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The Moduli of Reducible Vector Bundles

Yang-Hui He, Burt A. Ovrut, and René Reinbacher

Department of Physics, University of Pennsylvania

Philadelphia, PA 19104–6396, USA

Abstract

A procedure for computing the dimensions of the moduli spaces of reducible, holomorphic vector bundles on elliptically fibered Calabi-Yau threefolds X is presented. This procedure is applied to poly-stable rank $n+m$ bundles of the form $V \oplus \pi^* M$, where V is a stable vector bundle with structure group $SU(n)$ on X and M is a stable vector bundle with structure group $SU(m)$ on the base surface B of X. Such bundles arise from small instanton transitions involving five-branes wrapped on fibers of the elliptic fibration. The structure and physical meaning of these transitions are discussed.

*yanghe@physics.upenn.edu, ovrut@physics.upenn.edu, rene01@student.physics.upenn.edu

1 Introduction

When Hořava-Witten theory [1] is compactified on a smooth Calabi-Yau threefold one obtains, at intermediate energy scales, a five-dimensional theory called heterotic M-theory [2, 3]. Heterotic M-theory is characterized by a five-dimensional bulk space-time bounded on either side of the fifth-dimension by four-dimensional "end-of-the-world" orbifold fixed planes. These boundary three-branes descend from Hořava-Witten theory as the uncompactified part of nine-branes partially wrapped on the Calabi-Yau threefold.

Before compactification, each nine-brane carries an $\mathcal{N}=1$ supersymmetric E_8 Yang-Mills supermultiplet on its worldvolume. Upon compactification, however, a G-instanton, that is, a static solution of the hermitian Yang-Mills equations with structure group $G \subseteq E_8$, may be present on the Calabi-Yau threefold. In this case, the associated three-brane carries a reduced gauge group H, which is the commutant of G in E_8. In addition, its worldvolume theory is $\mathcal{N}=1$ supersymmetric and contains chiral fermions transforming in several representations of H. Therefore, the boundary three-branes in heterotic M-theory can potentially carry a realistic theory of particle physics on their worldvolumes.

To analyze this possibility, one must be able to explicitly construct these worldvolume theories. It is clear that their structure is almost entirely determined by the properties of the G-instanton on the Calabi-Yau threefold. Here, one confronts a seemingly insurmountable problem, namely that there are no known solutions of the hermitian Yang-Mills equations on these manifolds. This is due, in part, to the fact that explicit metrics on Calabi-Yau threefolds also have never been constructed. However, a way around this problem was found by Uhlenbeck and Yau [4] and Donaldson [5]. These authors showed that stable, holomorphic vector bundles with structure group G on a Calabi-Yau threefold always admit a unique connection that must satisfy the hermitian Yang-Mills equations. They also proved the converse. Therefore, finding a solution of the hermitian Yang-Mills equations is equivalent to constructing the appropriate vector bundles. Happily, it is relatively easy to find stable, holomorphic vector bundles with arbitrary structure group G on Calabi-Yau threefolds.

A major step was taken in that direction by Freedman, Morgan, Witten [6, 7, 8] and Donagi [9], who gave a concrete procedure for constructing such bundles on elliptically fibered Calabi-Yau threefolds. This work was used, and extended, in a number of papers [10, 11], where it was shown that bundles of this type could indeed produce phenomenologically relevant particle physics theories on a threefold. Specifically, these papers showed that viable grand unified theories with gauge groups E_6, $SO(10)$ and $SU(5)$, for example, can be so constructed. The results of [10, 11] were limited to Calabi-Yau threefolds with trivial homotopy. In [12, 13, 14, 15] and more recently [16, 17, 18], methods were introduced for constructing stable, holomorphic vector bundles on torus fibered Calabi-Yau threefolds with non-trivial homotopy. These results allow one to add flat bundles, Wilson lines, to the G-instanton. With this extended capability, standard-like models with gauge group $SU(3)_C \times SU(2)_L \times U(1)_Y$ [12, 13, 14, 15] and $SU(3)_C \times SU(2)_L \times U(1)_Y \times U(1)_{B-L}$ [16, 17, 18] have now been constructed.

There is another interesting generalization of the theory of holomorphic vector bundles on elliptically fibered Calabi-Yau threefolds which has been less studied. All of the above results constructed stable vector bundles with simple structure groups G. However, one can show that so-called poly-stable bundles, that is, those with semi-simple structure groups such as $G = SU(n) \times SU(m)$, also admit connections satisfying the hermitian Yang-Mills equations. These poly-stable bundles were introduced in this context in [10]. Such vector bundles, having different structure groups, produce a different pattern of E_8 symmetry breaking and, therefore, different low energy gauge groups H. Some breaking patterns in this context were presented in [10].

Significantly, it was shown in [10, 19] that poly-stable holomorphic vector bundles must always arise in small instanton transitions involving a five-brane wrapped on a vertical fiber in the elliptically fibered Calabi-Yau threefold. For example, if prior to a small instanton transition a three-brane has a stable bundle with structure group $SU(n)$, then after a transition involving k vertical fibers the bundle becomes poly-stable with structure group $SU(n) \times SU(m)$, where $1 < m \leq k$. The $SU(m)$ factor of the vector bundle, the part contributed by the k fibers wrapped by five-branes coalescing with the initial three-brane, has a very specific structure. It is the pull-back to the full elliptically fibered Calabi-Yau threefold of a bundle constructed from k points on the base surface. Since phase transitions of this type may arise in a physical context, such as in the Ekpyrotic theory of Big Bang cosmology [20], poly-stable holomorphic vector bundles are of potential interest.

In this paper, we will examine a fundamental property of poly-stable holomorphic vector bundles with structure group $G = SU(n) \times SU(m)$ on elliptically fibered Calabi-Yau threefolds, namely, the dimension of their moduli spaces. The structure and dimension of the moduli spaces of stable vector bundles on these manifolds, such as the pure $SU(n)$ part of our poly-stable bundle, is known [21]. In this paper, we will extend these results to compute the dimension of the moduli spaces of stable, holomorphic bundles with structure group $SU(m)$ on a surface. We will show that this is identical to the number of moduli of the pull-back of the bundle constructed from k points on the base of the Calabi-Yau space. This is the pure $SU(m)$ part of our poly-stable vector bundle. However, there can be, and there are, moduli associated with the relationship of the $SU(n)$ and $SU(m)$ bundles to each other. These moduli are much harder to compute. Be that as it may, we explicitly perform that calculation in this paper, restricting our base surface to be $d\mathbb{P}_9$ for concreteness. Putting everything together, we present a general formula for the dimension of the moduli space of generic poly-stable holomorphic vector bundles on elliptically fibered Calabi-Yau threefolds with base $d\mathbb{P}_9$. This result is easily generalized to all other base surfaces.

Specifically, in this paper we do the following. In Section 2, we remind the reader of the structure of holomorphic vector bundles on elliptically fibered Calabi-Yau threefolds over $d\mathbb{P}_9$. We also establish our notation and preparatory formulas which will be used in the computation. Next, in Section 3, we present the physical motivation of why we are interested in poly-stable vector bundles and their relation to small instanton transitions in heterotic M-theory. Section 4 contains the details of the calculation of the dimension of the moduli space of these poly-stable holomorphic vector bundles. Finally, in Section 5, we present our result, together with a sample calculation. As a check on the validity of key steps in our computations, alternative proofs in a restricted context are carried out. These are given in Appendices A and B. While this paper was in preparation, [22] appeared. There is some overlap between the topics discussed in [22] and the results presented here.

2 Reducible Bundles on Elliptically Fibered Calabi-Yau Threefolds

In this section, we briefly review the basic ingredients used in our calculation and establish the nomenclature. In particular, in Sub-section 2.1, we concern ourselves with Calabi-Yau threefolds X that are elliptically fibered over $d\mathbb{P}_9$ surfaces. We then review some of the rudiments of the spectral cover construction of stable holomorphic vector bundles V on X in Sub-section 2.2. Finally, in Sub-section 2.3, we define and discuss the construction of a specific class of poly-stable vector bundles $\hat{V}$ on X.

2.1 Elliptically Fibered Calabi-Yau Threefolds X over $d\mathbb{P}_9$

We will consider Calabi–Yau threefolds X which are elliptic curves fibered over a base surface B. In other words, there is a mapping $\pi : X \to B$ such that $\pi^{-1}(b)$ is a smooth torus, E_b, for each generic point $b \in B$. Moreover, this torus fibered threefold has a zero section. That is, there exists an analytic map $\sigma : B \longrightarrow X$ that assigns to every element b of B an element $\sigma(b) \in E_b$. The point $\sigma(b)$ acts as the zero element for an Abelian group and turns E_b into an elliptic curve and X into an elliptic fibration. Throughout this paper, we will, for concreteness, focus on the case where the base B is a rationally ruled elliptic surface, also called a $d\mathbb{P}_9$ surface. That is,

$$B = d\mathbb{P}_9 \ . \tag{1}$$

Such a surface is itself an elliptic fibration over a base $\mathbb{P}^1$ with projection map $\pi_B : d\mathbb{P}_9 \longrightarrow \mathbb{P}^1$ and zero section $\sigma_B : \mathbb{P}^1 \longrightarrow d\mathbb{P}_9$. Its structure is that of a $\mathbb{P}^2$ blown-up at nine generic points. In summary, our Calabi-Yau threefold X has the structure of a double elliptic fibration

$$X \xrightarrow{\pi} B \xrightarrow{\pi_B} \mathbb{P}^1 \ , \tag{2}$$

with zero sections σ and σ_B respectively. We denote the fiber classes of π in X and of π_B in B as F and f respectively.

Many properties of $d\mathbb{P}_9$ are described, for example, in [16]. Those properties which we will need in this paper are the following. First, the second homology group, $H_2(B, \mathbb{Z})$, is spanned by ℓ, the hyperplane class of the $\mathbb{P}^2$, as well as the nine exceptional divisors $E_i, i = 1, \ldots, 9$ of the blow-up. These classes are effective and have the following intersection numbers

$$E_i \cdot E_j = -\delta_{ij}, \qquad E_i \cdot \ell = 0, \qquad \ell \cdot \ell = 1 \ . \tag{3}$$

Second, the first Chern class of the tangent bundle TB is

$$c_1(TB) = -c_1(K_B) = 3\ell - \sum_{i=1}^{9} E_i \ , \tag{4}$$

where K_B is the canonical bundle [16]. In fact, the first Chern class of the anti-canonical bundle K_B^* is precisely the fiber class f of this fibration over $\mathbb{P}^1$. That is,

$$-c_1(K_B) = f \ . \tag{5}$$

It then follows from (4) that

$$f = 3\ell - \sum_{i=1}^{9} E_i \ . \tag{6}$$

Recall that a line bundle $\mathcal{O}_B(D)$ associated with a divisor D in any surface B is ample, or positive, if

$$D \cdot D > 0, \qquad D \cdot c > 0 \tag{7}$$

for every effective irreducible curve c in B. An effective class in homology simply means that it is an actual geometric object. For example, an effective class in $H_2(B,\mathbb{Z})$ is a curve. One also called D satisfying (7) an ample divisor. A surface B is called ample if its anti-canonical bundle K_B^* is. We will often refer to these conditions of ampleness and effectiveness. Now note, using (3) and (4), that

$$c_1(K_B)^2 = 0 \ . \tag{8}$$

Therefore, it follows from (7) that K_B^* and, hence, $d\mathbb{P}_9$, is not ample. This is why a $d\mathbb{P}_9$ surface is sometimes called a half-K3. See, for example, [23]. Next, let us consider $c_2(TB)$, the highest Chern class on $d\mathbb{P}_9$. This is nothing but the topological Euler characteristic, $\chi(d\mathbb{P}_9) = 12$. See, for example, [16]. Therefore,

$$c_2(TB) = 12 \ . \tag{9}$$

We will also need the following intersection numbers,

$$\sigma_B \cdot \sigma_B = -1, \qquad \sigma_B \cdot f = 1, \qquad f \cdot f = 0 \ . \tag{10}$$

The first intersection follows simply from the adjunction formula and (5). The second equality follows because we are intersecting the fiber with a section, and the last because we are intersecting a fiber with itself.

Third, we need to know the effective curves in B. The irreducible effective curves in $d\mathbb{P}_9$ are spanned by [24] $f = 3\ell - \sum_{i=1}^{9} E_i$ and the curves

$$y = \beta\ell + \sum_{i=1}^{9} \alpha_i E_i \tag{11}$$

with

$$\alpha_i \in \mathbb{Z}, \qquad \beta \in \mathbb{Z}_{\geq 0} \ , \tag{12}$$

satisfying the constraints

$$y^2 = -1, \qquad y \cdot f = 1 \ . \tag{13}$$

These constraints imply that

$$-\beta^2 + \sum_{i=1}^{9} \alpha_i^2 = 1, \qquad 3\beta + \sum_{i=1}^{9} \alpha_i = 1 \ . \tag{14}$$

Finally, the Chern classes for the elliptically fibered Calabi-Yau threefold X were presented in [11]. We will make use of them later. Note that

$$c_1(TX) = 0 \ , \tag{15}$$

since X is a Calabi-Yau manifold. The second Chern class is given by

$$c_2(TX) = 12\sigma \cdot \pi^*(c_1(TB)) + \pi^*(c_2(TB) + 11c_1^2(TB)) \ . \tag{16}$$

In our case, $B = d\mathbb{P}_9$. Then, it follows from (4), (5) and (9) that

$$c_2(TX) = 12\sigma \cdot \pi^*(f) + 12F \ . \tag{17}$$

2.2 Spectral Cover Construction of Stable $SU(n)$ Bundles over X

One ingredient we need in this paper is stable, holomorphic $SU(n)$ vector bundles over the Calabi–Yau threefold X. We will construct such bundles using the so-called spectral cover method [6, 7, 8, 9]. We adhere to the notation and presentation in [10]. Stable, holomorphic $SU(n)$ vector bundles V over X can be explicitly constructed from two objects, a divisor $\mathcal{C}_X$ of X, called the spectral cover, and a line bundle $\mathcal{N}_X$ on $\mathcal{C}_X$. The pair $(\mathcal{C}_X, \mathcal{N}_X)$ is called the spectral data for V.

The spectral cover, $\mathcal{C}_X$, is a surface in X that is an n-fold cover of the base B. Its general form is

$$\mathcal{C}_X = n\sigma + \pi^*\eta \ , \tag{18}$$

where σ is the zero section and η is a curve in B. If $\mathcal{C}_X$ is an effective, irreducible surface, then the associated vector bundle is stable. We will always impose this restriction. In addition to $\mathcal{C}_X$, we must also specify a line bundle $\mathcal{N}_X$. For $SU(n)$ vector bundles, this must satisfy

$$c_1(\mathcal{N}_X) = n(\frac{1}{2} + \lambda)\sigma + (\frac{1}{2} - \lambda)\pi^*\eta + (\frac{1}{2} + n\lambda)\pi^* c_1(TB) \ , \tag{19}$$

where λ is a rational number such that

$$\begin{aligned} \lambda &= p + \tfrac{1}{2}, \qquad && n \ \text{odd} \\ \lambda &= p, && n \ \text{even} \end{aligned} \tag{20}$$

and $p \in \mathbb{Z}$.

Finally, the stable $SU(n)$ vector bundle V on X is constructed from the spectral data $(\mathcal{C}_X, \mathcal{N}_X)$ as $V = \pi_{1*}(\pi_2^*\mathcal{N}_X \otimes \mathcal{P})$, where π_1 and π_2 are the projections of the fiber product $X \times_B \mathcal{C}_X$ onto the two factors X and $\mathcal{C}_X$ respectively and $\mathcal{P}$ is the associated Poincaré bundle. We refer the reader to [6, 11] for a detailed discussion. This is essentially the Fourier-Mukai transform, which works in reverse as well. In other words, there is a 1-1 correspondence between V and the spectral data

$$(\mathcal{C}_X, \mathcal{N}_X) \overset{\text{Fourier-Mukai}}{\longleftrightarrow} V \ . \tag{21}$$

For $SU(n)$ bundles,

$$c_1(V) = 0 \ . \tag{22}$$

The remaining Chern classes were computed in [6, 11] and found to be

$$c_2(V) = \sigma\cdot\pi^*(\eta) - \frac{1}{24}c_1(TB)^2(n^3-n) + \frac{1}{2}(\lambda^2-\frac{1}{4})n\,\eta\cdot(\eta-nc_1(TB))F, \quad c_3(V) = 2\lambda\eta\cdot(\eta-nc_1(TB)) \ . \tag{23}$$

For $B = d\mathbb{P}_9$, these simplify by virtue of (5) and (10). We find that

$$c_2(V) = \sigma\cdot\pi^*(\eta) + \frac{1}{2}(\lambda^2-\frac{1}{4})n\eta\cdot(\eta-nf)F, \quad c_3(V) = 2\lambda\eta\cdot(\eta-nf) \ . \tag{24}$$

Let us consider the structure of the spectral cover $\mathcal{C}_X$ for a $d\mathbb{P}_9$ base surface. As stated previously, $\mathcal{C}_X$ must be both effective and irreducible. These requirements put constraints on the curve η which, in the $d\mathbb{P}_9$ context, are complicated to solve. Happily, we need not do this in general. As we will see in Section 4, it is expedient to restrict $\mathcal{C}_X$ to be an ample divisor in X. $\mathcal{C}_X$ being ample immediately implies that it is effective and simplifies the proof that it is irreducible. As we will now show, the conditions for $\mathcal{C}_X$ to be ample and irreducible are relatively straightforward.

If $\mathcal{C}_X$ is ample, then so is η. Unfortunately, the converse is not true. However, we can prove a partial converse. Let η' be any ample curve in $d\mathbb{P}_9$ and consider

$$\eta = \eta' + (n+1)f \ . \tag{25}$$

Clearly, η is also an ample curve. Note, however, that not every ample curve in $d\mathbb{P}_9$ is of the form (25). We can show that for every ample curve of the form (25), $\mathcal{C}_X$ is both ample and irreducible, as required. For η' to be ample in $d\mathbb{P}_9$, it must satisfy

$$\eta'\cdot\eta' > 0 \tag{26}$$

and its intersection with all effective curves must be positive. That is,

$$\eta'\cdot f > 0, \quad \eta'\cdot y > 0 \ . \tag{27}$$

Expanding η' in the basis of $H_2(B,\mathbb{Z})$ as

$$\eta' = b'\ell + \sum_{i=1}^{9} a'_i E_i \ , \qquad a'_i, b' \in \mathbb{Z} \ , \tag{28}$$

and using

$$\eta'\cdot\eta' = b'^2 - \sum_{i=1}^{9} a'^2_i \ , \qquad \eta'\cdot f = 3b' + \sum_{i=1}^{9} a'_i \ , \tag{29}$$

we see that (26) and (27) translate into three constraints on the coefficients a'_i and b'. These are

$$b'^2 - \sum_{i=1}^{9} a_i'^2 > 0, \qquad 3b' + \sum_{i=1}^{9} a'_i > 0, \qquad b'\beta - \sum_{i=1}^{9} a'_i\alpha_i > 0 \tag{30}$$

for all $\alpha_i \in \mathbb{Z}$ and $\beta \in \mathbb{Z}_{\geq 0}$ satisfying (14). If we also expand η into the basis of $H_2(B,\mathbb{Z})$ as

$$\eta = b\ell + \sum_{i=1}^{9} a_i E_i \ , \qquad a_i, b \in \mathbb{Z} \ , \tag{31}$$

then it follows from (25) that

$$a_i = a'_i - (n+1), \quad b = b' + 3(n+1) \ . \tag{32}$$

To conclude this section, we present some useful intersection formulas which we will use frequently throughout this paper [10]. First, we have

$$\eta \cdot \eta = -\sum_{i=1}^{9} a_i^2 + b^2, \qquad \eta \cdot c_1(TB) = \eta \cdot f = 3b + \sum_{i=1}^{9} a_i \ . \tag{33}$$

Next, we have

$$\sigma \cdot \sigma = -(\pi^*(c_1(TB))) \cdot \sigma = -\pi^*(f) \cdot \sigma \ . \tag{34}$$

Furthermore, note, using (4), that the triple intersection,

$$\begin{aligned} \sigma^3 &= (-\pi^*(c_1(TB)) \cdot \sigma) \cdot \sigma = (-\pi^*(c_1(TB)) \cdot (-\pi^*(c_1(TB))) \cdot \sigma \\ &= \pi^*(c_1(TB) \cdot c_1(TB)) \cdot \sigma = 0 \ . \end{aligned} \tag{35}$$

Finally, since we can always choose a representative F of the fiber of X which will not intersect the pull-back of the curve η, we have

$$F \cdot \pi^*(\eta) = 0 \ . \tag{36}$$

Also, it is obvious, since σ is a section, that

$$F \cdot \sigma = 1 \ . \tag{37}$$

2.3 Reducible Vector Bundles on X

The previous sub-section reviewed the spectral cover construction of stable, holomorphic $SU(n)$ vector bundles V over elliptically fibered Calabi-Yau threefolds X with base $B = d\mathbb{P}_9$. Our main concern in this paper will be reducible, poly-stable rank $n+m$ holomorphic vector bundles over the same space X. In other words, we want to construct bundles with structure group $SU(n) \times SU(m)$. For physical reasons to be discussed in the next section, we will restrict the bundles associated with the factor group $SU(m)$ to be the pull-back to X of stable rank m holomorphic vector bundles on the $d\mathbb{P}_9$ base surface. This construction was carried out in Section 6 of [10].

2.3.1 $d\mathbb{P}_9$ Bundles and Their Pull-back to X

We want to construct stable, holomorphic $SU(m)$ vector bundles M with $m \geq 2$ over the base B with the Chern classes given by

$$c_1(M) = 0, \quad c_2(M) = k \in \mathbb{Z}, \quad c_3(M) = 0 \ . \tag{38}$$

In this paper, the base is $d\mathbb{P}_9$ with a zero section σ_B. Hence, we can use the spectral cover construction with spectral data $(\mathcal{C}_B, \mathcal{N}_B)$. Here, $\mathcal{C}_B$ is a curve in B which is an m-fold cover of the base $\mathbb{P}^1$. Generically, it is given by

$$\mathcal{C}_B = m\sigma_B + kf \ . \tag{39}$$

Recall that $\mathcal{C}_B$ must be both effective and irreducible in order for M to be stable. $\mathcal{C}_B$ will be effective if we choose

$$k \geq 0 \ . \tag{40}$$

To ensure that the spectral cover is irreducible, we must impose [10] the constraint that

$$1 < m \leq k \ . \tag{41}$$

$\mathcal{N}_B$ is the spectral line bundle over $\mathcal{C}_B$. In order for the vector bundle M corresponding to $(\mathcal{C}_B, \mathcal{N}_B)$ to have Chern classes (38), we need to require that [10]

$$c_1(\mathcal{N}_B) = \frac{1}{2}m(2k - 1 - m) \ . \tag{42}$$

One can now use (39) and (42) to construct the requisite vector bundles via the Fourier-Mukai transformation Schematically, we have the structure

$$(\mathcal{C}_B, \mathcal{N}_B) \overset{\text{Fourier-Mukai}}{\longleftrightarrow} M \ . \tag{43}$$

Having constructed the bundles M on $d\mathbb{P}_9$, we can easily lift them to the Calabi–Yau threefold X. The pull–back $\pi^* M$ of M to X is a stable holomorphic $SU(m)$ vector bundle with $m \geq 2$ over X. Its Chern classes are

$$c_1(\pi^* M) = 0, \quad c_2(\pi^* M) = kF, \quad c_3(\pi^* M) = 0 \ . \tag{44}$$

2.3.2 Reducible $SU(n) \times SU(m)$ Bundles over X

Equipped with two stable holomorphic vector bundles over X, namely the $SU(n)$ bundle V in (21) and the $SU(m)$ bundle $\pi^* M$, which is a pull-back of the bundle M in (43) on $d\mathbb{P}_9$, our preparatory work is done. We can now construct reducible, poly–stable, rank $n+m$ holomorphic vector bundles with structure group $SU(n) \times SU(m)$ over X, which we denote by $\hat{V}$, simply as the direct sum

$$\hat{V} = V \oplus \pi^* M \ . \tag{45}$$

The Chern classes of this reducible bundle are easily computed to be

$$c_1(\hat{V}) = 0, \quad c_2(\hat{V}) = c_2(V) + kF, \quad c_3(\hat{V}) = c_3(V) \ , \tag{46}$$

where $c_2(V)$ and $c_3(V)$ were given in (24). Schematically, the relation between the Calabi-Yau threefold X, the base $B = d\mathbb{P}_9$, and the bundles is

$$\begin{array}{cccc} & V \oplus \pi^* M & & M \\ & \downarrow & & \downarrow \\ \pi : & X & \longrightarrow & B \end{array} \tag{47}$$

3 Small Instanton Transitions

Having constructed reducible $SU(n) \times SU(m)$ vector bundles on X, we now recall why such objects are of interest to us. The Hořava-Witten vacuum [1] of M-theory is an $S^1/\mathbb{Z}_2$ interval with an 11-dimensional bulk space and two "end-of-the-world" 10-dimensional fixed planes, each carrying an $\mathcal{N} = 1$ E_8 Yang-Mills supermultiplet. This theory has chiral fermions on the orbifold planes. This vacuum can be further compactified on a Calabi-Yau threefold, leading to a five-dimensional "brane-world" scenario wherein one fixed plane, or three-brane, is our "observable" 4-dimensional world and the other is a "hidden" brane. This compactified theory is called heterotic M-theory [10].

A wrapped BPS five–brane in the bulk space of heterotic M-theory has a modulus corresponding to the translation of the five–brane in the orbifold direction. The following question was addressed in [10]. What happens to a wrapped bulk five–brane in heterotic M–theory when it is translated across the bulk space and comes into direct contact with one of the boundary three–branes? It was shown that in collisions of a bulk five–brane with the observable boundary three–brane, the wrapped five–brane disappears and its data is "absorbed" into a singular bundle, called a torsion free sheaf. This sheaf is localized on the Calabi–Yau threefold associated with the observable three–brane and is referred to as a "small instanton"[19]. This small instanton can then be "smoothed out" to a non-singular holomorphic vector bundle by moving in its moduli space. The physical picture is that the bulk five–brane disappears after the collision, but at the cost of altering the topology of the instanton vacuum on the boundary three–brane.

There are two different phase transitions associated with the above-mentioned collisions. First, there is a "chirality-changing" transition, where the number of lepton-quark families changes and second, there is a "gauge-changing" transition, where the gauge group on the boundary three-brane is altered. Let us briefly remind ourselves of the two transitions. Recall from [10] that for an elliptically fibered Calabi-Yau threefold with zero section σ and fiber class F, the bulk five–brane wraps a class

$$W = W_B\sigma + a_F F \ , \tag{48}$$

where W_B is the lift of a curve in the base and the fiber coefficient a_F is a non-negative integer. The chirality-changing transitions have the property that they absorb all, or part, of the base component, W_B, into the holomorphic vector bundle. Such transitions do not affect the fiber component, $a_F F$, of the five–brane curve, which is identical on either side of the small instanton transition. On the other hand, in gauge-changing transitions, all, or a portion, of $a_F F$ is absorbed via the small instanton phase transition into the vector bundle on the observable brane. In these transitions, it is the base component W_B of the five-brane curve that is left undisturbed. It was shown in [10] that these two processes lead to a different kind of holomorphic vector bundle after the small instanton transition. Let us assume that initially, prior to the collision, the observable brane has a stable holomorphic $SU(n)$ vector bundle V associated with the spectral data $(\mathcal{C}_X, \mathcal{N}_X)$. After the transition, the vector bundle becomes V' with spectral data $(\mathcal{C}'_X, \mathcal{N}'_X)$. As demonstrated in [10], in chirality-changing transitions the holomorphic vector bundle V' is stable with structure group $SU(n)$, exactly as for V. However, its spectral cover is altered by absorbing W_B in such a way as to induce a change in the third Chern class and, hence, in the number of lepton-quark families. On the other hand, gauge-changing transitions preserve the third Chern class. However, absorbing kF, where $k \leq a_F$, makes V' a reducible bundle with structure group $SU(n) \times SU(m)$. Furthermore, the bundles associated with the factor group $SU(m)$ are always a pull-back to X of stable rank m holomorphic vector bundles on the base. These bundles are constructed from k points, the projection onto the base B of the fibers kF, and satisfy $1 < m \leq k$. Both kinds of phase transitions alter the second Chern class of the holomorphic bundle in such a way as to preserve the over-all anomaly freedom. The structure of the spectral data, both before and after a small instanton transition, is listed in Table 3.

Before	After	
$SU(n), V \leftrightarrow (\mathcal{C}_X, \mathcal{N}_X)$	Chirality Changing	$SU(n), V' \leftrightarrow (\mathcal{C}_X \cup W_B, \mathcal{N}_X)$
	Gauge Changing	$SU(n) \times SU(m), V' \leftrightarrow (\mathcal{C}_X \cup m\sigma, \mathcal{N}_X \oplus \sigma_* M)$

Table 1: Vector bundles and the associated spectral data before and after the collision of the bulk five-brane.

In this paper, we will focus on gauge changing phase transitions, that is, the absorption via a small instanton phase transition of all, or a portion, of $a_F F$ by the vector bundle on the observable brane. This creates an $SU(n) \times SU(m)$ vector bundle that is reducible and poly-stable. This $SU(n) \times SU(m)$ bundle is precisely $\hat{V} = V \oplus \pi^* M$ introduced in [10] and reviewed earlier in (47) . The above discussions explains the physical relevance of these poly-stable bundles. We now proceed to compute their moduli.

4 Computing the Moduli of the Reducible Bundle

In this section, we compute the moduli for any reducible vector bundle $\hat{V} = V \oplus \pi^* M$, specifically, the dimension its space of deformations. The space of deformations of an arbitrary vector bundle U on a complex manifold X is given by [26]

$$H^1(X, \mathrm{End}(U)) \ , \tag{49}$$

where

$$\mathrm{End}(U) = U \otimes U^* \tag{50}$$

is the sheaf of endomorphisms of U. Therefore, in this paper, we wish to calculate

$$h^1(X, \mathrm{End}\hat{V}) \ , \tag{51}$$

with $\hat{V} = V \oplus \pi^* M$. We can readily express $H^1(X, \mathrm{End}(V))$ as four terms

$$H^1(X, \mathrm{End}\hat{V}) = H^1(X, \hat{V} \otimes \hat{V}^*) = I \oplus II \oplus III \oplus IV \ . \tag{52}$$

Terms I and IV are the moduli spaces for the bundles V and $\pi^* M$ respectively, and are defined as

$$I = H^1(X, V \otimes V^*), \quad IV = H^1(X, \pi^* M \otimes (\pi^* M)^*) \ . \tag{53}$$

On the other hand, terms II and III are given by

$$II = H^1(X, V \otimes (\pi^* M)^*), \quad III = H^1(X, \pi^* M \otimes V^*) \tag{54}$$

and contain the moduli associated with the cross terms between V and $\pi^* M$. We proceed, then, to calculate the dimensions of the four terms I, II, III and IV.

4.1 Term I: Moduli From V

We are familiar with the first term $I = H^1(X, V \otimes V^*)$. It contains the moduli associated with the stable holomorphic $SU(n)$ vector bundle V. This case was addressed in Section 4 of [21]. It was shown that with respect to the spectral data $(\mathcal{C}_X, \mathcal{N}_X)$,

$$\dim(I) = h^0(X, \mathcal{O}_X(\mathcal{C}_X)) - 1 \ . \tag{55}$$

Let us assume that $\mathcal{O}_X(\mathcal{C}_X)$ is positive, as in [21]. This condition means that $\mathcal{C}_X$ is ample which, as discussed earlier, can be implemented by imposing the positivity constraints (30) and (32). We can now easily evaluate (55) using the methods introduced in Section 4 of [21]. With the positivity assumption, all of the higher cohomology classes are zero by the Kodaira-Serre vanishing theorem and we have

$$\begin{aligned} h^0(X, \mathcal{O}_X(\mathcal{C}_X)) &= \chi(X, \mathcal{O}_X(\mathcal{C}_X)) \\ &= \int_X \mathrm{ch}(\mathcal{O}_X(\mathcal{C}_X)) \wedge \mathrm{td}(TX) \\ &= \frac{1}{6} \int_X c_1^3(\mathcal{O}_X(\mathcal{C}_X)) + \frac{1}{12} \int_X c_1(\mathcal{O}_X(\mathcal{C}_X)) \wedge c_2(TX) \\ &= \frac{1}{6} \int_X \mathcal{C}_X^3 + \frac{1}{12} \int_X \mathcal{C}_X \wedge (12\sigma \cdot \pi^*(f) + 12F) \ . \end{aligned} \tag{56}$$

In evaluating this expression, we have used the fact that $c_1(\mathcal{O}_X(\mathcal{C}_X)) = \mathcal{C}_X$, the Atiyah-Singer index theorem and the result for $c_2(TX)$ given in (17).

Finally, recalling that the pull-back of a point $b \in B$ gives the fiber class F, that is, $\pi^*(b) = F$ and using (31), (35), (36) and (37), we can immediately finish the computation of (56). We find that

$$h^0(X, \mathcal{O}_X(\mathcal{C}_X)) = (1 - \frac{1}{2}n^2) \sum_{i=1}^{9} a_i - \frac{1}{2}n \sum_{i=1}^{9} a_i^2 + (3 - \frac{3}{2}n^2)b + \frac{1}{2}nb^2 + n \ . \tag{57}$$

It follows that

$$\dim(I) = (1 - \frac{1}{2}n^2) \sum_{i=1}^{9} a_i - \frac{1}{2}n \sum_{i=1}^{9} a_i^2 + (3 - \frac{3}{2}n^2)b + \frac{1}{2}nb^2 + n - 1 \ , \tag{58}$$

where a_i and b satisfy the conditions given in (30) and (32).

4.2 Term IV: Moduli from π^*M

Next, we consider the term $IV = H^1(X, \pi^*M \otimes (\pi^*M)^*)$. This corresponds to the moduli associated with the stable $SU(m)$ vector bundle M on the base pulled back to X. We will show that this term essentially reduces to the deformations of M over the base B.

We evoke a Leray spectral sequence in this context [26]. This states that for $\pi : X \to B$ and any vector bundle U on X, we have the exact sequence

$$0 \to H^1(B, \pi_*(U \otimes U^*)) \to H^1(X, U \otimes U^*) \to H^0(B, R^1\pi_*(U \otimes U^*)) \to \dots \tag{59}$$

where $R^i\pi_*$ is the i-th right derived functor for the push-forward map π_*. See, for example, [27]. In our case, taking U to be π^*M, we have the sequence

$$0 \to H^1(B, \pi_*(\pi^*M \otimes (\pi^*M)^*) \to IV \to H^0(B, R^1\pi_*(\pi^*M \otimes (\pi^*M)^*)) \to \dots \tag{60}$$

We first use the projection formula [27] to simplify this expression. This formula states that for a morphism $f : X \to Y$, F an $\mathcal{O}_X$-module and E a locally free $\mathcal{O}_Y$-module of finite rank, we have

$$R^if_*(F \otimes f^*E) \simeq R^if_*(F) \otimes E \ . \tag{61}$$

Using this and the fact that $(\pi^*M)^* = \pi^*M^*$, we can write

$$\begin{aligned} R^1\pi_*(\pi^*M \otimes (\pi^*M)^*) &= R^1\pi_*(\pi^*M) \otimes M^* \\ &= R^1\pi_*(\mathcal{O}_X \otimes \pi^*M) \otimes M^* \ , \end{aligned} \tag{62}$$

where we have tensored with the trivial sheaf $\mathcal{O}_X$. We can use the projection formula again to further reduce this to

$$R^1\pi_*(\pi^*M \otimes (\pi^*M)^*) = R^1\pi_*\mathcal{O}_X \otimes M \otimes M^* \ . \tag{63}$$

Similarly,

$$\pi_*(\pi^*M \otimes (\pi^*M)^*) = M \otimes M^* \tag{64}$$

and the spectral sequence (60) becomes

$$0 \to H^1(B, M \otimes M^*) \to IV \to H^0(B, R^1\pi_*\mathcal{O}_X \otimes M \otimes M^*) \to \dots \tag{65}$$

We proceed to compute term IV by first focusing on the third term in (65) and showing that it is, in fact, zero. To do this, we evoke relative duality [28]. This states that for $\pi : X \to B$ and any sheaf $\mathcal{S}$ on X, we have

$$(R^1\pi_*\mathcal{S})^* \simeq R^0\pi_*(\mathcal{S} \otimes K_X \otimes \pi^*K_B^*) \ . \tag{66}$$

Taking $\mathcal{S}$ to be $\mathcal{O}_X$, and recalling that K_X is trivial since X is a Calabi-Yau manifold, we find

$$\begin{aligned}(R^1\pi_*\mathcal{O}_X)^* &= R^0\pi_*(\mathcal{O}_X \otimes \mathcal{O}_X \otimes \pi^* K_B^*) \\ &= \pi_*(\pi^* K_B^*) \\ &= K_B^* \ . \end{aligned} \tag{67}$$

Therefore, the third term in (65) becomes

$$H^0(B, K_B \otimes M \otimes M^*) \ , \tag{68}$$

which are the global sections of the bundle $K_B \otimes M \otimes M^*$. They correspond to global holomorphic maps from $\mathcal{O}_B \to K_B \otimes M \otimes M^*$, or, equivalently, to maps

$$K_B^* \to M \otimes M^* \ . \tag{69}$$

Now M and, hence, M^* are stable bundles of slope zero. Thus, their tensor product is poly-stable [29]. Now for the surface $d\mathbb{P}_9$, K_B^* is effective[1]. If there were any non-trivial maps (69) we would have a positive sub-bundle of $M \otimes M^*$, thus violating poly-stability. Therefore, we conclude that (68), that is, the third term in the sequence (65), vanishes. It follows that all moduli contributions to IV arise only from the base. Indeed, the sequence (65) now gives us the isomorphism

$$IV \simeq H^1(B, M \otimes M^*) \ . \tag{70}$$

We have, therefore, reduced the computation of term IV to that of the moduli of a stable rank m vector bundle over the base B.

The computation proceeds similarly to the one in Section 4 of [21]. Recalling the structure of the bundle M from (43), it is clear that the moduli of M arise from both the spectral curve $\mathcal{C}_B$ and the spectral line bundle $\mathcal{N}_B$. The number of such moduli was shown to be

$$\dim(IV) = [h^0(B, \mathcal{O}_B(\mathcal{C}_B)) - 1] + h^1(\mathcal{C}_B, \mathcal{O}_{\mathcal{C}_B}) \tag{71}$$

in [21], where the term in the square brackets describes moduli coming from $\mathcal{C}_B$ while the last term describes those coming from $\mathcal{N}_B$. The first term is in analogy with (55). Here, however, the spectral line bundle $\mathcal{N}_B$ also contributes moduli. Thus, we need to compute the two terms, $h^0(B, \mathcal{O}_B(\mathcal{C}_B))$ and $h^1(\mathcal{C}_B, \mathcal{O}_{\mathcal{C}_B})$.

[1] For all other del Pezzo surfaces, K_B^* is ample and a similar argument holds [23].

4.2.1 Spectral Curve Moduli

Let us first compute $h^0(B, \mathcal{O}_B(\mathcal{C}_B))$, the number of spectral curve moduli. Although we found it expedient to assume that $\mathcal{O}_X(\mathcal{C}_X)$ was ample in our computation of term I, we need not make this assumption for $\mathcal{O}_B(\mathcal{C}_B)$. However, we must require that $\mathcal{C}_B$ be irreducible which, by Bertini's Theorem, requires the constraint (41). For general $\mathcal{O}_B(\mathcal{C}_B)$, the following calculation is somewhat technical. However, the special case where $\mathcal{O}_B(\mathcal{C}_B)$ is ample can be readily computed along the same lines as in [21]. We present the computation for this simple and illustrative case in Appendix A. At the end of this sub-section, when we find the result for the general case, we will compare it with the expression given in Appendix A and see that they agree. Now

$$H^0(B, \mathcal{O}_B(\mathcal{C}_B)) = H^0(\mathbb{P}^1, \pi_{B*}\mathcal{O}_B(\mathcal{C}_B)) \ , \tag{72}$$

where we have pushed down to the base $\mathbb{P}^1$. Recalling from (39) that $\mathcal{C}_B = m\sigma_B + kf$, it follows that

$$\begin{aligned} \mathcal{O}_B(\mathcal{C}_B) &= \mathcal{O}_B(m\sigma_B) \otimes \mathcal{O}_B(kf) \\ &= \mathcal{O}_B(m\sigma_B) \otimes \pi^*\mathcal{O}_{\mathbb{P}^1}(k) \ . \end{aligned} \tag{73}$$

Using the projection formula again, we obtain

$$\pi_{B*}\mathcal{O}_B(\mathcal{C}_B) = \pi_{B*}\mathcal{O}_B(m\sigma_B) \otimes \mathcal{O}_{\mathbb{P}^1}(k) \ . \tag{74}$$

Let us focus on the first term $\pi_{B*}\mathcal{O}_B(m\sigma_B)$. The second term is the familar line bundle on $\mathbb{P}^1$. We can proceed inductively. First, we note that we have the exact sequence

$$0 \to \mathcal{O}_B \to \mathcal{O}_B(\sigma) \to \mathcal{O}_\sigma(\sigma) \to 0 \ . \tag{75}$$

Applying the functor π_{B*}, we find

$$0 \to \pi_{B*}\mathcal{O}_B \to \pi_{B*}\mathcal{O}_B(\sigma) \to \pi_{B*}\mathcal{O}_\sigma(\sigma) \to R^1\pi_{B*}\mathcal{O}_B \to R^1\pi_{B*}\mathcal{O}_B(\sigma) \to \dots \tag{76}$$

This will be the starting point of a calculation by induction. We note that

$$\pi_{B*}\mathcal{O}_B = \mathcal{O}_{\mathbb{P}^1} \tag{77}$$

on the base $\mathbb{P}^1$. Also, we have

$$R^1\pi_{B*}\mathcal{O}_B(\sigma) = 0 \ . \tag{78}$$

This follows from the Kodaira vanishing theorem since $\mathcal{O}_B(\sigma)$ is of positive degree. In fact, we have

$$R^1\pi_{B*}\mathcal{O}_B(n\sigma) = 0, \quad n \in \mathbb{Z}_{>0} \ . \tag{79}$$

Furthermore, it follows from relative duality [28] that

$$R^1\pi_{B*}\mathcal{O}_B = \mathcal{O}_{\mathbb{P}^1}(-1) \ . \tag{80}$$

Also, note that

$$\pi_{B*}\mathcal{O}_\sigma(\sigma) = \mathcal{O}_{\mathbb{P}^1}(-1) \ . \tag{81}$$

In general, we find that

$$\pi_{B*}\mathcal{O}_\sigma(n\sigma) = \mathcal{O}_{\mathbb{P}^1}(-n), \quad n \in \mathbb{Z}_{>0} \ . \tag{82}$$

Using (79), (80) and (81), the sequence (76) becomes

$$0 \to \mathcal{O}_{\mathbb{P}^1} \to \pi_{B*}\mathcal{O}_B(\sigma) \to \mathcal{O}_{\mathbb{P}^1}(-1) \to \mathcal{O}_{\mathbb{P}^1}(-1) \to 0 \ , \tag{83}$$

which implies that

$$\pi_{B*}\mathcal{O}_B(\sigma) = \mathcal{O}_{\mathbb{P}^1} \ . \tag{84}$$

We now proceed to the next short exact sequence in our induction. It is

$$0 \to \mathcal{O}_B(\sigma) \to \mathcal{O}_B(2\sigma) \to \mathcal{O}_\sigma(2\sigma) \to 0 \ , \tag{85}$$

from which we have the long exact sequence

$$0 \to \pi_{B*}\mathcal{O}_B(\sigma) \to \pi_{B*}\mathcal{O}_B(2\sigma) \to \pi_{B*}\mathcal{O}_\sigma(2\sigma) \to R^1\pi_{B*}\mathcal{O}_B(\sigma) \to R^1\pi_{B*}\mathcal{O}_B(2\sigma)\ldots \tag{86}$$

By virtue of (79), (82) and (84), this sequence then reads

$$0 \to \mathcal{O}_{\mathbb{P}^1} \to \pi_{B*}\mathcal{O}_B(2\sigma) \to \mathcal{O}_{\mathbb{P}^1}(-2) \to 0 \ , \tag{87}$$

which implies that

$$\pi_{B*}\mathcal{O}_B(2\sigma) = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2) \ . \tag{88}$$

The pattern of induction is now clear. Continuing onwards we find, in general, that

$$\pi_{B*}\mathcal{O}_B(m\sigma) = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2) \oplus \ldots \oplus \mathcal{O}_{\mathbb{P}^1}(-m) \tag{89}$$

for any positive integer m. Therefore, we see from (74) and (89) that

$$\pi_{B*}\mathcal{O}_B(\mathcal{C}_B) = \left(\mathcal{O}_{\mathbb{P}^1} \oplus \bigoplus_{j=2}^{m} \mathcal{O}_{\mathbb{P}^1}(-j)\right) \otimes \mathcal{O}_{\mathbb{P}^1}(k) \ . \tag{90}$$

It then follows from expression (72) that

$$
\begin{aligned}
h^0(B, \mathcal{O}_B(\mathcal{C}_B)) &= h^0(\mathbb{P}^1, \left(\mathcal{O}_{\mathbb{P}^1} \oplus \bigoplus_{j=2}^{m} \mathcal{O}_{\mathbb{P}^1}(-j)\right) \otimes \mathcal{O}_{\mathbb{P}^1}(k)) \\
&= h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(k) \oplus \bigoplus_{j=2}^{m} \mathcal{O}_{\mathbb{P}^1}(-j+k)) \\
&= (k+1) + \frac{1}{2}(2k-m)(-1+m) \\
&= 1 - \frac{1}{2}m^2 + km + \frac{1}{2}m \ . \qquad (91)
\end{aligned}
$$

In the above calculation, we have used the fact that

$$h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(n)) = n+1 \quad \text{for} \quad n \geq 0 \ . \qquad (92)$$

Note, from equation (41), that $k \geq m$. Therefore, expression (91) is always non-negative, as it must be. Let us compare (91) with the special case where $\mathcal{O}_B(\mathcal{C}_B)$ is ample. This result is given in equation (186) in Appendix A. We see that they agree. This is a very re-assuring consistency check.

4.2.2 Spectral Line Bundle Moduli

Next, we must compute $h^1(\mathcal{C}_B, \mathcal{O}_{\mathcal{C}_B})$, the number of spectral line bundle moduli. This is none other than the genus g of the curve $\mathcal{C}_B$, that is,

$$h^1(\mathcal{C}_B, \mathcal{O}_{\mathcal{C}_B}) = g \ . \qquad (93)$$

To find the genus, we use the adjunction formula [27]

$$2g - 2 = \mathcal{C}_B \cdot (\mathcal{C}_B + K_B) \ . \qquad (94)$$

We will henceforth use K_B and $c_1(K_B)$ in adjunction formulas interchangeably without ambiguity. Recalling from (5) that $K_B = \mathcal{O}_B(-f)$, we can use (10) and (39) to compute (94) and obtain

$$2g - 2 = -m^2 + 2km - m \ . \qquad (95)$$

Hence

$$h^1(\mathcal{C}_B, \mathcal{O}_{\mathcal{C}_B}) = 1 - \frac{1}{2}m^2 + km - \frac{1}{2}m \ . \qquad (96)$$

Having computed both $h^0(B, \mathcal{O}_B(\mathcal{C}_B))$ and $h^1(\mathcal{C}_B, \mathcal{O}_{\mathcal{C}_B})$, we can now determine $\dim(IV)$. Inserting (91) and (96) into (71), we find that

$$\dim(IV) = 1 - m^2 + 2km \ . \qquad (97)$$

Note that we actually know more than just the dimension of the moduli space. Using the spectral data, we see that the structure of the moduli space is actually

$$H^1(X, \pi^* M \otimes (\pi^* M)^*) = H^1(B, M \otimes M^*) \ . \tag{98}$$

Equation (71) then tells us that this moduli space is a direct product of a projective space with a torus.

The above results were computed directly within the context of $B = d\mathbb{P}_9$. However, if one wishes to know only the dimension of the moduli space, then one can give a more general computation using the Atiyah-Singer index theorem that is applicable to any base surface B. We present this computation in Appendix B. Happily, we see that the result obtained using the index theorem and given in (192) agrees with the expression (97). Again, this is a very re-assuring consistency check.

4.3 Terms II and III: The Cross Terms

We now turn to the computation of the two cross terms $II = H^1(X, V \otimes (\pi^* M)^*)$ and $III = H^1(X, \pi^* M \otimes V^*)$. We find that computing each term individually is very difficult. However, by relating the two terms, we can arrive at the requisite expressions for $\dim(II)$ and $\dim(III)$. In this subsection, we will proceed as follow. We first use the index theorem to compute the difference $\dim(II) - \dim(III)$. Next, using Leray spectral sequences, we will show that $\dim(II)$ and $\dim(III)$ each counts the number of global holomorphic sections on some sheaf over a support curve in the base. We will calculate the degrees of these sheafs and see that they can not both be simultaneously positive. This means that one of the two terms is always zero. Consequently, from our expression for the difference $\dim(II) - \dim(III)$, we can deduce expressions for $\dim(II)$ and $\dim(III)$, the terms we want.

4.3.1 The Difference Between Terms II and III

First, we find the difference between $\dim(II)$ and $\dim(III)$. We use the Atiyah-Singer index theorem for $V \otimes (\pi^* M)^*$ in term II, which implies that

$$\chi(X, V \otimes (\pi^* M)^*) = \int_X \mathrm{ch}(V \otimes (\pi^* M)^*) \wedge \mathrm{td}(TX) = \sum_{i=0}^{3} (-1)^i h^i(X, V \otimes (\pi^* M)^*) \ . \tag{99}$$

Now,

$$h^0(X, V \otimes (\pi^* M)^*) = 0 \tag{100}$$

since V is stable and, hence, there are no global sections. It then follows that

$$h^3(X, V \otimes (\pi^* M)^*) = 0 \tag{101}$$

by Serre duality. We can also use Serre duality to show that

$$H^2(X, V \otimes (\pi^* M)^*) = H^1(X, K_X \otimes (V \otimes (\pi^* M)^*)^*) = H^1(X, \pi^* M \otimes V^*) = III \ . \tag{102}$$

We have used the fact that K_X is trivial since X is a Calabi-Yau manifold and the definition of term III given in (54). Therefore, using (99), (100) and (101),

$$\int_X \mathrm{ch}(V \otimes (\pi^* M)^*) \wedge \mathrm{td}(TX) = -\mathrm{dim}(II) + \mathrm{dim}(III) \ . \tag{103}$$

Recall that for any vector bundle U,

$$\mathrm{ch}(U) = \mathrm{rk}(U) + \mathrm{ch}_1(U) + \mathrm{ch}_2(U) + \mathrm{ch}_3(U) + \ldots, \tag{104}$$

where

$$\begin{aligned} \mathrm{ch}_1(U) &= c_1(U), \quad \mathrm{ch}_2(U) = \frac{1}{2}(c_1(U)^2 - 2c_2(U)), \\ \mathrm{ch}_3(U) &= \frac{1}{6}(c_1(U)^3 - 3c_1(U)c_2(U) + 3c_3(U)) \ , \ldots \ . \end{aligned} \tag{105}$$

Similarly,

$$\mathrm{td}(U) = 1 + \mathrm{td}_1(U) + \mathrm{td}_2(U) + \mathrm{td}_3(U) + \ldots \tag{106}$$

with

$$\mathrm{td}_1(U) = \frac{1}{2}c_1(U), \quad \mathrm{td}_2(U) = \frac{1}{12}(c_2(U) + c_1(U)^2), \quad \mathrm{td}_3(U) = \frac{1}{24}(c_1(U)c_2(U)) \ , \ldots \ . \tag{107}$$

Using these expressions, the left hand side of (103) becomes

$$\int_X (n - c_2(V) + \frac{1}{2}c_3(V)) \wedge (m - c_2(\pi^* M)) \wedge (1 + \frac{1}{12}c_2(TX)) \ , \tag{108}$$

which is equal to

$$\frac{1}{2}m \int_X c_3(V) \ . \tag{109}$$

This result and (24) imply that expression (103) becomes

$$\mathrm{dim}(II) - \mathrm{dim}(III) = -m\lambda\eta \cdot (\eta - nf) \ . \tag{110}$$

4.3.2 Simplification for Term II

Having obtained the difference between the expressions for terms II and III in (110), we proceed to simplify each term individually. We will see that each counts the global holomorphic sections of a sheaf on some support curve in the base. We will then determine whether such sections exist by calculating the degree of this sheaf.

Let us address term II first. As in the calculation of term IV, we can fit term II into a Leray spectral sequence

$$0 \to H^1(B, \pi_*(V \otimes (\pi^* M)^*)) \to II \to H^0(B, R^1\pi_*(V \otimes (\pi^* M)^*)) \to H^2(B, \pi_*(V \otimes (\pi^* M)^*)) \to \dots \tag{111}$$

which, using the projection theorem

$$R^i\pi_*(V \otimes (\pi^* M)^*) = R^i\pi_* V \otimes M^* \qquad \text{for } i \geq 0 \ , \tag{112}$$

becomes

$$0 \to H^1(B, \pi_* V \otimes M^*) \to II \to H^0(B, R^1\pi_* V \otimes M^*) \to H^2(B, \pi_* V \otimes M^*) \to \dots \tag{113}$$

We will first simplify (113) substantially by arguing that $\pi_* V$ vanishes. Let us rewrite $\pi_* V$ in a more useful form. For some point b in the base B, we denote the sheaf of sections of $\pi_* V$ at b by $(\pi_* V)|_b$. Then, it is straight-forward to see that

$$(\pi_* V)|_b = V|_{\pi^{-1}(b)} = V|_F = H^0(F, V|_F) \ , \tag{114}$$

where F is the fiber class on X, which we recall is an elliptic curve, and $V|_F$ means the restriction of V to the fiber. We have used here the sheaf-theoretic interpretation of V which allows us to conveniently write $(\pi_* V)|_b$ in terms of $H^0(F, V|_F)$. Now, the n-fold spectral cover of V given in (18) intersects F precisely n times. We denote these points by p_i, with $i = 1, 2, \dots, n$. Moreover, let the zero section σ intersect F once at the point e. Over a generic point $b \in B$, e is distinct from the n points p_i. Therefore, at such a generic point, we can write,

$$(\pi_* V)|_b = H^0(F, V|_F) = \bigoplus_{i=1}^{n} H^0(F, \mathcal{O}_F(e - p_i)) \ . \tag{115}$$

Now, each bundle $\mathcal{O}_F(e - p_i)$ is clearly of degree zero, being the sheaf for the divisor of points $e - p_i$. It is also holomorphic. However, one can show [26] that a nontrivial bundle of degree zero over an elliptic curve admits no global sections. Therefore, $H^0(F, \mathcal{O}_F(e - p_i))$ is trivial for each i and, hence, it follows from (115) that

$$(\pi_* V)|_b = 0 \ . \tag{116}$$

But V is torsion free, which implies that $\pi_* V$ is also torision free [27]. Therefore, $(\pi_* V)|_b = 0$ for generic points $b \in B$ means that

$$\pi_* V = 0 \tag{117}$$

everywhere. Thus, our sequence (113) reduces to

$$0 \to II \to H^0(B, R^1\pi_* V \otimes M^*) \to 0 \ , \tag{118}$$

which implies that

$$II \simeq H^0(B, R^1\pi_* V \otimes M^*) \ . \tag{119}$$

Therefore,

$$\dim(II) = h^0(B, R^1\pi_* V \otimes M^*) \ . \tag{120}$$

We would now like to evaluate $R^1\pi_* V$. By definition III.8 of [27], we have

$$R^1\pi_* V = H^1(F, V|_F) \ . \tag{121}$$

Furthermore, one can show that

$$\mathrm{rk} R^1\pi_* V = \mathrm{rk} H^1(F, V|_F) = 0 \ . \tag{122}$$

Using arguments similar to the above, we see that

$$(R^1\pi_* V)|_b = 0 \tag{123}$$

for the generic points $b \in B$ where e is distinct from the n points p_i. However, we can not conclude, as we did for $\pi_* V$, that $R^1\pi_* V$ vanishes everywhere. This is because the first higher direct image functor, $R^1\pi_* V$, is not necessarily torsion free, even if V is [27]. It follows that at any point $b' \in B$ over which e is equal to one of the points p_i, $(R^1\pi_* V)|_{b'}$ need not vanish. The locus of such special points b' form a co-dimension one object in B, namely, a curve. This support curve, which we will denote by C, is given by

$$C = \pi_*(\mathcal{C}_X \cdot \sigma) \ . \tag{124}$$

The sequence (118) is clearly trivial everywhere except on this curve. Therefore, any non-zero contribution to $\dim(II)$ arises from restricting the sheaf $R^1\pi_* V \otimes M^*$ to the curve C. Note, using (25), that C is smooth. Let us be more specific about the form of C. Recalling the expression for $\mathcal{C}_X$ from (18) and using (5) and (34), the curve C defined in (124) becomes

$$\begin{aligned} C = \pi_*(\mathcal{C}_X \cdot \sigma) &= \pi_*(\pi^*(-nf + \eta) \cdot \sigma) \\ &= \eta - nf \ . \end{aligned} \tag{125}$$

When restricted to the support curve C, (120) becomes

$$\dim(II) = h^0(C, (R^1\pi_* V \otimes M^*)|_C) \ , \tag{126}$$

which is the number of the global holomorphic sections of the sheaf $(R^1\pi_*V \otimes M^*)|_C$ on C. To determine the number of global sections, it will suffice to compute the degree,

$$d = c_1\big((R^1\pi_* V \otimes M^*)|_C\big) \ , \tag{127}$$

of $(R^1\pi_*V\otimes M^*)|_C$. We will later use the fact that if this degree is negative then $h^0(C, (R^1\pi_*V\otimes M^*)|_C)$ vanishes [26].

4.3.3 The Degree d Associated to Term II

We now proceed to determine the degree d in (127). To do this, we invoke the Groethendieck-Riemann-Roch theorem, which states that for any map $f : X \to B$ and any sheaf $\mathcal{S}$ on X, we have

$$\mathrm{td}(TB)\mathrm{ch}(\sum_{i=0}^{2}(-1)^i R^i f_* \mathcal{S}) = f_*(\mathrm{ch}(\mathcal{S})\mathrm{td}(TX)) \ . \tag{128}$$

For the case at hand, $\mathcal{S}$ is the vector bundle V. Then, (128) becomes

$$\mathrm{td}(TB)\mathrm{ch}(R^0\pi_*V - R^1\pi_*V) = \pi_*(\mathrm{ch}(V)\mathrm{td}(TX)) \ . \tag{129}$$

From (117), we know that $R^0\pi_*V = \pi_*V = 0$. Therefore, this expression simplifies to

$$\mathrm{td}(TB)\mathrm{ch}(-R^1\pi_*V) = \pi_*(\mathrm{ch}(V)\mathrm{td}(TX)) \ . \tag{130}$$

Using (105) and (107), we can expand (130) as

$$\begin{aligned}(1 + \mathrm{td}_1(TB) + \mathrm{td}_2(TB))\big(\mathrm{ch}_1(-R^1\pi_*V) + \mathrm{ch}_2(-R^1\pi_*V)\big) =& \\ \pi_*\left((n + \mathrm{ch}_2(V) + \mathrm{ch}_3(V))\,(1 + \mathrm{td}_2(TX))\right) \ .&\end{aligned} \tag{131}$$

In writing (131), we have used the facts that $\mathrm{rk}(V) = n$, $\mathrm{td}_1(TX) = \mathrm{td}_3(TX) = 0$ since X is a Calabi-Yau manifold and $\mathrm{rk}(-R^1\pi_*V) = 0$ by (122). Furthermore, terms on the left hand side must terminate at order 2 since B is of dimension 2 and terminate at order 3 on the right hand side since X is of dimension 3. Multiplying out (131), we find that

$$\begin{aligned}\mathrm{ch}_1(-R^1\pi_*V) + \mathrm{ch}_2(-R^1\pi_*V) + \mathrm{ch}_1(-R^1\pi_*V)\mathrm{td}_1(TB)& \\ = \pi_*\left(n + n\mathrm{td}_2(TX) + \mathrm{ch}_2(V) + \mathrm{ch}_3(V)\right) \ .&\end{aligned} \tag{132}$$

Using (105) and (107), and identifying terms of equal order, (132) implies that

$$\mathrm{ch}_1(-R^1\pi_*V) = \pi_*\left(\mathrm{ch}_2(V) + n\ \mathrm{td}_2(TX)\right)\ , \quad \mathrm{ch}_2(-R^1\pi_*V) = \pi_*\mathrm{ch}_3(V) - \mathrm{td}_1(TB)\mathrm{ch}_1(-R^1\pi_*V)\ . \tag{133}$$

It then follows from (17), (24), the intersection (10), and using the fact that π_*F vanishes, that

$$\mathrm{ch}_1(-R^1\pi_*V) = nf - \eta\ , \quad \mathrm{ch}_2(-R^1\pi_*V) = \lambda\eta\cdot(\eta - nf) + \frac{1}{2}f\cdot\eta\ . \tag{134}$$

Having obtained these results, recall that we want to compute $c_1(R^1\pi_*V\otimes M^*)|_C)$. For convenience, let us define the sheaf

$$\mathcal{F} = R^1\pi_*V\otimes M^*\ . \tag{135}$$

Next, we recall the multiplicative property of the Chern character, that is, for sheafs A and B

$$\mathrm{ch}(A\otimes B) = \mathrm{ch}(A)\mathrm{ch}(B)\ . \tag{136}$$

Thus, from (135), we have that

$$\mathrm{ch}(-\mathcal{F}) = \mathrm{ch}(-R^1\pi_*V)\mathrm{ch}(M^*)\ , \tag{137}$$

which, using (105) and (107) can be expanded into

$$\begin{aligned}&\mathrm{rk}(-\mathcal{F}) + \mathrm{ch}_1(-\mathcal{F}) + \mathrm{ch}_2(-\mathcal{F})\\&= \left(\mathrm{rk}(-R^1\pi_*V) + \mathrm{ch}_1(-R^1\pi_*V) + \mathrm{ch}_2(-R^1\pi_*V)\right)\left(\mathrm{rk}(M^*) + \mathrm{ch}_1(M^*) + \mathrm{ch}_2(M^*)\right)\ .\end{aligned} \tag{138}$$

We can now make use of the facts that $\mathrm{rk}(-R^1\pi_*V) = 0$ from (122), $\mathrm{rk}(M^*) = \mathrm{rk}(M) = m$ and that $c_1(M) = 0$ from (38). Identifying terms of order 1 and 2 respectively, (138) becomes

$$\begin{aligned}\mathrm{rk}(-\mathcal{F}) &= 0\\ \mathrm{ch}_1(-\mathcal{F}) &= m\ \mathrm{ch}_1(-R^1\pi_*V) = m(nf-\eta)\\ \mathrm{ch}_2(-\mathcal{F}) &= m\ \mathrm{ch}_2(-R^1\pi_*V) = m(\lambda\eta\cdot(\eta - nf) + \frac{1}{2}f\cdot\eta)\ ,\end{aligned} \tag{139}$$

where we have used (134). We wish to compute $c_1(\mathcal{F}|_C)$. To do this, let us invoke the Groethendieck-Riemann-Roch theorem again, this time for the inclusion map

$$i: C\to B \tag{140}$$

and the sheaf $-\mathcal{F} = -R^1\pi_* V \otimes M^*$. Then

$$i_* \left(\mathrm{td}(TC)\mathrm{ch}(-\mathcal{F}|_C) \right) = \mathrm{ch}(\sum_{j=0}^{2} (-1)^j R^j i_*(-\mathcal{F}|_C)\mathrm{td}(TB) \ . \tag{141}$$

However, all the higher image functors $R^j i_*$ for $j > 0$ vanish because i is an inclusion map. Moreover, $R^0 i_*(-\mathcal{F}|_C) = i_*(-\mathcal{F}|_C)$ is simply $-\mathcal{F}$. Therefore, (141) becomes

$$i_* \left(\mathrm{td}(TC)\mathrm{ch}(-\mathcal{F}|_C) \right) = \mathrm{ch}(-\mathcal{F})\mathrm{td}(TB) \ . \tag{142}$$

Thus, expanding (142) and using the first expresssion in (139), we have

$$\begin{aligned} & i_* \left((1 + \mathrm{td}_1(TC))(\mathrm{ch}_0(-\mathcal{F}|_C) + \mathrm{ch}_1(-\mathcal{F}|_C)) \right) \\ = \ & \left(\mathrm{ch}_1(-\mathcal{F}) + \mathrm{ch}_2(-\mathcal{F}) \right) \left(1 + \mathrm{td}_1(TB) + \mathrm{td}_2(TB) \right) \ . \end{aligned} \tag{143}$$

Upon identifying terms of order 1 and 2 respectively, this implies

$$\mathrm{rk}(-\mathcal{F}|_C) i_*(1) = \mathrm{ch}_1(-\mathcal{F}) \tag{144}$$

and

$$\mathrm{ch}_1(-\mathcal{F}|_C) + \mathrm{ch}_0(-\mathcal{F}|_C)\mathrm{td}_1(TC) = \mathrm{ch}_1(-\mathcal{F})\mathrm{td}_1(TB) + \mathrm{ch}_2(-\mathcal{F}) \ . \tag{145}$$

Noting that

$$i_*(1) = C \ , \tag{146}$$

and using the middle equation in (139), it follows from (144) that

$$\mathrm{rk}(-\mathcal{F}|_C) C = -m(\eta - nf) \ , \tag{147}$$

This then implies that

$$\mathrm{rk}(-\mathcal{F}|_C) = -m, \qquad C = \eta - nf \ . \tag{148}$$

Both of these expressions are re-assuring. Note that

$$\mathrm{rk}(-\mathcal{F}|_C) = \mathrm{rk}(-R^1\pi_* V)\mathrm{rk}_C(M^*) = -m \tag{149}$$

which is consistent with the first equation in (148). Second, the result for C in (148) is identical to that found in (125). Emboldened, let us proceed to equation (145) which would give us what we are after. Now, we note that even though $\mathcal{F}$ is not a vector bundle, $\mathcal{F}|_C$, being supported on C, is. We recall that for any vector bundle U,

$$c_i(U^*) = (-1)^i c_i(U) \ . \tag{150}$$

Therefore, from (145) and (150), we have

$$c_1(\mathcal{F}|_C) = -c_1(-\mathcal{F}|_C) = -\Big(\mathrm{ch}_1(-\mathcal{F})\mathrm{td}_1(TB) + \mathrm{ch}_2(-\mathcal{F}) + m\mathrm{td}_1(TC)\Big) \ . \tag{151}$$

Using (139), we find that

$$c_1(\mathcal{F}|_C) \ = \ -m\lambda\eta\cdot(\eta - nf) - m \ \mathrm{td}_1(TC) \ . \tag{152}$$

Also, we know that

$$\mathrm{td}_1(TC) = \frac{1}{2}c_1(TC) = \frac{1}{2}\chi(C) = \frac{1}{2}(2-2g(C)) \ , \tag{153}$$

where $g(C)$ is the genus of the curve C. Recalling from (127) and (135) that $d = c_1(\mathcal{F}|_C)$, we find from (152) that

$$d = -m\lambda\eta\cdot(\eta - nf) - m(1-g(C)) \ . \tag{154}$$

We now need to find the genus of C. Since C is in B, to obtain the genus we again invoke the adjunction formula (94) in the base B,

$$g(C) = \frac{1}{2}C\cdot(C+K_B) + 1 \ . \tag{155}$$

From (33) and (125), we have

$$C\cdot C = (\eta - nf)^2 = -2n\eta\cdot f + \eta^2 \ . \tag{156}$$

Similarly, we have

$$C\cdot K_B = (\eta - nf)\cdot(-f) = -\eta\cdot f \ . \tag{157}$$

Therefore, we find that

$$g(C) = \frac{1}{2}(\eta^2 - (2n+1)\eta\cdot f) + 1 \ . \tag{158}$$

Upon substitution of (158) into (154), we find the degree

$$d = \frac{m}{2}((2\lambda - 1)\eta^2 + (1+2n-2n\lambda)\eta\cdot f) \ . \tag{159}$$

It is convenient to recast this expression into one depending on the base curve η' whose definition was given in (25). Doing this, (159) becomes

$$d = \frac{m}{2}\left((1-2\lambda)\eta'^2 + (1-2(n+2)\lambda)\eta'\cdot f\right) \ . \tag{160}$$

4.3.4 The Degree d' Associated with Term III

Having computed, in (160) the degree d of the sheaf $(R^1\pi_*V \otimes M^*)|_C$ arising in term II, we will now compute the analogous quantity for term III, which we will denote by d'. The combined knowledge of d and d' will allow us to determine, in conjunction with the index theorem result (110), the individual expression for $\dim(II)$ and $\dim(III)$.

First, we recall that $III = H^1(X, \pi^*M \otimes V^*)$. We can fit this into a Leray spectral sequence in precise analogy to what was done for term II. Indeed, term III will be again supported on some curve because the properties of V used earlier, namely that π_*V vanishes and $R^1\pi_*V$ vanishes except on a curve, hold for the dual bundle V^* as well. Furthermore, the spectral cover for V^* is identical to that of V [24]. Thus, (124) implies that the support curve for $R^1\pi_*V^* \otimes M$ remains C given in (124). Therefore, we have the analogue of equation (126),

$$\dim(III) = h^0(C, (R^1\pi_*V^* \otimes M)|_C) \ . \tag{161}$$

This is the number of global holomorphic sections of the sheaf $R^1\pi_*V^* \otimes M$ restricted to C. To find this, we need to determined the degree

$$d' = c_1\left((R^1\pi_*V^* \otimes M)|_C\right) \ . \tag{162}$$

We repeat the analysis of the previous subsections, and find that the only change in our expression (127) is that we now have $\mathrm{ch}_3(V^*)$ rather than $\mathrm{ch}_3(V)$. Using

$$\mathrm{ch}_3(V^*) = -\mathrm{ch}_3(V) \ , \tag{163}$$

(162) is found to be

$$d' = m\lambda\eta \cdot (\eta - nf) - m(1 - g(C)) \ . \tag{164}$$

In terms of the curve η' defined in (25), (164) becomes

$$d' = \frac{m}{2}\left((1+2\lambda)\eta'^2 + (1 + 2(n+2)\lambda)\eta' \cdot f\right) \ . \tag{165}$$

4.3.5 Comparing d and d'

We now compare (160) and (165). First, we note that η'^2 and $\eta' \cdot f$ are both positive by our constraints (26) and (27). Therefore, if $\lambda \geq \frac{1}{2}$, then $d < 0$ while $d' > 0$. Now, because there are no global holomorphic section to a sheaf of negative degree, this would imply that $\dim(II)$ vanishes. On the other hand, if $\lambda \leq -\frac{1}{2}$, then $d > 0$ while $d' < 0$. This would imply that $\dim(III)$ vanishes. This leaves the one single case of $\lambda = 0$, for which our arguments do not apply. When $\lambda = 0$, both d and d' are positive, meaning that both $\dim(II)$ and $\dim(III)$ are non-zero. However, their difference in this case, by (110), is zero. Therefore, in this special case $\dim(II) = \dim(III)$. For convenience, let us henceforth assume that

$$\lambda \neq 0 \ . \tag{166}$$

In summary, then

$$\begin{aligned} \lambda \geq \frac{1}{2}, \qquad & \dim(II) = 0 \\ \lambda \leq -\frac{1}{2}, \qquad & \dim(III) = 0 \ . \end{aligned} \tag{167}$$

That is, one of the two cross terms always vanishes if $\lambda \neq 0$.

4.3.6 Obtaining $\dim(II)$ and $\dim(III)$

Combining our discussions in the previous subsections, we at last can obtain explicit expressions for $\dim(II)$ and $\dim(III)$. Applying (167) with the index theorem result (110), we find that

$$\dim(II) = 0 \ , \quad \dim(III) = m\lambda\eta \cdot (\eta - nf), \tag{168}$$

for $\lambda \geq \frac{1}{2}$ and

$$\dim(II) = -m\lambda\eta \cdot (\eta - nf) \ , \quad \dim(III) = 0 \tag{169}$$

for $\lambda \leq -\frac{1}{2}$. It is re-assuring to see that all cases are non-negative, as they must be. Using (33), the individual expressions (168) and (169) can be combined to give

$$\dim(II) + \dim(III) = m|\lambda| \left(b^2 - 3nb + \sum_{i=1}^{9} (a_i - a_i^2) \right) \tag{170}$$

for $\lambda \neq 0$.

5 Final Result: The Moduli for $\hat{V} = V \oplus \pi^* M$

Collecting all four terms from (58), (97), and (170), we have our final result for the dimension of the moduli space of a reducible rank $n + m$ holomorphic vector bundle of the form $\hat{V} = V \oplus \pi^* M$ on a Calabi-Yau threefold X elliptically fibered over $d\mathbb{P}_9$. If we denote the moduli space of this bundle by $\mathcal{M}(V \oplus \pi^* M)$, then

$$\begin{aligned} \dim\left(\mathcal{M}(V \oplus \pi^* M)\right) \ = \ & (m|\lambda| + \frac{1}{2}n)b^2 + (3 - \frac{3}{2}n - 3mn|\lambda|)b + (n + 2km - m^2) + \\ & \sum_{i=1}^{9} \left((1 + m|\lambda| - \frac{1}{2}n^2)a_i - (\frac{1}{2}n + m|\lambda|)a_i^2 \right) \ , \end{aligned} \tag{171}$$

where

$$\begin{aligned} \lambda &= p+\tfrac{1}{2}, \qquad & n \;\; \text{odd} \\ \lambda &= p, \qquad & n \;\; \text{even} \end{aligned} \tag{172}$$

with $p \in \mathbb{Z}$,

$$\lambda \neq 0 \tag{173}$$

and

$$1 < m \leq k \; . \tag{174}$$

The parameters a_i and b are defined by

$$a_i = a'_i - (n+1), \quad b = b' + 3(n+1) \; , \tag{175}$$

where a'_i and b' obey the following constraints

$$b'^2 - \sum_{i=1}^{9} a'^2_i > 0, \qquad 3b' + \sum_{i=1}^{9} a'_i > 0, \qquad b'\beta - \sum_{i=1}^{9} a'_i \alpha_i > 0 \tag{176}$$

for all $\alpha_i \in \mathbb{Z}$ and $\beta \in \mathbb{Z}_{>0}$ such that

$$-\beta^2 + \sum_{i=1}^{9} \alpha_i^2 = 1, \qquad 3\beta + \sum_{i=1}^{9} \alpha_i = 1 \; . \tag{177}$$

To illustrate these results, let us present a sample calculation. An obvious solution to the positivity constraints (176) is

$$\begin{aligned} a'_i &= -n, \qquad i = 1, 2, \ldots, 9 \\ b' &= 3n+1 \; . \end{aligned} \tag{178}$$

This implies that

$$\begin{aligned} a_i &= -2n-1, \qquad i = 1, 2, \ldots, 9 \\ b &= 6n+4 \; . \end{aligned} \tag{179}$$

Substituting these coefficients into result (171) gives

$$\dim(\mathcal{M}(V \oplus \pi^* M)) = 3 + 2\,k\,m - 2\,|\lambda|\,m - m^2 - 18\,|\lambda|\,m\,n - 18\,|\lambda|\,m\,n^2 + \frac{9\,n^2}{2} + \frac{9\,n}{2} \; . \tag{180}$$

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Appendix A: Ample Case for the Spectral Curve Moduli of Term IV

In this Appendix, we compute $h^0(B, \mathcal{O}_B(\mathcal{C}_B))$ for the simple case where $\mathcal{O}_B(\mathcal{C}_B)$ is ample. We will compare the result with the general case (91) computed in the text of this paper and find that they agree.

Consider the bundle $\mathcal{O}_B(\mathcal{C}_B) \otimes K_B^{-1}$ and assume that it is ample. Then its positivity allows us to invoke the Kodaira vanishing theorem [26]

$$H^q(B, \Omega^p(\mathcal{O}_B(\mathcal{C}_B) \otimes K_B^{-1})) = 0, \qquad \text{for } p+q > \text{dim} B \ . \tag{181}$$

Taking $p = 2$ and using the fact that $\Omega^p(\mathcal{O}_B(\mathcal{C}_X)) = \Omega^2(TB) \otimes \mathcal{O}_B(\mathcal{C}_B) = K_B \otimes \mathcal{O}_B(\mathcal{C}_B)$, we have, for all $q > 0$, that

$$H^q(B, \mathcal{O}_B(\mathcal{C}_B)) = H^q(B, \Omega^2(TB) \otimes \mathcal{O}_B(\mathcal{C}_B) \otimes K_B^{-1}) = 0 \ . \tag{182}$$

Therefore

$$\chi(B, \mathcal{O}_B(\mathcal{C}_B)) = \sum_{i=0}^{2} (-1)^i h^i(B, \mathcal{O}_B(\mathcal{C}_B)) = h^0(B, \mathcal{O}_B(\mathcal{C}_B)) \ . \tag{183}$$

Now,

$$\chi(B, \mathcal{O}_B(\mathcal{C}_B)) = \int_B \text{ch}(\mathcal{C}_B) \wedge \text{td}(TB) \tag{184}$$

by the Groethendieck-Riemann-Roch theorem. For surfaces B, this reduces to

$$\begin{aligned} h^0(B, \mathcal{O}_B(\mathcal{C}_B)) &= \frac{1}{2}\mathcal{C}_B \cdot (\mathcal{C}_B - K_B) + \frac{1}{12}(K_B^2 + c_2(TB)) \\ &= \frac{1}{2}\mathcal{C}_B \cdot \mathcal{C}_B - \frac{1}{2}\mathcal{C}_B \cdot K_B + \frac{1}{12}c_2(TB) \ , \end{aligned} \tag{185}$$

where we have used the fact, from (4), that $K_B^2 = 0$ on a $d\mathbb{P}_9$. We also recall from (9) that $c_2(TB) = 12$, from (43) that $\mathcal{C}_B = m\sigma_B + kf$ and from (5) that $c_1(K_B) = -f$. Together with the intersection numbers given in (10), we find that

$$h^0(B, \mathcal{O}_B(\mathcal{C}_B)) = 1 - \frac{1}{2}m^2 + km + \frac{1}{2}m \ . \tag{186}$$

This expression indeed agrees with the result for the general case of $\mathcal{O}_B(\mathcal{C}_B)$ computed in (91).

Appendix B: $\dim(IV)$ from the Index Theorem

In this Appendix we calculate the dimension of term IV, that is, $h^1(B, M \otimes M^*)$, using the Atiyah-Singer index theorem. We then compare the result to (97) computed in the $d\mathbb{P}_9$ context and find perfect agreement.

First, we note that $H^0(B, M \otimes M^*) = 1$ because M is stable. Then, by Serre duality we have

$$H^2(B, M \otimes M^*) = H^0(B, K_B \otimes M^* \otimes M) \ , \tag{187}$$

which vanishes by the arguments following (68). Therefore, the Atiyah-Singer index theorem

$$\chi(B, M \otimes M^*) = \sum_{i=0}^{2} (-1)^i h^i(B, M \otimes M^*) = \int_B \mathrm{ch}(M \otimes M^*) \wedge \mathrm{td}(TB) \ , \tag{188}$$

now reduces to

$$h^1(B, M \otimes M^*) = 1 - \int_B \mathrm{ch}(M \otimes M^*) \wedge \mathrm{td}(TB) \ . \tag{189}$$

Using (38), (105) and (136), we have

$$\mathrm{ch}(M \otimes M^*) = (m - k \ \mathrm{pt}) \wedge (m - k \ \mathrm{pt}) = m^2 - 2km \ \mathrm{pt} \ , \tag{190}$$

where we have been careful in including the class pt of points. Similarly, from (4), (9) and (107), we have

$$\mathrm{td}(TB) = 1 + \frac{1}{2} f + 1 \ \mathrm{pt} \ . \tag{191}$$

Multiplying (190) with (191), and inserting into (189), we obtain the final result

$$\dim(IV) = h^1(B, M \otimes M^*) = 1 - m^2 + 2km \ , \tag{192}$$

which is in perfect agreement with (97).

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