *The group $G(a) := Z_k * D_d$ satisfies the BBM conditions if a is even and the relation (3.13) is obeyed. In this case its matrices actually furnish a non-faithful representation of a direct product $\tilde{G} := Z_{\tilde{k}} \times D_d$ and hence affords a BBM[h] of Type Z-D.*

This action of $G(a) \hookrightarrow \tilde{G}$ is what we mean by embedding. We conclude by saying that the simple example of §2 where the BBM is constructed for untwisted (direct-product) groups is in fact general and Type Z-D BBM's can be obtained for twisted groups by imbedding into such direct-product structures.

4 A New Class of $SU(3)$ Quivers

It would be nice to see whether the ideas presented in the above sections can be generalised to give the BBM of other types such as Type Z-E, Z-d or D-E whose definitions are obvious. Moreover, E refers to the exceptional groups $\widehat{E_{6,7,8}}$ and d the ordinary dihedral group. Indeed, we must first have the brane setups for these groups. Unfortunately as of yet the E groups still remain elusive. However we will give an account of the ordinary dihedral groups and the quiver theory thereof, as well as the ordinary E groups from a new perspectively from an earlier work [19]. These shall give us a new class of $SU(3)$ quivers.

[h] Though possibly disconnected with the number of components depending on the order of an Abelian subgroup $H \triangleleft \tilde{G}$.

We note that, as pointed out [19], the ordinary di-, tetra-, octa- and iscosa-hedral groups (or d, $E_6, 7, 8$ respectively) are excluded from the classification of the discrete finite subgroups of $SU(2)$ because they in fact belong to the centre-modded group $SO(3) \cong SU(2)/\mathbb{Z}_2$. However due to the obvious embedding $SO(3) \hookrightarrow SU(3)$, these are all actually $SU(3)$ subgroups. Now the d-groups were not discussed before [19] because they did not have non-trivial 3-dimensional irreps and are not considered as non-trivial (i.e., they are fundamentally 2-dimensional collineation groups) in the standard classification of $SU(3)$ subgroups; or in a mathemtical language [20], they are transitives. Moreover, E_6 is precisely what was called $\Delta(3 \times 2^2)$ earlier [19], E_7, $\Delta(6 \times 2^2)$ and E_8, Σ_{60} and were discussed there. However we shall here see all these groups together under a new light, especially the ordinary dihedral group to which we now turn.

4.1 The Group $d_{k'}$

The group is defined as

$$\beta^{k'} = \gamma^2 = 1, \qquad \beta\gamma = \gamma\beta^{-1},$$

and differs from its binary cousin $D_{k'}$ in subsection §2.1 only by having the orders of β, γ being one half of the latter. Indeed, defining the normal subgroup $H := \{1, \beta^{k'}\} \triangleleft D_{k'}$ we have

$$d_{k'} \cong D_{k'}/H.$$

We can subsequently obtain the character table of $d_{k'}$ from that of $D_{k'}$ by using the theory of subduced representations, or simply by keeping all the irreps of $D_{k'}$ which are invariant under the equivalence by H. The action of H depends on the parity of k'. When it is even, the two conjugacy classes $(\gamma\beta^{even})$ and $(\gamma\beta^{odd})$ remain separate. Furthermore, the four 1-dimensional irreps are invariant while for the 2-dimensionals we must restrict the index l as defined in subsection §2.1 to $l = 2, 4, 6, ..., k' - 2$ so as to observe the fact that the two conjugacy classes $\{\beta^a, \beta^{-a}\}$ and $\{\beta^{k-a}, \beta^{a-k}\}$ combine into a single one. All in all, we have 4 1-dimensional irreps and $\frac{k'-2}{2}$ 2-dimensionals. On the other hand, for k' odd, we have the two classes $(\gamma\beta^{even})$ and $(\gamma\beta^{odd})$ collapsing into a single one, whereby we can only keep χ^1, χ^3 in the 1-dimensionals and restrict $l = 2, 4, 6, ..., k' - 1$ for the 2-dimensionals. Here we have a total of 2 1-dimensional irreps and $\frac{k'-1}{2}$ 2-dimensionals.

In summary then, the character tables are as follows:

	1	2	2	$\cdots$	2	n
Γ_1	1	1	1	$\cdots$	1	1
Γ_2	1	1	1	$\cdots$	1	-1
Γ_3	2	$2\cos\phi$	$2\cos 2\phi$	$\cdots$	$2\cos m\phi$	0
Γ_4	2	$2\cos 2\phi$	$2\cos 4\phi$	$\cdots$	$2\cos 2m\phi$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$
$\Gamma_{\frac{k'+3}{2}}$	2	$2\cos m\phi$	$2\cos 2m\phi$	$\cdots$	$2\cos m^2\phi$	0

k' odd
$m = \frac{k'-1}{2}$
$\phi = \frac{2\pi}{k'}$

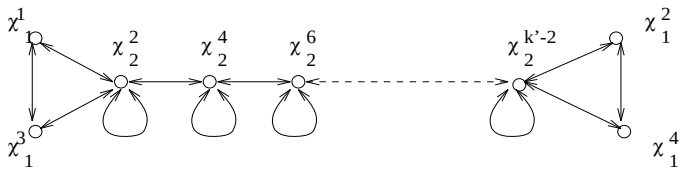

Figure 2: The quiver diagram for $d_{k=\text{even}}$. Here the notation of the irreps placed on the nodes is borrowed from D_k in §2.1. Notice that it gives a finite theory with non-chiral matter content.

	1	2	2	$\cdots$	2	1	$n/2$	$n/2$
Γ_1	1	1	1	$\cdots$	1	1	1	1
Γ_2	1	1	1	$\cdots$	1	1	-1	-1
Γ_3	1	-1	1	$\cdots$	$(-1)^{m-1}$	$(-1)^m$	1	-1
Γ_4	1	-1	1	$\cdots$	$(-1)^{m-1}$	$(-1)^m$	-1	1
Γ_5	2	$2\cos\phi$	$2\cos 2\phi$	$\cdots$	$2\cos(m-1)\phi$	$2\cos m\phi$	0	0
Γ_6	2	$2\cos 2\phi$	$2\cos 4\phi$	$\cdots$	$2\cos 2(m-1)\phi$	$2\cos 2m\phi$	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\Gamma_{\frac{k'+6}{2}}$	2	$2\cos(m-1)\phi$	$2\cos 2(m-1)\phi$	$\cdots$	$2\cos(m-1)^2\phi$	$2\cos m(m-1)\phi$	0	0

k' even
$m = \frac{k'}{2}$
$\phi = \frac{2\pi}{k'}$

4.2 A New Set of Quivers

Now we must choose an appropriate $SU(3)$ decomposition of the **3** for our group in order to make physical sense for the bifundamentals. The choice is

$$\mathbf{3} \longrightarrow \chi_1^3 + \chi_2^2.$$

Here, we borrow the notation of the irreps of d_k from D_k in §2.1. The relationship between the irreps of the two is discussed in the previous subsection. The advantage of using this notation is that we can readily use the tabulated tensor decompositions of D_k in §2.1. With this chosen decomposition, we can immediately arrive at the matter matrices a_{ij} and subsequent quiver diagrams. The $k' =$ even case gives a quiver which is very much like the affine $\widehat{D_{k'+2}}$ Dynkin Diagram, differing only at the two ends, where the nodes corresponding to the 1-dimensionals are joined, as well as the existence of self-joined nodes. This is of course almost what one would expect from an $\mathcal{N} = 2$ theory obtained from the binary dihedral group as a finite subgroup of $SU(2)$; this clearly reflects the intimate relationship between the ordinary and binary dihedral groups. The quiver is shown in Figure 2. On the other hand, for k' odd, we have a quiver which looks like an ordinary $D_{k'+1}$ Dynkin Diagram with 1 extra line joining the 1-nodes as well as self-adjoints. This issue of the dichotomous appearance of affine and ordinary Dynkin graphs of the D-series in brane setups has been raised before [18,16]. The diagram for k' odd is shown in Figure 3.

For completeness and comparison we hereby also include the 3 exceptional groups of $SO(3) \subset SU(3)$. For these, we must choose the **3** to be the unique (up to automorphism among the conjugacy classes) 3-dimensional irrep. Any other decompostion leads to non-faithful representations of the action and subsequently, by our rule discussed earlier, to disconnected quivers. This is why when they were considered as $SU(2)/\mathbb{Z}_2$ groups with $\mathbf{3} \to \mathbf{1} \oplus \mathbf{2}$ chosen, uninteresting and disconnected quivers were obtained [19]. Now under this new light, we present the quivers for these 3 groups in Figure 4.

There are two points worth emphasising. All the above quivers correspond to theories which are finite and non-chiral. By **finite** we mean the condition [12] for anomaly cancelation, that the matter matrix a_{ij}^R

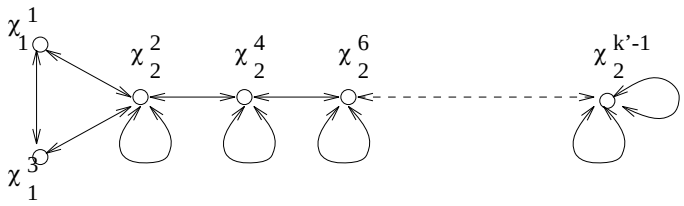

Figure 3: The quiver diagram for $d_{k=\text{odd}}$. Here again we use the notation of the irreps of D_k to index the nodes. Notice that the theory is again finite and non-chiral.

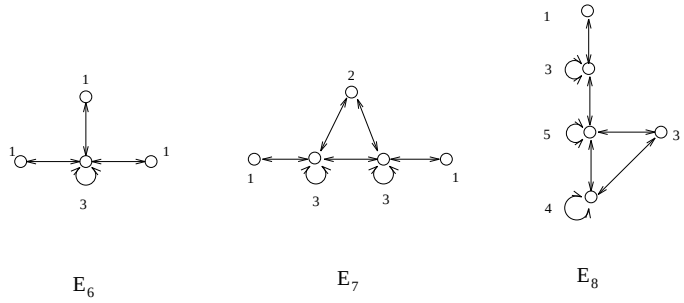

Figure 4: The quiver diagrams for $E_6 = A_4 = \Delta(3 \times 2^2)$, $E_7 = S_4 = \Delta(6 \times 2^2)$ and $E_8 = A_5 = \Sigma_{60}$. The theories are finite and non-chiral.

must satisfy

$$\sum_j a_{ij}^R \dim(r_j) = \sum_j a_{ji}^R \dim(r_j)$$

What this mean graphically is that for each node, the sum of the indices of all the neighbouring nodes flowing thereto (i.e., having arrows pointing to it) must equal to the sum of those flowing therefrom, and must in fact, for an $\mathcal{N} = 1$ theory, be equal to 3 times the index for the node itself. We observe that this condition is satisfied for all the quivers presented in Figure 3 to Figure 4.

On the other hand by **non-chiral** we mean that for every bi-fundamental chiral multiplet $(N_i, \bar{N}_j)$ there exists a companion $(N_j, \bar{N}_i)$ (such that the two combine together to give a bi-fundamental hypermultiplet in the sense of $\mathcal{N} = 2$). Graphically, this dictates that for each arrow between two nodes there exists another in the opposite direction, i.e., the quiver graph is unoriented. Strangely enough, non-chiral matter content is a trademark for $\mathcal{N} = 2$ theories, obtained from $\mathbb{C}^2/\Gamma \subset SU(2)$ singularities, while $\mathcal{N} = 1$ usually affords chiral (i.e., oriented quivers) theories. We have thus arrived at a class of finite, non-chiral $\mathcal{N} = 1$ super Yang-Mills theories. This is not that peculiar because all these groups belong to $SO(3)$ and thus have real representations; the reality compel the existence of complex conjugate pairs. The more interesting fact is that these groups give quivers that are in some sense in between the generic non-chiral $SU(2)$ and chiral $SU(3)$ quiver theories. Therefore we expect that the corresposnding gauge theory will have better properties, or have more control, under the evolution along some energy scale.

4.3 An Interesting Observation

Having obtained a new quiver, for the group d_k, it is natural to ask what is the corresponding brane setup. Furthermore, if we can realize such a brane setup, can we apply the ideas in the previous sections to realize the BBM of Type *Z-d*? We regrettably have no answers at this stage as attempts at the brane setup have met great difficulty. We do, however, have an interesting brane configuration which gives the correct matter content of d_k but has a different superpotential. The subtle point is that d_k gives only $\mathcal{N} = 1$ supersymmetry

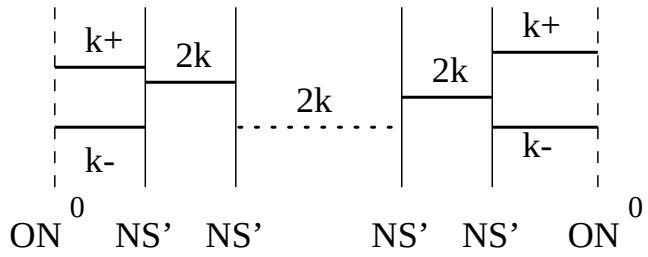

Figure 5: The brane configuration which gives the same matter content as the $d_{k=\text{even}}$ quiver.

and unlike $\mathcal{N}=2$, one must specify both the matter content and the superpotential. Two theories with the same matter content but different superpotential usually have different low-energy behavior.

We now discuss the brane configuration connected with d_k, which turns out to be a rotated version of the configuration for D_k as given by Kapustin [6] (related examples [16,21] on how rotating branes breaks supersymmetry further may be found). In particular we rotate all NS5-branes (along direction (12345)) between the two ON^0-plane as drawn in Figure 1 of Kapustin [6] to NS5$'$-branes (along direction (12389)). The setup is shown in Figure 5. Let us analyse this brane setup more carefully. First when we end D4-branes (extended along direction (1236)) on the ON^0-plane, they can have two different charges: positive or negtive. With the definition of the matrix

$$\Omega = \begin{pmatrix} 1_{k+\times k+} & 0 \\ 0 & -1_{k-\times k-} \end{pmatrix},$$

the projection on the Chan-Paton matrix of the D4-branes is as follows. The scalar fields in the D4-worldvolume are projected as

$$\phi^\alpha = \Omega\phi^\alpha\Omega^{-1} \quad \text{and} \quad \phi^i = -\Omega\phi^i\Omega^{-1}$$

where α runs from 4 to 5 and describes the oscillations of the D4-branes in the directions parallel to the ON^0-plane while i runs from 7 to 9 and describes the transverse oscillations. If we write the scalars as matrice in block form, the remaining scalars that survive the projection are

$$\phi^\alpha = \begin{pmatrix} U_{k+\times k+} & 0 \\ 0 & U_{k-\times k-} \end{pmatrix} \quad \text{and} \quad \phi^i = \begin{pmatrix} 0 & U_{k+\times k-} \\ U_{k-\times k+} & 0 \end{pmatrix}.$$

From these we immediately see that ϕ^α give scalars in the adjoint representation and ϕ^i, in the bifundamental representation. Next we consider the projection conditions when we end the other side of our D4-brane on the NS-brane. If we choose the NS5-brane to extend along (12345), then the scalars ϕ^α will be kept while ϕ^i will be projected out and we would have an $\mathcal{N}=2$ D_k quiver (see Figure 6).

However, if we choose the NS5-branes to extend along (12389), then ϕ^α and $\phi^{i=7}$ will be projected out while $\phi^{i=8,9}$ will be kept. It is in this case that we see immediately that we obtain the same matter content as one would have from a $d_{k=\text{even}}$ orbifold discussed in the previous subsection (see Figure 7).

Now we explain why the above brane configuration, though giving the same matter content as the $d_{k=\text{even}}$, is insufficient to describe the full theory. The setup in Figure 5 is obtained by the rotation of NS-branes to NS$'$-branes; in this process of rotation, in general we change the geometry from an orbifold to a conifold. In other words, by rotating, we break the $\mathcal{N}=2$ theory to $\mathcal{N}=1$ by giving masses to scalars in the $\mathcal{N}=2$ vector-multiplet. After integrating out the massive adjoint scalar in low energies, we usually get quartic terms in the superpotential (for more detailed discussion of rotation see Erlich et al. [21]). Indeed Klebanov and Witten [22] have explained this point carefully and shows that the quartic terms will exist even at the limiting case when the angle of rotation is $\frac{\pi}{2}$ and the NS5-branes become NS5$'$-branes. On the other

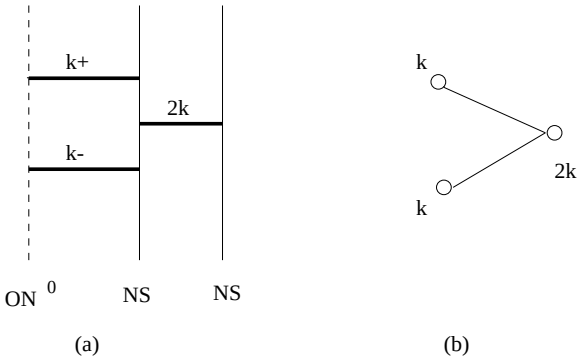

Figure 6: (a). The brane configuration of the projection using NS5-branes. (b). The quiver diagram for the brane configuration in (a).

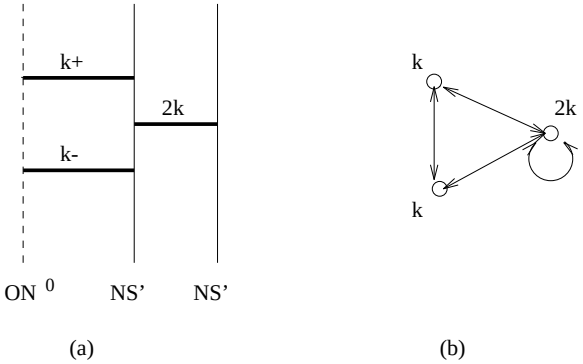

Figure 7: (a). The brane configuration of projection using NS5′-branes. (b). The quiver diagram for the brane configuration in (a).

hand, the superpotential for the orbifold singularity of d_k contains only cubic terms as required by Lawrence et al. [12] and as we emphasized in §2. It still remains an intersting problem to construct consistent brane setups for d_k that also has the right superpotential; this would give us one further stride toward attacking non-Abelian brane configurations.

5 Conclusions and Prospects

As inspired by the Brane Box Model (BBM) constructions [16] for the group $Z_k * D_{k'}$ generated by (1.1), we have discussed in this paper a class of groups which are generalisations thereof. These groups we have called the twisted groups (that satisfy BBM conditions). In particular we have analysed at great length, the simplest memeber of this class, namely the direct product $Z_k \times D_d$, focusing on how the quiver theory, the BBM construction as well as the inverse problem (of recovering the group by reading the brane setup) may be established. The brane configuration for such an example, as in Figure 1, we have called a **BBM of Type** Z-D; consisting generically of a grid of NS5-branes with the horizontal direction bounded by 2 ON-planes and the vertical direction periodically identified. We have also addressed, as given in Proposition 2.1 the issue of how non-faithful representations lead to disconnected quivers graphs, or in other words several disjunct pieces of the BBM setup.

What is remarkable is that the twisted groups, of which the one in our recent paper [16] is a special case, can under certain circumstances be embedded into a direct product structure (by actually furnishing a non-faithful representation thereof). This makes our naïve example of $Z_k \times D_d$ actually possess great generality as the twisted cases untwist themselve by embedding into this, in a sense, universal cover in the fashion of Proposition 3.2. What we hope is that this technique may be extended to address more non-Abelian singularities of $\mathbb{C}^3$, whereby the generic finite discrete group $G \subset SU(3)$ maybe untwisted into a direct-product cover. In order to do so, it seems that G needs to obey a set of what we call **BBM conditions**. We state these in a daring generality: (1) That G maybe written as a semi-direct product $A \rtimes B$, (2) that the structure of the irreps of G preserves those of the factors A and B and (3) that there exists a decomposition into the irreps of G consistent with the unitarity and determinant 1 constraints of $SU(3)$.

Indeed it is projected and hoped, with reserved optimism, that if A, B are $SU(2)$ subgroups for which a brane setup is known, the techniques presented above may inductively promote the setup to a BBM (or perhaps even brane cube for $SU(4)$ singularities). Bearing this in mind, we proceeded further to study more examples, hoping to attack for example, BBM's of the Z-d type where d is the ordinary dihedral group. Therefrom arose our interest in the ordinary groups $d, E_{6,7,8}$ as finite subgroups of $SO(3) \subset SU(3)$. These gave us a new class of quiver theories which have $\mathcal{N} = 1$ but non-chiral matter content. Brane setups that reproduce the matter content, but unfortunately not the superpotential, have been established for the ordinary dihedral groups. These give an interesting brane configuration involving rotating NS5-brane with respect to ON-planes.

Of course much work remains to be done. In addition to finding the complete brane setups that reproduce the ordinary dihedral quiver as well as superpotential, we have yet to clarify the BBM conditions for groups in general and head toward that beacon of brane realisations of non-Abelian orbifold theories.

Acknowledgements

We would like to extend our sincere gratitute to A. Uranga for valuable discussions and comments, N. Moeller and J. S. Song for suggestions and help. Indeed we are further indebted to the CTP and LNS of MIT for their gracious patronage and YHH, to the NSF for her generous support. A. H. is supported in part by an A. P. Sloan foundation fellowship and by a DOE OJI Award.

1. A. Hanany and E. Witten, "Type IIB Superstrings, BPS monopoles, and Three-Dimensional Gauge Dynamics," hep-th/9611230.
2. A. Giveon and D. Kutasov, "Brane Dynamics and Gauge Theory," hep-th/9802067.
3. K. Intriligator, N. Seiberg, "Mirror Symmetry in Three Dimensional Gauge Theories," Phys.Lett. B387 (1996) 513-519, hep-th/9607207.
4. M. Porrati, A. Zaffaroni, "M-Theory Origin of Mirror Symmetry in Three Dimensional Gauge Theories", Nucl.Phys. B490 (1997) 107-120, hep-th/9611201.
5. Jan de Boer, Kentaro Hori, Hirosi Ooguri, Yaron Oz and Zheng Yin, "Mirror Symmetry in Three-dimensional Gauge Theories, $SL(2,Z)$ and D-Brane Moduli Spaces," hep-th/9612131.
6. A. Kapustin, "D_n Quivers from Branes," hep-th/9806238.
7. S. Elitzur, A. Giveon and D. Kutasov, "Branes and N=1 Duality in String Theory," Phys.Lett. B400 (1997) 269-274, hep-th/9702014.
   S. Elitzur, A. Giveon, D. Kutasov, E. Rabinovici and A. Schwimmer, "Brane dynamics and $N = 1$ supersymmetric gauge theory," Nucl.Phys.B **505**(1997) 202-250.
8. E. Witten, "Solutions of Four-Dimensional Field Theories Via M Theory," hep-th/9703166.
9. Karl Landsteiner, Esperanza Lopez, David A. Lowe, " N=2 Supersymmetric Gauge Theories, Branes and Orientifolds", Nucl.Phys. B507 (1997) 197-226. hep-th/9705199.
   Karl Landsteiner, Esperanza Lopez, "New Curves from Branes", Nucl.Phys. B516 (1998) 273-296, hep-th/9708118.
10. M. Douglas and G. Moore, "D-Branes, Quivers, and ALE Instantons," hep-th/9603167.
11. Clifford V. Johnson, Robert C. Myers, "Aspects of Type IIB Theory on ALE Spaces", Phys.Rev. D55 (1997) 6382-6393.hep-th/9610140.
12. A. Lawrence, N. Nekrasov and C. Vafa, "On Conformal Field Theories in Four Dimensions," hep-th/9803015.
13. A. Hanany and A. Zaffaroni, "On the Realisation of Chiral Four-Dimenaional Gauge Theories Using Branes," hep-th/9801134.
14. A. Hanany and A. Uranga, "Brane Boxes and Branes on Singularities," hep-th/9805139.
15. Amihay Hanany, Matthew J. Strassler and Angel M. Uranga, "Finite Theories and Marginal Operators on the Brane," hep-th/9803086, JHEP 9806 (1998) 011.
16. Bo Feng, Amihay Hanany, Yang-Hui He, "The $Z_k \times D_{k'}$ Brane Box Model," hep-th/9906031.
17. Andreas Karch, Dieter Lust, Douglas J. Smith, "Equivalence of Geometric Engineering and Hanany-Witten via Fractional Branes", Nucl.Phys. B533 (1998) 348-372, hep-th/9803232.
18. A. Hanany and A. Zaffaroni, "Issues on Orientifolds: On the Brane Construction of Gauge Theories with $SO(2n)$ Global Symmetry," hep-th/9903242.
19. A. Hanany and Y.-H. He, "Non-Abelian Finite Gauge Theories," hep-th/9811183, JHEP 9902 (1999) 013.
20. A. Hanany and Y.-H. He, "A Monograph on the Classification of the Discrete Subgroups of $SU(4)$," hep-th/9905212.
    S.-T. Yau and Y. Yu, "Gorenstein Quotients Singularities in Dimension Three," Memoirs of the AMS, 505, 1993.
21. J. Erlich, A. Hanany, and A. Naqvi, "Marginal Deformations from Branes," hep-th/9902118.
22. Igor R. Klebanov, Edward Witten, "Superconformal Field Theory on Threebranes at a Calabi-Yau Singularity", Nucl.Phys. B536 (1998) 199-218,hep-th/9807080.