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Thermodynamic Bethe Ansatz with Haldane Statistics

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Abstract

We derive the thermodynamic Bethe ansatz equation for the situation in which the statistical interaction of a multi-particle system is governed by Haldane statistics. We formulate a macroscopical equivalence principle for such systems. Particular CDD-ambiguities play a distinguished role in compensating the ambiguity in the exclusion statistics. We derive Y-systems related to generalized statistics. We discuss several fermionic, bosonic and anyonic versions of affine Toda field theories and Calogero-Sutherland type models in the context of generalized statistics.

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1 Introduction

The Bethe ansatz is a technique which is based upon the quantum mechanical description of a many particle system by a wave function. The interaction of the individual particles is assumed to be relativistic, short-range and characterized by a factorizable scattering matrix. The boundary condition for the many particle wave function leads to what is commonly referred to as the Bethe ansatz equation, which provides the quantization condition for possible momenta of this system. Taking the thermodynamic limit of this equation, that is taking the size of the quantizing system to infinity, leads to the so-called thermodynamic Bethe ansatz (TBA). The origins of this analysis trace back to the seminal papers by Yang and Yang [1] and the technique has been refined and applied to different situations in numerous works [2-6] thereafter. The TBA constitutes an interface between massive integrable models and conformal field theories. One may extract different types of information from it, where the ultraviolet behaviour, i.e. ultimately the central charge of the conformal field theory, is the most accessible.

In the derivation of the TBA-equation the underlying statistical interaction (also called exclusion statistics) is usually taken to be either of bosonic or fermionic type. Seven years ago Haldane [7] proposed a generalized statistics based upon a generalization of Pauli's exclusion principle. This type of statistics (anyonic) has many important applications, in particular in the description of the fractional quantum Hall effect [8]. The main purpose of this manuscript is to implement systematically the Haldane statistics into the analysis of the thermodynamic Bethe ansatz. Hitherto, attempts in this direction [9, 10] were mainly based on the consideration of particular statistical interaction which only involves one species, like the Calogero-Sutherland model [11]. Our approach will cover a general choice of statistical interaction described by some, in general non-diagonal, matrix g_{ij}. We put particular emphasis on the ultraviolet region, which corresponds to the high temperature regime. We also continue our investigation started in [12] and clarify the role of the anyonic S-matrix (25) in the context of the TBA.

We formulate a macroscopical equivalence principle in the sense that the macroscopical nature of a multi-particle system is only governed by the combination (13) of the dynamical- and statistical interactions. This means in particular that two multi-particle systems differing on the microscopical level, i.e. in the S-matrix, may be made macroscopically equivalent by tuning the statistical interaction.

Our manuscript is organized as follows: In section 2 we recall the derivation of several thermodynamic quantities from Haldane statistics. In section 3 we derive the thermodynamic Bethe ansatz equation for a multi-particle system in which the statistical interaction is governed by Haldane statistics. In section 4 we argue that certain multi-particle systems may be transformed into macroscopically equivalent systems by tuning the statistical- and the dynamical interaction. In section 5 we demonstrate that certain scattering matrices, leading to equivalent thermodynamical systems differ only by CDD-ambiguities and comment on the ambiguity in choosing a particular statistical interaction. In section 6 we discuss the ultraviolet limit of the generalized TBA-equation. In section 7 we derive Y-systems related to generalized statistics. In section 8 we illustrate our general statements by some explicit examples. Our conclusions are stated in section 9.

2 Thermodynamics from Haldane Statistics

The object of our consideration is a multi-particle system containing l different species confined to a finite region of size L. We denote by n_i the number of particles, by N_i the dimension of the Fock-space related to the species i and by d_i the number of available states (holes) before the n_i-th particle has been added to the system. When treating bosons the number of available states naturally equals the total dimension of the Fock space, i.e. $N_i = d_i$, whereas when treating fermions there will be restrictions due to Pauli's exclusion principle and one has $N_i = d_i + n_i - 1$. With these relations in mind the total dimension of the Hilbert space may be written for both cases as

$$W = \prod_{i=1}^{l} \frac{(d_i + n_i - 1)!}{n_i!(d_i - 1)!} \quad . \tag{1}$$

Conventionally one employs in (1) the Fock-space dimension rather than the number of available states before the n_i-th particle has been added to the system. However, besides being a unified formulation, equation (1) has the virtue that it allows for a generalization to Haldane statistics.

By introducing a statistical interaction g_{ij}, Haldane [7] proposed the following generalized Pauli exclusion principle

$$\frac{\Delta d_i}{\Delta n_j} = -g_{ij} \quad . \tag{2}$$

Relation (2) means that the number of available states should be regarded as a function of the particles present inside the system. In this proposal a statistical interaction between different particle species is conceivable. In the bosonic case there will be no restriction such that $g_{ij} = 0$, whereas in the fermionic case the number of available states reduces by one if a particle is added to the system, hence one chooses $g_{ij} = \delta_{ij}$. One assumes [7, 9], that the total dimension of the Hilbert space is still given by (1), where the quantities involved are related to each other by (2).

We now want to analyze the multi-particle system in its thermodynamic limit, that is we let the size of the confining region approach infinity, $L \to \infty$. It is then a common assumption that the ratio of the particle (hole or state) numbers over the system size L remains finite. For the fraction of particles (holes or states) of species i with rapidities between $\theta - \Delta\theta/2$ and $\theta + \Delta\theta/2$ it is convenient to introduce densities

$$\Delta N_i = \rho_i(\theta)\, \Delta\theta\, L \tag{3}$$
$$\Delta n_i = \rho_i^r(\theta)\, \Delta\theta\, L \tag{4}$$
$$\Delta d_i = \rho_i^h(\theta)\, \Delta\theta\, L \quad . \tag{5}$$

The rapidity θ parameterizes as usual the two-momentum $\vec{p} = m\,(\cosh\theta, \sinh\theta)$. Integration of the generalized Pauli exclusion principle (2) then yields a relation between the different types of densities

$$\rho_i(\theta) = \rho_i^h(\theta) + \sum_{j=1}^{l} g_{ij}\rho_j^r(\theta) \quad . \tag{6}$$

The constant of integration has been identified with the Fock space dimension. The reason for this identification is based on fact that in this way one recovers for $g_{ij} = 0$ and $g_{ij} = \delta_{ij}$ the usual bosonic and fermionic relations, respectively. Notice that in the finite case the constant is in general slightly different from N_i, for instance for fermionic statistics it has to be chosen as $N_i + 1$ in order to recover the relation $N_i = d_i + n_i - 1$. However, in the thermodynamic limit this difference is negligible.

Let us remark, that equation (6) suggests from a physical point of view that g_{ij} should be non-negative. However, if there are additional symmetries then this requirement can be weakened. For instance for two conjugate particles, say i and $\bar{\imath}$ (see section 8 for examples), one naturally assumes that $\rho_i^r(\theta) = \rho_{\bar{\imath}}^r(\theta)$, such that only the combination $g_{ji} + g_{j\bar{\imath}}$ has to be taken non-negative.

We are now in the position to construct further thermodynamic quantities. First of all we may sum up all contributions from occupied states in order to obtain the total energy

$$E\,[\rho^r\,] = L \sum_{i=1}^{l} \int_{-\infty}^{\infty} d\theta \rho_i^r(\theta) m_i \cosh\theta \quad . \tag{7}$$

Furthermore, we obtain from (1), upon using Stirling's formula $\ln n! \approx n \ln n$, (3)-(5) and (6), the entropy $S = k \ln W$ as a functional of the particle- and Fock-space density

$$S\,[\rho, \rho^r\,] = kL \sum_{i=1}^{l} \int_{-\infty}^{\infty} d\theta \left[(\rho_i - g_{ij}\rho_j^r) \ln\left(\frac{\rho_i + h_{ij}\rho_j^r}{\rho_i - g_{ij}\rho_j^r} \right) + \rho_i^r \ln\left(\frac{\rho_i + h_{ij}\rho_j^r}{\rho_i^r} \right) \right] \quad . \tag{8}$$

We introduced here $h_{ij} = \delta_{ij} - g_{ij}$ and use the sum convention (sum over j) to avoid bulky expressions. k is Boltzmann's constant. According to the fundamental postulates of thermodynamics the equilibrium state of a system is found by minimizing the free energy F. Hence, keeping the temperature constant we may obtain the equilibrium condition by minimizing $F[\rho, \rho^r] = E[\rho^r] - TS[\rho, \rho^r]$ with respect to ρ^r. The equilibrium condition reads

$$\frac{\delta F}{\delta \rho_i^r} = \frac{\delta E}{\delta \rho_i^r} - T\frac{\delta S}{\delta \rho_i^r} - T\sum_{j=1}^{l} \frac{\delta S}{\delta \rho_j}\frac{\delta \rho_j}{\delta \rho_i^r} = 0 \quad . \tag{9}$$

So far we did not provide any information about the admissible momenta in the system, which are restricted by the boundary conditions.

3 Thermal Equilibrium with Boundaries

Boundary conditions may be accounted for by the Bethe ansatz equations. Recall that the Bethe ansatz equation is simply the equation which results from taking a particle in the multi-particle wave function on a trip through the whole system [1]. The particle will scatter with all other particles in the system, described by a factorizable S-matrix, such that

$$\exp(iLm_i \sinh\theta_i)\prod_{j\neq i}^{l} S_{ij}(\theta_i - \theta_j) = 1 \quad , \tag{10}$$

has to hold for consistency [1]. To simplify notations we may assume here that the scattering matrix is diagonal, such that the subscripts only label particle species. This set of transcendental equations determines which rapidities are admissible in the system due to the quantization as a result of restricting the size of the system. Taking the logarithmic derivative of the Bethe ansatz equation (10) and employing densities as in (3) and (4) one obtains (see for instance [2], [3] or [4] for more details)

$$\frac{1}{2\pi}m_i\cosh\theta + \sum_{j=1}^{l}\left(\varphi_{ij} * \rho_j^r\right)(\theta) = \rho_i(\theta) \quad . \tag{11}$$

Here we introduced as usual the notation $\varphi_{ij}(\theta) = -i\frac{d}{d\theta}\ln S_{ij}(\theta)$ and denote the convolution by $(f * g)(\theta) := 1/(2\pi)\int d\theta' f(\theta-\theta')g(\theta')$. The equilibrium condition (9) together with (11) yields the desired thermodynamic Bethe ansatz equation for a system in which statistical interaction is governed by Haldane statistics

$$\frac{1}{kT}m_i\cosh\theta = \ln(1 + x_i(\theta)) + \sum_{j=1}^{l}\left(\Phi_{ij} * \ln(1 + x_j^{-1})\right)(\theta) \quad . \tag{12}$$

We use here the abbreviations $x_i(\theta) := \rho_i^h(\theta)/\rho_i^r(\theta)$ and

$$\Phi_{ij}(\theta) := \varphi_{ij}(\theta) - 2\pi g_{ij}\delta(\theta) \quad . \tag{13}$$

In the derivation of (12) we assumed that $g_{ij} = g_{ji}$. We assume that $x_i(\theta)$ is symmetric* in the rapidity throughout the manuscript. In general one is only able to solve equations (12) numerically as we demonstrate in section 8. In the formulation of the TBA-equation of bosonic or fermionic type, it is common to introduce here as an additional quantity the so-called pseudo-energies $\varepsilon_i(\theta)$. In general one may employ

$$\ln(1 + x_i(\theta)) - \sum_{j=1}^{l} g_{ij} \ln(1 + x_j^{-1}(\theta)) = \varepsilon_i(\theta) \quad , \tag{14}$$

for this purpose. For the finite case the same relation was obtained by Wu [9], there however, the quantity $x_i(\theta)$ has a slightly different meaning. Clearly it is not possible to give a general solution of (14). However, for the bosonic and fermionic case it is solved easily, we obtain $x_i(\theta) = \exp(\varepsilon_i(\theta)) - 1$ and $x_i(\theta) = \exp(\varepsilon_i(\theta))$. Substitution of these solutions into equations (12) turns them into the well-known TBA-equations of bosonic and fermionic type [2], respectively. With (6) the ratios of the related particle- and Fock-space densities

$$\frac{\rho_i^r(\theta)}{\rho_i(\theta)} = \frac{1}{\exp(\varepsilon_i(\theta)) \mp 1} \tag{15}$$

become the usual Bose-Einstein (upper sign) and Fermi-Dirac (lower sign) distributions. From our point of view it does not seems to be necessary to introduce pseudo-energies and one should rather view $x_i(\theta) = \rho_i^h(\theta) / \rho_i^r(\theta)$ as a more fundamental entity. In addition one avoids the problem of solving (14). From a numerical point of view however, it appears sometimes useful to formulate the TBA-equation in different variables.

We now substitute back the equilibrium condition into the expression for the free energy and obtain together with the general expressions for the total energy (7) and the entropy (8), the generalized thermodynamic Bethe ansatz equation (12) and (11)

$$F(T) = -\frac{LkT}{2\pi} \sum_{i=1}^{l} \int_{-\infty}^{\infty} d\theta m_i \cosh\theta \ln\left(1 + x_i^{-1}(\theta)\right) \quad . \tag{16}$$

The relation between the free energy and the finite size scaling function is well-known to be $c(T) = -6F(T)/(\pi L T^2)$ [14]. As usual we now identify the temperature with the inverse of one radial size of a torus† $T = 1/r$ and choose now Boltzmann's constant to be one. Then

$$c(r) = \frac{6r}{\pi^2} \sum_{i=1}^{l} m_i \int_0^{\infty} d\theta \cosh\theta \ln\left(1 + x_i^{-1}(\theta)\right) \quad . \tag{17}$$

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*Taking into account that $\Phi_{ij}(\theta)$ is symmetric in θ (due to (21)), this is equivalent to the assumption that (12) may be solved iteratively (see e.g. [13]).

†For more details on the physical picture see for instance [2, 3, 4, 14].

Once more for $x_i(\theta) = \exp(\varepsilon_i(\theta)) - 1$ and $x_i(\theta) = \exp(\varepsilon_i(\theta))$ we recover the well-known expressions for the scaling functions of bosonic and fermionic statistics. In the ultraviolet limit the scaling function becomes the effective central charge of a conformal field theory [14], i.e. $\lim_{r\to 0} c(r) = c_{eff} = c - 24h'$. Here c is the usual conformal anomaly and h' denotes the lowest conformal dimension [15].

To summarize: For a given statistical interaction (2) and dynamical interaction described by a factorizable scattering matrix, we may solve in principle the generalized thermodynamic Bethe ansatz equation (12) for $x_i(\theta)$. This solution together with the knowledge of the mass spectrum of the theory, allows (up to a one dimensional integral which may always be carried out by simple numerics) the calculation of the entire scaling function.

4 Equivalent Multi-Particle Systems

From a thermodynamic point of view, systems which have the same expressions for the free energy show the same behaviour. Hence multi-particle systems which possess the same scaling function over the entire range of the scaling parameter r are to be considered as equivalent. This implies that two multi-particle systems are equivalent, if the mass spectra and the quantities Φ_{ij} are identical. As a consequence of this we may achieve that two systems equal each other from a macroscopical point of view, despite the fact that they involve different scattering matrices describing the dynamical particle interaction. The apparent difference can be compensated by a different choice of the underlying statistical interaction. Considering for instance the well-known TBA-equations of bosonic- and fermionic type (see for instance [2]) involving $\varphi^b_{ij}(\theta)$ and $\varphi^f_{ij}(\theta)$, respectively, we observe that

$$\varphi^f_{ij}(\theta) = \varphi^b_{ij}(\theta) + 2\pi\delta_{ij}\delta(\theta) \tag{18}$$

transforms both equations into each other.

The question of whether relations between two scattering matrices leading to (18) are at all conceivable immediately comes to mind. Such relations emerge in several places. For example in [5] the authors assume

$$S'_{ij}(\theta) = S_{ij}(\theta)\exp(-2\pi i\delta_{ij}\Theta(\theta))\,. \tag{19}$$

The function $\Theta(\theta)$ was taken to be the usual step-function, with the property that $\Theta(0) = 1/2$, such that $S_{ij}(\theta)$ and $S'_{ij}(\theta)$ only differ at the origin of the complex rapidity plane. This modification was necessary in order to extend the validity of certain identities ((2.18) in [5] leading to eq. (40) below) involving the scattering matrices of ADE-affine Toda field theories (see for instance [16]), i.e. $S_{ij}(\theta)$, also to the origin. In particular, in some special cases these identities become equal to the bootstrap equations. A further motivation to introduce (19) was to derive so-called Y-systems proposed by Zamolodchikov (last ref. in [2]) for ADE-affine

Toda field theories. In this case the system which involves $S_{ij}(\theta)$ with fermionic statistical interaction is equivalent to the system involving $S'_{ij}(\theta)$ with bosonic statistical interaction.

As a further example we may also consider a system in which the dynamical interaction is described by the scattering matrix $S(\theta) = -\exp(-i\pi\lambda\epsilon(\theta))$, where λ denotes the coupling constant and $\epsilon(\theta) = \Theta(\theta) - \Theta(-\theta)$, of the Calogero-Sutherland model [11] with a statistical interaction of bosonic type, i.e. $g = 0$. This system is equivalent to a system with a constant S-matrix, e.g. $S'(\theta) = -1$, and a statistical interaction of the form $g' = \lambda$.

In general we have the following equivalence principle. Multi-particle systems, involving quantities such that the relation

$$\varphi'_{ij}(\theta) = \varphi_{ij}(\theta) + 2\pi \left(g'_{ij} - g_{ij} \right) \delta(\theta) \tag{20}$$

holds and the masses for the same species are identical, are thermodynamically equivalent.

5 Microscopical Ambiguities

5.1 On CDD-Ambiguities

We now want to elaborate on the question in which sense the scattering matrices leading to (20) differ. The analysis of analytic properties of the scattering matrix leads to a set of consistency equations, which have to be satisfied by any S-matrix related to integrable models in 1+1 dimensions. These equations are so restrictive, that they determine the S-matrices of a particular model up to what is usually referred to as CDD-ambiguities [17, 18]. In this section we argue that certain S-matrices related to each other in such a way, that they lead to (20), differ precisely by such ambiguities.

A scattering matrix $S_{ij}(\theta)$ is usually assumed to be a meromorphic function in the strip $0 < \mathrm{Im}\,\theta < \pi, -\infty < \mathrm{Re}\,\theta < +\infty$. This region is considered as physical, meaning that all singularities occurring in this sheet acquire a physical interpretation. The scattering matrix in (10) is usually regarded as the one which results from an analysis of the so-called bootstrap equations

$$\begin{aligned} S_{ij}(\theta) S_{ij}(-\theta) &= 1 & (21)\\ S_{ij}(\theta) S_{\bar{\imath}j}(\theta - i\pi) &= 1 & (22)\\ S_{li}(\theta + i\eta) S_{lj}(\theta + i\eta') &= S_{l\bar{k}}(\theta) \quad . & (23) \end{aligned}$$

Equation (21) is a result of unitarity and analytic continuation, (22) a consequence of crossing invariance and (23) (strictly speaking this is the boostrap equation) expresses the factorization property for the fusing process $i + j \to \bar{k}$. The so-called fusing angles η, η' are specific to each model depending on the mass spectrum. It was found in [18] that the most general solution to these consistency equations will always be of the form

$$\prod_{\alpha\in A} \frac{\tanh\frac{1}{2}(\theta+\alpha)}{\tanh\frac{1}{2}(\theta-\alpha)} \quad , \tag{24}$$

where A is a set of complex numbers which characterizes a particular model. When particles are not self-conjugate one should replace tanh by sinh in (24). There is however the freedom to multiply these expressions with a so-called CDD-ambiguity [17] (also of the form (24), but related to a different set A'), which satisfies by itself all the consistency requirements without introducing any additional poles into the physical sheet. A well-known example for such an ambiguity are for instance the coupling constant dependent blocks. One seeks solutions of the general form (24) involving two sets A and A' such that the product related to the set A (the so-called minimal S-matrix) already closes the bootstrap and accounts for the whole particle spectrum independent of the coupling constant. Then the additional factors related to A' constitute a CDD-ambiguity depending on the coupling constant β in such a way, that in the limit $\beta \to 0$ (in theories which admit duality also $\beta \to \infty$) the S-matrix becomes free, that is one.

As a particular case of two S-matrices related to each other such that they may lead to an equation of the type (20) we will now consider

$$S'_{ij}(\theta) = S_{ij}(\theta)\exp(-2\pi i\Delta^+_{ij}\Theta(\mathrm{Re}\,\theta) - 2\pi i\Delta^-_{ij}\Theta(-\mathrm{Re}\,\theta)) \quad , \tag{25}$$

which we discussed in [12]. Similarly as in [5] we choose $\Theta(0) = 1/2$. Here the $\Delta^{\pm}_{ij}$ are related to the asymptotic phases of the S-matrix $S_{ij}(\theta)$

$$\lim_{\mathrm{Re}\,\theta\to\pm\infty} S_{ij}(\theta) = \exp\left(2\pi i\Delta^{\pm}_{ij}\right) . \tag{26}$$

The asymptotic phases are well defined, since the limit $\mathrm{Re}\,\theta \to \pm\infty$ of (24) does not depend on the imaginary part of θ. This property is in particular needed to obtain (29). The transformation (25) compensates the asymptotic phases and creates a non-trivial phase at $\theta \to 0$, i.e. the anyonic situation. It was argued [19], that from a physical point of view, S-matrices which possess a non-trivial asymptotic phase should be regarded rather as auxiliary objects. Scattering matrices of the type (25) should be considered as the genuine physical quantities, since they lead to the correct physical properties, including the exchange statistics.

We shall demonstrate that the system of equations (21)-(23) does not have to be altered for the anyonic matrix $S'_{ij}(\theta)$. For this purpose we will first derive some properties concerning the phases. Taking the limit $\mathrm{Re}\,\theta \to \infty$ in (21)-(23) yields immediately several relations between the asymptotic phases

$$\begin{aligned} \Delta^+_{ij} + \Delta^-_{ij} &= n && (27)\\ \Delta^{\pm}_{\bar{i}j} + \Delta^{\pm}_{ij} &= n'_{\pm} && (28)\\ \Delta^{\pm}_{li} + \Delta^{\pm}_{lj} &= \Delta^{\pm}_{l\bar{k}} + n''_{\pm} \quad , && (29) \end{aligned}$$

with $n, n'_\pm, n''_\pm$ being some integers. Using now these relations for the phases together with the fact that $S_{ij}(\theta)$ satisfies the consistency equations (21)-(23), it is straightforward to derive the related equations for the anyonic scattering matrix $S'_{ij}(\theta)$

$$
\begin{aligned}
S'_{ij}(\theta)S'_{ij}(-\theta) &= 1 && (30)\\
S'_{ij}(\theta)S'_{\bar{i}j}(\theta - i\pi) &= 1 && (31)\\
S'_{li}(\theta + i\eta)S'_{lj}(\theta + i\eta') &= S'_{l\bar{k}}(\theta) \quad . && (32)
\end{aligned}
$$

Comparing $S_{ij}(\theta)$ and $S'_{ij}(\theta)$, we conclude, taking (27) into account, that the additional factor in (25) has altered the behaviour at the imaginary axis of the rapidity plane only up to a sign. Thus $S_{ij}(\theta)$ and $S'_{ij}(\theta)$ differ by a CDD-factor.

It is straightforward to generalize the previous argument to the case when the scattering matrix is non-diagonal, such that in addition to (21)-(23) one also has to satisfy the Yang-Baxter equation as a consequence of factorization.

5.2 On statistical Ambiguities

In this subsection we recall the argument which leads to a particular choice of the statistical interaction. In general one considers the value of the scattering matrix at $\theta = 0$ in order to deduce the symmetry properties of the Bethe wave function. Then together with the a priori (or e.g. from a Lagrangian) knowledge of the nature of the particles one deduces the statistical interaction. For example for $S_{ii}(0) = -1$ the Bethe wave function is antisymmetric. If in this case one describes bosons, one is forced to choose a fermionic statistical interaction. However, this way of arguing seems somewhat ambiguous as a simple example demonstrates. Considering for instance a system in which the dynamical interaction is described by the affine Toda scattering matrix, one usually selects [4] fermionic statistics due to the fact that $S_{ij}(0) = (-1)^{\delta_{ij}}$. In the limit $\beta \to 0$ (because of strong-weak duality one may also take $\beta \to \infty$) the theory becomes free and one obtains $S_{ij}(\theta) = 1$. Using now the same arguments one has to deduce, from the symmetry of the Bethe wave function and the fact that one still describes bosons, that the statistical interaction has to be bosonic. Concerning the scaling function there is no problem here since in both cases we obtain the same ultraviolet limit. However, there is a change in the statistics. The apparent paradox is resolved by making use of the macroscopical equivalence. Using the scattering matrix (19) instead one has now $S_{ij}(0) = 1$. Making then use of the bosonic-fermionic transformation (18) we obtain by means of the same arguments a unique, that is bosonic, statistical interaction for the entire range of the coupling constant β.

Scattering matrices related to each other as in (25) play a distinguished role in this context since they only differ by a CDD-ambiguity as argued in the previous section. Considering such expressions the relation between the corresponding quantities $\varphi'_{ij}(\theta)$ and $\varphi_{ij}(\theta)$ are fixed. In this case we achieve thermodynamical equivalence by demanding

$$g'_{ij} - g_{ij} = \Delta^{-}_{ij} - \Delta^{+}_{ij} \quad . \tag{33}$$

We may consider a few examples. For instance in the case of affine Toda field theory we have $\Delta^{\pm}_{ij} = \pm\delta_{ij}/2$ [4], such that $g'_{ij} - g_{ij} = -\delta_{ij}$, which is a transformation from fermionic exclusion statistics to bosonic one. In case we only consider the minimal part of the scattering matrix for ADE-affine Toda field theory we obtain from the asymptotic behaviour observed in [4], i.e. $\Delta^{\pm}_{ij} = \pm(\delta_{ij}/2 - (C^{-1})_{ij})$, that is $g'_{ij} - g_{ij} = C^{-1}I$, where C denotes the Cartan matrix and I the incidence matrix of the related Lie algebra. Assuming now that g_{ij} is fixed by the arguments presented above, we obtain for the later case an interesting expression for the statistical interaction g'_{ij} in terms of Lie algebraic quantities. Therefore one may formulate the generalized Pauli principle in this context in a Lie algebraic form.

6 The ultraviolet Limit

One of the interesting quantities which may be extracted from the thermodynamic Bethe ansatz is the effective central charge of the conformal field theory when taking the ultraviolet limit [14], i.e. $r \to 0$ in (17). The integral equation (12) simplifies in this case to a set of constant coupled non-linear equations

$$\ln(1 + x_i) = \sum_{j=1}^{l} (N_{ij} + g_{ij}) \ln(1 + x_j^{-1}) \quad , \tag{34}$$

where $N_{ij} = \Delta^{-}_{ij} - \Delta^{+}_{ij}$. In this limit one may approximate $rm_i \cosh\theta$ in (12) and (17) by $\exp\theta \; rm_i/2$ [‡]. Taking thereafter the derivative of (12) we obtain, upon the substitution of the result into (17), for the effective central charge

$$c_{eff} = \frac{6}{\pi^2} \sum_{i=1}^{l} L\left(\frac{1}{1 + x_i}\right) \quad . \tag{35}$$

Here $L(x) = -\frac{1}{2}\int_0^x dt \left[\frac{\ln(1-t)}{t} + \frac{\ln t}{1-t}\right] = \sum_{n=1}^{\infty} \frac{x^n}{n^2} + \frac{1}{2}\ln x \ln(1-x)$ denotes Rogers dilogarithm [21]. In these definitions it is assumed that x takes its values between 0 and 1, which in turn implies that all x_i in (35) are to be non-negative. This is in agreement with the physical interpretation of the x_i as ratios of densities. Once again with $x_i = \exp(\varepsilon_i) - 1$ and $x_i = \exp(\varepsilon_i)$ we recover the well-known expressions for the bosonic and fermionic type of statistical interaction, respectively.

[‡] Of course these approximations rely upon certain assumptions (for details see for instance [20, 2, 4]).

Obviously the transformation properties discussed in section 4 also survive this limit process. As follows directly from (34) they read now

$$N'_{ij} = N_{ij} + g_{ij} - g'_{ij} \quad . \tag{36}$$

Apparently this condition is weaker than (20). It guarantees equivalence multi-particle systems only at the conformal point. The transformation from bosonic to fermionic statistics is compatible with equations (50) and (51) in [12] for the case b=1, where such transformations where obtained purely on the conformal level, that is by means of manipulations of certain characters.

It is instructive to consider a few examples. For instance having the situation that the difference of the phases of the scattering matrix equals the negative of the statistical interaction, i.e. $N_{ij} = -g_{ij}$ we always obtain

$$c_{eff} = \frac{6}{\pi^2} \sum_{i=1}^{l} L(1) = l \quad . \tag{37}$$

An example for this situation is to consider a system involving the scattering matrix related to ADE-affine Toda field theory (recall that in this case $N_{ij} = -\delta_{ij}$ [4]) and choose the statistical interaction to be fermionic.

An interesting structure emerges when considering a system in which the dynamical interaction is described by a direct product of Calogero-Sutherland scattering matrices, i.e. $S_{ij}(\theta) = -\exp(-i\pi\lambda_i\delta_{ij}\epsilon(\theta))$, and the statistical interaction is taken to be of fermionic type. Then (34) reduces to

$$x_i^{1+\lambda_i} = (1 + x_i)^{\lambda_i} \quad . \tag{38}$$

The same equation is of course obtained when the statistical interaction is chosen to be bosonic and the coupling constants shifted by one. In some cases (38) is solved easily analytically and we can employ (35) to compute the effective central charge. We may decompose the effective central charge into contributions coming from different choices for the λ_i: $c_{eff} = \sum_{i=1}^{l} c_{eff}^i$. For instance for $\lambda_i = 1$ the solution of (38) is $x_i = (1+\sqrt{5})/2$ and for $\lambda_i = -1/2$ we obtain $x_i = (\sqrt{5}-1)/2$. Then for $\lambda_i = -1/2$ and $\lambda_i = 1$ we obtain, with the help of (35),

$$c_{eff}^i = \frac{6}{\pi^2} L\left(\frac{2}{1+\sqrt{5}}\right) = \frac{3}{5} \qquad \text{and} \qquad c_{eff}^i = \frac{6}{\pi^2} L\left(\frac{2}{3+\sqrt{5}}\right) = \frac{2}{5}, \tag{39}$$

respectively. As already mentioned the computation of the effective central charge by means of (34) and (35) is not always rigorously justified. However, the explicit analytic computations of the full scaling functions provided in the next section confirm these results. It is intriguing to note that we recover in this way the effective central charges for certain minimal models [15]. The case $\lambda = -1/2$ corresponds to the minimal model $\mathcal{M}(3,5)$, whilst $\lambda = 1$ corresponds to the

Yang-Lee model, i.e. $\mathcal{M}(2,5)$. Equation (38) may also be solved for different values of λ and we may compute the effective central charge by means of (35). The dependence of c_{eff} on λ for the one-particle contribution is depicted in figure 1. [§] This figure suggests immediately that one may find other minimal models as ultraviolet limits of (17) for different values of λ. However, apart from $\lambda = 0, -1, \infty$, the two values presented are the only possible choices for a model with solely one particle leading to accessible[¶] relations for dilogarithms. This implies that other rational λ do not lead to rational values of c_{eff}.

7 Y-systems

For some class of models it has turned out to be possible to carry out certain manipulations on the TBA-equations such that the original integral TBA-equations acquire the form of a set of functional equations in new variables Y_i (last ref. in [2]). These functional equations have the further virtue that unlike the original TBA-equations they do not involve the mass spectrum and are commonly referred to as Y-systems. In these new variables certain periodicities in the rapidities are exhibited more clearly. These periodicities may then be utilized in order to express the quantity Y as a Fourier series, which in turn is useful to find solution of the TBA-equations and expand the scaling function as a power series in the scaling parameter. We will now demonstrate that similar equations may be derived for a multi-particle system in which the dynamical scattering is governed by the scattering matrix related to ADE-affine Toda field theories and the statistical interaction is of general type.

We consider the modified version (in the sense of (19)) of the minimal part of the scattering matrix of ADE-affine Toda field theories. As was shown in [5] these S-matrices lead to the identity

$$\varphi_{ij}\left(\theta + \frac{i\pi}{h}\right) + \varphi_{ij}\left(\theta - \frac{i\pi}{h}\right) = \sum_{k=1}^{r} I_{ik}\varphi_{kj}(\theta) - 2\pi I_{ij}\delta(\theta) \quad , \tag{40}$$

where h denotes the Coxeter number, r the rank and I the incidence matrix of the Lie algebra. It is then straightforward to derive the "$YZ-$system"

$$Y_i\left(\theta + \frac{i\pi}{h}\right) Y_i\left(\theta - \frac{i\pi}{h}\right) = \prod_{j=1}^{r} Z_j(\theta)^{I_{ij}} . \tag{41}$$

[§]The plot is presented only for $\lambda \geq -1$ since eq. (38) possesses non-negative solutions only at this interval.

[¶]In general a dilogarithmic identity is called accessible if it is of the form $\sum_{i=1}^{N} L(y_i) = k\pi^2/6$, with k being rational and y_i algebraic [21].

The quantities $Y(\theta)$ and $Z(\theta)$ involve now the statistical interaction in the form

$$Y_i(\theta) = (1 + x_i(\theta)) \prod_{j=1}^{r} (1 + x_j^{-1}(\theta))^{-g_{ij}} \quad \text{and} \qquad Z_i(\theta) = Y_i(\theta)\,(1 + x_i^{-1}(\theta)) \quad . \tag{42}$$

Equations (41) follow upon first adding (12) at $\theta + \frac{i\pi}{h}$ and $\theta - \frac{i\pi}{h}$ and subtracting I times (12) at θ from the sum. Thereafter we employ the fact that the masses of affine Toda field theory are proportional to the Perron-Frobenius vector of the Cartan matrix, i.e. $\sum_{j=1}^{r} C_{ij} m_j = 4\sin^2(\pi/(2h)) m_i$ [22]. Then, with the help of (40) and (42), the equations (41) follow.

In comparison with (12) the equations (41) have already the virtue that they are simple functional equations and do not involve the mass spectrum. However, in order to solve them we still have to express Z in terms of Y or vice versa, which is not possible in general. However, once the statistical interaction is specified this may be achieved. For instance, for $g_{ij} = \delta_{ij}$ we obtain $Z_i = Y_i + 1$ and recover the known fermionic Y-system. In the bosonic case, i.e. $g_{ij} = 0$ we obtain $Z_i = Y_i^2/(Y_i - 1)$ and for "semionic" statistical interaction with $g_{ij} = \delta_{ij}/2$ we obtain $Z_i = \left(\sqrt{1 + 4Y_i^2} + 1\right)^2 / (4Y_i)$.

8 Examples

8.1 Ising model and Klein-Gordon Theory

The most elementary examples which illustrate the features outlined above more concretely is simply to consider the Ising model (A_1-minimal affine Toda field theory) $S(\theta) = -1$ or the Klein-Gordon theory $S(\theta) = 1$. Then for both cases equation (12) is solved trivially. We obtain

$$x(\theta) = \exp(rm\cosh\theta) - 1 \quad \text{and} \quad x(\theta) = \exp(rm\cosh\theta) \tag{43}$$

for bosonic- and fermionic statistical interaction, respectively. With the help of these solutions we may then compute the entire scaling function. For fermionic statistics we obtain from (17)

$$c(r) = \frac{6}{\pi^2} rm \sum_{n=1}^{\infty} (-1)^{n+1} \frac{K_1(nrm)}{n} \tag{44}$$

whilst bosonic statistics yields

$$c(r) = \frac{6}{\pi^2} rm \sum_{n=1}^{\infty} \frac{K_1(nrm)}{n} \quad , \tag{45}$$

where $K_1(x)$ is a modified Bessel function. We depict these functions in figure 3(b), referring to both of them by a slight abuse of notation as A_1 . One observes, that the difference in the statistical interaction is effecting most severely the ultraviolet region. The two scaling functions converge relatively fast towards each other in the infrared regime. We shall encounter these feature also in other models.

Using the well-known property for the asymptotic behaviour of the modified Bessel function $\lim_{x \to 0} x K_1(x) = 1$ leads to

$$\lim_{r \to 0} c(r) = -\frac{6}{\pi^2} Li(-1) = \frac{1}{2} \quad \text{and} \quad \lim_{r \to 0} c(r) = \frac{6}{\pi^2} Li(1) = 1 \tag{46}$$

for the fermionic and bosonic type equations, respectively. Here $Li(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}$ denotes Euler's dilogarithm [21].

We may also compute some less trivial cases. For instance taking the statistical interaction to be of "semionic" type, i.e. $g = 1/2$, we obtain, after solving (12), for the scaling function

$$\begin{aligned} c(r) &= \frac{12r}{\pi^2} m \int_0^{\infty} d\theta \, \mathrm{arsinh} \left[\frac{1}{2} \exp(-rm \cosh\theta) \right] \cosh\theta \\ &= \frac{12r}{\pi^2} m \sum_{n=0}^{\infty} (-1)^n \frac{(2n-1)!!}{(2n)!!(2n+1)2^{2n+1}} K_1((2n+1)rm) \quad . \end{aligned} \tag{47}$$

Once more we carry out the ultraviolet limit with the help of the asymptotics of the modified Bessel function

$$c(0) = \frac{12}{\pi^2} \sum_{n=0}^{\infty} (-1)^n \frac{(2n-1)!!}{(2n)!!(2n+1)^2 2^{2n+1}} = \frac{12}{\pi^2} \int_0^{1/2} dt \, \frac{\mathrm{arsinh}\, t}{t} = \frac{3}{5} \quad . \tag{48}$$

This value of the effective central charge corresponds to the minimal model $\mathcal{M}(3,5)$. This is of course what we expect from section 6, since with the help of (20) it can be easily seen that we have thermodynamical equivalence between the Calogero-Sutherland model with coupling $\lambda = -1/2$ and fermionic statistical interaction and the Ising model with "semionic" statistical interaction.

For $g_{ij} = 2\delta_{ij}$ we carry out a similar computation and obtain for the scaling function

$$c(r) = \frac{6r}{\pi^2} m \int_0^{\infty} d\theta \ln \left(\frac{1 + \sqrt{1 + 4\exp(-rm\cosh\theta)}}{2} \right) \cosh\theta \quad . \tag{49}$$

Once again we may perform the ultraviolet limit and obtain in this case $c_{eff} = 2/5$, which corresponds to the minimal model $\mathcal{M}(2,5)$. This is in agreement with the results in section 6, since the Calogero-Sutherland model with coupling $\lambda = 1$ and fermionic statistical interaction and the Ising model with statistical interaction $g_{ij} = 2\delta_{ij}$ are thermodynamically equivalent.

8.2 Scaling Potts and Yang-Lee Models

Next we investigate the scaling Potts model in this context, which was previously studied by Zamolodchikov [2] with regard to conventional fermionic statistics. The S-matrix of the scaling Potts model equals the minimal S-matrix of A_2-affine Toda field theory [23] and reads

$$S_{11}(\theta) = S_{22}(\theta) = \frac{\sinh\left(\frac{\theta}{2}+\frac{i\pi}{3}\right)}{\sinh\left(\frac{\theta}{2}-\frac{i\pi}{3}\right)} \quad \text{and} \quad S_{12}(\theta) = -\frac{\sinh\left(\frac{\theta}{2}+\frac{i\pi}{6}\right)}{\sinh\left(\frac{\theta}{2}-\frac{i\pi}{6}\right)} \quad . \tag{50}$$

As commented in section 4, the S-matrix does not satisfy the bootstrap at $\theta = 0$. From our point of view it seems therefore more natural to use its modification in the sense of (19) and employ bosonic statistics, which of course by the equivalence principle leads to the same TBA-equations. The two particles in the model are conjugate to each other, i.e. $1 = \bar{2}$, and consequently the masses are the same $m_1 = m_2 = m$. The conjugate particle occurs as a bound state when two particles of the same species scatter, for instance 1+1→2. For the TBA-equation we need

$$\varphi_{11}(\theta) = \varphi_{22}(\theta) = \frac{-\sqrt{3}}{2\cosh\theta + 1} \quad \text{and} \quad \varphi_{12}(\theta) = \frac{\sqrt{3}}{1 - 2\cosh\theta} \quad . \tag{51}$$

Then for the fermionic statistics, equation (12) can be solved iteratively

$$\ln\left(x^{(n+1)}(\theta)\right) = rm\cosh\theta + \frac{2\sqrt{3}}{\pi}\int_{-\infty}^{\infty} d\theta' \frac{\cosh(\theta-\theta')}{1+2\cosh 2(\theta-\theta')} \ln\left(1 + \frac{1}{x^{(n)}(\theta')}\right). \tag{52}$$

Here we assumed that the Z_2-symmetry of the model will be preserved such that $x_1(\theta) = x_2(\theta)$. Once more for $x(\theta) = \exp(\varepsilon(\theta))$ we recover the well-known TBA-equation of fermionic type (first ref. in [2]). It appears to be impossible to find analytic solutions to this equation, but it is straightforward to solve it numerically. Taking $\ln\left(x^{(0)}(\theta)\right) = rm\cosh\theta$ one can iterate this equation as indicated by the superscripts. Depending on the value of mr and θ, convergence is achieved relatively quickly (typically n<50). The result is shown in figure 2(a) and appears to be in complete agreement with the calculation in [2]. To make contact with the literature we introduced the quantity $L(\theta) = \ln\left(1 + x^{-1}(\theta)\right)$. One observes the typical behaviour of $\lim_{mr\to 0} L(\theta) = const$ for some region of θ, which is required to derive (34).

We now consider a model with fermionic statistical interaction which involves an S-matrix related to the scaling Potts model in the sense of (25). According to the discussion in sections 4 and 5, this is equivalent to considering the scaling Potts model in which the statistical interaction is taken to be $g_{ij} = \delta_{ij} - N_{ij}$. Since

$$N = \Delta^- - \Delta^+ = -\frac{1}{2\pi}\int_{-\infty}^{\infty} d\theta\varphi(\theta) = C_{A_2}^{-1} \cdot I_{A_2} = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}, \tag{53}$$

(C_{A_2} and I_{A_2} denote here the Cartan matrix and the incidence matrix of the A_2-Lie algebra), we obtain

$$g = \begin{pmatrix} \frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} \end{pmatrix}, \tag{54}$$

and the generalized TBA-equation (12) becomes

$$\ln(x(\theta)) = rm\cosh\theta - \ln\left(1 + x^{-1}(\theta)\right) + \frac{2\sqrt{3}}{\pi}\int_{-\infty}^{+\infty} d\theta' \frac{\cosh(\theta-\theta')\ln(1+x^{-1}(\theta'))}{1+2\cosh 2(\theta-\theta')}. \tag{55}$$

At first sight g_{ij} in (54) seems to be inappropriate since its off-diagonal elements are negative. However, due to the Z_2-symmetry this does not pose a problem (c.f. section 2), since $g_{11} + g_{12} = 0$ and $x_1(\theta) = x_2(\theta)$. In fact, this system may be thought of as a bosonic system with one particle species. Equation (55) may be solved easily numerically in a similar fashion as (52), whereas in comparison with (52) convergence is now achieved much faster (typically $n < 20$). The result is shown in figure 2(b).

Having solved the generalized TBA-equation for $x(\theta)$ we may also compute the entire scaling function. The result of the numerical computation is depicted in figure 3(b) for the two types of statistical interactions presented in this subsection. Notice that the conformal limit for the exotic statistics corresponds to $c = 1$, which is in agreement with (37). Hence in this limit we obtain a one particle bosonic system.

Due to the fact that the scattering matrices of the scaling Potts model and the scaling Yang-Lee model are related as $S^{YL}(\theta) = S_{11}^{A_2}(\theta)S_{12}^{A_2}(\theta)$, the TBA-equations for the two models are identical. The only difference occurs due the fact that in the scaling Yang-Lee model there is only one instead of two particles present and therefore the scaling functions equal each other up to a factor 2.

We finish the discussion of the anyonic scaling Potts model with a brief remark on other possible choices of exotic statistics. For example, we may take the S-matrix of the scaling Potts model and choose the statistical interaction to be of form $g_{ij} = g\delta_{ij}$. For this choice of g_{ij} the Z_2-symmetry is also present, so that $x_1(\theta) = x_2(\theta)$. This leads, in particular, to a simplification of the system (34). In fact, it reduces to two equations of type (38) with $i = 1, 2$ and $\lambda_1 = \lambda_2 = g$. Therefore, the effective central charge describing conformal limit of this anyonic version of the scaling Potts model is given by $c_{SP}(g) = 2c_{CS}(\lambda = g)$, where $c_{CS}(\lambda)$ is the effective central charge of the Calogero-Sutherland type model discussed above (see figure 1). This also means that the conformal limit of the scaling Yang-Lee model with statistical interaction g is described by $c_{YL}(g) = c_{CS}(\lambda = g)$ (of course, as we discussed in section 6, this does not imply global equivalence with the Calogero-Sutherland model).

8.3 A_3-minimal affine Toda Theory

Finally we present the fermionic computation for the easiest model, which contains at least two different mass values , i.e. A_3-affine Toda field theory. The minimal S-matrix reads [23]

$$
\begin{aligned}
S_{11}(\theta) = S_{33}(\theta) &= \frac{\sinh\left(\frac{\theta}{2}+\frac{i\pi}{4}\right)}{\sinh\left(\frac{\theta}{2}-\frac{i\pi}{4}\right)} & S_{13}(\theta) &= -\frac{\sinh\left(\frac{\theta}{2}+\frac{i\pi}{4}\right)}{\sinh\left(\frac{\theta}{2}-\frac{i\pi}{4}\right)} \\
S_{12}(\theta) = S_{23}(\theta) &= \frac{\sinh\left(\frac{\theta}{2}+\frac{i\pi}{8}\right)}{\sinh\left(\frac{\theta}{2}-\frac{i\pi}{8}\right)} \frac{\sinh\left(\frac{\theta}{2}+\frac{3\pi i}{8}\right)}{\sinh\left(\frac{\theta}{2}-\frac{3\pi i}{8}\right)} & S_{22}(\theta) &= -\left(\frac{\sinh\left(\frac{\theta}{2}+\frac{i\pi}{4}\right)}{\sinh\left(\frac{\theta}{2}-\frac{i\pi}{4}\right)}\right)^2 .
\end{aligned} \tag{56}
$$

Hence

$$
\varphi_{11}(\theta) = \varphi_{33}(\theta) = \varphi_{13}(\theta) = \frac{1}{2}\varphi_{22}(\theta) = -\frac{1}{\cosh\theta} \tag{57}
$$

$$
\varphi_{12}(\theta) = \varphi_{23}(\theta) = -\frac{2\sqrt{2}\cosh\theta}{\cosh 2\theta} . \tag{58}
$$

The masses are given by $m_1 = m_3 = m/\sqrt{2}$ and $m_3 = m$. Again we assume that the Z_2-symmetry is preserved, such that $x_1(\theta) = x_3(\theta)$. The TBA-equations for fermionic statistics read

$$
\begin{aligned}
\ln\left(x_1(\theta)\right) &= \frac{rm}{\sqrt{2}}\cosh\theta + \int_{-\infty}^{\infty} d\theta' \left(\frac{L_1(\theta')}{\pi\cosh(\theta-\theta')} + \frac{\sqrt{2}\cosh(\theta-\theta')L_2(\theta')}{\pi\cosh 2(\theta-\theta')}\right), \\
\ln\left(x_2(\theta)\right) &= rm\cosh\theta + \int_{-\infty}^{\infty} d\theta' \left(\frac{L_2(\theta')}{\pi\cosh(\theta-\theta')} + \frac{\sqrt{8}\cosh(\theta-\theta')L_1(\theta')}{\pi\cosh 2(\theta-\theta')}\right),
\end{aligned}
$$

where $L_i(\theta) = \ln\left(1 + x_i^{-1}(\theta)\right)$ for $i = 1, 2$. Once again we may solve these equations numerically and compute the entire scaling function. The result is depicted in figure 3(b). The functions $L_i(\theta)$ exhibit the typical plateau as mr approaches zero. In comparison with the other models we observe that scaling functions for systems with the same statistical interaction have qualitatively the same shape. Similarly as in the previous subsection we could now also consider the anyonic system with $g_{ij} = \delta_{ij} - N_{ij}$. However, in this case we would have an overall negative contribution of ρ_1^r to the density ρ_2 in (6), which as discussed in section 2 seems inappropriate from a physical point of view. We also observed that in the numerical computations singularities occur. However, we may perfectly well choose a different type of statistics which leads to satisfactory equations. For instance, we may choose $g_{ij} = g\delta_{ij}$. In this case equations (34) and (35) predict that for $g = 0,\ 1/2,\ 1,\ 2$ the conformal limit is described by c_{eff} taking the following values: $c_{eff}(0) \approx 1.16$, $c_{eff}(1/2) \approx 1.07$, $c_{eff}(1) = 1$ and $c_{eff}(2) \approx 0.895$. One may conjecture that $c_{eff}(g)$ is a monotonically decreasing function. A detailed investigation of whole scaling functions for admittable choices of statistical interaction is left for future studies.

9 Conclusion

For a multi-particle system which involves a factorizable scattering matrix describing the dynamical interaction and a statistical interaction governed by Haldane statistics we derived the TBA-equations. These equations may be solved by the same means as the conventional TBA-equations of fermionic and bosonic type and allow the computation of the entire scaling function.

The behaviour of the scaling functions depicted in figure 3(b) suggests the validity of the conjecture [2], that the series for the scaling function in the scaling parameter r commences with a constant and thereafter involves quadratic and higher powers in r, for fermionic type of statistics. For exotic statistics these features don't seem to be evident.

The question of how to select a particular statistics without the prior knowledge of the nature of the particles remains to be clarified.

It would be very interesting to generalize these kind of considerations also to the situation in which the dynamical scattering is described by non-diagonal S-matrices (e.g. affine Toda field theory with purely imaginary coupling constant), to the excited TBA-equations [2, 6] or even more exotic statistics [24].

After the completion of our manuscript Dr. Ilinski pointed out to us, that there exists an ongoing dispute [25] about how to achieve compatibility between equations (1) and (2), that is to provide a prescription for the counting of the states involving (2) which would lead to (1). There are even doubts whether it is at all possible to achieve compatibility between the two equations. What our manuscript concerns, this is not a crucial issue, since we could also start with a much weaker assumption. We need only (1) with d_i replaced according to (2).

We are also grateful to Dr. Hikami for pointing out reference [26] to us, in which one also finds a discussion on ideal g-on gas systems with fractional exclusion statistics. In particular our figure 1 is similar to the figure in there.

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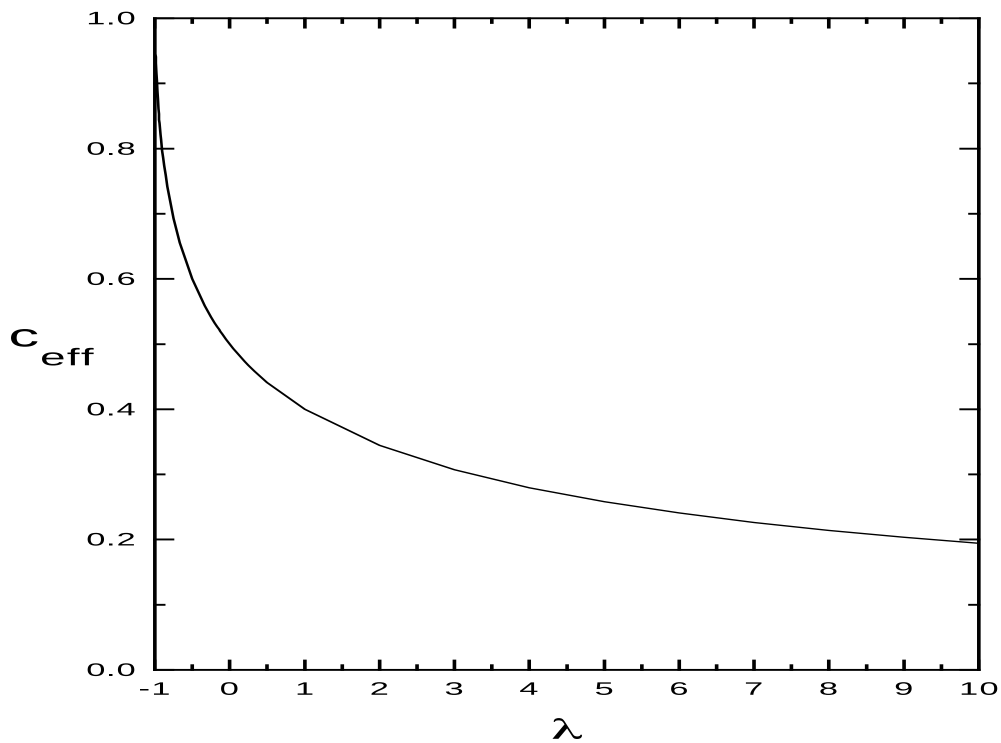

Figure 1: The effective central charge for the ultraviolet limit of the Calogero-Sutherland model versus its coupling constant λ.

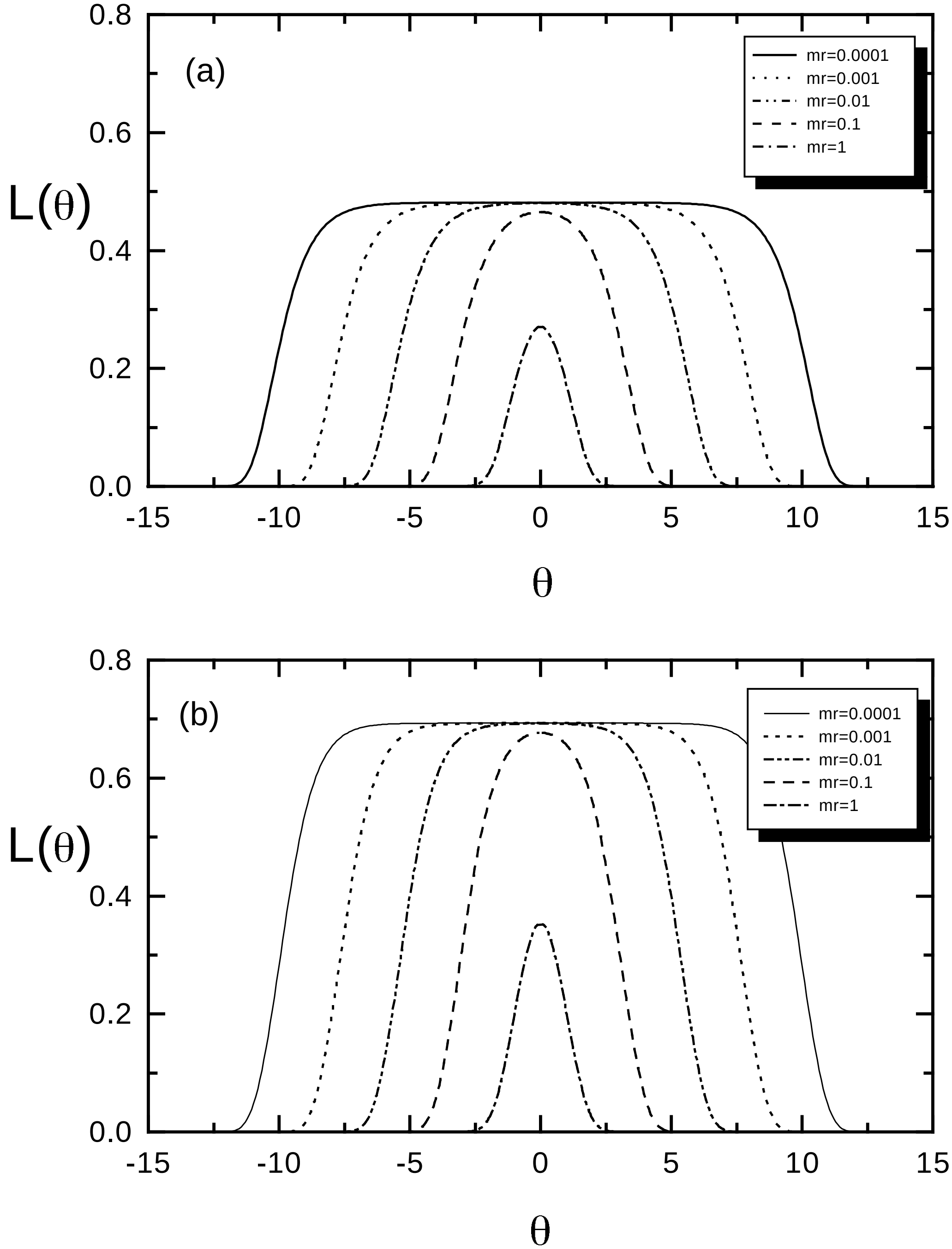

Figure 2: Solution of the thermodynamic Bethe ansatz for the scaling Potts model with fermionic statistical interaction (a) and exotic statistical interaction, i.e. $g_{ij} = \delta_{ij} - N_{ij}$, figure (b).

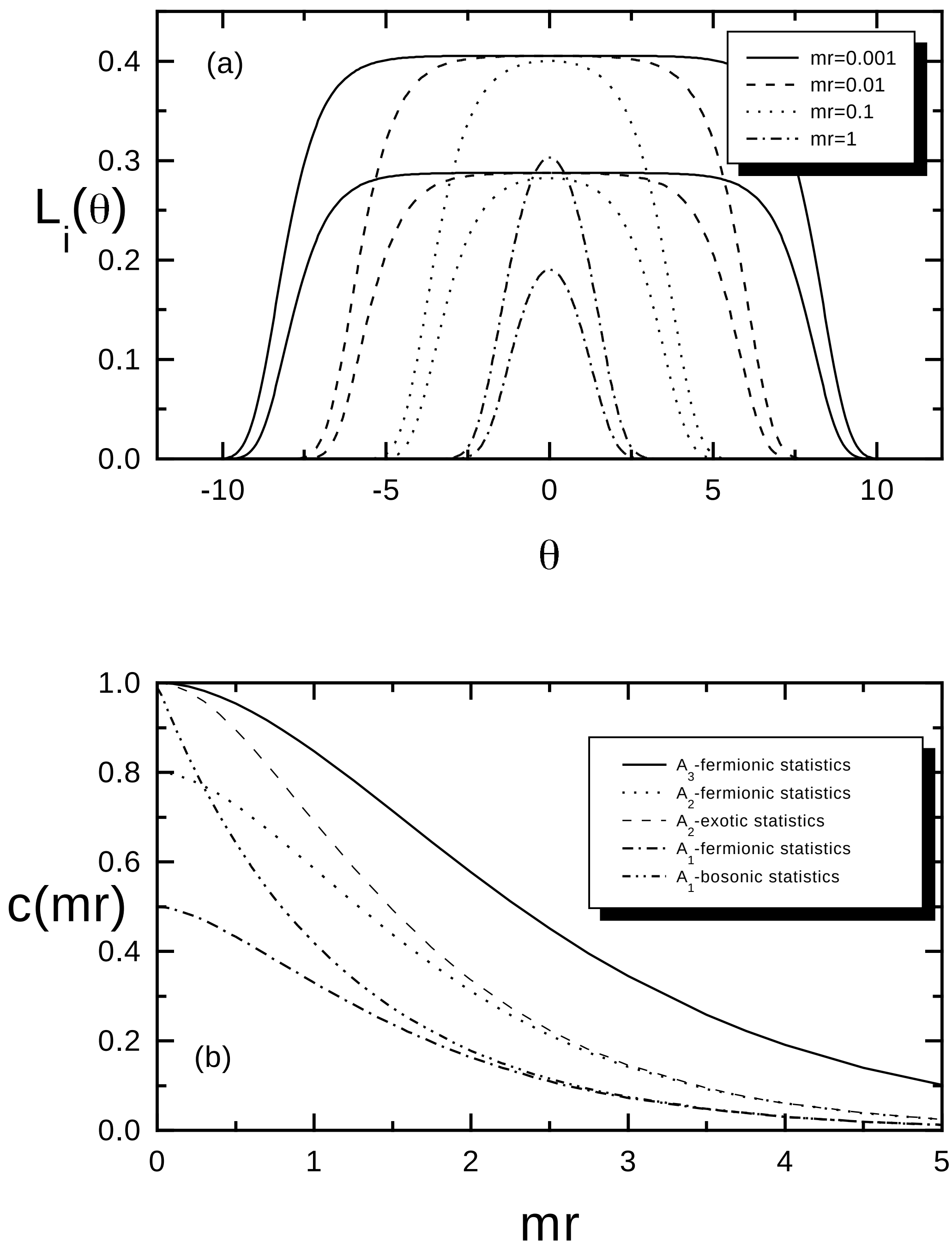

Figure 3: Solution of the TBA-equation for the A_3-minimal affine Toda field theory (a). For the same value of mr the curve $L_2(\theta)$ is always below $L_1(\theta)$. Scaling function for systems involving different types of dynamical and statistical interaction (b). The exotic statistics for A_2 means $g_{ij} = \delta_{ij} - N_{ij}$.