Counting Gauge Invariants: the Plethystic Program

Bo Feng[1,2*] **Amihay Hanany**[3†], **Yang-Hui He**[4,5‡],

[1]*Blackett Lab. & Inst. for Math. Science, Imperial College, London, SW7 2AZ, U.K.*

[2]*Center of Mathematical Science, Zhejiang University, Hangzhou 310027, P. R. China*

[3]*Perimeter Institute, 31 Caroline St. N., Waterloo, ON, N2L 2Y5, Canada*

[4]*Collegii Mertonensis in Academia Oxoniensis, Oxford, OX1 4JD, U.K.*

[5]*Mathematical Institute, Oxford University, 24-29 St. Giles', Oxford, OX1 3LB, U.K.*

Abstract

We propose a programme for systematically counting the single and multi-trace gauge invariant operators of a gauge theory. Key to this is the plethystic function. We expound in detail the power of this plethystic programme for world-volume quiver gauge theories of D-branes probing Calabi-Yau singularities, an illustrative case to which the programme is not limited, though in which a full intimate web of relations between the geometry and the gauge theory manifests herself. We can also use generalisations of Hardy-Ramanujan to compute the entropy of gauge theories from the plethystic exponential. In due course, we also touch upon fascinating connections to Young Tableaux, Hilbert schemes and the MacMahon Conjecture.

*b.feng@imperial.ac.uk

†ahanany@perimeterinstitute.ca

‡hey@maths.ox.ac.uk

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1 Introduction and Recapitulation

Given a supersymmetric quantum field theory, one of the first quantities one wishes to determine is the spectrum of BPS operators. Such a desire becomes particularly manifest for the class of theories which arise in the AdS/CFT correspondence in string theory. Of special interest are chiral BPS mesonic operators of the 4-dimensional, $\mathcal{N} = 1$ SUSY gauge theory living on D3-branes probing a Calabi-Yau (CY) singularity. Such a setup has been archetypal in the aforementioned AdS/CFT correspondence and when the transverse CY space is trivially $\mathbb{C}^3$, we are in the paradigmatic $\mathcal{N} = 4$ CFT and $AdS_5 \times S^5$ situation of [1]. When the transverse CY is non-trivial, we have new classes of so-called quiver gauge theories, pioneered by [2], which has been extensively developed over the past decade (for a review, q.v. e.g. [3]).

Of vital geometrical significance is the the fact that the BPS mesonic operators form a chiral ring whose relations determine the transverse Calabi-Yau geometry. More technically, the syzygy amongst these gauge invariant operators (GIO's) (modulo F-flatness) gives the equation of the Calabi-Yau threefold as an affine variety. This correspondence is guaranteed by the fact, *per construtio*, the D3-brane probe is a point in the transverse CY. Thus an intimate relation is established between the gauge theory and the algebraic geometry of the transverse space.

In our recent paper [4], we solved the problem of counting these mesonic GIO's for arbitrary singularities, both single-trace and multi-trace, and for both large and finite number of D3-branes. Using results from combinatorics, commutative algebra and number theory, we advocate a **plethystic programme** wherein such counting problem is not only systematically addressed, but also intrinsically linked to the underlying geometry. With a brief recapitulation of this over-arching programme let us first occupy the reader.

To set notation, let a stack of N parallel coincident D3-branes probe a Calabi-Yau singularity $\mathcal{M}$. The mesonic BPS gauge invariant operators fall into two categories: single- and multi-trace. The former consists of words in operators, with gauge-indices contracted but only a single overall trace and the latter, various products of the single-trace GIO's. We let the generating function of the single-trace GIO's be $f_N(t;\ \mathcal{M})$, and that of the multi-trace be $g_N(t;\ \mathcal{M})$. The n-th coefficient in the power expansion for f and g would then give the number of GIO's at level n (where level can be construed as some representative $U(1)$ charge such as the R-charge, in the problem. For simple cases like $\mathbb{C}^n$ or the conifold, a good $U(1)$ charge is the number of operators, but generically it is not a good qauntum number and we will refer to a typical U(1) charge). When there are enough isometries, such as in the case of $\mathcal{M}$ being a toric variety, we can refine the counting and extend f and g to $f_N(t_1, t_2, t_3;\ \mathcal{M})$ and $g_N(t_1, t_2, t_3;\ \mathcal{M})$. Power expansion in the variables $t_{1,2,3}$ again gives the number of GIO's, with the multi-degree now related to global $U(1)$ charges of the problem, including R-charge and other flavour charges. Some of the main results of [4] are then as follows.

- The generating functions obey (we can easily generalise from a single variable t to the tuple $t_{i=1,2,3,\ldots}$):

  $$\boxed{g_1(t) = f_\infty(t); \qquad f_\infty(t) = PE[f_1(t)], \quad g_\infty(t) = PE[g_1(t)]; \qquad g_N(t) = PE[f_N(t)]}$$

  where PE is the **plethystic exponential** function defined as

  $$f(t) = \sum_{n=0}^{\infty} a_n t^n \quad \Rightarrow \quad g(t) = PE[f(t)] = \exp\left(\sum_{n=1}^{\infty} \frac{f(t^n) - f(0)}{n}\right) = \frac{1}{\prod\limits_{n=1}^{\infty}(1-t^n)^{a_n}} .$$

- The quantity $f_\infty = g_1$ is the geometric *point d'appui* and can be directly computed from properties of $\mathcal{M}$. We have called it the (Hilbert-)Poincaré series. In [4], we referred to this as the Poincaré series; it is, in fact, more appropriate, for reasons which shall become clear in §4.4, to call it the **Hilbert series**, an appelation to which we henceforth adhere. When $\mathcal{M}$ is an orbifold $\mathbb{C}^3/G$ for some finite group G, f_∞ is the Molien series [6] (We remark that Molien series and plethysms have appeared in the context of four-dimensional dualities in [5]). When $\mathcal{M}$ is a toric variety, f_∞ can be obtained from the toric diagram [9] (see also related [20, 21, 46]). When $\mathcal{M}$ is a manifold of complete intersection f_∞ can be directly computed by the defining equations of the manifold.

- The inverse function to PE is the plethystic logarithm, given by

$$f(t) = PE^{-1}(g(t)) = \sum_{k=1}^{\infty} \frac{\mu(k)}{k} \log(g(t^k)) , \qquad \mu(k) := \begin{cases} 0 & k \text{ has repeated prime factors} \\ 1 & k = 1 \\ (-1)^n & k \text{ is a product of } n \text{ distinct primes} \end{cases}$$

  where $\mu(k)$ is the Möbius function. The plethystic logarithm of the Hilbert series gives the syzygies of $\mathcal{M}$, i.e.,

  $$\boxed{f_1(t) = PE^{-1}[f_\infty(t)] = \text{defining equation of } \mathcal{M}.}$$

In particular, if $\mathcal{M}$ were complete-intersection, $f_1(t)$ is a polynomial.

- For finite N, define the function $g(\nu;t)$ such that

$$f_\infty(t) = \sum_{n=0}^{\infty} a_n t^n \quad \Rightarrow \quad \boxed{g(\nu;t) := \prod_{n=0}^{\infty} \frac{1}{(1-\nu\, t^n)^{a_n}} = \sum_{N=0}^{\infty} g_N(t)\nu^N} \ .$$

In other words, the ν-expansion of $g(\nu;t)$ gives the generating function $g_N(t)$ of multi-trace GIO's for finite number N of D3-branes. The single-trace generating function $f_N(t)$ is then retrieved from $g_N(t)$ by PE^{-1}. This qualifies ν as the chemical potential for the number of D3-branes.

Crucial to the derivation of the above expression is the almost tautological yet very important fact that

$$g_N(t;\mathcal{M}) = g_1(t;\mathrm{Sym}^N(\mathcal{M})), \qquad \mathrm{Sym}^N(\mathcal{M}) := \mathcal{M}^N/S_N \ .$$

That is to say, the moduli space of a stack of N D3-branes is the N-th symmetrised product of that of a single D3-brane, viz., the Calabi-Yau space $\mathcal{M}$.

The above points highlight the key constituents of the plethystic programme and inter-relates the D-brane quiver gauge theory and the geometry of $\mathcal{M}$. Indeed, one function distinguishes herself, viz., f_∞, which, as a Hilbert series, can be obtained directly from the geometry. Henceforth, as was in [4], we will often denote the fundamental generating function f_∞ and its associated g_∞ simply as f and g.

We emphasise that the applicability of the plethystic programme is not limited to world-volume theories of D-brane probes on Calabi-Yau singularities. Indeed, if we knew the geometry of the classical moduli space of a gauge theory, which may not even be $\mathcal{N} = 1$, and especially if this vacuum space is a complete intersection variety, we could obtain the Hilbert series and thenceforth use the plethystic exponential to find the gauge invariants.

Without much further ado, let us outline the contents of our current paper. In §2 we derive explicit expressions for the plethystic exponential. We will see how to recursively write all g_N generating functions in terms of the fundamental Hilbert series; natural connexions with Young tableaux will arise. Of great importantance will also be the asymptotic behaviour of the multi-trace generating functions g_N and we will see how a result due to Haselgrove and Temperley may be used to generalise the Meinardus theorem. Thus armed, we can estimate the entropy of our gauge theory; this is the subject of §2.5. We will explicitly see the dependence of the critical exponents on the dimension of the geometry and the volume of the Sasaki-Einstein manifold.

With all this technology, we move on to concrete classes of examples. In §3, we analytically compute the number of single-trace operators for the ADE-singularities and give the expressions for the asymptotic behaviour of the number of multi-trace operators. As a passing curiosity, we point out intimate relations to the MacMahon Conjecture. Then, in §4, we compute all fundamental generating functions for Calabi-Yau threefold orbifolds, again, in explicit detail. Subsequently, one can allow discrete torsion in these cases, and see how the plethystic programme also encompasses these classes of theories in §5. As a mathematical aside, we see how the plethystics relate to Hilbert schemes of points in §6. Finally, moving onto toric varieties, we see how the plethystic programme lends itself to deriving the equations for wide classes of moduli spaces, exemplifying with the $Y^{p,q}$ spaces.

2 Explicit Expressions for Plethystics

With the plethystic programme thus outlined above, it is expedient to present some useful results concerning the generating functions f and g. First, let us take a closer look at the fundamental relation of the plethystic inversion formula:

$$\begin{aligned} g(t) &= PE[f(t)] := PE[\sum_{k=0}^{\infty} a_k t^k] = \exp\left[\sum_{p=1}^{\infty} \frac{1}{p}(f(t^p) - f(0))\right] = \prod_{m=1}^{\infty} \frac{1}{(1-t^m)^{a_m}} \Leftrightarrow \\ f(t) - f(0) &= PE^{-1}[g(t)] = \sum_{l=1}^{\infty} \frac{\mu(l)}{l} \log(g(t^l)) \ . \end{aligned} \tag{2.1}$$

The above expression is a central motif for the plethystic programme and the proof of which was not presented in [4], nor, for that matter, could one find it, within a body of literature often obscured by mathematical sophistry, in an explicit fashion. The proof is, in fact, rather straight-forward, which we shall presently see.

Taking the logarithm of the product form of PE in (2.1) and series-expanding, we have

$$\log(g(t)) = \sum_{k=1}^{\infty}(-a_k) \sum_{m=1}^{\infty} -\frac{1}{m}(t^k)^m \ . \tag{2.2}$$

Whence,

$$PE^{-1}[g(t)] = \sum_{l=1}^{\infty} \frac{\mu(l)}{l} \log(g(t^l)) \;=\; \sum_{l=1}^{\infty} \frac{\mu(l)}{l} \left(\sum_{k=1}^{\infty} a_k \sum_{m=1}^{\infty} \frac{1}{m} (t^{lk})^m \right)$$
$$= \sum_{k=1}^{\infty} a_k \sum_{n=1}^{\infty} \sum_{l|n} \mu(l) \frac{1}{n} (t^k)^n \ , \qquad (2.3)$$

where we have re-written the double sum on m and l as the alternative sum on $n = m\,l$ and its divisors l. Using a fundamental theorem of analytic number theory, viz., the Möbius inversion formula [10]

$$\sum_{d|n} \mu(d) = \delta_{n,1} \ ,$$

the double sum $\sum\limits_{n=1}^{\infty} \left(\sum\limits_{l|n} \mu(l) \right) \frac{1}{n} (t^k)^n$ simply reduces to t^k, whereby making the RHS of (2.3) equal to $\sum\limits_{k=1}^{\infty} a_k t^k = f(t) - f(0)$, as is required.

Next, the ν-inserted version of PE is of vital importance:

$$g(\nu, t) = \prod_{m=0}^{\infty} \frac{1}{(1 - \nu\, t^m)^{a_m}} = \sum_{N=0}^{\infty} g_N(t) \nu^N \ . \qquad (2.4)$$

This simple insertion gives us, almost miraculously, the powerful generating functions g_N which capture the multi-trace GIO's for any finite N and from which the counting f_N for the single-trace GIO's can be extracted by the plethystic logarithm, i.e., $f_N = PE^{-1}[g_N(t)]$. The remarkable fact is that $g_N(t)$ requires only the knowledge of the Hilbert series $f(t) := f_\infty(t) = \sum\limits_{m=0}^{\infty} a_m t^m$, which we recall from our outline above, is the fundamental object obtained purely from the geometry of the Calabi-Yau singularity $\mathcal{M}$. Explicit expressions fot g_N, especially its large-N behaviour, are certainly important in, for example, entropy-counting of bulk black-hole states.

2.1 All g_N as Functions of g_1

Now, from the series expansion (2.4), we can find recursion relations among the coefficients of expansion, whereby expressing our desired g_N in terms of the basic Hilbert series $g_1 = f_\infty$. As an enticement, for example, we notice that:

$$\begin{aligned}
\frac{\partial^2 g(\nu,t)}{\partial \nu^2} &= \left(\sum_{k=0}^{\infty} \frac{a_k t^k}{(1-\nu\, t^k)}\right)^2 g(\nu,t) + g(\nu,t)\sum_{k=0}^{\infty} \frac{a_k t^{2k}}{(1-\nu\, t^k)^2}\ , \\
\frac{\partial^3 g(\nu,t)}{\partial \nu^3} &= \left(\sum_{k=0}^{\infty} \frac{a_k t^k}{(1-\nu\, t^k)}\right)^3 g(\nu,t) + 3g(\nu,t)\left(\sum_{k=0}^{\infty} \frac{a_k t^k}{(1-\nu\, t^k)}\right)\left(\sum_{k=0}^{\infty} \frac{a_k t^{2k}}{(1-\nu\, t^k)^2}\right) + \\
&\quad + g(\nu,t)\left(\sum_{k=0}^{\infty} \frac{2a_k t^{3k}}{(1-\nu\, t^k)^3}\right)\ .
\end{aligned}$$

From this we have

$$\begin{aligned}
g_2(t) &= \frac{1}{2!}\frac{\partial^2 g}{\partial \nu^2}|_{\nu=0} = \frac{1}{2}[g_1^2(t) + g_1(t^2)]\ , \\
g_3(t) &= \frac{1}{3!}\frac{\partial^3 g}{\partial \nu^3}|_{\nu=0} = \frac{1}{6}[g_1^3(t) + 3g_1(t)g_1(t^2) + 2g_1(t^3)]\ .
\end{aligned} \tag{2.5}$$

A Systematic Approach: We can obtain the above results more systematically. Recalling that the fundamental definition of PE has two equivalent expressions, as a sum or as a product (q.v. (2.1)), we have that

$$g(\nu,t) = \prod_{m=0}^{\infty} \frac{1}{(1-\nu\, t^m)^{a_m}} = \exp\left(\sum_{k=1}^{\infty} \frac{1}{k} g_1(t^k)\nu^k\right)\ , \tag{2.6}$$

where $g_1(t) = f_\infty(t) = \sum\limits_{m=0}^{\infty} a_m t^m$. Hence,

$$\sum_{N=0}^{\infty} g_N(t)\nu^N = \exp\left(\sum_{k=1}^{\infty} \frac{1}{k} g_1(t^k)\nu^k\right)\ .$$

Expanding the exponential in the RHS gives a series in powers of ν:

$$\begin{aligned}
g(\nu,t) &= 1 + g_1(t)\,\nu + \frac{\left(g_1(t)^2 + g_1(t^2)\right)\ \nu^2}{2} + \frac{\left(g_1(t)^3 + 3\,g_1(t)\,g_1(t^2) + 2\,g_1(t^3)\right)\ \nu^3}{6} + \\
&\quad + \frac{\left(g_1(t)^4 + 6\,g_1(t)^2\,g_1(t^2) + 3\,g_1(t^2)^2 + 8\,g_1(t)\,g_1(t^3) + 6\,g_1(t^4)\right)\ \nu^4}{24} + \mathcal{O}(\nu^5)
\end{aligned}$$

Thus, very straight-forwardly, we obtain the expressions for $g_N(t)$ by simply reading off the coeffcients of ν^N; giving us the desired generating function $g_N(t)$ in terms of the Poincaré series g_1 with powers of its argument t. The results for $N = 2, 3$ are seen to agree with those in (2.5).

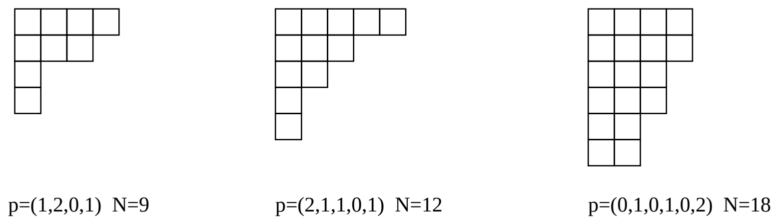

Figure 1: Examples of Young Tableaux with partition $p = \{p_1, p_2, p_3, ..., p_k, ...\}$ and N. The constraint is that $N = \sum\limits_{k=1} p_k k$.

2.2 Relation to Young Tableaux

One can proceed further with the above expansion for g_N, and obtain interesting connections to Young tableaux. From (2.6), one can series-expand the exponential as:

$$\sum_{N=0}^{\infty} g_N(t)\nu^N = \exp\left(\sum_{k=1}^{\infty} \frac{1}{k} g_1(t^k)\nu^k\right) = \prod_{k=1}^{\infty} e^{\left(\nu^k \frac{g_1(t^k)}{k}\right)} = \prod_{k=1}^{\infty}\left(\sum_{p_k=0}^{\infty} \nu^{kp_k} \frac{g_1(t^k)^{p_k}}{p_k!\ k^{p_k}}\right) . \tag{2.7}$$

Now, which terms contribute to ν^N? We see that this is whenever

$$\sum_{k=1}^{\infty} p_k k = N . \tag{2.8}$$

Under this constraint we have the explicit expression for $g_N(t)$ as

$$g_N(t) = \sum_{\substack{p_1,\, p_2,\, ..\\ \sum\limits_{k=1}^{\infty} p_k k \,=\, N}} \prod_{k=1}^{\infty} \frac{(g_1(t^k))^{p_k}}{p_k!\ k^{p_k}} . \tag{2.9}$$

The relation (2.8) is a familiar combinatorial problem: the partition of N into increasing components $k = 1, 2, 3\ldots$ of respective multiplicity p_k. This is, of course, just the Young Tableau; to see it we just draw p_k columns of length k from right to left with k increasing. For clarity, we have drawn a few illustrations in Fig. 1 with given $p = \{p_1, p_2, p_3, ..., p_k, ...\}$. For example, for the first tableau, there is a total of 9 boxes. The vector $(1, 2, 0, 1)$ means that $p_1 = 1, p_2 = 2, p_3 = 0$ and $p_4 = 1$. Now, $p_1 = 1$ means that there is 1 column with only one box; this is the first column from the right. Similarly, there are $p_2 = 2$ columns with 2 boxes and $p_3 = 0$ means there are no columns with 3 boxes. Finally, $p_4 = 1$ means there is one column with 4 boxes, the one to the far left.

We wish to emphasize that the natural emergence of Young Tableaux is not an accident and has deep connections to Hilbert scheme which we will touch upon later.

The reader is referred to the recent works of [11]. At a superficial level, we have related each term in the sum (2.9) to a given Young Tableau. In other words, given a Young Tableau we can count the number of columns with length k, say it is p_k; then we can assign one factor $\frac{(g_1(t^k))^{p_k}}{p_k!\ k^{p_k}}$. Multiplying all factors together we get contribution for the particular Young Tableau. Finally we sum up all Young Tableaux with box number N, giving us the g_N we need.

A Fermionic Version? As a brief digression, one notices that the expression for the plethstic exponential, in its product form, is a generating function for a bosonic oscillator. One might wonder what the fermionic counter-part signifies. In other words, we could define, for $f(t) = \sum\limits_{n=0}^{\infty} a_n t^n$,

$$\widetilde{PE}[f(t)] := \prod_{k=1}^{\infty}(1+t^k)^{a_k}, \qquad \widetilde{PE}_\nu[f(t)] := \prod_{k=0}^{\infty}(1+\nu t^k)^{a_k} \ .$$

It would be interesting to find what these may count in the D-brane gauge theory and what nice inverse functions they possess.

2.3 Generalising Meinardus

The asymptotic expressions for the generating functions are clearly of importance. In [4], we discussed at length the so-called Meinardus theorem [12] which generalises the Hardy-Ramanujan formula for the partition of integers and gives the asymptotics of the function $g_\infty(t)$. Now, what about the aymptotic expressions of $g_N(t)$ where we have a finite number N of D3-branes? In other words, we wish to know, as $n \to \infty$ in the expansion

$$g_N(t) = \sum_{n=0}^{\infty} g_N(n) t^n \ , \qquad (2.10)$$

the behaviour of $g_N(n)$ for a given N.

Thus, we need a generalisation of Meinardus to include ν-insertions. Luckily, there is a result due to Haselgrove-Temperley [13] with certain relaxation of conditions in [14]. The fermionic version mentioned above has its asymptotics studied in detail by [15]. The key result of [13, 14] is, under certain convergence conditions into which we shall not delve, that

THEOREM 2.1. *For* $G(\nu,t) = \prod_{r=1}^{\infty}(1-\nu t^{\lambda_r})^{-1} = \sum_{n,N=0}^{\infty} g_N(n) t^n \nu^N$, *define*

$$\begin{aligned}
\Psi(x) &:= \log G(x) = -\sum_{r=1}^{\infty}\log(1-e^{x\lambda_r}),\\
K(x) &:= \prod_{r=1}^{\infty}\left(1+\frac{x}{\lambda_r}\right)^{-1} e^{x/\lambda_r}, \quad F(y) := \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} K(x) e^{xy} dx,\\
\xi &:= \text{a root of } \Psi'(\xi)+n=0\ , \quad N_0 := \sum_{r=1}^{\infty}(e^{\xi\lambda_r}-1)^{-1},
\end{aligned}$$

then, the asymptotics (for n *large and* N *fixed) are:*

$$g_N(n) \sim \xi F((N-N_0)\xi)\ g(n), \qquad g(n) \sim (2\pi\Psi''(\xi))^{-\frac{1}{2}} e^{\Psi(\xi)+n\xi}\ .$$

Of course, we need to recast our $g(\nu,t)$ in (2.4) into the form which the theorem addresses; this is a redefinition of the λ_r in terms of the a_m to eliminate repetitions:

$$\lambda_r = \begin{cases} 1 & r = a_0,\ \ldots,\ a_1; \\ 2 & r = a_1+1,\ \ldots,\ a_1+a_2; \\ 3 & r = a_1+a_2+1,\ \ldots,\ a_1+a_2+a_3; \\ \ldots & \end{cases} \tag{2.11}$$

We see that the function $G(x) = \exp(\Psi(x))$ is when the ν-insertion is absent (note that here counting does start from $r=1$) and should capture the original Meinardus result for the plethystic exponential. Importantly, a key property of $\Psi(x)$, in terms of the a_m coefficients (cf. [16]), is that its asymptotic behaviour is

$$G(x) = e^{\Psi(x)} = \prod_{r=1}^{\infty}(1-e^{xr})^{-a_r} \sim \exp\left[A\Gamma(\alpha)\zeta(\alpha+1)x^{-\alpha} - D(0)\log x + D'(0)\right]\ , \tag{2.12}$$

where $D(s) := \sum_{m=1}^{\infty}\frac{a_m}{m^s}$ is the Dirichlet series which has only 1 simple pole at $s=\alpha\in\mathbb{R}_+$ with residue A.

Using (2.12) and its derivative, we see that the quantities ξ and $g(n)$ in Theorem 2.1 explicitly evaluate to, for large n,

$$\begin{aligned}
\xi &\sim \text{Root}\left[-A\Gamma(\alpha+1)\zeta(\alpha+1)x^{-\alpha-1} - D(0)/x + n = 0\right] \sim \left(\frac{1}{n}A\Gamma(\alpha+1)\zeta(\alpha+1)\right)^{\frac{1}{\alpha+1}}\\
g(n) &\sim C_1 n^{C_2}\exp\left[n^{\frac{\alpha}{\alpha+1}}(1+\frac{1}{\alpha})\left(A\Gamma(\alpha+1)\zeta(\alpha+1)\right)^{\frac{1}{\alpha+1}}\right]\\
C_1 &:= e^{D'(0)}\frac{1}{\sqrt{2\pi(\alpha+1)}}\left(A\Gamma(\alpha+1)\zeta(\alpha+1)\right)^{\frac{1-2D(0)}{2(\alpha+1)}}\ , \quad C_2 := \frac{D(0)-1-\frac{\alpha}{2}}{\alpha+1}\ .
\end{aligned} \tag{2.13}$$

We see that $g(n)$ above is exactly the Meinardus result [12, 16] for the asymptotics of the plethystic exponential without ν-insertion (cf. also, Section 6 of [4]). In other words, the content of Theorem 2.1 is that the pre-factor

$$\xi F\left((N-N_0)\xi\right) \tag{2.14}$$

encodes the effects of ν-insertion, i.e., the N-dependence, to the classical Meinardus asymptotic formula for $g(n)$ in (2.13). For values of $n<N$ the expression for $g_N(n)$ should coincide precisely with that of $g_\infty(n)$ as the pre-factor tends to 1. On the other hand, for $n>N$ there will be corrections and the $g_N(n)$ is expected to be smaller than $g_\infty(n)$; this is because the counting should be less at finite N since there are constraints which vanish at infinite N.

Example: $\mathbb{C}$ Let us first check a simple case. Let $a_m=1$ for all $m\in\mathbb{Z}_{\geq 0}$. This is where the Hilbert series is equal to $f_\infty(t)=(1-t)^{-1}$ and we recall [4] that the geometry is just $\mathbb{C}$. The conversion (2.11) makes $\lambda_r=r$, which is a specific example considered on p238 of [13], giving us

$$K(x)=\prod_{r=1}^{\infty}\left(1+\frac{x}{r}\right)^{-1}e^{x/r}=e^{\gamma x}\Gamma(x+1), \qquad F(y)=\exp\left(-(\gamma+y)-e^{-(\gamma+y)}\right)\ ,$$

where $\gamma:=\lim\limits_{n\to\infty}\left(\sum\limits_{j=1}^{n}j^{-1}-\log(n)\right)$ is the Euler constant[1]. The Dirichlet series is here $D(s)=\sum\limits_{m=1}^{\infty}m^{-s}=\zeta(s)$; whence $\alpha=A=1$ with $D(0)=-\frac{1}{2}$ and $D'(0)=-\frac{1}{2}\log(2\pi)$. Therefore, (2.12) dictates that

$$\Psi(x)\sim\frac{\pi^2}{6x}+\frac{1}{2}\log x-\frac{1}{2}\log 2\pi\Rightarrow\Psi'(x)\sim-\frac{\pi^2}{6x^2}+\frac{1}{2x},\ \ \Psi''(x)\sim\frac{\pi^2}{3x^3}-\frac{1}{2x^2}\ .$$

By (2.13) we thus have

$$\xi\sim\frac{-3+\sqrt{24n\pi^2+9}}{12n}\sim\frac{\pi}{\sqrt{6n}}\ ,\quad g(n)\sim\left(2\pi\Psi''(\xi)\right)^{-\frac{1}{2}}e^{\Psi(\xi)+n\xi}\sim\frac{1}{4\sqrt{3}n}e^{\pi\sqrt{2n/3}}\ . \tag{2.15}$$

Indeed, $g(n)$ is exactly the famous Hardy-Ramanujan asymptotic behaviour for the η-function. The effect of the ν-insertion is then apparent in the pre-factor governed by

[1] Indeed, we can see this since $F(y)=\sum\limits_{n=-1,-2,-3,\ldots}\operatorname*{Res}\limits_{z\to n}\Gamma(z+1)e^{\gamma z+zy}=\sum\limits_{n=0}^{\infty}\frac{(-1)^n}{n!}e^{-(n-1)(\gamma+y)}=e^{-1/a}/a$, for $a=\exp(\gamma+y)$.

the function F. Now, we see that, for small x,

$$N_0(x) = \sum_{r=1}^{\infty} (\exp(rx)-1)^{-1} \sim \sum_{r=1}^{1/x} \frac{1}{rx} + \sum_{r=1/x}^{\infty} \exp(-rx) = -\frac{H(1/x)}{x} + \frac{e^{x-1}}{e^x-1} \sim -\frac{\log x}{x} \ , \tag{2.16}$$

where we have series-expanded for the first part of the sum ($H(x)$ is the Harmonic number) and neglected the small contribution of the -1 in the denominator for the second sum. Therefore, since n is large, we can apply (2.16) to give us

$$N_0 = N_0(\xi) \sim \frac{\sqrt{6n}}{\pi} \log \frac{\sqrt{6n}}{\pi} \ .$$

Thus, we can write the pre-factor in (2.14) (n is large and N is fixed) as

$$\begin{aligned} \xi F\left((N-N_0)\xi\right) \ &\sim \ \tfrac{\pi}{\sqrt{6n}} \exp\left(-(\gamma + (N-N_0)\tfrac{\pi}{\sqrt{6n}}) - e^{-(\gamma+(N-N_0)\frac{\pi}{\sqrt{6n}})}\right) \\ &\sim \ \tfrac{\pi}{\sqrt{6n}} \exp\left(\log \tfrac{\sqrt{6n}}{\pi} - \tfrac{N\pi}{\sqrt{6n}} - e^{\log \frac{\sqrt{6n}}{\pi} - \frac{N\pi}{\sqrt{6n}}}\right) \sim \exp(-\tfrac{N\pi}{\sqrt{6n}} - \tfrac{\sqrt{6n}}{\pi} e^{-\frac{N\pi}{\sqrt{6n}}}) \ . \end{aligned}$$

In summary, we have the asymptotic expansion of $g_N(n)$ as

$$g_N(n;\mathbb{C}) \sim \frac{1}{4\sqrt{3}n} \exp\left(\pi\sqrt{\frac{2n}{3}}\right) \exp\left[-\frac{N\pi}{\sqrt{6n}} - \frac{\sqrt{6n}}{\pi} e^{-\frac{N\pi}{\sqrt{6n}}}\right] \ . \tag{2.17}$$

We have actually reproduced a classical result of [17], which is also studied recently in Bose-Einstein condensates in [18]. Specifically, the above result agrees completely with Eq.(13) of [18], wherein they have simplified the expression to $\frac{g(n)}{\sqrt{n}} \exp(-\frac{2}{c}\exp(x_N(n)) - x_N(n))$ with $c = \sqrt{\frac{2}{3}}\pi$, $g(n)$ given in (2.15) and $x_N(n) := \frac{cN}{2\sqrt{n}} - \log(\sqrt{n})$.

2.4 A Large Class of Examples

Thus emboldened, we may proceed to more examples. Since the plethystic exponential has a singularity at $t = 1$ at $\nu = 1$, it is expedient to study contributions of the form

$$f(t) = f_\infty(t;\mathcal{M}) = g_1(t;\mathcal{M}) = \frac{V_3}{(1-t)^3} + \frac{V_2}{(1-t)^2} + \frac{V_1}{1-t} + V_0 + \mathcal{O}(1-t) \ ; \tag{2.18}$$

we go up to poles of order 3 because $\mathcal{M}$ is at most 3-dimensional in the cases of concern. Physically, V_3 can be thought of as the volume of the dual AdS horizon, i.e., the normalised volume of the Sasaki-Einstein manifold (cf. [9, 20, 21, 22]), and V_i are related to the components of the Reeb vectors.

It turns out, for what we shall shortly describe in the next section, that we do not need as refined an attack as Haselgrove-Temperley, but, rather, a leading order analysis. Indeed, the results of [13] for $d > 1$ require a regularisation into whose subtleties we presently do not wish to venture. We shall, instead, follow the saddle-point method in the physics literature, such as [19]. Indeed we are essentially studying a contour integral

$$g_N(n) = \frac{1}{(2\pi i)^2} \oint_{\Gamma_{\nu=0}} d\nu \oint_{\Gamma_{t=0}} dt \frac{g(\nu,t)}{\nu^{N+1} t^{n+1}} \ ,$$

which picks up the residues at the poles and the form in (2.18) will be dominant. The statement, with the same notations as above, is as follows. For both N and n large (note that Haselgrove-Temperley only requires that n be large),

$$\begin{array}{l} g_N(n) \sim g(\nu_0, t_0) \nu_0^{-N-1} t_0^{-n-1}, \quad \text{where} \\ \left[N+1 = \nu \frac{\partial}{\partial \nu} \log g(\nu,t) \right]_{\nu_0, t_0} \ , \left[n+1 = t \frac{\partial}{\partial t} \log g(\nu,t) \right]_{\nu_0, t_0} \ . \end{array} \tag{2.19}$$

We can first directly evaluate $\log g(\nu,t)$. From (2.18), we have that

$$\begin{aligned} a_n &= V_3 \frac{(n+1)(n+2)}{2} + V_2(n+1) + V_1 + V_0 \delta_{n,0} \qquad \Rightarrow \\ \log g(\nu,t) &= -\sum_{n=0}^{\infty} a_n \log(1-\nu t^n) \\ &= -V_0 \log(1-\nu) + \sum_{k=1}^{\infty} \frac{\nu^k}{k} \left[\frac{1}{2} V_3 Li_{-2}(t^k) + (V_2 + \frac{3}{2} V_3) Li_{-1}(t^k) + (V_1+V_2+V_3)(1+Li_0(t^k)) \right] \ , \end{aligned}$$

where we have used the definition of the Polylogarithmic function $Li_d(x) = \sum\limits_{n=1}^{\infty} \frac{x^n}{n^d}$. In fact, for $d \in \mathbb{Z}_{\leq 0}$, these are simply rational functions.

Recall now that we wish to study the behaviour of $g(\nu,t)$ near $t=1$ and $\nu=1$. Hence, we can define $t := e^{-q}$ and $\nu = e^{-w}$ and will study the behaviour near $q, w \to 0$. Series expanding $\log g(\nu,t)$ and keeping dominant contributions in the inverses of q and w, we find that

$$\begin{aligned} \log g(w,q) &\sim \sum_{k=1}^{\infty} \frac{\nu^k}{k} \left[V_0 + \frac{V_1}{2} + \frac{5\,V_2}{12} + \frac{3\,V_3}{8} + \frac{V_1+V_2+V_3}{k\,q} + \frac{2\,V_2 + 3\,V_3}{2\,k^2\,q^2} + \frac{V_3}{k^3\,q^3} \right] \\ &\sim \frac{V_3}{q^3}(\zeta(4) - \zeta(3)w) - (V_0 + \frac{V_1}{2} + \frac{5\,V_2}{12} + \frac{3\,V_3}{8}) \log(w) \ . \end{aligned} \tag{2.20}$$

We are now ready to solve for the saddle points given in (2.19). Since $t = e^{-q}, \nu =$

e^{-w}, we have $t\frac{\partial}{\partial t} = -\frac{\partial}{\partial q}$ and $\nu\frac{\partial}{\partial \nu} = -\frac{\partial}{\partial w}$ and the saddle equations read:

$$\begin{aligned} n+1 &= -\frac{\partial \log g(w,q)}{\partial q} \sim 3\zeta(4)V_3 q^{-4} \ , \\ N+1 &= -\frac{\partial \log g(w,q)}{\partial w} \sim \zeta(3)V_3 q^{-3} + (V_0 + \frac{V_1}{2} + \frac{5\,V_2}{12} + \frac{3\,V_3}{8})w^{-1} \ . \end{aligned}$$

Therefore, the saddle points are

$$q_0 \sim \left(\frac{3V_3\zeta(4)}{n}\right)^{\frac{1}{4}} \ , \qquad w_0 \sim (V_0 + \frac{V_1}{2} + \frac{5\,V_2}{12} + \frac{3\,V_3}{8})\left(N - \zeta(3)V_3 q_0^{-3}\right)^{-1} \tag{2.21}$$

These results are encouraging. For $V_{0,1,2} = 0$ and $V_3 = 1$, the case was studied in nice detail in [19]. The expressions in (2.21), to leading order, agree exactly with their Eq. (17-19), in *cit. ibid.* Substituting back into (2.19), we conclude that, to leading order,

$$\begin{aligned} \log g_N(n) &\sim \log g(\nu_0, t_0) + Nw + nq \\ &\sim C_0 n^{\frac{3}{4}} + C_1 \left[\frac{N}{N - C_2 n^{\frac{3}{4}}} + \log\left(N - C_2 n^{\frac{3}{4}}\right)\right] \ ; \qquad (2.22) \\ C_0 &:= 3^{-\frac{3}{4}} 4 (V_3\zeta(4))^{\frac{1}{4}}, \quad C_1 := V_0 + \frac{V_1}{2} + \frac{5\,V_2}{12} + \frac{3\,V_3}{8}, \quad C_2 := \zeta(3)V_3^{\frac{1}{4}}(3\zeta(4))^{-\frac{3}{4}} \ . \end{aligned}$$

Once more, we are re-assured. The first term, which only depends on n, should be the classical Meinardus result while the second is the pre-factor (2.14) discussed above. We have done the Meinardus analysis for $\mathbb{C}^3$ in [4]; substituting $\zeta(4) = \frac{\pi^4}{90}$ gives us the first term as $\frac{2\cdot 2^{\frac{3}{4}}\pi}{3\cdot 15^{\frac{1}{4}}} n^{\frac{3}{4}}$, precisely the exponent of p_1 in Eq (6.7) of [4].

Another interesting limit to consider is $t \sim 1$ and $\nu \sim 0$. Here, we expand about $q = -\log t$ and ν directly and (2.20) becomes

$$\log g(\nu, t) \sim \nu V_3 q^{-3} \ , \tag{2.23}$$

giving us the saddle points $q_0 \sim 3N/n$ and $\nu_0 \sim (3N/n)^3 N/V_3$. Thus,

$$\log g_N(n) \sim 4N\left(\log(N^{-1}n^{\frac{3}{4}}) + 1 - \frac{1}{4}\log\frac{27}{V_3}\right) \ . \tag{2.24}$$

Note that in order for $\nu \sim 0$, we need $N \ll n^{\frac{3}{4}}$.

2.5 The Entropy of Quiver Theories

Having expounded upon a collection of examples and demonstrated the explicit power of the Halselgrove-Temperley result as well as saddle-point evaluations in generalising

Meinardus, let us now address a problem of great physical interest. A chief motivation for finding explicit expressions, in paricular the asymptotic behaviour, of our generating functions is to determine the number of degrees of freedom, i.e., the entropy of the gauge theory. Indeed, as the Hardy-Ramanujan formula is central to the determining the entropy of the bosonic critical string, the results presented in the previous section will be essential to that of D-brane probe theories.

The growth of the number of our mesonic BPS operators in the gauge theory can be a good estimate for the entropy of the system. More generally, it serves as a lower bound for the total number of operators in the gauge theory, regardless of whether they are BPS or not. Thus if we are looking for an underlying black hole entropy, the discussions above will be greatly pertinent. Specifically, in our context of the gauge theory of N D-branes probing a geometry $\mathcal{M}$, we can define the entropy $S_N(n)$ as

$$S_N(n) = \log g_N(n) \quad \text{where} \quad g(\nu, t; \mathcal{M}) := \sum_{N,n=0}^{\infty} g_N(n) t^n \nu^N \ , \tag{2.25}$$

and we recall that $g(\nu, t; \mathcal{M})$ is the ν-inserted plethystic exponential of the Hilbert series (the fundamental generating function f) of the geometry of $\mathcal{M}$.

Now, we would like to compute critical exponents depending on dimensionality. Therefore, we need to consider the generalisation of (2.18) to

$$f(t) = \frac{V_d}{(1-t)^d} + \ldots + \frac{V_2}{(1-t)^2} + \frac{V_1}{1-t} + V_0 + \mathcal{O}(1-t) \ . \tag{2.26}$$

Following the computation performed above, we easily see that the saddle points are (in the $t, \nu \sim 1$ limit) now

$$q_0 \sim \left(\frac{dV_d \zeta(d+1)}{n}\right)^{\frac{1}{d+1}} \ , \qquad w_0 \sim \left[V_0 + \sum_{j=1}^{d} V_j (1 + \sum_{i=0}^{j-1} \beta_i \zeta(-i))\right] \left(N - \zeta(d) V_d q_0^{-d}\right)^{-1} \ , \tag{2.27}$$

where β_i are coefficients such that

$$\binom{n+d-1}{d-1} := \sum_{i=0}^{d-1} \beta_i n^i \ . \tag{2.28}$$

Substituting into the saddle point equation, we find the entropy to be

$$S_N(n) \sim C_0 n^\alpha + C_1 \left[\frac{N}{N - C_2 n^\alpha} + \log\left(N - C_2 n^\alpha\right)\right] \ , \tag{2.29}$$

where the critical exponent is $\alpha = \frac{d}{d+1}$ and the constants are

$$\begin{aligned}
C_0 &= d^{-\frac{d}{d+1}}(d+1)(V_d\zeta(d+1))^{\frac{1}{d+1}} \ , \\
C_1 &= V_0 + \sum_{j=1}^{d} V_j(1+\sum_{i=0}^{j-1}\beta_i\zeta(-i)) \ , \\
C_2 &= \zeta(d)V_d^{\frac{1}{d+1}}(d\zeta(d+1))^{-\frac{d}{d+1}} \ .
\end{aligned}$$

We remark, upon obtaining a similar expression as (2.24) for $\nu \sim 0$, that our treatment gives rise to a critical regime in which there is a cross over between $\nu \sim 0$ and $\nu \sim 1$. This critical regime is given by the order parameter $N \sim n^{d/d+1}$ or, alternatively, $n \sim N^{1+1/d}$. When the two sides are of the same order we are in the $\nu \sim 1$ regime and the number of operators is controlled by n essentially. When the order parameter is small the number of operators depends on N.

3 $SU(2)$ Subgroups: ADE Revisited

We have, in the above, discussed extensively the various general properties of the generating functions, the recursions, relations to Young tableaux, and especially the asymptotics. Now, let us move on to some specific examples. An extensively studied class of CY singularities are orbifold theories. Of particular mathematical interest has been the local-K3 singularities, viz., $\mathbb{C}^2/G$ where G is a discrete, finite subgroup of $SU(2)$. Such groups fall under an ADE-pattern and the quivers are central to the McKay Correspondence.

In [4], we computed the fundamental generating functions, i.e., the Hilbert series $g_1 = f_\infty$. We recall that for orbifolds of finite group G, the Hilbert series is computed by the so-called Molien series

$$f_\infty(t;G) = M(t;G) = \frac{1}{|G|}\sum_{g\in G}\frac{1}{\det(\mathbb{I}-tg)} \ . \tag{3.1}$$

A natural question to ask is what explicit expressions can be derived for g_N at finite N. Using the prescription in the previous section, we can readily expand a few terms of (2.7) to see what we obtain. Take the example of $G = \hat{D}_4$, the binary dihedral group of order 8, which was investigated in detail in [4], the Hilbert series is the Molien series

$$g_1(t) = M(t;\ \hat{D}_4) = \frac{1+t^6}{(1-t^4)^2} \ . \tag{3.2}$$

Substituting (3.2) into (2.7), we obtain ($g_0(t) = 1$ automatically):

$$g_2(t) = \frac{1 - t^2 + t^6 + t^8 - t^{12} + t^{14}}{(1-t^2)\,(1-t^4)^2\,(1-t^8)}\,, \qquad g_3(t) = \frac{1 - t^2 + 2\,t^8 + t^{12} + t^{18} + 2\,t^{22} - t^{28} + t^{30}}{(1-t^2)\,(1-t^4)^2\,(1-t^6)\,(1-t^8)\,(1-t^{12})}\,, \quad \ldots$$

We see that these coefficients quickly become complicated. Nevertheless, the algorithm is clear and one may extract g_N *ad libertum.*

3.1 Recursion Relations and Difference Equations

Let us entice the reader with some immediately noticeable curiosities for the series-coefficients for the Hilbert (Molien) series for the ADE orbifolds. Take the A-family (where $\hat{A}_{n-1} := \mathbb{Z}_n$). We recall from [4] that

$$f_\infty(t;\ \hat{A}_{n-1}) = \frac{(1+t^n)}{(1-t^2)(1-t^n)}\ .$$

We see that

$$\begin{array}{rcl} f_\infty(t;\ \hat{A}_1) & = & 1 + 3\,t^2 + 5\,t^4 + 7\,t^6 + 9\,t^8 + 11\,t^{10} + 13\,t^{12} + 15\,t^{14} + 17\,t^{16} + 19\,t^{18} + 21\,t^{20} + \mathcal{O}(t^{21}) \\ f_\infty(t;\ \hat{A}_3) & = & 1 + t^2 + 3\,t^4 + 3\,t^6 + 5\,t^8 + 5\,t^{10} + 7\,t^{12} + 7\,t^{14} + 9\,t^{16} + 9\,t^{18} + 11\,t^{20} + \mathcal{O}(t^{21}) \\ f_\infty(t;\ \hat{A}_5) & = & 1 + t^2 + t^4 + 3\,t^6 + 3\,t^8 + 3\,t^{10} + 5\,t^{12} + 5\,t^{14} + 5\,t^{16} + 7\,t^{18} + 7\,t^{20} + \mathcal{O}(t^{21}) \end{array} \tag{3.3}$$

Thus, for $n = 2k$ even, the pattern of the coefficients is $\{1, \ldots, 1;\ 3, \ldots, 3;\ 5, \ldots, 5; \ldots\}$.

In fact, we will now proceed to find analytic expressions for the series-coefficients, i.e., the number of single-trace GIO's, of f_∞ for all the discrete, finite subgroups of $SU(2)$. This indeed places our generating function in full power and provide us with invariants of arbitrary degree immediately. The reason we can do so is because the Molien series is a rational function in t and indeed, for any rational function, one could systematically obtain recursion relations, which can then be solved. It is easiest to start with the exceptionals, i.e., the E-family, with which we shall commence our illustration.

The $\hat{E}_6$ Singularity: For $\hat{E}_6$, we recall from [4] that

$$f = \frac{1 - t^4 + t^8}{1 - t^4 - t^6 + t^{10}} = 1 + t^6 + t^8 + 2\,t^{12} + t^{14} + \mathcal{O}(t^{16}) := \sum_{k=0}^{\infty} a_k t^k\ . \tag{3.4}$$

Multiplying through by the denominator gives us

$$
\begin{aligned}
1-t^4+t^8 &= \sum_{k=0}^{\infty} a_k t^k - \sum_{k=4}^{\infty} a_{k-4} t^k - \sum_{k=6}^{\infty} a_{k-6} t^k + \sum_{k=10}^{\infty} a_{k-10} t^k \qquad (3.5)\\
&= \sum_{k=0}^{9} a_k t^k - \sum_{k=4}^{9} a_{k-4} t^k - \sum_{k=6}^{9} a_{k-6} t^k + \sum_{k=10}^{\infty} \left(a_k - a_{k-4} - a_{k-6} + a_{k-10}\right) \ .
\end{aligned}
$$

Identifying the coefficients of powers of t, this readily gives us the recursion relation:

$$
a_k = a_{k-4} + a_{k-6} - a_{k-10}, \quad k \geq 10 \ . \qquad (3.6)
$$

There should be 10 initial conditions for a_k, which could be obtained by matching the 1 as well as the $-t^4$ and t^8 terms in the LHS with the various finite sum pieces in the RHS of (3.5). Alternatively, it is easier to simply read off the first 10 values of a_k in the series expansion in (3.4), giving us

$$
a_{0,6,8} = 1, \ \text{else,} \ a_{k<10} = 0 \ .
$$

Of course, all linear homogeneous difference equations of this kind can be solved. Upon substitution of the ansatz $a_k = t^k$ for some $t \in \mathbb{C}$, one obtains the eigen-equation for t which is simply the denominator $1 - t^4 - t^6 + t^{10}$ in (3.4). This has 10 roots: $\{\omega_6^{i=0,\ldots,5}, \pm i\}$ with double roots at 1 and -1. Using the usual trick that for each multiple root λ of order m, there are extra roots $k^{j=1,\ldots,m-1}\lambda^k$, the solution is reaily found to be

$$
\begin{aligned}
a_k \ = \ & (-1)^k \ (c(1) + k\, c(2)) + c(3) + k\, c(4) + c(5) \ \cos(\frac{k\,\pi}{3}) + \\
& c(6) \ \cos(\frac{k\,\pi}{2}) + c(7) \ \cos(\frac{2\,k\,\pi}{3}) + c(8) \ \sin(\frac{k\,\pi}{3}) + c(9) \ \sin(\frac{k\,\pi}{2}) + c(10) \ \sin(\frac{2\,k\,\pi}{3}) \ ,
\end{aligned}
$$

with initial constants $c(i)$, $i = 1, \ldots, 10$. Matching these with the 10 initial conditions in (3.6) gives us the final solution

$$
\begin{aligned}
a_k \ = \ & \frac{1}{72}\left[3\,\left(1+(-1)^k\right)\,(1+k) + 18\,\cos(\frac{k\,\pi}{2}) + 24\,\left(\cos(\frac{k\,\pi}{3}) + \cos(\frac{2\,k\,\pi}{3})\right) + \right.\\
& \left. +8\,\sqrt{3}\,\left(\sin(\frac{k\,\pi}{3}) - \sin(\frac{2\,k\,\pi}{3})\right)\right], \qquad k = 0, 1, \ldots
\end{aligned}
$$

There is an obvious cyclicity of 12 and $a_{12m} = 1 + m$ for $m \in \mathbb{Z}_{\geq 0}$. We will shortly see this in another guise in §3.2.

The $\hat{E}_7$ Singularity: For $\hat{E}_7$, we have [4] that

$$f = \frac{1-t^6+t^{12}}{1-t^6-t^8+t^{14}} = 1+t^8+t^{12}+t^{16}+t^{18}+t^{20}+2\,t^{24}+t^{26}+t^{28}+t^{30}+2\,t^{32}+\mathcal{O}(t^{34}) \ ,$$

giving us the recursion relations

$$a_k = a_{k-6}+a_{k-8}-a_{k-14}, \quad k \geq 14, \qquad a_{0,8,12} = 1, \text{ else, } a_{k<14} = 0 \ . \tag{3.7}$$

This can be readily solved using the above methods to be, for $k = 0, 1, \ldots$,

$$\begin{aligned} a_k &= \frac{1}{144}\left[3\left(1+(-1)^k\right)(1+k)+2\cos(\frac{k\,\pi}{2})\left(27+24\cos(\frac{k\,\pi}{6})+\right.\right. \\ &\qquad \left.\left. 18\left(\cos(\frac{k\,\pi}{4})-\sin(\frac{k\,\pi}{4})\right)-8\sqrt{3}\sin(\frac{k\,\pi}{6})\right)\right] \ . \end{aligned}$$

Again, there is an obvious cyclicity of 24 and $a_{24m} = 1+m$ for $m \in \mathbb{Z}_{\geq 0}$.

The $\hat{E}_8$ Singularity: For $\hat{E}_8$, we have that [4]

$$\begin{aligned} f = \frac{1+t^2-t^6-t^8-t^{10}+t^{14}+t^{16}}{1+t^2-t^6-t^8-t^{10}-t^{12}+t^{16}+t^{18}} &= 1+t^{12}+t^{20}+t^{24}+t^{30}+t^{32}+t^{36}+t^{40}+t^{42}+ \\ & \quad t^{44}+t^{48}+t^{50}+t^{52}+t^{54}+t^{56}+2\,t^{60}+t^{62}+\mathcal{O}(t^{64}) \ , \end{aligned}$$

giving us the recursion relations

$$a_k = -a_{k-2}+a_{k-6}+a_{k-8}+a_{k-10}+a_{k-12}-a_{k-16}-a_{k-18} \quad k \geq 18, \qquad a_{0,12} = 1, \text{ else, } a_{k<18} = 0 \ .$$

Again, this can be solved exactly, giving us, for $k = 0, 1, \ldots$,

$$\begin{aligned} a_k &= \frac{1}{1800}\left[15\left(1+(-1)^k\right)(1+k)+36\sqrt{5\left(5-2\sqrt{5}\right)}\left(\sin(\frac{2\,k\,\pi}{5})-\sin(\frac{3\,k\,\pi}{5})\right)+\right. \\ &\quad +36\sqrt{5\left(5+2\sqrt{5}\right)}\left(\sin(\frac{k\,\pi}{5})-\sin(\frac{4\,k\,\pi}{5})\right)+ \\ &\quad \left.+10\cos(\frac{k\,\pi}{2})\left(45+36\left(\cos(\frac{k\,\pi}{10})+\cos(\frac{3\,k\,\pi}{10})\right)+60\cos(\frac{k\,\pi}{6})-20\sqrt{3}\sin(\frac{k\,\pi}{6})\right)\right] \ . \end{aligned}$$

Once more, there is an obvious cyclicity of 60 and $a_{60m} = 1+m$ for $m \in \mathbb{Z}_{\geq 0}$.

The $\hat{A}_n$ Family: Now, let us move on to the infinite families. For, $\hat{A}_{n-1}$, we have that, letting $f_\infty(t;\ \hat{A}_n) = \frac{(1+t^n)}{(1-t^2)(1-t^n)} := \sum\limits_{k=0}^{\infty} a_k t^k$,

$$\begin{aligned} 1+t^n &= \sum_{k=0}^{\infty} a_k t^k - \sum_{k=2}^{\infty} a_{k-2} t^k - \sum_{k=n}^{\infty} a_{k-n} t^k + \sum_{k=n+2}^{\infty} a_{k-n-2} t^k \\ &= (a_0+a_1 t+\ldots+a_{n+1}t^{n+1}) - (a_0 t^2+a_1 t^3+\ldots+a_{n-1}t^{n+1}) - \\ &\quad -(a_0 t^n + a_1 t^{n+1}) + \sum_{k=n+2}^{\infty}(a_k - a_{k-2} - a_{k-n} + a_{k-n-2})\,t^k \ . \end{aligned} \tag{3.8}$$

Identifying coefficients of t, we have that

$$a_k = a_{k-2} + a_{k-n} - a_{k-n-2}, \quad k \geq n+2 \; ; \tag{3.9}$$

we still need $n+2$ initial conditions. One is obvious, $a_0 = 1$, the remaining can be obtained by solving for the system of associated equations above for $a_1, \ldots, a_{n+1}$.

Now, we could solve this recursion equation, which is rather difficult because of the determination of these initial conditions. However, in this case, it is far easier to simply observe the pattern and conclude that

$$\begin{array}{ll} n = \text{odd} & a_k = \text{floor}(\frac{k}{n}) + \frac{1}{2}\left(1 + (-1)^{\bmod (k,n)}\right) \\ n = \text{even} & a_k = \left(\text{floor}(\frac{k}{n}) + \frac{1}{2}\right)\left(1 + (-1)^{\bmod (k,n)}\right) \end{array} \tag{3.10}$$

Again, the cyclicities are apparent: for odd n, $a_{k=2\beta n} = 2\beta$ and for even n, $a_{k=2\beta n} = 4\beta + 1$ for $\beta \in \mathbb{Z}_{\geq 0}$. We will write these coefficients explicitly later using the MacMahon and Dedekind functions in §3.2.

The $\hat{D}_n$ Family: For the $\hat{D}_{n+2}$ groups, the recursion relation reads

$$f_\infty(t;\, \hat{D}_{n+2}) = \frac{(1+t^{2n+2})}{(1-t^4)(1-t^{2n})} := \sum_{k=0}^{\infty} a_k t^k, \qquad a_k = a_{k-4} + a_{k-2n} - a_{k-2n-4}; \quad k \geq 2n+4 \,, \tag{3.11}$$

together with $2n+4$ initial conditions.

Once again, it is easier to directly observe the pattern here. First, we notice that, upon making the substitution $t^2 \to t$, the Hilbert series becomes quite analogous to the A-series. Indeed, in analogy to (3.3), we find that the coefficients for even n come in periodicity of order n and that for the k-th period the even coefficients are $2k+1$ and the odd coefficients are 0. We will use this in writing expressions for the generating function in §3.2.

In summary, we may observe the pattern of the expansion coefficients as:

$$\frac{(1+t^{n+1})}{(1-t^2)(1-t^n)} := \sum_{k=0}^{\infty} b_k t^k \quad \Rightarrow \quad b_k = \frac{1}{2}\left(1 + (-1)^k\right) + \text{floor}\left(\frac{1}{n} \bmod (k, 2n)\right) + 2\,\text{floor}\left(\frac{k}{2n}\right) \,. \tag{3.12}$$

Therefore, upon restoring $t \to t^2$, we only have even powers; whence, for all $k = 0, 1, \ldots$,

$$a_k = \begin{cases} 0, & k \text{ odd;} \\ \frac{1}{2}\left(1 + (-1)^{k/2}\right) + \text{floor}\left(\frac{1}{n} \bmod (k, 4n)\right) + 2\,\text{floor}\left(\frac{k}{4n}\right), & k \text{ even.} \end{cases} \tag{3.13}$$

3.2 Full Generating Functions: MacMahon and Euler

Having obtained analytic expressions for the counting of single-trace GIO's, i.e., the coefficient of the fundamental generating function f_∞, the Hilbert series, we can say something further about the plethystic exponentials. The expressions for the ADE orbifolds can be represented as infinite sums. Such sums appear in different counting formulae for integer partitions under special restrictions. For example, it is not surprising to find that the multi-trace generating function for $\mathbb{C}^2$,

$$g_\infty(t;\mathbb{C}^2) = \exp\left(\sum_{n=1}^{\infty}\frac{1}{n}\left((1-t^n)^{-2}-1\right)\right) = 1+2t^2+6t^3+14t^4+33t^5+70t^6+\ldots \quad (3.14)$$

generates the sequence of the number of partitions of n objects with 2 colors [24]. It would be interesting to find similar results for the ADE series.

Now, we can use an alternative representation for the generating functions such as (2.4). For the exmaple of $\mathbb{C}^2$, we recall that

$$g(\nu;t;\mathbb{C}^2) = \prod_{n=0}^{\infty}(1-\nu t^n)^{-(n+1)}.$$

We note that the coefficients a_n have a linear piece and a constant piece. This property turns out to be generic for all 2 dimensional singular manifolds. We will therefore define two basic functions. First, let the generalized MacMahon function be:

$$M(\nu;t) := \prod_{n=1}^{\infty}(1-\nu t^n)^{-n} \ ; \quad (3.15)$$

next, let the generalized Dedekind Eta function (in this form it is actually the generalised Euler function, which differs from the Eta function by the famous factor of $t^{-1/24}$) be defined as:

$$\eta(\nu;t) := \prod_{n=0}^{\infty}(1-\nu t^n)^{-1} \ . \quad (3.16)$$

In terms of these functions we can now rewrite

$$g(\nu;t;\ \mathbb{C}^2) = M(\nu;\ t)\eta(\nu;\ t) \ .$$

We now wonder if this form of the expression can be done for the ADE orbifolds due to the fact that the coefficients a_n for the Hilbert (Molien) series, as we recall from [4], are always of a linear and a constant form, corresponding to the functions M and η, respectively. This turns out to be correct.

Let us look, for example, at the generating function for $\mathbb{C}^2/\mathbb{Z}_2$. We find that

$$g(\nu;t;\ \mathbb{C}^2/\mathbb{Z}_2) = \prod_{n=0}^{\infty}(1-\nu t^{2n})^{-(2n+1)} \ ,$$

which can be easily rewritten as

$$g(\nu;t;\ \mathbb{C}^2/\mathbb{Z}_2) = M(\nu;t^2)^2\eta(\nu;t^2) \ .$$

To proceed with the full A-family, we now use the periodic pattern for the coefficient a_n which was obtained above in (3.10), and obtain the following succint expression for the generating function $g(\nu,\ t)$:

$$\begin{aligned} g(\nu;t;\mathbb{C}^2/\mathbb{Z}_{2k}) &= \prod_{j=0}^{k-1} M(\nu t^{2j};t^{2k})^2\ \eta(\nu t^{2j};t^{2k}) \\ g(\nu;t;\mathbb{C}^2/\mathbb{Z}_{2k+1}) &= \prod_{j=0}^{2k} M(\nu t^{j};t^{2k+1}) \prod_{j=0}^{k}\eta(\nu t^{2j};t^{2k+1}) \end{aligned}$$

Similarly, we can obtain the full-generating function for the D-family:

$$\begin{aligned} g(\nu,t;\mathbb{C}^2/\hat{D}_{2k}) &= \prod_{j=0}^{2k-3} M(\nu t^{2j};t^{4k-4}) \prod_{j=0}^{k-2}\eta(\nu t^{4j};t^{4k-4}) \\ g(\nu,t;\mathbb{C}^2/\hat{D}_{2k+1}) &= \prod_{j=0}^{4k-3} M(\nu t^{2j};t^{8k-4})^2 \prod_{j=0}^{2k-2} \eta(\nu t^{4j};t^{8k-4}) \prod_{j=0}^{2k-2} \eta(\nu t^{2j+4k-2};t^{8k-4}) \ . \end{aligned}$$

Finally, for the E-family, as mentioned above we find that each of the Hilbert series come with a quasi-periodicity of 12, 24, and 60 for $\hat{E}_{6,7,8}$, respectively, which can be seen from the explicit expressions for the coefficients in the various equations for a_k given in the previous subsection. The growth of the coefficients is always linear in these periods. Furthermore, odd powers never appear. Therefore, one can write vectors of length 6, 12, and 30, which will denote the starting powers of the coefficients. Explicitly, we have:

$$\begin{aligned} v_{E6} &= \{1,0,0,1,1,0\} \\ v_{E7} &= \{1,0,0,0,1,0,1,0,1,1,1,0\} \\ v_{E8} &= \{1,0,0,0,0,0,1,0,0,0,1,0,1,0,0,1,1,0,1,0,1,1,1,0,1,1,1,1,1,0\} \ . \end{aligned}$$

The generating functions then take the form

$$\begin{aligned}
g(\nu,t;\mathbb{C}^2/\hat{E}_6) &= \prod_{j=0}^{5} M(\nu t^{2j};t^{12})\eta(\nu t^{2j};t^{12})^{v^j_{E_6}} \\
g(\nu,t;\mathbb{C}^2/\hat{E}_7) &= \prod_{j=0}^{11} M(\nu t^{2j};t^{24})\eta(\nu t^{2j};t^{24})^{v^j_{E_7}} \\
g(\nu,t;\mathbb{C}^2/\hat{E}_8) &= \prod_{j=0}^{29} M(\nu t^{2j};t^{60})\eta(\nu t^{2j};t^{60})^{v^j_{E_8}} \ .
\end{aligned}$$

Two curiosities are perhaps worthy of note. First, the periodicities of the coefficients in the Hilbert series are, respectively, one half the order of the finite groups themselves. Second, for each of the vectors $v_{E6,E7,E8}$ above, one can draw a line at the middle, then upon mirror reflection about this line, a zero is mapped to a one, and vice versa.

3.3 Asymptotic Expansions for g_∞

As was emphasised in [4] as well as the proceeding discussions, the asymptotic behaviour of g_∞ is of great interest. Using the Meinardus Theorem, we can estimate the asymptotic behaviour for g_∞ for the ADE-singularities. Though the expressions for the a_k are, evidently, quite involved, the large k behaviour is dominated by the term proportional to k, which can be directly observed; other eigenvalues have less than unit modulus and decay *ad nullam*. We wish to find d_m in

$$\prod_{k=1}^{\infty} \frac{1}{(1-\ t^k)^{a_k}} := \sum_{m=0}^{\infty} d_m t^m$$

for large m.

It suffices to see the large k behaviour of a_k for the ADE-orbifolds. We see, from the expressions above, that all a_k are essentially linear in k. For $\hat{A}_{n-1}$, n odd, the coefficient of the linearity is simply $1/n$. For all other cases, the coefficient is the reciprocal of $1/2$ the order of the group. However, for all these cases, exactly $1/2$ of the terms are zero and contribute 1 to the product. Therefore, overall, the effective large k-behaviour is still simply the reciprocal of the order of the group. Hence, we conclude that

$$\text{For } G = ADE, \qquad a_k \sim \frac{k}{|G|} \ . \tag{3.17}$$

Now, we are at liberty to use the Meinardus analysis. For $a_k \sim k$, we recall from [4] that this is the case of the MacMahon function, whose behaviour goes as

$$\prod_{k=1}^{\infty} \frac{1}{(1-\ t^k)^k} := \sum_{m=0}^{\infty} \varphi(m) t^m, \quad \Rightarrow \quad \varphi(m) \sim \frac{2^{-\frac{11}{36}} \zeta(3)^{\frac{7}{36}} e^{\frac{1}{12}}}{G_l \sqrt{3}\,\pi} m^{-\frac{25}{36}} \exp\left(\frac{3}{2} (2\,\zeta(3))^{\frac{1}{3}} m^{\frac{2}{3}}\right), \tag{3.18}$$

with $G_l := \frac{1}{12} - \zeta'(-1)$ being the Glaisha constant. Thus, we see that for G being an ADE-group,

$$d_m \sim \varphi(m)^{\frac{1}{|G|}} \ . \tag{3.19}$$

In fact, taking the logarithm of this expression will give us the entropy of the quiver gauge theory as discussed in §2.5. We conclude that the entropy is reduced by a factor of $|G|$ and this is the natural expectation from an extensive parameter like the entropy since, by the orbifold action, we are losing $|G|$ of the degrees of freedom.

Incidentally, the MacMahon function is the generating function for the plane-partition problem which is a generalisation of the Young Tableaux to 2-dimensions. That is, consider an integer m, how many ways are there to write

$$m = \sum_{i,j} n_{i,j} \quad \text{such that} \quad n_{i+1,j} \geq n_{i,j},\ n_{i,j+1} \geq n_{i,j}; \quad n_{i,j} \in \mathbb{Z}_+ \ .$$

The answer was shown in [25] to be precisely $\varphi(m)$. The 1-dimensional partition problem, i.e., how many Young Tableaux (also called Ferrers Diagram) are there of a given total number of squares, is simply the standard partitioning problem. By this we mean how many ways, irrespectively ordering, are there to write a given integer m as sums of integers. This is because we could always order the parts in decreasing fashion and arrive at a Young Tableau. The generating function here is simply the famous Euler function $\prod\limits_{k=1}^{\infty} \frac{1}{(1-t^k)}$. It is a curious fact that 3 and higher dimensional analogues of the problem remain unsolved. A conjecture was made in [25] which was later shown to be incorrect.

We see that counting GIO's for the ADE gauge theories is related to the 2-dimensional counting problem in a simple fashion: generating function, asymptotically, is simply that of the MacMahon to the $|G|$-th root. This can be conceived of tiling, asymptotically, not the whole plane, but rather, a $|G|$-th fraction of the plane, as the orbifold indeed requires. However, to which exact partition problems the ADE results correspond remains elusive.

3.4 The MacMahon Conjecture

One could imagine what the result for solid-partitions, which, as mentioned above, is unknown, might actually be. Let us tabulate the result for the gauge theories for $\mathbb{C}$ and $\mathbb{C}^2$. We recall from [4] that

$$
\begin{aligned}
f_\infty(t;\ \mathbb{C}) &= \frac{1}{1-t} = \sum_{k=0}^{\infty} t^k, \quad \Rightarrow \quad g_\infty(t;\ \mathbb{C}) = \prod_{k=1}^{\infty} \frac{1}{(1-\ t^k)};\\
f_\infty(t;\ \mathbb{C}^2) &= \frac{1}{(1-t)^2} = \sum_{k=0}^{\infty} (k+1)t^k, \quad \Rightarrow \quad g_\infty(t;\ \mathbb{C}^2) = \prod_{k=1}^{\infty} \frac{1}{(1-\ t^k)^{k+1}} \ .
\end{aligned}
$$

Thus we see that the multi-trace problem for $\mathbb{C}$ counts the 1-dimensional partition; that for $\mathbb{C}^2$, when shifted by 1, counts the 2-dimensional problem. It is perhaps natural to guess that the one for $\mathbb{C}^3$, when shifted by one, would give the generating function for the 3-dimensional partition problem, i.e.,

$$
\begin{aligned}
& f_\infty(t;\ \mathbb{C}^3) = \frac{1}{(1-t)^3} = \sum_{k=0}^{\infty} \frac{(k+1)(k+2)}{2} t^k, \quad \text{shift} \Rightarrow a_k = \frac{k(k+1)}{2} \quad \Rightarrow \\
& g_\infty = \prod_{k=1}^{\infty} \frac{1}{(1-\ t^k)^{\frac{k(k+1)}{2}}} = 1 + t + 4\,t^2 + 10\,t^3 + 26\,t^4 + 59\,t^5 + 141\,t^6 + 310\,t^7 + 692\,t^8 + \mathcal{O}(t^9) \ .
\end{aligned}
\tag{3.20}
$$

Unfortunately, this leads us back to MacMahon's erroneous guess [25]. The correct numbers, as generated by exhaustive computer simulation of the explicit partitions, should be (cf. e.g. [26]):

$$
1, 1, 4, 10, 26, 59, 140, 307, 684, 1464, 3122, 6500, 13426, 27248, 54804, 108802, \ldots \tag{3.21}
$$

One sees that starting from the term 141, the generating function g_∞ in (3.20) over-counts. The actual series a_k which *does* generate the correct numbers can be easily found, by taking the plethystic logarithm, to be (cf. also [26])

$$
\begin{aligned}
a_{k=1,2,\ldots} \quad = \quad & \{1, 3, 6, 10, 15, 20, 26, 34, 46, 68, 97, 120, 112, 23, -186, -496, \\
& -735, -531, 779, 3894, 9323, 16472, 23056, 23850, 10116, \ldots\} \ .
\end{aligned}
\tag{3.22}
$$

As it is evident, the negative entries complicate things; it suggests that a_k itself cannot be a Hilbert series. Could it be the plethystic logarithm of a Hilbert series? Recall that for these, there are often negative entries, signifying relations among fundamental invariants. Well, it certainly is the plethystic logarithm of something; this is after all, how the series (3.22) was obtained from (3.21), but this brings us to to where we started. What if we took the plethystic log of (3.22) itself? Unfortunately, we obtain nothing particularly enlightening.

4 All $SU(3)$ Subgroups

We have discussed the ADE-groups above in some detail; of perhaps more physical interest are the orbifolds of $\mathbb{C}^3$. These are local Calabi-Yau threefolds that give rise to $\mathcal{N}=1$ 4-dimensional chiral gauge theories on the D3-brane world-volume. The quiver theories were studied in [23] and using the notation therein, the discrete finite subgroups of $SU(3)$ are:

(I) The infinite family $\mathbb{Z}_m \times \mathbb{Z}_n$;

(II) The infinite families $\Delta(3n^2)$ and $\Delta(6n^2)$;

(III) The exceptionals Σ_{60}, Σ_{108}, Σ_{168}, Σ_{216}, Σ_{648}, and Σ_{1080}.

The theory of Molien series and algebraic invariants is nicely exposed in [6], wherein some explicit Molien series are also computed for the discrete subgroups of $SL(3;\mathbb{C})$.

In order to explicitly write the generators of the groups, first, define

$$\omega_n := \exp(\frac{2\pi i}{n}),$$

and the matrices

$$\begin{array}{c}
S := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega_3 & 0 \\ 0 & 0 & {\omega_3}^2 \end{pmatrix}, \quad S_1 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & {\omega_5}^4 & 0 \\ 0 & 0 & \omega_5 \end{pmatrix}, \quad S_2 := \begin{pmatrix} \omega_7 & 0 & 0 \\ 0 & {\omega_7}^2 & 0 \\ 0 & 0 & {\omega_7}^4 \end{pmatrix}; \\
T := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad T_1 := \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 1 & 1 \\ 2 & \frac{1}{2}(-1-\sqrt{5}) & \frac{1}{2}(-1+\sqrt{5}) \\ 2 & \frac{1}{2}(-1+\sqrt{5}) & \frac{1}{2}(-1-\sqrt{5}) \end{pmatrix}, \quad T_2 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}; \\
R := -\frac{1}{\sqrt{-7}} \begin{pmatrix} -{\omega_7}^3 + {\omega_7}^4 & {\omega_7}^2 - {\omega_7}^5 & \omega_7 - {\omega_7}^6 \\ {\omega_7}^2 - {\omega_7}^5 & \omega_7 - {\omega_7}^6 & -{\omega_7}^3 + {\omega_7}^4 \\ \omega_7 - {\omega_7}^6 & -{\omega_7}^3 + {\omega_7}^4 & {\omega_7}^2 - {\omega_7}^5 \end{pmatrix}; \\
U := \begin{pmatrix} {\omega_9}^4 & 0 & 0 \\ 0 & {\omega_9}^4 & 0 \\ 0 & 0 & \omega_3\,{\omega_9}^4 \end{pmatrix}, \quad U_1 := \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}; \\
V := \frac{1}{\sqrt{-3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega_3 & {\omega_3}^2 \\ 1 & {\omega_3}^2 & \omega_3 \end{pmatrix}, \quad V_1 := \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & \frac{1}{4}(-1+\sqrt{-15}) & \frac{1}{4}(-1+\sqrt{-15}) \\ \frac{1}{2}(-1-\sqrt{-15}) & \frac{1}{2}(-1-\sqrt{5}) & \frac{1}{2}(-1+\sqrt{5}) \\ \frac{1}{2}(-1-\sqrt{-15}) & \frac{1}{2}(-1+\sqrt{5}) & \frac{1}{2}(-1-\sqrt{5}) \end{pmatrix}; \\
A(m) := \begin{pmatrix} \omega_m & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \omega_m^{-1} \end{pmatrix}, \quad B(n) := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega_n & 0 \\ 0 & 0 & \omega_n^{-1} \end{pmatrix}.
\end{array} \tag{4.1}$$

Finally, we adhere to the usual notation that

$$G = \langle g_1, \ldots, g_k \rangle$$

is the finite group G generated by matrices $g_1, \ldots g_k$.

4.1 The Abelian Series: $\mathbb{Z}_m \times \mathbb{Z}_n$

The first of our series is simply $\mathbb{Z}_m \times \mathbb{Z}_n = \langle A(m),\ B(n) \rangle$. The Molien series is given by

$$f(t;\ \mathbb{Z}_m \times \mathbb{Z}_n) = \frac{1}{mn} \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} \det \left(\mathbb{I}_{3\times 3} - t \begin{pmatrix} \omega_m^i & 0 & 0 \\ 0 & \omega_n^j & 0 \\ 0 & 0 & \omega_m^{-i}\omega_n^{-j} \end{pmatrix} \right)^{-1} \tag{4.2}$$

$$= \frac{1}{mn} \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} \frac{1}{(1 - t\omega_m^i)(1 - t\omega_n^j)(1 - t\omega_m^{-i}\omega_n^{-j})} = \frac{1}{mn} \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} \sum_{p,q,r=0}^{\infty} t^p \omega_m^{ip} t^q \omega_n^{jq} t^r \omega_m^{-ir} \omega_n^{-jr} \ .$$

Using the identity

$$\sum_{i=0}^{m-1} \omega_m^{ix} = m \delta_{x, m\mathbb{Z}} \ ,$$

where the Kronecker-Delta is 1 whenever x is a multiple of n, we can see that non-zero contributions come from

$$p = r + \widetilde{p}\ m \ \text{ for } \widetilde{p} = -[\frac{r}{m}], -[\frac{r}{m}] + 1, \ldots; \qquad q = r + \widetilde{q}\ n \ \text{ for } \widetilde{q} = -[\frac{r}{n}], -[\frac{r}{n}] + 1, \ldots.$$

Here, $[\frac{r}{m}]$ means to take the integer part (i.e., floor(r/m)). Whence,

$$f(t;\ \mathbb{Z}_m \times \mathbb{Z}_n) = \sum_{r=0}^{\infty} \sum_{\widetilde{p}=-[\frac{r}{m}]}^{\infty} \sum_{\widetilde{q}=-[\frac{r}{n}]}^{\infty} t^{3r + \widetilde{p}\ m + \widetilde{q}\ n} = \sum_{r=0}^{\infty} \frac{t^{3r - [\frac{r}{m}]m - [\frac{r}{n}]n}}{(1 - t^m)(1 - t^n)} \ .$$

To go further, we can write $r = \widetilde{r} + LCM(m,n)z$ for $z = 0, 1, ..., \infty$ and $\widetilde{r} = 0, 1, ..., LCM(m,n) - 1$, where LCM is the lowest common multiple. Using this parametrization, the sum reduces to

$$\begin{aligned} f(t;\ \mathbb{Z}_m \times \mathbb{Z}_n) &= \frac{1}{(1 - t^m)(1 - t^n)} \sum_{\widetilde{r}=0}^{LCM(m,n)-1} \sum_{z=0}^{\infty} t^{3\widetilde{r} - [\frac{\widetilde{r}}{m}]m - [\frac{\widetilde{r}}{n}]n} t^{LCM(m,n)z} \\ &= \frac{1}{(1 - t^m)(1 - t^n)(1 - t^{LCM(m,n)})} \sum_{\widetilde{r}=0}^{LCM(m,n)-1} t^{3\widetilde{r} - [\frac{\widetilde{r}}{m}]m - [\frac{\widetilde{r}}{n}]n} \ . \end{aligned} \tag{4.3}$$

The above expression becomes particularly simple in the case of perhaps greatest interest, viz, when $m = n$; here $LCM(m,n) = m$ and within the range of summation of $\widetilde{r}$, $[\frac{\widetilde{r}}{m}]$ is zero, hence

$$f(t;\ \mathbb{Z}_m \times \mathbb{Z}_m) = \frac{1 - t^{3m}}{(1 - t^3)(1 - t^m)^3} \ . \tag{4.4}$$

Taking the plethystic logarithm of (4.4) gives polynomials, suggesting that $\mathbb{C}^3/(\mathbb{Z}_m\times\mathbb{Z}_m)$ are all complete intersections! Explicitly, we have that

$$f_1(t;\ \mathbb{Z}_m \times \mathbb{Z}_m) = PE^{-1}[f(t;\ \mathbb{Z}_m \times \mathbb{Z}_m)] = \left\{ \begin{array}{cc} 3t & m=1, \\ 3t^2+t^3-t^6 & m=2, \\ 4t^3-t^9 & m=3, \\ t^3+3t^m-t^{3m} & m\geq 4. \end{array} \right. \tag{4.5}$$

The $m=1$ case is a good check; this is simply the result for the parent $\mathbb{C}^3$ theory.

Refinement: As was pointed out in [4], where there are enough isometries, as certainly is the case with toric varieties, refinements can be made to the Molien series. The above Abelian series are indeed toric, hence one could write the refined Molien (Hilbert) series as

$$f(t_1,t_2,t_3;\ \mathbb{Z}_m \times \mathbb{Z}_n) = \frac{1}{mn}\sum_{i=0}^{m-1}\sum_{j=0}^{n-1} \det\left(\mathbb{I}_{3\times 3} - \begin{pmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & t_3 \end{pmatrix}\begin{pmatrix} \omega_m^i & 0 & 0 \\ 0 & \omega_n^j & 0 \\ 0 & 0 & \omega_m^{-i}\omega_n^{-j} \end{pmatrix}\right)^{-1} .$$

This can, using the above reparametrisation of the summation variables, be re-written as

$$f(t_1,t_2,t_3;\ \mathbb{Z}_m \times \mathbb{Z}_n) = \frac{1}{(1-t_1^m)(1-t_2^n)(1-t_3^{LCM(m,n)})}\sum_{\tilde{r}=0}^{LCM(m,n)-1} (t_1t_2t_3)^{\tilde{r}} t_1^{-[\frac{\tilde{r}}{m}]m} t_2^{-[\frac{\tilde{r}}{n}]n} .$$

Once again, for the case of $m=n$, the expression simplifies considerably:

$$f(t_1,t_2,t_3;\ \mathbb{Z}_m \times \mathbb{Z}_m) = \frac{1-(t_1t_2t_3)^m}{(1-t_1t_2t_3)(1-t_1^m)(1-t_2^m)(1-t_3^m)} . \tag{4.6}$$

The plethystic logarithm of this expression becomes particularly simple:

$$f_1(t_1,t_2,t_3;\ \mathbb{Z}_m \times \mathbb{Z}_m) = t_1^m + t_2^m + t_3^m + t_1t_2t_3 - (t_1t_2t_3)^m , \qquad m=1,2,3,\ldots \tag{4.7}$$

4.2 Non-Abelian Subgroups

Having expounded upon the $\mathbb{Z}_m\times\mathbb{Z}_n$ series in detail, we can proceed to the non-Abelian groups. The Molien series for the exceptionals can be quite simply computed by [28] and are presented in the next subsection. The two Delta-series maybe dealt with in much the same manner as the abovementioned $\mathbb{Z}_m\times\mathbb{Z}_m$.

The elements of $\Delta(3n^2) := \langle A(n), B(n), T\rangle$ fall into three classes, viz, the orbits of $\mathbb{Z}_n^2 \simeq \langle A(n), B(n)\rangle$ under $\{\mathbb{I}, T, T^2\}$ since the matrix T, which we recall from (4.1), is of order 3. Therefore,

$$\begin{aligned} f(t;\ \Delta(3n^2)) &= \frac{1}{3n^2}[\sum_{i,j=0}^{n-1} \det\left(\mathbb{I}_{3\times 3} - t\left(\begin{smallmatrix} \omega_n^i & 0 & 0 \\ 0 & \omega_n^j & 0 \\ 0 & 0 & \omega_n^{-i-j} \end{smallmatrix}\right)\right)^{-1} + \det\left(\mathbb{I}_{3\times 3} - t\left(\begin{smallmatrix} 0 & \omega_n^j & 0 \\ 0 & 0 & \omega_n^{-i-j} \\ \omega_n^i & 0 & 0 \end{smallmatrix}\right)\right)^{-1} \\ &\quad + \det\left(\mathbb{I}_{3\times 3} - t\left(\begin{smallmatrix} 0 & 0 & \omega_n^{-i-j} \\ \omega_n^i & 0 & 0 \\ 0 & \omega_n^j & 0 \end{smallmatrix}\right)\right)^{-1}] \\ &= \frac{1}{3n^2}\left[n^2 \frac{1-t^{3n}}{(1-t^3)(1-t^n)^3} + n^2\frac{1}{1-t^3} + n^2\frac{1}{1-t^3}\right] \\ &= \frac{1-t^n+t^{2n}}{(1-t^3)(1-t^n)^2}\ . \end{aligned} \tag{4.8}$$

One could in fact take the plethystic logarithm and see that these are complete intersections:

$$f_1(t;\ \Delta(3n^2)) = PE^{-1}[f(t;\ \Delta(3n^2))] = \begin{cases} t+t^2+2\,t^3-t^6 & n=1, \\ t^2+t^3+t^4+t^6-t^{12} & n=2, \\ 2\,t^3+t^6+t^9-t^{18} & n=3, \\ t^3+t^n+t^{2n}+t^{3n}-t^{6n} & n\geq 4. \end{cases} \tag{4.9}$$

In complete analogy, $\Delta(6n^2) := \langle A(n), B(n), T, T_2\rangle$. In fact $\Delta(6n^2) \simeq \Delta(6(2n)^2)$, thus it suffices to consider only odd n, and we have that

$$f(t;\ \Delta(6n^2)) = \frac{1+t^{6n+3}}{(1-t^6)(1-t^{2n})(1-t^{4n})}, \qquad n = 1, 3, 5, \ldots\ .$$

Again, taking the plethystic logarithm shows these to be complete intersections

$$f_1(t;\ \Delta(6n^2)) = PE^{-1}[f(t;\ \Delta(6n^2))] = \begin{cases} t^2+t^4+t^6+t^9-t^{18} & n=1, \\ 2\,t^6+t^{12}+t^{21}-t^{42} & n=3, \\ t^6+t^{2n}+t^{4n}+t^{6n+3}-t^{12n+6} & n\geq 5. \end{cases} \tag{4.10}$$

4.3 Summary of $SU(3)$ Subgroups

We now summarise and tabulate the relevant results for all the discrete subgroups of $SU(3)$. The Molien series for the exceptional ones have been computed in [6]. The above results for the infinite families are new. In addition, we compute the plethystic logarithm of the Molien series, which should give us the defining equations; a remarkable fact is that all of them (except $\mathbb{Z}_m \times \mathbb{Z}_n$ for $m \neq n$, whose Molien series we have not been able to simplify further) are complete intersections. We use the notation (cf. [35] for gauge theory moduli spaces of this type)

$$(p, (d_1, \ldots, d_k);\ (\ell_1^{a_1}, \ldots, \ell_q^{a_q}))$$

to denote the intersection of k equations, of degrees $d_1, \ldots, d_k$ in $\mathbb{C}^p$, composed of a_1 invariants of degree ℓ_1, a_2 invariants of degree ℓ_2, etc.

The generating functions (Molien series) $f = f_\infty(t)$ and the associated f_1 which encode the syzygies (defining equations) for the discrete, finite subgroups of $SU(3)$ are:

$G \subset SU(3)$	Generators	Molien $f(t;G)$	$f_1 = PE^{-1}(f)$	Defining Equation
$\mathbb{Z}_m \times \mathbb{Z}_n$	$\langle A(m),\ B(n)\rangle$	q.v. (4.3)	q.v. (4.5)	$-$
$\Delta(3n^2)$	$\langle A(n),\ B(n),\ T\rangle$	$\frac{1-t^n+t^{2n}}{(1-t^3)(1-t^n)^3}$	q.v. (4.9)	q.v. (4.12)
$\Delta(6n^2)$	$\langle A(n),\ B(n),\ T,\ T_2\rangle n$ odd	$\frac{1+t^{6n+3}}{(1-t^6)(1-t^{2n})(1-t^{4n})}$	q.v. (4.10)	q.v. (4.13)
Σ_{60}	$\langle S_1,\ T_1,\ U_1\rangle$	$\frac{1+t^{15}}{(1-t^2)(1-t^6)(1-t^{10})}$	$t^2+t^6+t^{10}+t^{15}-t^{30}$	$(4,(30);\ (2,6,10,15))$
Σ_{108}	$\langle S,\ T,\ V\rangle$	$\frac{1+t^9+t^{12}+t^{21}}{(1-t^6)^2(1-t^{12})}$	$2\,t^6+t^9+2\,t^{12}-t^{18}-t^{24}$	$(5,(18,24);\ (6^2,9,12^2))$
Σ_{168}	$\langle S_2,\ T,\ R\rangle$	$\frac{1+t^{21}}{(1-t^4)(1-t^6)(1-t^{14})}$	$t^4+t^6+t^{14}+t^{21}-t^{42}$	$(4,(42);\ (4,6,14,21))$
Σ_{216}	$\langle S,\ T,\ V,\ UVU^{-1}\rangle$	$\frac{1+t^{12}+t^{24}}{(1-t^6)(1-t^9)(1-t^{12})}$	$t^6+t^9+2\,t^{12}-t^{36}$	$(4,(36);\ (6,9,12^2))$
Σ_{648}	$\langle S,\ T,\ V,\ U\rangle$	$\frac{1+t^{18}+t^{36}}{(1-t^9)(1-t^{12})(1-t^{18})}$	$t^9+t^{12}+2\,t^{18}-t^{54}$	$(4,(54);\ (9,12,18^2))$
Σ_{1080}	$\langle S_1,\ T_1,\ U_1,\ V_1\rangle$	$\frac{1+t^{45}}{(1-t^6)(1-t^{12})(1-t^{30})}$	$t^6+t^{12}+t^{30}+t^{45}-t^{90}$	$(4,(90);\ (6,12,30,45))$

(4.11)

In the table, the defining equations for the Delta-series are:

$$\Delta(3n^2) \simeq \begin{cases} (4,(6);\ (1,2,3,3)) & n=1, \\ (4,(12);\ (2,3,4,6)) & n=2, \\ (4,(18);\ (3,3,6,9)) & n=3, \\ (4,(6n);\ (3,n,2n,3n)) & n \geq 4. \end{cases} \tag{4.12}$$

and

$$\Delta(6n^2)_{n \text{ odd}} \simeq \begin{cases} (4,(18);\ (2,4,6,9)) & n=1, \\ (4,(42);\ (6,6,12,21)) & n=3, \\ (4,(12n+6);\ (6,2n,4n,6n+3)) & n \geq 5. \end{cases} \tag{4.13}$$

The exceptional groups are addressed in [6] and our defining equations, obtained from f_1, agrees completely with Theorem C of p7 therein. The forms of the actual equations, with the coefficients, are very complicated and the reader is referred to the aforementioned theorem in *cit. ibid.*

4.4 The Fundamental Generating Function: The Hilbert Series

Before we proceed to discuss other fascinating features of $\mathbb{C}^3$-orbifolds in the ensuing section, let us venture on a small digression. In many expressions above, we have seen the power of the plethystic programme: how the plethystic logarithm of the Molien series encodes the geometrical information of the orbifold, with the situation even more conspicuous for complete intersections. We advertised in the introduction and in [4], the paramountcy of the fundamental generating function $f_\infty = g_1$, here we shall explain why it should capture the geometry.

Let us give the formal definition of the Hilbert Series (cf. e.g. [29]). Let $M := \bigoplus_i M_i$ be a graded module over $K[x_1, \ldots, x_n]$ (for K some field) with respect to weights $w_1, \ldots w_n$, then the Hilbert Series is the generating function for the dimension of the graded pieces:

$$H(t; M) := \sum_i \dim_K(M_i) t^i \ .$$

Usually, we take K to be $\mathbb{C}$ and are working over polynomials in n variables; in this case, the grading i can be taken to be the total degree and $\dim_K(M_i)$ is simply the number of independent polynomials at degree i. The fundamental property of the Hilbert Series is that it is, in fact, a rational function, of the form

$$H(t; M) = \frac{Q(t)}{\prod_{i=1}^{n}(1 - t^{w_i})} \ , \tag{4.14}$$

where $Q(t)$ is some, in general rather complicated, polynomial.

In the case of orbifolds, the Molien series counts the invariant polynomials of a given degree. Since the syzygies (relations) of these invariants define the orbifold as a variety, the Molien series is therefore the Hilbert series for the orbifold [6]. It just so happens that in this case, we have a nice way to compute the Hilbert series, using the data of the finite group, viz., expression (3.1). In the case of toric singularities, the situation is similar, the equivariant index of [9] and the equivalent sum over vertices in (p, q)-webs in [4], reduces the evaluation of the Hilbert series to combinatorics of the toric diagram.

Let us illustrate the foregoing generalities. Take our familiar $\Delta(27)$ orbifold; we recall from (4.8) that $f_\infty(t; \Delta(27) = \frac{1-t^3+t^6}{(1-t^3)^3}$. Now, this can in fact be re-written into what was dubbed "Euler form" in [4], i.e.,

$$f_\infty(t; \Delta(27) = \frac{1 - t^3 + t^6}{(1 - t^3)^3} = \frac{1 - t^{18}}{(1 - t^3)^2 \ (1 - t^6) \ (1 - t^9)} \ . \tag{4.15}$$

In this form, both numerator and denominator are products of $(1-t^{w_i})$ factors. Then, from (4.14), geometrically, $\mathbb{C}^3/\Delta(27)$ could be realised in $\mathbb{C}[x_1,\ldots,x_4]$ with weights $(3,3,6,9)$. This choice of weights arises because the 4 primitive invariants polynomials in the coordinates (x,y,z) of $\mathbb{C}^3$ are respectively of degrees 3,3,6 and 9. Recall further, from §5.2 of [4], that ($m_i, w_j \geq 0$ and not necessarily distinct)

$$PE^{-1}\left[\frac{\prod_i(1-t^{m_i})}{\prod_j(1-t^{w_j})}\right] = \sum_i t^{m_j} - \sum_j t^{w_j} \ ,$$

we have from (4.15) that

$$f_1(t;\Delta(27)) = PE^{-1}[f_\infty(t;\Delta(27))] = 3\,t^3 + t^6 + t^9 - t^{18} \ ,$$

in agreement with (4.9). Therefore, indeed the Hilbert series has the promised properties and indeed we see why f_1 should encode the geometric information of the variety.

Two cautionary notes. Though the denominator of the Hilbert series is always in Euler form, specifiying essentially the information about the embedding space, the numerator $Q(t)$ is in general complicated. When $Q(t)$ can indeed be placed into Euler form, f_1 terminates and $\mathcal{M}$ is a complete intersection; otherwise, f_1 is an infinite series, encoding progressively higher syzygies. Second, the form of the Hilbert series is sensitively dependent on the choice of embedding. Had we not chosen the weights $(3,3,6,9)$ for the above example, but, rather, have simply tried to find relations among the 4 primitive invariants, we would have found a complete intersection whose Hilbert series is $\frac{(1-t^{18})}{(1-t)^4}$, which would not have given enough information about the geometry of the orbifold.

5 Discrete Torsion

One might wonder what happens if one were turn on discrete torsion for the orbifold probe theories. In the D-brane probe context, this was initiated by [30, 31]. In [32], it was realised that the most systematic approach is to compute the so-called covering group $\tilde{G}$ of the orbifold group G. The discrete torsion then corresponds to the second group-cohomology $A = H^2(G, U(1))$, which is an Abelian group (so-called Schur multiplier) such that $\tilde{G}/A \simeq G$.

For all subgroups of $SU(2)$, the Schur multiplier is trivial and hence the corresponding $\mathcal{N}=2$ gauge theories do not admit discrete torsion. For the subgroups of $SU(3)$, however, the situation is more interesting and the discrete-torsion and the corresponding Schur multipliers and covering groups have been computed and classified in [32].

The moduli space for the discrete torsion theories for $\mathbb{Z}_n \times \mathbb{Z}_n$ has been expounded in detail in [31] (cf. also [33] for a non-commutative perspective). In general (cf. Section 3.2 of [31]), for N D3-branes, it is a $U(N)$ theory with 3 adjoints $\phi_{i=1,2,3}$ which are $N \times N$ matrices, and with a superpotential

$$W = \mathrm{Tr}\left[\phi_1(\phi_2\phi_3 - \omega_n^{-1}\phi_3\phi_2)\right] \ . \tag{5.1}$$

As an illustration, let us first study the simplest case of $N = 1$. Here, the superpotential is $W = (1-\omega_n^{-1})\phi_1\phi_2\phi_3$ and the gauge invariants are simply the 3 numbers $\phi_{i=1,2,3}$. The moduli space is therefore just the F-flat solutions, which are $\phi_1\phi_2 = \phi_2\phi_3 = \phi_3\phi_1 = 0$. Hence, the moduli space $\mathcal{M}$ consists of 3 branches, all touching at the origin: the first parametrized by ϕ_1 non zero and $\phi_2 = \phi_3 = 0$, and the other 2 being cyclic permutations.

To construct the generating function we will use a notion called surgery [34]. It is trivial for this case but is generically powerful for more involved cases. Since each branch of $\mathcal{M}$ is the complex line we have three $U(1)$ isometries (this holds true for higher N as well) and g_1 gets a contribution $1/(1 - t_i)$ for each $i = 1, 2, 3$. We sum all together as the spaces are not intersecting at generic points but need to subtract the intersection spaces which here is just the one point at the origin. The result for the fundamental generating function is thus $g_1 = 1/(1 - t_1) + 1/(1 - t_2) + 1/(1 - t_3) - 2$ Setting $t_i = t$ gives $f_\infty(t) = g_1(t) = 3/(1 - t) - 2$. Taking the plethystic logarithm gives us an infinite series $3t - 3t^2 + 2t^3 - \ldots$, whose first two terms whereby agrees with the F-flat equation above.

5.1 Example: $\mathbb{Z}_2 \times \mathbb{Z}_2$

Now, how do we reproduce the above quantities using our Hilbert series and plethystic programme? Let us consider in detail the $Z_2 \times Z_2$ example. With our discrete torsion, we have one $U(N)$ gauge group with three chiral fields and superpotential $W = \mathrm{Tr}(XYZ + XZY)$. Thus the F-terms induce anti-commutative relations, viz., $XY = -YX, XZ = -ZX, YZ = -ZY$. This is the simplest example which allows discrete torsion and [31] claims that the solution (F-terms plus D-terms) is given by

$$X = X_1 \otimes \sigma_1, \quad Y = Y_1 \otimes \sigma_2, \quad Z = Z_1 \otimes \sigma_3, \tag{5.2}$$

where σ_i are Pauli matrices and X_1, Y_1, Z_1 all commute and so can be chosen to be all diagonal.

To construct the single-trace gauge invariant mesonic operators, we write down the general form as $\mathrm{Tr}(X^{n_1}Y^{n_2}Z^{n_3})$ with $n_i = 0, ..., \infty$. Because $\sigma_i^2 = I$ we can divide such operators into eight cases

- (1) All n_i are even, i.e., $n_i = 2k_i$. In this case we have the sum as
$$\sum_{k_i=0}^{\infty}(t_1^2)^{k_1}(t_2^2)^{k_2}(t_3^2)^{k_3} = \frac{1}{(1-t_1^2)(1-t_2^2)(1-t_3^2)} \ ;$$

- (2) One of the n_i is odd. We will have three sub-cases. Let us focus on the case $n_1 = 2k_1, n_2 = 2k_2, n_3 = 2k_3 + 1$. It is easy to see that $\mathrm{Tr}(\sigma_3) = 0$. Thus, this category does not give non-zero meson operators.

- (3) One of n_i is even. Again there are three sub-cases and we focus on the case where $n_1 = 2k_1, n_2 = 2k_2 + 1, n_3 = 2k_3 + 1$. It is easy to see that $\mathrm{Tr}(\sigma_2\sigma_3) = 0$. Thus the contribution in this category is again zero.

- (4) The last case is that all n_i are odd, $n_1 = 2k_1 + 1, n_2 = 2k_2 + 1, n_3 = 2k_3 + 1$ Using $\mathrm{Tr}(\sigma_1\sigma_2\sigma_3) \sim \mathrm{Tr}(I) \neq 0$, we have the counting
$$\sum_{k_i=0}^{\infty} t_1t_2t_3(t_1^2)^{k_1}(t_2^2)^{k_2}(t_3^2)^{k_3} = \frac{t_1t_2t_3}{(1-t_1^2)(1-t_2^2)(1-t_3^2)} \ .$$

- (5) It can be shown that $\mathrm{Tr}(X^{2k}Y^{2k}Z^{2k}) = \mathrm{Tr}((XYZ)^{2k})$, thus we need to be careful about double-counting. However, categories (1) and (4) have different powers (even or odd), so we do not have a double-counting problem here.

Adding the above two together we have the final counting to be,
$$f_\infty(t_1, t_2, t_3;\ \mathbb{C}^3/\mathbb{Z}_2^2)_{\text{torsion}} = \frac{1 + t_1t_2t_3}{(1-t_1^2)(1-t_2^2)(1-t_3^2)} \ .$$
We should take the plethystic logarithm to check the equation for moduli space. It should be the form $xyz = t^2$. To see this let us first set $t_1 = t_2 = t_3 = t$ and take pletytistic logarithm and indeed we get (terminating) polynomial expression, $3t^2 + t^3 - t^6$, which is exactly what we should have, as one could see from case $m = 2$ of (4.5). Indeed, for $N = 1$, it does not give a three-dimension moduli space, but, rather, a degenerate one-dimensional one which is what was argued above from [31], viz., $\frac{3}{(1-t)^2}$. We can be more refined and actually compute the full plethystic logarithm with all three variables, giving us $t_1^2 + t_2^2 + t_3^2 + t_1t_2t_3 - (t_1t_2t_3)^2$.

5.2 The General $\mathbb{Z}_n \times \mathbb{Z}_n$ Case

Let us proceed to the general case. For the group $\mathbb{Z}_n \times \mathbb{Z}_n$, with action

$$\begin{aligned} g_1: \quad (z_1, z_2, z_3) &\rightarrow (z_1, e^{-\frac{2\pi i}{n}} z_2, e^{\frac{2\pi i}{n}} z_3), \quad (5.3) \\ g_2: \quad (z_1, z_2, z_3) &\rightarrow (e^{\frac{2\pi i}{n}} z_1, z_2, e^{-\frac{2\pi i}{n}} z_3) \ , \quad (5.4) \end{aligned}$$

the discrete torsion is $\mathbb{Z}_n$, with the 2-cocycle class given by $\widetilde{\epsilon}^m((a,b),(a',b')) = \zeta^{m(ab'-a'b)}$ and $\zeta := e^{\frac{\pi i}{n}}$ for n even or $\zeta = e^{\frac{2\pi i}{n}}$ for n odd. We will consider the case that $gcd(m,n) = 1$, for which the projective representation is given by

$$\gamma_1(g_1) = P, \qquad \gamma_1(g_2) = Q \ , \quad (5.5)$$

with P and Q being the following $n \times n$ matrix (where $\epsilon = \zeta^{2m}$ and $\epsilon^n = 1$)

$$P = \left(\begin{array}{ccccc} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{array} \right), \qquad Q = \left(\begin{array}{ccccc} 0 & \epsilon & 0 & \cdots & 0 \\ 0 & 0 & \epsilon^2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & \epsilon^{n-1} \\ 1 & 0 & 0 & \cdots & 0 \end{array} \right)$$

for n odd. For n even, P is the same, while Q is (with $\delta^2 = \epsilon$):

$$Q = \left(\begin{array}{ccccc} 0 & \delta & 0 & \cdots & 0 \\ 0 & 0 & \delta^3 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & \delta^{2n-3} \\ \delta^{2n-1} & 0 & 0 & \cdots & 0 \end{array} \right).$$

We have the following properties

$$PQ = \epsilon QP, \quad P^n = 1 = Q^n, \quad \mathrm{Tr}(P^k) = \mathrm{Tr}(Q^k) = \mathrm{Tr}(Q^r Q^{k-r}) = 0, \quad if \quad k \neq nZ \quad (5.6)$$

Under the condition $gcd(m,n) = 1$, the theory has gauge group $U(M)$, with three chiral adjoint fields ϕ_i and superpotential $\mathrm{Tr}(\phi_1\phi_2\phi_3 - \epsilon^{-1}\phi_1\phi_3\phi_2)$. This gives F-term condition

$$\phi_i\phi_j - \epsilon^{-1}\phi_j\phi_i, \quad (i,j) = (1,2),(2,3),(3,1) \ . \quad (5.7)$$

Again, the solution of F-terms and D-terms relation is given by

$$\phi_1 = X \otimes Q, \quad \phi_2 = Y \otimes P, \quad \phi_3 = Z \otimes (QP)^{-1} \ , \quad (5.8)$$

where X, Y, Z all commute, just like parent $\mathcal{N}=4$ theory. Now, we consider the mesonic operators $\mathrm{Tr}(\phi_1^{n_1}\phi_2^{n_2}\phi_3^{n_3})$ and write $n_i = nk_i + s_i$ with $k_i = 0, ..., \infty$ and $s_i = 0, ..., n-1$. The key part is to see if $\mathrm{Tr}(Q^{s_1}P^{s_2}(QP)^{-s_3})$ is zero (where we have used the fact that $P^n = Q^n = \mathbb{I}$).

To see the properties of P, Q we use the following observation. We take P, Q as the action of n-dimensional vector space with basis $e_0, ..., e_{n-1}$. Then the action is as follows (for simplicity we assume n is odd)

$$P(e_i) = e_{i+1}, \qquad Q(e_i) = \epsilon^{i+1} e_{i+1} \ , \tag{5.9}$$

whence we have

$$\begin{aligned}
P^k(e_i) &= e_{i+k}, \qquad \mathrm{Tr}(P^k) = 0, \quad if \quad k \neq nZ \\
Q^k(e_i) &= (\prod_{r=1}^{k} \epsilon^{i+r})(e_{i+k}) = \epsilon^{ki+\frac{k(k+1)}{2}}(e_{i+k}), \quad \mathrm{Tr}(Q^k) = 0, \quad if \quad k \neq nZ \\
P^{r_1}Q^{r_2}(e_i) &= P^{r_1}\epsilon^{r_2 i+\frac{r_2(r_2+1)}{2}}(e_{i+r_2}) = \epsilon^{r_2 i+\frac{r_2(r_2+1)}{2}}(e_{i+r_1+r_2}), \\
\mathrm{Tr}(P^{r_1}Q^{r_2}) &= \delta(r_1+r_2-nZ)\sum_{i=0}^{n-1}\epsilon^{r_2 i+\frac{r_2(r_2+1)}{2}} = \epsilon^{r_2 i+\frac{r_2(r_2+1)}{2}}\delta(r_1+r_2-nZ)\delta(r_2-n\widetilde{Z}) \\
&= \epsilon^{r_2 i+\frac{r_2(r_2+1)}{2}}\delta(r_1-nZ_1)\delta(r_2-nZ_2) \ .
\end{aligned}$$

The last equation is very important; it tells us that we need r_1, r_2 to be integer when multiplying by n.

Now, we calculate

$$\begin{aligned}
(QP)^r &= \epsilon^{\frac{r(r-1)}{2}} Q^r P^r \\
Q^{s_1}P^{s_2}(QP)^{-s_3} &= Q^{s_1}P^{s_2}\epsilon^{\frac{s_3(s_3+1)}{2}} Q^{-s_3}P^{-s_3} \\
&\sim P^{s_2-s_3}Q^{s_1-s_3} \ .
\end{aligned}$$

From this we can see that $\mathrm{Tr}(Q^{s_1}P^{s_2}(QP)^{-s_3}) \neq 0$ when and only when $s_1 - s_3 = nZ_1, s_2 - s_3 = nZ_2$. Also, because $s_i \in [0, n-1]$ we have the only possibility that $Z_1 = Z_2 = 0$.

With all these analysis we have following counting

$$\begin{aligned}
f_\infty(t_1,t_2,t_3;\ \mathbb{C}^3/\mathbb{Z}_n^2)_{\mathrm{torsion}} &= \sum_{k_i=0}^{\infty}\sum_{s_i=0}^{n-1}\sum_{Z_1,Z_2} t_1^{nk_1+s_1}t_2^{nk_2+s_2}t_3^{nk_3+s_3}\delta(s_1-s_3-nZ_1)\delta(s_2-s_3-nZ_2) \\
&= \frac{1}{(1-t_1^n)(1-t_2^n)(1-t_3^n)}\frac{(1-(t_1t_2t_3)^n)}{(1-t_1t_2t_3)} \ .
\end{aligned}$$

First, for simplicity we can set all t_i to be equal to t and get $\frac{(1-t^{3n})}{(1-t^3)(1-t^n)^3}$. For $n = 2$ it goes back to our previous result. Taking the plethytistic logarithm we get the (terminated) polynomial $t^3 + 3t^n - t^{3n}$ which is the equation of $xyz = t^n$ with t degree three and x, y, z degree n. We can also compute the full plethystic logarithm including the three variables and obtain $t_1^n + t_2^n + t_3^n + t_1t_2t_3 - (t_1t_2t_3)^n$.

It is very interesting to notice that, comparing with (4.6) and (4.7), the result for $\mathbb{Z}_n \times \mathbb{Z}_n$ with torsion is same as the one without discrete torsion. This is consistent with our claim that at least for complete intersection geometries we can get the fundamental invariant $g_1(t)$ using defining equation directly. And indeed, the defining equation does not distinguish if there is torsion or not.

In fact, for an orbifold action, there are three parts: (1) The adjoint action on Chan-Paton factors; (2) The space-time action on the three chiral multiples X, Y, Z; and (3) The projected superpotential coming from Clebesh-Gordon coefficients. To get the quiver, we need to know the information of the first two parts only. The space-time action is always a faithful representation while the Chan-Paton action could be projective. However, because it is an adjoint action, the cocycle factor does not affect the discussion and this is why we can use the covering group to get the quiver diagram. The difference between faithful representation and projective representation is that the dimension of matrix is different, thus, given N D3-brane probes we have less number of gauge groups under projective representation. For example, for $\mathbb{Z}_n \times \mathbb{Z}_n$, in the minimum case we have only one gauge group for projective representation while for the faithful one (without discrete torsion), we have $U(1)^{n^2}$ gauge groups.

To determine the theory completely, we need to know the superpotential as well; now, the difference between projective and faithful representations gives different F-term relations. However, as we have emphasized that the space-time action is the same with or without discrete torsion, it is reasonable that we get the same answer and the Molien (Hilbert) series seems to apply. Although we have not checked all cases where the geometry may or may not be complete intersection, we do conjecture that counting will be same in all such cases.

6 Hilbert Schemes and Symmetric Products

We have delved into orbifolds and quotients quite intensively in the foregoing discussions. Of key significance in our derivation in [4] for $g(\nu, t)$ is that the full generating function for all finite N relies on an important quotient, viz., the space

$$\mathrm{Sym}^N(\mathcal{M}) := \mathcal{M}^N / S_N \ , \tag{6.1}$$

the N-th symmetric product of a space $\mathcal{M}$. This is important because if $\mathcal{M}$ is the vacuum moduli space of a single D3-brane probe (i.e., the transverse Calabi-Yau singularity), then $\mathcal{M}^N/S_N$ is that of a stack thereof. In general, $\mathcal{M}^N/S_N$ may have singularities, however, there is a canonical resolution called the Hilbert scheme [37], which is much richer in structure than the mere moduli space of N (non-coinciding) points in $\mathcal{M}$ as captured by $\mathcal{M}^N/S_N$.

Formally, the Hilbert scheme is defined to be the set of all sub-schemes (we switch liberally between schemes and ideals using the algebra-geometry correspondence) of a variety X of length N, i.e.,

$$\mathrm{Hilb}^n(X) := \{\text{ideals } I \subset X | \dim(X/I) = N\} \ . \tag{6.2}$$

In dimension 1, everything is easy and we have that

$$\mathrm{Hilb}^N(\mathbb{C}) = \mathrm{Sym}^N(\mathbb{C}), \qquad \mathrm{Hilb}^N(\mathbb{P}^1) = \mathbb{P}^N \ . \tag{6.3}$$

6.1 The Second Symmetric Product of $\mathbb{C}^m$

Let us see what we can say about $\mathbb{C}^m$. We know that

$$g_N(t;\ \mathcal{M}) = g_1(t;\ \mathrm{Sym}^N(\mathcal{M})), \tag{6.4}$$

and that

$$g_1(t;\ \mathbb{C}^m) = f_\infty(t;\ \mathbb{C}^m) = \frac{1}{(1-t)^m} \ . \tag{6.5}$$

In fact, the expression is refined as $\prod\limits_{i=1}^{m} \frac{1}{(1-t_i)}$. Some immediate results can be read off for $N = 2$. Here, $S_2 \simeq \mathbb{Z}_2$. Take $m = 2$, we have that

$$\mathrm{Sym}^2(\mathbb{C}^2) \simeq \mathbb{C}[x_1, y_1, x_2, y_2]^{\mathbb{Z}_2}, \qquad \mathbb{Z}_2 \simeq \langle \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \rangle \ ; \tag{6.6}$$

so the Molien series and the corresponding plethystic logarithm are simply

$$M(t) = f_\infty(t) = \frac{1+t^2}{(1-t)^4\,(1+t)^2}\ , \qquad f_1(t) = PE^{-1}[f(t)] = 2\,t + 3\,t^2 - t^4\ . \tag{6.7}$$

Again, this has a refinement and $f_1 = t_1 + t_2 + t_1^2 + t_1 t_2 + t_2^2 - t_1^2 t_2^2$. This is consistent with the fact that the defining equation for $\mathbb{C}^4/\mathbb{Z}_2$ with our chosen action, is a complete intersection. It is given as a single relation (of degree 4) amongst 5 primitive (2 linear and 3 quadratic) invariants (the syzygies can be readily computed using [36]):

$$\begin{aligned} Y_{1,\dots,5} &:= \{x_1+x_2, y_1+y_2, {x_1}^2+{x_2}^2, x_1\,y_1+x_2\,y_2, {y_1}^2+{y_2}^2\}; \\ &\Rightarrow\ {Y_2}^2\,Y_3 - 2\,Y_1\,Y_2\,Y_4 + 2\,{Y_4}^2 + {Y_1}^2\,Y_5 - 2\,Y_3\,Y_5 = 0\ . \end{aligned} \tag{6.8}$$

It is clear here that the refinement comes from an independent counting of x's and y's. Set t_1 to count x's, t_2 to count y's and the equation for f_1 follows.

Next, we can take $m = 3$ and $N = 2$. Now, we have that

$$\mathrm{Sym}^2(\mathbb{C}^3) \simeq \mathbb{C}[x_1,y_1,z_1,x_2,y_2,z_2]^{\mathbb{Z}_2}, \qquad \mathbb{Z}_2 \simeq \langle \begin{pmatrix} 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1\\ 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0 \end{pmatrix} \rangle\ , \tag{6.9}$$

giving us

$$M(t) = f_\infty(t) = \frac{1+3\,t^2}{(1-t)^6\,(1+t)^3}\ ; \qquad f_1(t) = 3\,t + 6\,t^2 - 6\,t^4 + 8\,t^6 - 18\,t^8 + \mathcal{O}(t^{10})\ . \tag{6.10}$$

Already, here, we see from the expression for f_1 that the space is not a complete intersection. Here, the refinement is interesting:

$$g_1 = \frac{1 + t_1 t_2 + t_2 t_3 + t_3 t_1}{\prod_{i=1}^3 (1-t_i)(1-t_i^2)}\ , \tag{6.11}$$

giving

$$t_1 + t_2 + t_3 + \sum_{i\le j} t_i t_j - \sum_{i<j} (t_i t_j)^2 - t_1 t_2 t_3 (t_1 + t_2 + t_3) + \dots \tag{6.12}$$

A general formula for g_2 of any m is the following:

$$g_2(t_1,\dots,t_m;\mathbb{C}^m) = \left(\prod_{i=1}^m (1-t_i)(1-t_i^2)\right)^{-1} \left(1 + \sum_{i<j} t_i t_j + \sum_{i<j<k<l} t_i t_j t_k t_l + \begin{array}{c}\text{Similar order}\\ \text{6 term}\end{array} + \dots\right)$$

As a historical digression, the series coefficients of $M(t)$ for Sym^2 of $\mathbb{C}^m$ are known as Paraffin (or Alkane) Numbers, having to do with reading off diagonals of Losanitsch's generalisation of Pascal's triangle [26, 27].

6.2 The n-th Symmetric Product of $\mathbb{C}^2$

Alternatively, one could analyse the family

$$\mathrm{Sym}^n(\mathbb{C}^2) \simeq \mathbb{C}[x_1, y_1;\ x_2, y_2;\ \ldots;\ x_n, y_n]/S_n \ ,$$

where (x, y) are the coordinates of $\mathbb{C}^2$ and S_n permutes the n-tuple of points (x_i, y_i). This is very much in the spirit of Hilbert scheme of points on surfaces as detailed in [37]. In fact, in this case, [38] has given, in our notation, the ν-inserted plethystic exponential (cf. Eq (4.5), *cit. Ibid.*):

$$g(\nu,\ t_1, t_2;\ \mathbb{C}^2) = PE_\nu[((1-t_1)(1-t_2))^{-1}] = \exp\left[\sum_{k=1}^{\infty} \frac{\nu^k}{k(1-t_1^k)(1-t_2^k)}\right] \ . \tag{6.13}$$

Unrefining by setting $t_1 = t_2 = t$ and power expanding in ν, gives, for the coefficient of ν^n, the Hilbert series for $\mathrm{Sym}^n(\mathbb{C}^2)$:

$$PE_\nu[\frac{1}{(1-t)^2}] = 1 + \frac{1}{(1-t)^2}\nu + \frac{1+t^2}{(1-t)^4(1+t)^2}\nu^2 + \frac{1+t^2+2\,t^3+t^4+t^6}{(1-t)^4\,(1+t)^2\,(1-t^3)^2}\nu^3 + \mathcal{O}(\nu^4) \tag{6.14}$$

We see that the ν-term is indeed that of $\mathrm{Sym}^1(\mathbb{C}^2) = \mathbb{C}^2$ and the ν^2-term is what we calculated in (6.7) for $\mathrm{Sym}^2(\mathbb{C}^2)$.

This is, of course, in perfect congruence with our proposal in [4]: that the ν-inserted plethystic exponential of the Hilbert series should give the generating function for g_N, the multi-trace generating function for N D3-branes. Indeed, the coefficient to ν^N in (6.14) is the Hilbert series $f_\infty = g_1$ of $\mathrm{Sym}^N(\mathbb{C}^2)$. However, recalling from (6.4) that $g_1(t;\ \mathrm{Sym}^N(\mathbb{C}^2)) = g_N(t;\ \mathbb{C}^2)$, the agreement is re-assuring.

In fact, [26], the series-expansion in t for the n-th coeffcient, i.e., the t-expansion for the Hilbert series of $\mathrm{Sym}^n(\mathbb{C}^2)$ gives the planar integer partitions (Young-Tableaux) of trace n. That is to say, the coefficient of t^m corresponds to the number of ways of writing the given integer m as $\sum_{i,j} z_{i,j}$ with $z_{i,j} \in \mathbb{Z}_+$ such that $z_{i+1,j} > z_{i,j}$, $z_{i,j+1} > z_{i,j}$ and $\sum_{i,i} z_{i,i} = n$.

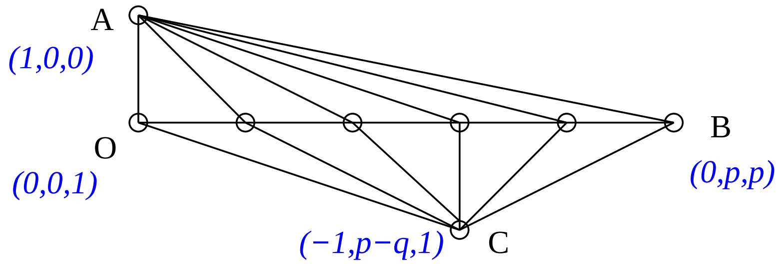

Figure 2: The toric diagram for the space $Y^{p,q}$.

7 A Detailed Analysis of $Y^{p,q}$

Having indulged ourselves with quotient spaces, let us change our palatte awhile to toric varieties. The space $Y^{p,q}$ (cf. e.g. [7, 8, 39, 40, 41]) was studied in detail in [4]. The toric data is given by the integer lattice points $O = (0,0,1)$, $A = (1,0,1)$, $B = (0,p,1)$ and $C = (-1, p-q, 1)$ as drawn in Fig. 2. As indicated, we take the trianglization of the toric diagram by connecting the point $T_a := (0,a,1)$ to A and C with $a = 1, ..., p$ (so that $T_p = B$). Thus, we have $2p$ triangles given by T_aAT_{a+1} and T_aCT_{a+1}, $a = 0, ..., p-1$. From this we can read out the fundamental generating function f_∞ (cf. [4]) as

$$f(x,y,z;\,Y^{p,q}) = \sum_{a=0}^{p-1} \frac{1}{(1-x)\,\left(1-\frac{x^a\,y}{z^a}\right)\left(1-\frac{x^{-1-a}\,z^{1+a}}{y}\right)} + \frac{1}{\left(1-\frac{1}{x}\right)\left(1-\frac{x^{-a+p-q}\,y}{z^a}\right)\left(1-\frac{x^{1+a-p+q}\,z^{1+a}}{y}\right)} \,. \tag{7.15}$$

To get a more compact expression for f, we need to sum up the two series. This can be done as follows. First, for each term in the series OAB we can write it as

$$\frac{-xz^{-1}}{(1-x)(1-xz^{-1})}\left(-\frac{1}{1-x^{-a}z^a y^{-1}} + \frac{1}{1-x^{-a-1}z^{a+1}y^{-1}}\right) \,.$$

Summing up from $a = 0$ to $a = p-1$ we get

$$\frac{xy(z^p - x^p)}{(1-x)(z-x)(1-y)(x^p y - z^p)} \,.$$

For the series OBC, each term can be written as

$$\frac{-x}{(1-x)(1-xz)}\left(\frac{1}{1-x^{-a+p-q}yz^{-a}} - \frac{1}{1-x^{-a-1+p-q}yz^{-a-1}}\right) \,,$$

and summing up we get

$$\frac{xy(1-x^p z^p)}{(1-x)(1-xz)(1-x^{p-q}y)(x^q z^p - y)} \,.$$

Putting all together we have

$$f(x,y,z;\,Y^{p,q}) = \frac{xy}{(1-x)}\left(\frac{(1-x^p z^p)}{(1-xz)(1-x^{p-q}y)(x^q z^p - y)} + \frac{(z^p - x^p)}{(z-x)(1-y)(x^p y - z^p)}\right) . \tag{7.16}$$

Now, let us try to find the defining equation for $Y^{p,q}$ using the plethystic programme. The basic invariants and relations were counted in [42] and we wish to use our plethystic programme to check those results. We therefore need the generating function f_1 by taking the plethystic logarithm of (7.16).

For this purpose let us first study the structure of the dual cone, which in toric geometry will give us the relations. It is easy to find that in our coordinates the following dual vectors are generators of dual cone:

$$e_1 = (0,1,0), \quad e_2 = (-p,-1,p), \quad e_3 = (q,-1,p), \quad e_4 = (p-q,1,0) .$$

However, these are the generators of the dual cone over $\mathbb{R}_+$, we need the generators over $\mathbb{Z}_+$. This is to say that the above four vectors are not complete in the sense of lattice points, and there will exist some integer vectors, which are linear combinations of these four with positive real number coefficients, but not with positive integer coefficients.

To find these missing vectors, first we notice that $e_3 - e_2 = (p+q,0,0)$, thus the following $(p+q+1)$ vectors

$$e_{23,m} = e_2 + (m,0,0) = (-p+m,-1,p), \quad m = 0,...,p+q$$

must be included into the $\mathbb{Z}_+$-generators of the dual cone. Similarly, since $e_4 - e_1 = (p-q,0,0)$, the following $(p-q+1)$ vectors should also be included:

$$e_{14,m} = (m,1,0), \quad m = 0,...,p-q .$$

We are not finished yet. From the fact that $e_3 + e_4 = (p,0,p)$ we find that we need $(1,0,1)$ and from $e_1 + e_2 = (-p,0,p)$, we need $(-1,0,1)$. Finally, from these two we get $(0,0,2)$ so we need $(0,0,1)$ as generator. Thus we have three more generators

$$e_5 = (1,0,1), \quad e_6 = (-1,0,1), \quad e_7 = (0,0,1) .$$

Putting all together we have $2p+5$ generators $e_{23,m}, e_{14,m}, e_5, e_6, e_7$ as claimed by [42].

To have a simple result for the plethystic logarithm, we want as many as generators having same degree. One simple choice could be the following scaling: $x \to 1, y \to$

$t^p, z \to t^2$ under which the generators $e_{23,m}$ and $e_{14,m}$ have same degree. However, when p is even, there is an interference between the number of variables and the number of equations for the definition of geometry. By this we mean that in expanding the expression for the plethystic logarithm, positive terms signify invariants while negative terms signify relations, these could potentially cancel and confuse the counting; this situation was encountered in the example of the non-complete intersection $\mathbb{C}^3/\mathbb{Z}_3$ in [4]. Thus the above scaling is not a good choice. Another choice will be the $x \to 1, y \to t^{2p-1}, z \to t^4$. In this case, $e_{23,m}$ and $e_{14,m}$ do not have same degree, but the interference is avoided.

However, we can do better by having multiple variables in the plethystic logarithm. To do so, first we need to set $x \to 1$. The reason is very simple because we want to set $e_{23,m}$ (as well as $e_{14,m}$) to have the same degree. After this we have

$$\begin{aligned}
f(x \to 1, y, z;\ Y^{p,q}) &= \frac{yA + y^2 B + y^3 C}{(1-y)^2(1-z)^2(y-z^p)^2}\ ; \\
A &:= -z^p(z^{p+1}(p-q-1) - z^p(p-q+1) + z(p+q+1) - (p+q-1)) \\
B &:= (1+z)(1-z^{2p}) - 4pz^p(1-z) \\
C &:= -z^{p+1}(p+q-1) + z^p(p+q+1) - z(p-q+1) + (p-q-1)\ .
\end{aligned}$$

In this form, it is not suitable to take the plethystic logarithm because of the overall factor y in the numerator as well as the factor $(y - z^p)$ in the denominator, which would give a logarithmic singularity in at $y = 0$. To amend this, we change variables as $y \to y$ and $z \to ty$. Thus

$$\begin{aligned}
&f(x \to 1, y, ty;\ Y^{p,q}) = \frac{A + B + yC}{(1-y)^2(1-ty)^2(1-t^p y^{p-1})^2} \\
&A := -t^p y^{p-1}((ty)^{p+1}(p-q-1) - (ty)^p(p-q+1) + ty(p+q+1) - (p+q-1)) \\
&B := (1+ty)(1-(ty)^{2p}) - 4p(ty)^p(1-ty) \\
&C := -(ty)^{p+1}(p+q-1) + (ty)^p(p+q+1) - ty(p-q+1) + (p-q-1)\ .
\end{aligned}$$

Now, we can take the plethystic logarithm and get the right answer! It is easy to see that under the above scaling we have $(p+q+1)$ variables with scaling $e_{23,m} \to y^{p-1}t^p$, $(p-q+1)$ variables with scaling $e_{14,m} \to y$ and three variables with scaling $e_{5,6,7} \to yt$.

Let us check the above result with the tabulation of the following several examples:

(p,q)	$f_1(t,y;Y^{p,q})$
$(p=1,q=0)$	$(2t+2y)-ty$
$(p=1,q=1)$	$(3t+y)-t^2$
$(p=2,q=0)$	$(3t^2y+3y+3ty)-y^2-4ty^2-10t^2y^2-4t^3y^2-t^4y^2+...$
$(p=2,q=1)$	$(4t^2y+2y+3ty)-2ty^2-9t^2y^2-6t^3y^2-3t^4y^2+...$
$(p=2,q=2)$	$(5t^2y+y+3ty)-6t^2y^2-8t^3y^2-6t^4y^2+...$
$(p=3,q=0)$	$(4t^3y^2+4y+3ty)-3y^2-6ty^2-t^2y^2-16t^3y^3-6t^4y^3-3t^6y^4+...$
$(p=3,q=1)$	$(5t^3y^2+3y+3ty)-y^2-4ty^2-t^2y^2-15t^3y^3-8t^4y^3-6t^6y^4+...$
$(p=3,q=2)$	$(6t^3y^2+2y+3ty)-2ty^2-t^2y^2-12t^3y^3-10t^4y^3-10t^6y^4+...$
$(p=3,q=3)$	$(7t^3y^2+y+3ty)-t^2y^2-7t^3y^3-12t^4y^3-15t^6y^4+...$

The interpretation of f_1 was outlined in [4]. For $p=2, q=0$, for example, we should have 3 variables with scaling t^2y, 3 variables with scaling y and another 3 with scaling ty; this is given exactly inside the first bracket. The remaining part is the information about relations, i.e, defining equations. The term $-y^2$ means there is one relation among 3 variables with scaling y. Similarly the term $-4ty^2$ tells us that there are four relations between variables with scaling y and variables with scaling yt.

Checking all examples we find that the total number of defining equations is given by $p^2+10p-4$. It is different from the claim given by [42] where it is claimed that the number of equations should be $p^2+10p-q^2-4$. However, in *cit. ibid.*, only minimum relations are counted. In other words, if $a=b$ and $b=c$ then $a=c$ will not be counted as a new relation. However, in our case, $a=c$ would be counted separately.

8 Conclusions and Prospects

We have proposed a **plethystic programme** for the counting of gauge invariant operators in gauge theories. Though we have restricted our attention to chiral BPS mesonic operators in world-volume theories of D-branes probing Calabi-Yau singularities, the programme should be of wider applicability. In the case of our present focus, an intimate web of connexions between geometry, gauge theory and combinatorics emerges. This field of quiver theories is where the plethystics fully blossom. In a way, this does not surprise us. Indeed, for D-brane quiver theories, the mesonic gauge invariants in the chiral ring, modulo the F-term constraints, should give a classical moduli space that by construction is the Calabi-Yau variety which the brane probes.

What is beautiful about the plethystic programme is that the plethystic exponential function and its inverse provide the explicit link between the geometry and the gauge invariants. One only needs to construct a fundamental invariant of the Calabi-Yau manifold $\mathcal{M}$, which we have called $f = f_\infty = g_1$ and which mathematically corresponds to the Hilbert series. This is the generating function for the single-trace operators. The plethystic logarithm, PE^{-1}, gives all the syzygies of $\mathcal{M}$. In the case of $\mathcal{M}$ being complete intersection, $f_1 = PE^{-1}[f]$ is a polynomial from which one immediately reads out the defining equation of $\mathcal{M}$. On the other hand, the plethystic exponential gives $g_\infty = PE[f]$, the generating function for the multi-trace operators.

Continuing with [4], we have provided a host of examples to demonstrate the power of the plethystic programme, ranging from orbifold theories to toric singularities, from discrete torsion to Hilbert schemes, touching upon such interesting curios as relations to Young Tableaux and to the MacMahon Conjecture. Importantly, we have also, using results of Temperley and Haselgrove, generalised the formulae of Hardy-Ramanujan and Meinardus, in estimating the asymptotic behaviour of the number of such operators. This is an estimate of the degrees of freedom of the gauge theory, whereby providing us with explicit expressions for the entropy for an arbitrary number N of D-branes.

We have, of course, only touched upon the fringe of a fertile ground. How do plethystics teach us about other branches of the moduli space? For example, it will be interesting to count baryonic branch and to see if syzygies of divisors wrapped by D3-branes be captured by plethystics. Including of baryonic operators into the counting should correspond to the geometry of cycles in the Sasaki-Einstein manifold, can the syzygies of these divisors be captured by plethystics? Recently, progress in counting baryonic operators has been made in [22]. Along similar lines is the mixed branch studied in [34]. Can our programme be extended to study such other branches? What about 1/4 or 1/8 BPS states?

Moreover, in [4], we have performed the counting for geometries whose quiver theories have not yet been constructed; we could also do so for non-quiver and even non-SUSY theories. Indeed, there has also been much study of certain indices of superconformal theories [47, 48, 49, 50, 51, 52] as well as the counting of instantons such as in [53]. How does our plethystic programme relate to these counting problems? The portals to a Grecian mansion have been opened to us, to fully explore the plethora of her plethystic secrets shall be our continued goal.

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References

[1] J. M. Maldacena, "The large N limit of superconformal field theories and supergravity," Adv. Theor. Math. Phys. **2**, 231 (1998) [Int. J. Theor. Phys. **38**, 1113 (1999)] [arXiv:hep-th/9711200].

[2] M. R. Douglas and G. W. Moore, "D-branes, Quivers, and ALE Instantons," arXiv:hep-th/9603167.

[3] Y. H. He, "Lectures on D-branes, gauge theories and Calabi-Yau singularities," arXiv:hep-th/0408142; Y. H. He, "On algebraic singularities, finite graphs and D-brane gauge theories: A string theoretic perspective," arXiv:hep-th/0209230.

[4] S. Benvenuti, B. Feng, A. Hanany and Y. H. He, "Counting BPS operators in gauge theories: Quivers, syzygies and plethystics," arXiv:hep-th/0608050.

[5] P. Pouliot, "Molien function for duality," JHEP **9901**, 021 (1999) [arXiv:hep-th/9812015].

[6] Stephen S.-T. Yau and Y. Yu, "Gorenstein Quotient Singularities in Dimension Three," Memoirs of the AMS, no. 505, Providence, 1993.

[7] J. P. Gauntlett, D. Martelli, J. Sparks and D. Waldram, "Supersymmetric AdS(5) solutions of M-theory," Class. Quant. Grav. **21** (2004) 4335 [arXiv:hep-th/0402153].

[8] J. P. Gauntlett, D. Martelli, J. Sparks and D. Waldram, "Sasaki-Einstein metrics on S(2) x S(3)," Adv. Theor. Math. Phys. **8** (2004) 711 [arXiv:hep-th/0403002].

[9] D. Martelli, J. Sparks and S. T. Yau, "Sasaki-Einstein manifolds and volume minimisation," arXiv:hep-th/0603021.

[10] J.-P. Serre, "A Course in Arithmetic," GTM, Springer, 1996

[11] Y. Noma, T. Nakatsu and T. Tamakoshi, "Plethystics and instantons on ALE spaces," arXiv:hep-th/0611324.

[12] G. Meinardus, "Asymptotische Aussagen über Partitionen," Mathematische Zeitschrift 59 (1954) 388–398.

[13] C. B. Haselgrove and H. N. V. Temperley, "Asymptotic formulae in the theory of partitions," Proc. Cambridge Phil. Soc., 50 (1954), 225-241.

[14] L. B. Richmond, "Some General Problems on the Number of Parts in Partitions," Acta Arithmetica, LXVI.4 (1994), p297-313.

[15] H. Hwang, "Distributions of integer partitions with large number of summands," Acta Arithmetica, 78, 351–365 (1997).

[16] A. A. Actor, "Infinite products, partition functions, and the Meinardus theorem," J. Mathematical Physics, Vol 35, 11, Nov. 1994, pp.5749-5764.

[17] F.C. Auluck, S. Chowla, and H. Gupta, J. Ind. Math. Soc. 6, 105; 1942.

[18] S. Grossmann and M. Holthaus, "Microcanonical fluctuations of a Bose systems ground state occupation number," Phys. Rev. E 54, 3495 (1996).

[19] M. Gajda and K. Rzazewski, "Fluctuations of Bose-Einstein Condensate," Phys. Rev. Letters, 78, no.14, (1997).

[20] D. Martelli and J. Sparks, "Dual giant gravitons in Sasaki-Einstein backgrounds," Nucl. Phys. B **759**, 292 (2006) [arXiv:hep-th/0608060].

[21] A. Butti, D. Forcella and A. Zaffaroni, "Deformations of conformal theories and non-toric quiver gauge theories," arXiv:hep-th/0607147.

[22] A. Butti, D. Forcella and A. Zaffaroni, "Counting BPS baryonic operators in CFTs with Sasaki-Einstein duals," arXiv:hep-th/0611229.

[23] A. Hanany and Y. H. He, "Non-Abelian finite gauge theories," JHEP **9902**, 013 (1999) [arXiv:hep-th/9811183].

[24] P. Cameron, T. Prellberg, D. Stark, "Asymptotics for incidence matrix classes," math.CO/0510155.

[25] P. A. MacMahon, "Memoir on the Theory of the Partitions of Numbers. V: Partitions in Two-Dimensional Space." Phil. Trans. Roy. Soc. London Ser. A 211, 75-110, 1912a; – "Memoir on the Theory of the Partitions of Numbers. VI: Partitions in

Two-Dimensional Space, to which is Added an Adumbration of the Theory of Partitions in Three-Dimensional Space." Phil. Trans. Roy. Soc. London Ser. A 211, 345-373, 1912b.

[26] N. Sloane, "The On-Line Encyclopedia of Integer Sequences," http://www.research.att.com/∼njas/sequences/index.html

[27] S. M. Losanitsch, "Die Isomerie-Arten bei den Homologen der Paraffin-Reihe," Chem. Ber. 30 (1897), 1917-1926.

[28] The GAP Group, *GAP – Groups, Algorithms, and Programming, Version 4.4*; 2006, (\protect\vrule width0pt\protect\href{http://www.gap-system.org}{http://www.ga

[29] D. Eisenbud, "Commutative Algebra With A View Toward Algebraic Geometry," GTM 150, Springer-Verlag, 1995.

[30] M. R. Douglas, "D-branes and discrete torsion," arXiv:hep-th/9807235.

[31] M. R. Douglas and B. Fiol, "D-branes and discrete torsion. II," JHEP **0509**, 053 (2005) [arXiv:hep-th/9903031].

[32] B. Feng, A. Hanany, Y. H. He and N. Prezas, "Discrete torsion, non-Abelian orbifolds and the Schur multiplier," JHEP **0101**, 033 (2001) [arXiv:hep-th/0010023]. B. Feng, A. Hanany, Y. H. He and N. Prezas, "Discrete torsion, covering groups and quiver diagrams," JHEP **0104**, 037 (2001) [arXiv:hep-th/0011192].

[33] D. Berenstein, V. Jejjala and R. G. Leigh, "Marginal and relevant deformations of N = 4 field theories and non-commutative moduli spaces of vacua," Nucl. Phys. B **589**, 196 (2000) [arXiv:hep-th/0005087].

[34] A. Hanany and C. Romelsberger, "Counting BPS Operators in the Chiral Ring of N=2 Supersymmetric Gauge Theories or N=2 Braine Surgery," arXiv:hep-th/0611346.

[35] J. Gray, Y. H. He, V. Jejjala and B. D. Nelson, "Exploring the vacuum geometry of N = 1 gauge theories," Nucl. Phys. B **750**, 1 (2006) [arXiv:hep-th/0604208]. "The geometry of particle physics," Phys. Lett. B **638**, 253 (2006) [arXiv:hep-th/0511062].

[36] D. Grayson and M. Stillman, "Macaulay 2, a software system for research in algebraic geometry," Available at http://www.math.uiuc.edu/Macaulay2/

[37] Hiraku Nakajima, "Lectures on Hilbert schemes of points on surfaces," American Mathematical Society, Providence, RI, 1999.

[38] Hiraku Nakajima, Kota Yoshioka, "Instanton counting on blowup. I. 4-dimensional pure gauge theory," math.AG/0306198.

[39] D. Martelli and J. Sparks, "Toric geometry, Sasaki-Einstein manifolds and a new infinite class of AdS/CFT duals," Commun. Math. Phys. **262**, 51 (2006) [arXiv:hep-th/0411238].

[40] S. Benvenuti, S. Franco, A. Hanany, D. Martelli and J. Sparks, "An infinite family of superconformal quiver gauge theories with Sasaki-Einstein duals," JHEP **0506**, 064 (2005) [arXiv:hep-th/0411264].

[41] S. Benvenuti, A. Hanany and P. Kazakopoulos, "The toric phases of the Y(p,q) quivers," JHEP **0507**, 021 (2005) [arXiv:hep-th/0412279].
S. Benvenuti and A. Hanany, "Conformal manifolds for the conifold and other toric field theories," JHEP **0508**, 024 (2005) [arXiv:hep-th/0502043].
A. Hanany, P. Kazakopoulos and B. Wecht, "A new infinite class of quiver gauge theories," JHEP **0508**, 054 (2005) [arXiv:hep-th/0503177].
H. Kihara, M. Sakaguchi and Y. Yasui, "Scalar Laplacian on Sasaki-Einstein manifolds Y(p,q)," Phys. Lett. B **621**, 288 (2005) [arXiv:hep-th/0505259].

[42] S. Pinansky, "Quantum deformations from toric geometry," JHEP **0603**, 055 (2006) [arXiv:hep-th/0511027].

[43] S. Lee, "Superconformal field theories from crystal lattices," arXiv:hep-th/0610204.

[44] T. Oota and Y. Yasui, "New example of infinite family of quiver gauge theories," arXiv:hep-th/0610092.

[45] A. Sinha, J. Sonner and N. V. Suryanarayana, "At the horizon of a supersymmetric AdS(5) black hole: Isometries and half-BPS giants," arXiv:hep-th/0610002.

[46] A. Basu and G. Mandal, "Dual giant gravitons in AdS(m) x Y**n (Sasaki-Einstein)," arXiv:hep-th/0608093.

[47] J. Kinney, J. M. Maldacena, S. Minwalla and S. Raju, "An index for 4 dimensional super conformal theories," arXiv:hep-th/0510251.

[48] C. Romelsberger, "Counting chiral primaries in N = 1, d=4 superconformal field theories," Nucl. Phys. B **747**, 329 (2006) [arXiv:hep-th/0510060].

[49] I. Biswas, D. Gaiotto, S. Lahiri and S. Minwalla, "Supersymmetric states of N = 4 Yang-Mills from giant gravitons," arXiv:hep-th/0606087.

[50] G. Mandal and N. V. Suryanarayana, "Counting 1/8-BPS dual-giants," arXiv:hep-th/0606088.

[51] A. Sinha and N. V. Suryanarayana, "Two-charge small black hole entropy: String-loops and multi-strings," JHEP **0610**, 034 (2006) [arXiv:hep-th/0606218].

[52] Y. Nakayama, "Index for orbifold quiver gauge theories," Phys. Lett. B **636**, 132 (2006) [arXiv:hep-th/0512280]. Y. Nakayama, "Index for supergravity on AdS(5) x T**(1,1) and conifold gauge theory," Nucl. Phys. B **755**, 295 (2006) [arXiv:hep-th/0602284].

[53] F. Fucito, J. F. Morales and R. Poghossian, "Instanton on toric singularities and black hole countings," arXiv:hep-th/0610154.