On the blocks of semisimple algebraic groups and associated generalized Schur algebras

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Abstract

In this paper we give a new proof for the blocks of any semisimple simply connected algebraic group when the characteristic of the field is greater than 5. The first proof was given by Donkin and works in arbitrary characteristic. Our new proof has two advantages. First we obtain a bound on the length of a minimum chain linking two weights in the same block. Second we obtain a sufficient condition on saturated subsets π of the set of dominant weights which ensures that the blocks of the associated generalized Schur algebra are simply the intersection of the blocks of the algebraic group with the set π. However, we show that this is not the case in general for the symplectic Schur algebras, disproving a conjecture of Renner.

Keywords: Blocks, semisimple algebraic groups, Schur algebras.

1 Introduction and notation

Let G be a semisimple simply connected algebraic group over an algebraically closed field k. We are interested in the category of all G-modules. When the field k has characteristic zero, this category is semisimple, in other words every G-modules splits as a direct sum of simple modules. Over a field of characteristic $p > 0$, the category of G-modules is no longer semisimple. Nevertheless, it can be split into 'blocks' such that every G-modules can be written as a direct sum of indecomposable (but not necessarily simple) modules belonging to these blocks. Thus in order to study the category of G-modules it is enough to study each block separately.

We recall the basic definitions and notation needed here. More details can be found in [10, Part II].

Let T be a maximal torus of G and let $W = N_G(T)/T$ be the Weyl group. Let B be a Borel subgroup containing T. Let $X = X(T)$ be the weight lattice and fix a non-singular, symmetric positive definite W-invariant form on $X \otimes_{\mathbb{Z}} \mathbb{R}$, denoted by $\langle ., . \rangle$. Let R be the root system, R^+ the set of positive roots which makes B the negative Borel and let S be the set of simple roots. Define the set of dominant weights by

$$X^+ = \{\lambda \in X \mid \langle \lambda, \check{\alpha} \rangle \geq 0 \ \forall \alpha \in S\}$$

where $\check{\alpha} = 2\alpha / \langle \alpha, \alpha \rangle$ for $\alpha \in R$. For $r \geq 1$ define also the set of p^r-restricted weights X_r by

$$X_r = \{\lambda \in X^+ \mid \langle \lambda, \check{\alpha} \rangle < p^r \ \forall \alpha \in S\}.$$

The weight lattice has a natural partial ordering: for $\lambda, \mu \in X$ we write $\lambda \geq \mu$ if and only if $\lambda - \mu$ is a sum of simple roots. Let w_0 be the longest element in the Weyl group W. We denote by β_0 the highest short root of R and by ρ half the sum of the positive roots. For each root $\beta \in R^+$ and each integer m, define the (affine) reflection $s_{\beta,m}$ on $X \otimes_{\mathbb{Z}} \mathbb{R}$ by

$$s_{\beta,m}(\lambda) = \lambda - (\langle \lambda, \check{\beta} \rangle - m)\beta.$$

Define the affine Weyl group W_p to be the group generated by all reflections $s_{\beta,mp}$ for $\beta \in R^+$, $m \in \mathbb{Z}$. Similarly, for any positive integer r we define W_{p^r} to be the group generated by all s_{β,mp^r} with $\beta \in R^+$ and $m \in \mathbb{Z}$. In this paper we always consider the dot action $w \cdot \lambda = w(\lambda + \rho) - \rho$ of W_p on X or $X \otimes_{\mathbb{Z}} \mathbb{R}$. So we view $s_{\beta,mp}$ as a reflection through the hyperplane

$$\{\lambda \in X \otimes_{\mathbb{Z}} \mathbb{R} \mid \langle \lambda + \rho, \check{\beta} \rangle = mp\}.$$

This action of the affine Weyl group W_p on $X \otimes_{\mathbb{Z}} \mathbb{R}$ defines a system of facets. A facet is a non-empty subset of the form

$$\begin{aligned} F = \{\lambda \in X \otimes_{\mathbb{Z}} \mathbb{R} \mid & \langle \lambda + \rho, \check{\alpha} \rangle = n_\alpha p \quad \forall \alpha \in R_0^+(F) \\ & (n_\alpha - 1)p < \langle \lambda + \rho, \check{\alpha} \rangle < n_\alpha p \quad \forall \alpha \in R_1^+(F)\} \end{aligned}$$

for some $n_\alpha \in \mathbb{Z}$ and some disjoint decomposition $R^+ = R_0^+(F) \cup R_1^+(F)$. The closure $\bar{F}$ of F is equal to

$$\begin{aligned}\bar{F} = \{\lambda \in X \otimes_{\mathbb{Z}} \mathbb{R} \mid \langle \lambda + \rho, \check{\alpha} \rangle = n_\alpha p \quad \forall \alpha \in R_0^+(F) \\ (n_\alpha - 1)p \leq \langle \lambda + \rho, \check{\alpha} \rangle \leq n_\alpha p \quad \forall \alpha \in R_1^+(F)\}\end{aligned}$$

and the upper closure $\hat{F}$ of F is equal to

$$\begin{aligned}\hat{F} = \{\lambda \in X \otimes_{\mathbb{Z}} \mathbb{R} \mid \langle \lambda + \rho, \check{\alpha} \rangle = n_\alpha p \quad \forall \alpha \in R_0^+(F) \\ (n_\alpha - 1)p < \langle \lambda + \rho, \check{\alpha} \rangle \leq n_\alpha p \quad \forall \alpha \in R_1^+(F)\}\end{aligned}$$

A facet F is called an alcove if $R_0^+(F) = \emptyset$. If F is an alcove then $\bar{F}$ is a fundamental domain for the action of W_p on $X \otimes_{\mathbb{Z}} \mathbb{R}$. We will often use a particular alcove, called the fundamental alcove, given by

$$C = \{\lambda \in X \otimes_{\mathbb{Z}} \mathbb{R} \mid 0 < \langle \lambda + \rho, \check{\beta} \rangle < p \quad \forall \beta \in R^+\}.$$

We now refine the partial order $\leq$ to the partial order $\uparrow$ using the affine Weyl group by setting $\lambda \uparrow \mu$ if and only if there are weights $\mu_1, \mu_2, \ldots, \mu_r \in X$ and reflections $s_1, s_2, \ldots, s_{r+1} \in W_p$ with

$$\lambda \leq s_1 \cdot \lambda = \mu_1 \leq s_2 \cdot \mu_1 = \mu_2 \leq \ldots \leq s_{r+1} \cdot \mu_r = \mu$$

or if $\lambda = \mu$.

For $\lambda \in X$, let k_λ be the one-dimensional B-module on which T acts via λ and denote by $\nabla(\lambda)$ the induced module $\mathrm{Ind}_B^G k_\lambda$. Then $\nabla(\lambda)$ is finite dimensional and it is non-zero if and only if $\lambda \in X^+$. For $\lambda \in X^+$, the socle $L(\lambda)$ of $\nabla(\lambda)$ is simple and furthermore $\{L(\lambda) \mid \lambda \in X^+\}$ is a complete set of non-isomorphic simple G-modules. A rational G-module M is said to have a good filtration if it has a filtration

$$\{0\} = M_0 \subseteq M_1 \subseteq \ldots \subseteq M_k = M$$

such that each quotient M_i/M_{i+1} is isomorphic to an induced module $\nabla(\mu_i)$ for some $\mu_i \in X^+$. A rational G-module T is called a tilting module if both T and T^* have a good filtration. Indecomposable tilting modules have been classified (see Ringel [11] and Donkin [4]), they are parametrized by the set of dominant weights X^+. For each $\lambda \in X^+$, we denote the corresponding indecomposable tilting module by $T(\lambda)$. This module has highest weight λ.

Note that for $\lambda = (p^r - 1)\rho$ we have $\nabla((p^r-1)\rho) = L((p^r-1)\rho) = T((p^r - 1)\rho)$, we call this module the r-th Steinberg module and denote it by St_r.

Define an equivalence relation $\sim$ on the set of simple G-modules, or on the set of dominant weights X^+, to be generated by

$$\mathrm{Ext}_G^1(L(\lambda), L(\mu)) \neq 0 \quad \Longrightarrow \quad L(\lambda) \sim L(\mu) \quad (\text{or } \lambda \sim \mu)$$

The equivalence classes of the relation $\sim$ are called the *blocks* of G.

Equivalently, the dominant weights λ and μ are in the same block if and only if there exists a chain of indecomposable G-modules $M_1, M_2, \ldots, M_s$ and dominant weights

$$\lambda = \lambda_0, \lambda_1, \ldots, \lambda_s = \mu \tag{1}$$

such that

$$[M_i : L(\lambda_{i-1})] \neq 0 \quad \text{and} \quad [M_i : L(\lambda_i)] \neq 0$$

for $i = 1, 2, \ldots, s$.

Using [7, (2.5.4)] we get a third equivalent definition for the blocks of G by replacing the indecomposable modules M_i above by induced modules $\nabla(\mu_i)$.

The linkage principle (see [10, Part II, 6.17]) tells us that the intersection of X^+ with the W_p-orbits on X are unions of blocks (see Step 1 in section 2 below). This was already observed by Humphreys and Jantzen in [9, Section 2.4]. They also proved that the W_p-orbits consisting of weights inside alcoves are in fact single blocks (using the representation theory of certain finite dimensional subalgebras of the corresponding hyperalgebra). Donkin then proved in [1] that the W_p-orbits consisting of primitive weights (see Step 2 in section 2 below) are single blocks and he then deduced a complete description of the blocks of G.

Theorem 1 [1, Theorem 5.8]
Let $\lambda \in X^+$. Define the integer $r(\lambda)$ by $\lambda + \rho \in p^{r(\lambda)}X \setminus p^{r(\lambda)+1}X$. Then the block containing λ is given by

$$\mathcal{B}(\lambda) = W_{p^{r(\lambda)+1}} \cdot \lambda \cap X^+.$$

Following Donkin's proof one might have to consider arbitrarily large dominant weights in order to show that two weights are in the same block. In particular, it does not give a bound on the length of a chain of the form (1) linking two weights in the same block.

In this paper, we use the knowledge of some composition factors of particular tilting modules to give a new proof of Theorem 1 except for a few blocks of G when the prime $p = 2$ and the root system of G has a component of type B_n,C_n,D_n,$E_{6,7,8}$,F_4 or G_2; $p = 3$ and the root system of G has a component of type $E_{6,7,8}$, F_4 or G_2 or $p = 5$ and the root system of G has a component of type E_8. Apart from the fact that this new proof is much shorter, it has the advantage of giving a bound on the length of a chain linking two weights in the same block. Moreover, the same proof gives the blocks of the associated generalized Schur algebras $S_G(\pi)$ provided the finite saturated subset $\pi \subset X^+$ is large enough to contain the highest weight of the tilting modules used in our proof.

We will assume from now on and without loss of generality that the root system of G is indecomposable.

2 New proof of the blocks of semisimple algebraic groups

We proceed in the following 5 steps. The first two are identical to those used by Donkin in his proof.

Step 1: Linkage principle gives unions of blocks.
The linkage principle states that if $\mathrm{Ext}^1_G(L(\lambda), L(\mu)) \neq 0$ then $\mu \in W_p \cdot \lambda$ (see [10, Part II,6.17]). Thus if two dominant weights are in the same block then they are in the same W_p-orbit and hence the W_p-orbits are unions of blocks for the group G.

Step 2: Reduction to primitive weights. (See [9], [1, Corollary 2.3]).
Let $\lambda \in X^+$. Define the integer $r(\lambda)$ by $\lambda + \rho \in p^{r(\lambda)}X \setminus p^{r(\lambda)+1}X$. Define

$$X^+_{(r)} = \{\lambda \in X^+ \mid r(\lambda) = r\}.$$

Note that $X^+_{(r)} = p^r(X^+_{(0)} + \rho) - \rho$. If $\lambda \in X^+_{(r)}$ with $r \geq 1$ then λ can be written as

$$\lambda = p^r(\lambda' + \rho) - \rho = (p^r - 1)\rho + p^r\lambda'$$

for some $\lambda' \in X^+_{(0)}$ and the functor $\Phi : M \mapsto M^{[r]} \otimes \mathrm{St}_r$ (where $M^{[r]}$ denotes the twist of M with the r-th power of the Frobenius endomorphism) is an equivalence of categories from the category of G-modules belonging to $X^+_{(0)}$ to the category of G-modules belonging to $X^+_{(r)}$ (see [10, Part II,10.5]). In particular, $\Phi(L(\lambda)) = L((p^r - 1)\rho + p^r\lambda)$ and $\Phi(\nabla(\lambda)) = \nabla((p^r - 1)\rho + p^r\lambda)$ so the corresponding map

$$\theta : X^+_{(0)} \to X^+_{(r)} : \lambda \mapsto (p^r - 1)\rho + p^r\lambda$$

takes blocks of G to blocks of G.

We say that a weight $\lambda \in X^+$ (or a block $\mathcal{B}(\lambda)$ of G) is primitive if $\lambda \in X^+_{(0)}$. Using the map θ it is enough to find the blocks for primitive weights.

Thus it is enough for us to show that if λ and η are primitive weights in the same W_p-orbit, then they are in the same block.

Step 3: Reduction to restricted weights.
From [9, 2.5 Corollary] , we know that if λ is primitive and $\nabla(\lambda)$ is simple then $\lambda \in X_1$. Thus if λ is primitive and $\lambda \notin X_1$ we can find $\mu \in X^+$ with $[\nabla(\lambda), L(\mu)] \neq 0$ and so $\mu \uparrow \lambda$ ($\mu \neq \lambda$) and μ is in the same block as λ. Hence we can assume that our two weights λ and η are restricted.

Step 4: Move away from outside walls of dominant region.
For $\lambda \in X_1$ we consider the indecomposable tilting module $T(2(p-1)\rho + w_0\lambda)$. We know (see [10, Part II,11.9(3) and 11.11] and [4, 2.5 Theorem]) that

$$[T(2(p-1)\rho + w_0\lambda) : L(\lambda)] \neq 0.$$

(In fact $L(\lambda)$ occurs in the socle of this tilting module). Thus λ is in the same block as $2(p-1)\rho + w_0\lambda = (p-1)\rho + ((p-1)\rho + w_0\lambda) \in (p-1)\rho + X_1$.

Hence we can restrict to the case where our weights λ and η belong to $(p-1)\rho + X_1$.

Step 5: Linking close weights.
This last step requires that the characteristic of the field k is not too small. We use homomorphisms between 'close' weights to show that λ and η are both in the same block as the unique representative of their W_p-orbit contained in $p\rho + \bar{C}$. We will use the following propositions.

For a dominant alcove C' we denote by $d(C')$ the number of hyperplanes for the affine Weyl group W_p separating C' from the fundamental alcove C.

Proposition 2.1 *Let C' be a dominant alcove. Then there exists dominant alcoves $C = C_r, C_{r-1}, \ldots, C_1, C_0 = C$ with*

$$C = C_r \uparrow C_{r-1} \uparrow \ldots \uparrow C_1 \uparrow C_0 = C$$

and $d(C_i) = d(C') - i$.

Proof: (See proof in [10, Part II Proposition 6.8]). We use induction on $d(C')$. We can always find a wall F of C' such that the hyperplane containing this wall separates C' from C. It has an equation of the form $\langle \lambda + \rho, \check{\beta} \rangle = mp$ for some $\beta \in R^+$ and some $m \in \mathbb{Z}$. As C' is dominant we have $m \geq 0$ and as the hyperplane separates C' and C we have that $m > 0$ and $\langle \lambda + \rho, \check{\beta} \rangle > mp$ for all $\lambda \in C'$. Thus $s_F \cdot C' \uparrow C'$ and $d(s_F \cdot C') = d(C') - 1$. Furthermore, $s_F \cdot C'$ is also dominant. We can now apply induction. □

For a facet F we denote by $W_p^0(F)$ the stabiliser of F in the affine Weyl group W_p, so $W_p^0(F) = \{w \in W_p \mid w \cdot \lambda = \lambda \quad \forall \lambda \in F\}$.

Proposition 2.2 *Let $\lambda \in X^+$ be contained in the facet F_1. Pick a facet $F \subset \bar{F}_1$ and consider $\mu = w \cdot \lambda$ with $w \in W_p^0(F)$ and $\mu \uparrow \lambda$. If $\langle x + \rho, \check{\beta} \rangle > 0$ for all $\beta \in R^+$ and for all $x \in F$ then $[\nabla(\lambda) : L(\mu)] \neq 0$.*

Proof: See [10, Part II Proposition 6.23] .

Combining Proposition 2.1 and Proposition 2.2, we get that any dominant weight contained in an alcove is in the same block as the unique representative of its W_p-orbit contained in the fundamental alcove. Hence this shows that the W_p-orbits containing weights inside alcoves are single blocks as already observed by Humphreys and Jantzen in [9, Section 2.4].

Now take a dominant weight λ which does not lie in an alcove. Then it belongs to a wall of some dominant alcove C' say and we can still apply to λ the sequence of reflections given in Proposition 2.1. (Note that we do not require $C' \cap X^+$ to be non-empty and so we do not make any restriction on the prime

p at this stage). There are two different problems arising if we try to apply the above argument to λ. First when we apply the sequence of reflections given in Proposition 2.1 we may obtain weights which are on the outside walls of the dominant region and so are no longer dominant. However, Step 4 allows us to move away from these walls by shifting everything by $p\rho$. Thus using the following shifted version of Proposition 2.1 by $p\rho$ solves this problem.

Proposition 2.3 *Let C' be an alcove in $p\rho + X^+$. Then there exist alcoves $C_r, C_{r-1}, \ldots, C_1, C_0$ contained in $p\rho + X^+$ with*

$$p\rho + C = C_r \uparrow C_{r-1} \uparrow \ldots \uparrow C_1 \uparrow C_0 = C'$$

and $d(C_i - p\rho) = d(C' - p\rho) - i$.

Thus assuming we can apply Proposition 2.2 at each stage we get that the primitive weight $\lambda \in (p-1)\rho + X^+$ is in the same block as the unique representative of its W_p-orbit contained in $p\rho + \bar{C}$.

We still need to make sure that following what happens to our weight when we apply the sequence of reflections given by Proposition 2.3, two consecutive weights satisfy the hypotheses of Proposition 2.2. This can be done most of the time as we now explain.

At each step, the alcove is reflected through one of its walls F say. First note that as the alcove belong to $p\rho + X^+$ we always have that $\langle x+\rho, \check{\beta}\rangle > 0$ for all $\beta \in R^+$, $x \in F$. Now, each alcove C' is a simplex, so if we pick a refection s_F through one of its wall F the only weight $\lambda \in \bar{C}'$ (if it exists) for which λ and $s_F \cdot \lambda$ do not satisfy the hypothesis of Proposition 2.2 is the vertex not contained in that wall. If such a weight exists, it is often a Steinberg weight, i.e. not primitive. But in some cases, when the prime p is very small, it is a primitive weight and our argument does not apply then.

We now give an explicit description of when such cases occur. As every alcove can be obtained from the fundamental alcove by a sequence of reflections and as the property under consideration only depends on the geometry of the alcove, not on its position, it is enough to consider the fundamental alcove C. Denote the simple roots by $\alpha_1, \alpha_2, \ldots, \alpha_n$. Then the $n+1$ walls of C are given by

$$\langle \lambda + \rho, \check{\alpha}_i\rangle = 0 \qquad i = 1, 2, \ldots n$$

$$\langle \lambda + \rho, \check{\beta}_0\rangle = p$$

We now run through the various types of root systems and give a complete list of the integral weights on vertices of the fundamental alcove in each case. We write $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ where $\lambda_i = \langle \lambda, \check{\alpha}_i\rangle$.

Type A_n:We have $\langle \lambda, \check{\beta}_0\rangle = \lambda_1 + \lambda_2 + \ldots + \lambda_n$ so in this case the $n+1$ vertices $(0, 0, \ldots, 0) - \rho$, $(p, 0, \ldots, 0) - \rho$, $(0, p, 0, \ldots, 0) - \rho$, $\ldots$, $(0, 0, \ldots, 0, p) - \rho$ are all Steinberg weights.

Type B_n: We have $\langle \lambda, \check{\beta_0} \rangle = 2\lambda_1 + 2\lambda_2 + \ldots + + 2\lambda_{n-1} + \lambda_n$ so we get two vertices $(0, 0, \ldots, 0) - \rho$ and $(0, 0, \ldots, 0, p) - \rho$ which are Steinberg weights. When $p = 2$ we also have $n - 1$ weights on vertices $(1, 0, \ldots, 0) - \rho$, $(0, 1, 0, \ldots, 0) - \rho$, $\ldots$, $(0, 0, \ldots, 0, 1, 0) - \rho$ which are primitive.

Type C_n: We have $\langle \lambda, \check{\beta_0} \rangle = \lambda_1 + 2\lambda_2 + \ldots + + 2\lambda_{n-1} + 2\lambda_n$ so we get two vertices $(0, 0, \ldots, 0) - \rho$ and $(p, 0, 0, \ldots, 0) - \rho$ which are Steinberg weights. When $p = 2$ we also have $n - 1$ weights on vertices $(0, 1, 0, \ldots, 0) - \rho$, $\ldots$, $(0, 0, \ldots, 0, 1) - \rho$ which are primitive.

Type D_n: We have $\langle \lambda, \check{\beta_0} \rangle = \lambda_1 + 2\lambda_2 + \ldots + 2\lambda_{n-2} + \lambda_{n-1} + \lambda_n$ so we get four vertices $(0, 0, \ldots, 0) - \rho$, $(p, 0, 0, \ldots, 0) - \rho$, $(0, 0, \ldots, 0, p, 0) - \rho$ and $(0, 0, \ldots, 0, p) - \rho$ which are Steinberg weights. When $p = 2$ we also have $n - 3$ weights on vertices $(0, 1, 0, \ldots, 0) - \rho$, $(0, 0, 1, 0, \ldots, 0) - \rho$, $\ldots$, $(0, 0, \ldots, 1, 0, 0) - \rho$ which are primitive.

Type E_6: We have $\langle \lambda, \check{\beta_0} \rangle = \lambda_1 + 2\lambda_2 + 2\lambda_3 + 3\lambda_4 + 2\lambda_5 + \lambda_6$ so we get three vertices $(0, 0, 0, 0, 0, 0) - \rho$, $(p, 0, 0, 0, 0, 0) - \rho$, $(0, 0, 0, 0, 0, p) - \rho$ which are Steinberg weights. When $p = 2$ we also have weights on vertices $(0, 1, 0, 0, 0, 0) - \rho$, $(0, 0, 1, 0, 0, 0) - \rho$, $(0, 0, 0, 0, 1, 0) - \rho$ which are primitive. When $p = 3$ we have one weight on a vertex $(0, 0, 0, 1, 0, 0) - \rho$ which is primitive.

Type E_7: We have $\langle \lambda, \check{\beta_0} \rangle = 2\lambda_1 + 2\lambda_2 + 3\lambda_3 + 4\lambda_4 + 3\lambda_5 + 2\lambda_6 + \lambda_7$ so we get two vertices $(0, 0, 0, 0, 0, 0, 0) - \rho$ and $(0, 0, 0, 0, 0, 0, p) - \rho$ which are Steinberg weights. When $p = 2$ we also get weights on vertices $(1, 0, 0, 0, 0, 0, 0) - \rho$, $(0, 1, 0, 0, 0, 0, 0) - \rho$ and $(0, 0, 0, 0, 0, 1, 0) - \rho$ which are primitive. When $p = 3$ we also get weights on vertices $(0, 0, 1, 0, 0, 0, 0) - \rho$ and $(0, 0, 0, 0, 1, 0, 0) - \rho$ which are primitive.

Type E_8: We have $\langle \lambda, \check{\beta_0} \rangle = 2\lambda_1 + 3\lambda_2 + 4\lambda_3 + 6\lambda_4 + 5\lambda_5 + 4\lambda_6 + 3\lambda_7 + 2\lambda_8$ so we get one vertex $(0, 0, 0, 0, 0, 0, 0, 0) - \rho$ which is a Steinberg weight. When $p = 2$ we also get weights on vertices $(1, 0, 0, 0, 0, 0, 0, 0) - \rho$ and $(0, 0, 0, 0, 0, 0, 0, 1) - \rho$ which are primitive. When $p = 3$ we also get weights on vertices $(0, 1, 0, 0, 0, 0, 0, 0) - \rho$ and $(0, 0, 0, 0, 0, 0, 1, 0) - \rho$ which are primitive. Finally, when $p = 5$ we also get a weight on the vertex $(0, 0, 0, 0, 1, 0, 0, 0) - \rho$ which is primitive.

Type F_4: We have $\langle \lambda, \check{\beta_0} \rangle = 2\lambda_1 + 3\lambda_2 + 4\lambda_3 + 2\lambda_4$ so we get one vertex $(0, 0, 0, 0) - \rho$ which is a Steinberg weight. When $p = 2$ we also have two weights on vertices $(1, 0, 0, 0) - \rho$ and $(0, 0, 0, 1) - \rho$ which are primitive. When $p = 3$ we also have one weight on a vertex $(0, 1, 0, 0) - \rho$ which is primitive.

Type G_2: We have $\langle \lambda, \check{\beta_0} \rangle = 2\lambda_1 + 3\lambda_2$ so we get one vertex $(0, 0) - \rho$ which is a Steinberg weight. For $p = 2$ we also have one weight on the vertex $(1, 0) - \rho$ which is primitive. For $p = 3$ we also have one weight on the vertex $(0, 1) - \rho$ which is primitive.

Thus we have given a new proof for the blocks of semisimple simply connected algebraic groups except for some blocks of G when $p = 2$ and G has type $B_n, C_n, D_n, E_{6,7,8}, F_4, G_2$; $p = 3$ and G has type $E_{6,7,8}, F_4, G_2$ or $p = 5$ and G has type E_8.

3 Bound on the length of a chain linking two weights in the same block

Any $\lambda \in X \otimes_{\mathbb{Z}} \mathbb{R}$ belongs to the upper closure of exactly one alcove, we denote this alcove by C_λ. So we have $\lambda \in \hat{C}_\lambda$.

Corollary 3.1 *Assume that $p \neq 2$ if G has type $B_n, C_n, D_n, E_{6,7,8}, F_4, G_2$, $p \neq 3$ if G has type $E_{6,7,8}, F_4, G_2$ and $p \neq 5$ if G has type E_8. Let λ and η be two weights in the same primitive G-block. Then there is a chain of dominant weights*

$$\lambda = \mu_0, \mu_1, \mu_2, \ldots, \mu_s = \eta$$

and indecomposable G-modules $M_1, M_2, \ldots, M_s$ such that

$$[M_i : L(\mu_{i-1})] \neq 0 \quad \textit{and} \quad [M_i : L(\mu_i)] \neq 0$$

for all $i = 1, 2, \ldots, s$, with

$$s \leq d(C_\lambda) + d(C_\eta) + 2 + 2d(w_0 \cdot C + p\rho).$$

Proof: We follow the steps given in section 2. Step 3 tells us that if $\lambda \notin X_1$ then there exists $\mu_1 \in X^+$ with $\mu_1 \uparrow \lambda$, $\mu_1 \neq \lambda$ and $[\nabla(\lambda) : L(\mu_1)] \neq 0$. Now if $\mu_1 \notin X_1$ we can repeat this process and get μ_2 etc. until we reach a restricted weight. So at each step we apply at least one reflection to μ_i through a hyperplane separating μ_i from C. Thus the number of reflections applied is at most $d(C_\lambda)$. Similarly we get at most $d(C_\eta)$ steps to link η to a restricted weight. Call $\lambda^{(1)}$ and $\eta^{(1)}$ the two restricted weights obtained.

Step 4 link $\lambda^{(1)}$ to $2(p-1)\rho + w_0\lambda^{(1)}$ and $\eta^{(1)}$ to $2(p-1)\rho + w_0\eta^{(1)}$, this adds 2 weights in our chain.

Finally in Step 5 we apply a sequence of reflections to $2(p-1)\rho + w_0\lambda^{(1)}$ and $2(p-1)\rho + w_0\eta^{(1)}$ that link them both to the unique representative of their W_p-orbit in $p\rho + \bar{C}$. As we have no control over $\lambda^{(1)}$ and $\eta^{(1)}$ we take them as far away as possible from $p\rho + \bar{C}$, i.e. in the upper closure of $w_0 \cdot C + 2p\rho$. Now the length of the chain linking $2p\rho + w_0 \cdot C$ to $p\rho + C$ (as in Proposition 2.3) is equal to the length of the chain linking $p\rho + w_0 \cdot C$ to C. This gives the final term. $\square$

Remark 3.2 Corollary 3.1 can easily be generalized to all blocks (not necessarily primitive) by defining the d function in terms of the affine Weyl group $W_{p^{r+1}}$.

4 Blocks of generalized Schur algebras

As a consequence to our new proof we obtain a sufficient condition for the blocks of a generalized Schur algebra to be naturally inherited from the blocks of the corresponding algebraic group.

Generalized Schur algebras were introduced by Donkin in [2]. We briefly recall their contruction here. We say that a subset π of X^+ is saturated if whenever $\lambda \in \pi$ and $\mu \uparrow \lambda$ we have $\mu \in \pi$. Let $\pi \subseteq X^+$ be a finite saturated subset of X^+. Denote by $C(\pi)$ the full subcategory of all G-modules M such that all composition factors of M have the form $L(\mu)$ with $\mu \in \pi$.

For any G-module M we define $O_\pi(M)$ to be the largest submodule of M belonging to $C(\pi)$. Consider the module $O_\pi(k[G])$. As π is finite it is finite dimensional, moreover it is a subcoalgebra of $k[G]$ and thus its linear dual $S_G(\pi) = O_\pi(k[G])^*$ has a natural structure of finite dimensional algebra. It is called the generalised Schur algebra associated to the subset π. Note that in Donkin's original definition he used the partial ordering $\leq$ instead of $\uparrow$. So we actually get more generalized Schur algebras here.

The category of all $S_G(\pi)$-modules is equivalent to the category $C(\pi)$ (see [2], [10, Part II, A.17(2)]). The notion of blocks for $S_G(\pi)$ can be defined in the same way as blocks for G, defined in section 1, simply by replacing all G-modules by $S_G(\pi)$-modules. It is clear that if two dominant weights are in the same $S_G(\pi)$-block then there are in the same G-block but the converse is false in general.

Example 4.1 Take $G = GL_n(k)$ the general linear group of degree n over k. Then $X^+ = \{\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \mid \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n\}$. Now consider the subset of X^+ given by

$$\pi = \pi(n,d) = \{\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \mid \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq 0, \ \sum_i \lambda_i = d\}.$$

Then $S_G(\pi) = S(n,d)$ is the usual Schur algebra as defined by Green (see [8]). Donkin proved in [5] that the blocks for the Schur algebra $S(n,d)$ are simply the intersection of the block of GL_n with $\pi(n,d)$.

As shown by the following example, this is clearly not the case for arbitrary finite saturated subset π of X^+.

Example 4.2 Take $G = SL_4(k)$ with $\operatorname{char} k \geq 3$. Then $X^+ = \{\lambda = (\lambda_1, \lambda_2, \lambda_3) \mid \lambda_i \geq 0\}$ where $\lambda_i = \langle \lambda, \check{\alpha_i} \rangle$. Now consider the subset of X^+ given by

$$\pi = \{\lambda^{(1)} = (p-1, 0, p-3), \lambda^{(2)} = (p-3, 0, p-1)\}.$$

First note that $\lambda^{(1)}$ and $\lambda^{(2)}$ are primitive and in the same W_p-orbit as $\lambda^{(1)} = s_{\alpha_2+\alpha_3,p} \cdot (p-2, 1, p-2)$ and $\lambda^{(2)} = s_{\alpha_1+\alpha_2,p} \cdot (p-2, 1, p-2)$. Thus $\lambda^{(1)}$ and $\lambda^{(2)}$ are in the same G-block. On the other hand, as $\lambda^{(1)}$ and $\lambda^{(2)}$ are both minimal (with respect to $\uparrow$), we have that the subset π is trivially saturated and moreover $\nabla(\lambda^{(1)})$ and $\nabla(\lambda^{(2)})$ are both simple modules. Thus the corresponding generalized Schur algebra $S_G(\pi)$ has two blocks $\{\lambda^{(1)}\}$ and $\{\lambda^{(2)}\}$. However, if we increase the size of π by including $\lambda^{(3)} = (p-2, 1, p-2)$ as well then the three weights are in the same block for the generalized Schur algebra as well.

Corollary 4.3 *Assume that $p \neq 2$ if G has type $B_n, C_n, D_n, E_{6,7,8}, F_4, G_2$, that $p \neq 3$ if G has type $E_{6,7,8}, F_4, G_2$, and $p \neq 5$ if G has type E_8. Let π be a finite saturated subset of X^+ and let $\lambda \in \pi \cap X^+_{(r-1)}$ $(r \geq 1)$. If for all $\eta \in \pi \cap X^+_{(r-1)} \cap X_r$ $(r \geq 1)$ we also have $2(p^r-1)\rho + w_0\eta \in \pi$ then the $S_G(\pi)$-block containing λ is equal to the intersection of the G-block containing λ with the set π.*

Proof: First consider primitive weights. In this case the proof is exactly the same as the one given here in section 2 for the algebraic group as at each step, except Step 4 (hence our assumption), we only link a weight λ with a weight μ satisfying $\mu \uparrow \lambda$, so if $\lambda \in \pi$ then so is μ. Note that in Step 4, if $2(p-1)\rho + w_0\eta$ belongs to π then the G-module $T(2(p-1)\rho + w_0\eta)$ belongs to the category $C(\pi)$ as all composition factors $L(\mu)$ of this module satisfy $\mu \uparrow 2(p-1)\rho + w_0\eta$.

The non-primitive case can be reduced to the primitive case using the equivalence of categories given in Step 2 of section 2. More precisely, if λ is non-primitive then $\lambda = (p^r-1)\rho + p^r\lambda'$ for some $\lambda' \in X^+_{(0)}$ and some $r \geq 1$. We have $\mu' \uparrow \lambda'$ if and only if $(p^r-1)\rho + p^r\mu' \uparrow \lambda$. Note also that $(p^r-1)\rho + p^r(2(p-1)\rho + w_0\lambda') = 2(p^{r+1}-1)\rho + w_0\lambda$. □

We now work through the example of the symplectic Schur algebra in details (see [3] and [6]). Consider the algebraic groups $G = GSp_{2m}(k) \subset GL_{2m}(k)$ defined by

$$G = \{g \in GL_{2m}(k) \mid (gv, gv') = \gamma(g)(v, v') \ \forall v, v' \in k^{2m}, \text{ for some } \gamma(g) \in k\}$$

where $(v, v') = v^T J v'$ with $2m \times 2m$ matrix J defined by

$$J_{ij} = \begin{cases} 1 & \text{if } j = 2m+1-i \text{ and } 1 \leq i \leq m \\ -1 & \text{if } j = 2m+1-i \text{ and } m+1 \leq i \leq 2m \end{cases}$$

The torus T of G is given by

$$T = \{t = \operatorname{diag}(t_1, t_2, \ldots, t_{2m}) \mid t_i t_{2m+1-i} = t_j t_{2m+1-j} \text{ for all } 1 \leq i, j \leq m\}.$$

Define $\epsilon_i : T \to k : \operatorname{diag}(t_1, t_2, \ldots t_{2m}) \mapsto t_i$ for $1 \leq i \leq 2m$. Then the root system of G is given by

$$R = \{\epsilon_i - \epsilon_j \mid 1 \leq i, j \leq 2m, \, i \neq j\}$$

and we take to set of positive roots to be

$$R^+ = \{\epsilon_i - \epsilon_j \mid 1 \leq i < j \leq 2m\}$$

Note that the roots are not all distinct as $\epsilon_i + \epsilon_{2m+1-i} = \epsilon_j + \epsilon_{2m+1-j}$ for $1 \leq i, j \leq m$. We write $\epsilon = \epsilon_i + \epsilon_{2m+1-i}$. The set of simple roots is given by

$$S = \{\alpha_1 = \epsilon_1 - \epsilon_2, \alpha_2 = \epsilon_2 - \epsilon_3, \ldots, \alpha_{m-1} = \epsilon_{m-1} - \epsilon_m, \alpha_m = 2\epsilon_m - \epsilon\}.$$

The set of dominant weights is given by

$$X^+ = \{\lambda = \lambda_1\epsilon_1 + \lambda_2\epsilon_2 + \ldots + \lambda_m\epsilon_m + a\epsilon \mid \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_m \geq 0\}.$$

Consider the set of polynomial weights given by

$$\pi(m) = \{\lambda = \lambda_1\epsilon_1 + \lambda_2\epsilon_2 + \ldots + \lambda_m\epsilon_m + a\epsilon \in X^+ \mid a \geq 0\}.$$

Define the degree of a polynomial weight λ by $|\lambda| = \sum_{i=1}^m \lambda_i + 2a$ and set $\pi(m,r)$ to be the set of all polynomial weights of degree r. Note that these are finite saturated subsets of X^+. We write $S_G(m,r)$ for the generalized Schur algebra $S_G(\pi(m,r))$, called the symplectic Schur algebras. I am not aware of any complete description of the blocks for these Schur algebras. But Corollary 4.3 applies to this case as it can easily be deduced from the corresponding result for the semisimple simply connected symplectic group. This result tells us that (for $p \geq 3$) if the degree r is large enough (with explicit bound) then the primitive blocks of these Schur algebras are given by the intersection of the blocks for G with the set of polynomial weights of degree r.

We now go further and give a complete description of the primitive blocks of the symplectic Schur algebras for $m = 2$ and $p \geq 3$. In particular we show that, surprisingly, the blocks of $S_G(2,r)$ are not always the intersection of the blocks of G with the set of polynomial weights. So this result disproves a conjecture of Renner [12, Conjecture 4.2]. This is joint work with Stephen Donkin and I thank him for allowing me to include it in this paper.

Proposition 4.4 *Assume $p \geq 3$. The primitive blocks of $S_G(2,r)$ are given by the intersection of the W_p-orbits with $\pi(2,r)$ except for the orbits $W_p\cdot(p-3-i,i;a)$ where $0 \leq i \leq \frac{p-3}{2}$ and $i+1 \leq a \leq p-1$ which decompose into two blocks as follows*

$$(i)\ \{(p-3-i,i;a)\} \quad \text{and} \quad \{(p-1+i,i;a-i-1)\}$$

for $1+i \leq a \leq p-1-i$, and

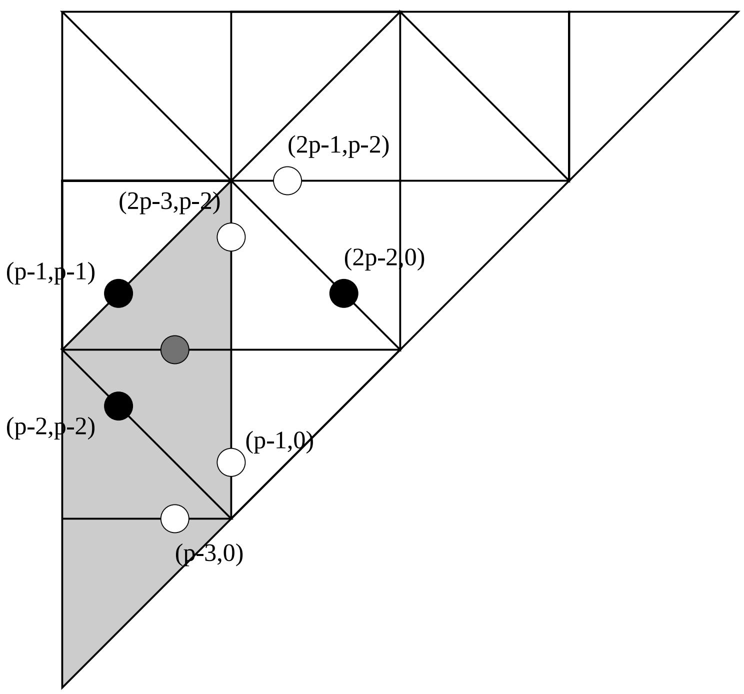
$$(ii)\ \{(p-3-i,i;a)\} \quad \text{and} \quad \{(p-1+i,i;a-i-1),(2p-i-3,p-i-2;a-p-1-i)\}$$

for $p-1-i \leq a \leq p-1$.

Proof: It is clear that the intersection of the W_p-orbits with the set of polynomial weights of degree r is a union of blocks. We just need to show that these orbits split into blocks as decribed in the proposition.

First note that if, for some degree r, all restricted weights in the same W_p-orbit are in the same $S_G(2,r)$-block then for all $r' \geq r$, the primitive $S_G(2,r')$-blocks are just the intersection of the W_p-orbit with $\pi(2,r')$. This follows from Step 3 of section 2.

Now using the Morita equivalences defined by the translation functors (see [10, Part II, 7.9]) it is enough to prove the result for one $W_p\cdot\lambda$ in each facet. As already noted after Proposition 2.2, the result is clear if λ belongs to an alcove, namely $W_p\cdot\lambda \cap \pi(2,r)$ is just a single block. We now consider the various cases when λ is not in an alcove. The restricted region is represented in light grey in the figure below. We will write (λ_1,λ_2) for $(\lambda_1,\lambda_2;a)$ as a is automatically determined by the fixed degree r (note that in the standard notation for simply connected symplectic group this corresponds to $(\lambda_1-\lambda_2,\lambda_2)$).



First consider $W_p\cdot\lambda$ where λ is represented by the grey dot in figure 1. Then $W_p\cdot\lambda \cap X_1$ contains a unique element, namely λ itself and so $W_p\cdot\lambda \cap \pi(2,r)$ is a single block (or the empty set).

We now turn to the orbit $W_p\cdot\lambda$ where λ is represented be a black dot in the figure.

Claim 1: For any $r \geq 2p-2$ with $r \equiv 2p-2 (\bmod 2)$ the dominant weights $(p-1,p-1)$ and $(p-2,p-2)$ are in the same $S_G(2,r)$-block.

Denote by $\chi(\lambda)$ the Weyl character corresponding to λ. For λ dominant, $\chi(\lambda)$ is the character of $\nabla(\lambda)$. As $\chi(1,0)$ is the character of the natural 4-dimensional G-module, we have

$$\chi(1,0) = e(1,0) + e(0,1) + e(0,-1) + e(-1,0).$$

Using Brauer's formula (see [10, Part II, 5.8]) we get

$$\begin{aligned}
\chi(1,0)^F\chi(p-2,0) &= \chi(2p-2,0) + \chi(p-2,p) + \chi(p-2,-p) + \chi(-2,0) \\
&= \chi(2p-2,0) - \chi(p-1,p-1) - \chi(p-2,p-2) - \chi(0,-1) \\
&= \chi(2p-2,0) - \chi(p-1,p-1) - \chi(p-2,p-2)
\end{aligned}$$

as $\chi(\lambda) = -\chi(s_\alpha\cdot\lambda)$ for all simple roots α. Thus we have

$$\chi(2p-2,0) = \chi(1,0)^F\chi(p-2,0) + \chi(p-1,p-1) + \chi(p-2,p-2).$$

Hence the standard module $\nabla(2p-2,0)$ contains both $L(p-1,p-1)$ and $L(p-2,p-2)$ as composition factors and so $(p-1,p-1)$ and $(p-2,p-2)$ are in the same block.

Finally we turn to the orbit $W_p\cdot\lambda$ where λ is represented by a white dot in the figure.

Claim 2: For any $r \geq 3p-3$ with $r \equiv 3p-3(\bmod 2)$ the dominant weights $(p-3,0)$, $(p-1,0)$ and $(2p-3,p-2)$ are in the same $S_G(2,r)$-block.

As $r \geq 3p-3$ we have that $\nabla(2p-1,p-2)$ is a module for $S_G(2,r)$. Using the same method as in the proof of Claim 1 we see that the standard module $\nabla(2p-1,p-2)$ contains both $L(p-1,0)$ and $L(p-3,0)$ as composition factors. On the other hand, using Proposition 2.2 we know that $L(p-1,0)$ is a composition factor of $\nabla(2p-3,p-2)$. Hence $(p-3,0)$, $(p-1,0)$ and $(2p-3,p-2)$ are all in the same block.

Claim 3: Let $p-1 \leq r \leq 3p-5$ with $r \equiv p-1(\bmod 2)$. The set $W_p\cdot(p-3,0)\cap\pi(2,r)$ decomposes as a union of $S_G(2,r)$-blocks

$$\{(p-3,0)\} \quad \text{and} \quad \{(p-1,0)\} \quad \text{for } p-1 \leq r < 3p-5$$

and

$$\{(p-3,0)\} \quad \text{and} \quad \{(p-1,0),(2p-3,p-2)\} \quad \text{for } r = 3p-5.$$

Clearly $\nabla(p-3,0)$ is simple as $(p-3,0)$ is minimal in its orbit. Moreover, using Jantzen's sum formula (see [10, Part II,8.19]), we easily obtain that $\nabla(p-1,0)$ is simple as well. This proves the first part. Using Jantzen's sum formula again, we see that $\nabla(2p-3,p-2)$ has only two composition factors, namely $L(2p-3,p-2)$ and $L(p-1,0)$. This proves the claim. □

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