Two Applications of U-Statistic Type Processes to Detecting Failures in Risk Models and Structural Breaks in Linear Regression Models

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Abstract

This dissertation is concerned with detecting failures in Risk Models and in detecting structural breaks in linear regression models. By applying Theorem 2.1 of Szyszkowicz on U-statistic type process, a number of weak convergence results regarding three weighted partial sum processes are established. It is shown that these partial sum processes share certain invariance properties; estimation risk does not affect their weak convergence results and they are also robust to asymmetries in the error process in linear regression models. There is also an application of the methods developed here to a four factor Capital Asset Pricing model where it is shown via the methods developed in Chapter 3 that manager stock selection abilities vary over time.

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Chapter 1

Introduction

This dissertation looks at statistical methods for detecting failure of financial risk models to report accurately the risks in financial positions that financial companies routinely take in the day-to-day operations of their businesses and in detecting structural breaks in linear regression models. The approach taken in this thesis is to apply change-point detection techniques, an unexplored alternative, to detect failure of risk models rather than the simple non-parametric tests presently used. A second application of these change-point detection techniques is to detecting structural breaks in linear regression models.

The literature on change-point techniques has grown rapidly since a number of papers were published in the late 80's and early 90's. These papers provided mathematical content to some of the observations pointed out in simulation studies of the CUSUM tests of Brown, Durbin and Evans (1975). Later work by Hansen (1992) introduced new methods of detecting structural breaks in linear models; contributions by Andrews (1993) introduced much improved tests based on traditional Wald, LM and Maximum Likelihood tests statistics, however, no optimality conditions were be established. Asymptotically optimal tests were later established by Andrews and Ploberger (1993). Even though these results have addressed some of the apparent deficiencies of older methods, many gaps in the literature on structural breaks in linear regression models remain. Current tests are unable to detect changes in parameters when they occur early/later on in the sample, avoid the common practice of trimming the sample, a practice that many tests routinely do, and allow for estimation of the timing of the structural break.

1.0.1 Aims, Objectives and Contribution of the Thesis

The financial crisis of 2007 revealed weaknesses in current backtesting methods to validate risk models implemented by internationally active financial companies. Indeed, backtesting methods that relied on counting the number of exceedances of actual trading results to reported results were unable to detect failures of risk models when they happened or when failure of these models was detected it was well after substantial financial losses were incurred by institutions. Given the apparent failure of traditional statistical methods, it has become imperative then to develop statistical methods that are more sensitive to failures in risk models and that also allow for detection of this failure as early as possible in out-of-sample evaluation of risk models. As an additional feature of the methods developed in this thesis is their insensitivity to estimation risk.

These considerations regarding risk models led to further investigations into the possibility of applying similar statistical methods to detecting structural breaks in linear regression models. These statistical methods were adapted so they could successfully address some of the deficiencies in traditional econometric methods commonly used in testing for structural breaks. The econometric methods used to detect structural breaks in regression models usually require trimming the sample; trimming essentially ignores some of the data which usually impairs statistical tests. Another notable limitation of current econometric methods is that they are often unable to detect structural breaks in the intercept of linear regression models; one aim of this thesis is to develop statistical tests that can separate a change in intercept parameter from a change in slope parameter.

Traditional backtesting methods have not exploited some of the advances in change-point detection methods to detecting failure of risk models. One of the objectives of this thesis is to use some of the statistical methods and corresponding limit theorems developed in the large literature on change-point methods to offer more powerful, in the statistical sense, tests for detecting failure of risk models. The tests so developed will allow for detecting failure of risk models early on in the out-of-sample evaluation period and as an byproduct of these methods allow for estimation of the timing of this failure. This is an important addition to the backtesting toolkit as risk managers are now able to locate the time of failure of their risk model. With this knowledge they can investigate factors which may have contributed to its failure. It will be shown in the following chapters the advantages to viewing failure of risk models as a change-point and the substantial improvement that can be obtained.

The framework developed in detecting failure of risk models is then applied to detecting structural breaks in parameters of linear regression models. The statistical methods are adapted to the linear model context and as such allow for disentangling breaks in the intercept from breaks in the slope parameters. Two stochastic process are developed; one process can be used to fashion a test that is capable of detecting a change in intercept and the other process can be used to fashion a test that is capable of detecting a change in slope parameters. The test statistics fashioned from these process are shown to be more powerful, in the statistical sense, then the asymptotically optimal tests via a Monte-Carlo experiment. The tests statistics are shown to asymptotically independent which allows for control of global error rates; exploit higher moments of the distribution; and can detect changes in the parameters when they occur early/late in the sample.

1.0.2 Structure of Thesis

Chapter 2 details the contribution made by this dissertation to the literature on backtesting risk models by applying change-point detection techniques to monitor risk models. In particular, Chapter 2 proposes a richer set of alternatives to standard backtesting methods that are based on U-statistic type processes. These processes allow for detecting failure in risk models and offer the additional benefit that they permit estimation of the timing of this failure. The test statistics constructed here are indexed by certain weight functions that improve the power of the statistics constructed here when compared to more traditional change-point detection methods. A second consideration in developing new methods is that they be easy for practitioners to implement and not over complicated, rely on complex distributions and re-sampling methods to approximate. The tests statistics developed in this chapter are weighted functions of Brownian bridges which are readily simulated and many of the limiting distributions are contained in Section 2.7 to this chapter.

The third chapter, Chapter 3, a second contribution is made to detecting structural breaks in linear regression models that is similar to Hansen's (1992) contribution. Hansen (1992) is interested in detecting a one-time change in each parameter of linear regression models. As in Hansen, the two test-statistics developed here are devised to separate a change in intercept from a change in slope parameters; and can be combined to form one test statistic that detects simultaneously for a change in intercept or slope in linear regression models.

1.0.3 Relevant Background Literature

Test statistics developed in both chapters exploit weight functions to improve their asymptotic power. In fact, Chapter 3, shows that the test-statistics constructed there have nontrivial power for detecting a one-time change in these parameters. As an additional attraction the test statistics are shown to be robust to estimation risk.

To establish weak convergence of the processes constructed in Chapters 2 and 3, we make use of Theorem 2.1 of Szyszkowicz (1991) which details optimal results for the asymptotic weighted sup-norm behaviour of a certain partial sum process. Theorem 2.1 provides an asymptotic approximation to the following process

$$Z_k = \sum_{i=1}^{k} \sum_{j=k+1}^{T} h(X_i, X_j), \quad 1 \le k < T, \tag{1.1}$$

where $h(x,y)$ is an anti-symmetric function ($h(x,y) = -h(x,y)$). Under the assumption that $\{X_1,\ldots,X_T\}$ is a random sample along with the additional restriction

$$Eh^2(X_1,X_2) < \infty$$

and $0 < \sigma^2 = E\tilde{h}^2(X_1)$, Theorem 2.1 can be stated as

Theorem 2.1 (Szyszkowicz) Let $\{X_t\}_{t=1}^T$ be a random sample; Let $h(x,y)$ be anti-symmetric and let $q(t)$ be continuous and satisfy $\inf_{\delta<t<1-\delta} q(\tau) > 0$ and $\delta \in (0,1/2)$. Then a sequence of Brownian bridges $\{B_T(\tau) : 0 \leq \tau \leq 1\}$ can be defined such that

a) $\sup\limits_{0<\tau<1} \frac{|\frac{Z_{[T\tau]}}{\sigma T^{3/2}} - B_T(\tau)|}{q(\tau)} = o_p(1)$

if and only if $\int_0^1 (\tau(1-\tau))^{-1} \exp(-c(\tau(1-\tau))^{-1})q(\tau)d\tau < \infty$ for all $c > 0$.

b) $\sup_{0<\tau<1} \frac{|Z_{[T\tau]}|}{\sigma T^{3/2}} \xrightarrow{\mathcal{D}} \sup_{0<\tau<1} |B(\tau)|$

if and only if $\int_0^1 (\tau(1-\tau))^{-1} \exp(-c(\tau(1-\tau))^{-1})q(\tau)d\tau < \infty$ for some $c > 0$. $B(\tau)$ is a Brownian bridge.

Theorem 2.1 of Szyszkowicz (1991) exploits the integral

$$\int_0^1 (\tau(1-\tau))^{-1} \exp(-c(\tau(1-\tau))^{-1})q(\tau)d\tau < \infty$$

to establish weak convergence of the process $Z_{[T\tau]}$. If the integral is finite for all $c > 0$ weak convergence follows. If the integral is finite for only some $c > 0$, only convergence in distribution of the supremum functional of $Z_{[T\tau]}$, suitably normalised, to the supremum functional of a Brownian bridge can be established. Integral conditions of this nature have been studied by many individuals in statistics and probability. First Chibisov (1964) and later O'Rielly (1974) introduced and studied the integral

$$\int_0^1 \tau^{-1} \exp(-c\tau^{-1}q^2(\tau))d\tau < \infty. \tag{1.2}$$

As long as this integral is finite for all $c > 0$, both show that the sequence of probability measures generated by the Empirical process converges to the probability measure generated by a Brownian bridge.

Throughout Chapters 2 and 3, the partial sum process constructed therein are described as U-statistic type processes. The partial sum process $Z_{[T\tau]}$ is not a U-statistic but, as shown by Csörgő and Horváth (1988), can be expressed as a linear combination of U-statistics. How does this relate to the process considered in Chapters 2 and 3? If, in the process (1.1), the kernel $h(x, y) = x - y$, then we arrive at the partial sum process

$$Z_{h,[T\tau]} = \sum_{i=1}^{[T\tau]} X_i - \tau \sum_{i=1}^{T} X_i, \tag{1.3}$$

that figures so prominently in what is to come in Chapters 2 and 3.

The field of econometrics/statistics that deals with detecting structural breaks in linear regression models has a long history in statistics and economics. One of the first contributions made to this literature was by Page (1994, 1955) who studied changes in location of independent and identically distributed random variables. Let $\{X_1, \ldots, X_{k-1}, X_k, X_{k+1}, \ldots, X_T\}$ be independent random variables, where X_t for $i = 1, \ldots, k$ have cumulative distribution function (CDF) $F(x)$, and X_t for $t = k+1, \ldots, T$ has CDF $F(x - \Delta)$, $-\infty < \Delta < \infty$. Δ is referred to as a location or shift parameter. As specified, this is considered to be a two sample problem with unknown location parameter Δ. However, when the integer k is unknown, this is no longer a standard two sample problem but becomes what is now referred to as a *change-point problem*. In particular, he considered testing

$$H_O^{(1)}\colon\ \Delta = 0$$

versus

$$H_A^{(1)}\colon\ \Delta \neq <> 0,$$

Page introduced $S_k^* = \sum_{t=1}^{k} V_t$, where $S_0^* = 0$,

$$V_j = \begin{cases} a & \text{if } X_t > \theta_o \\ -b & \text{if } X_t \leq \theta_o, \end{cases}$$

and $a > 0$ and $b > 0$ are constants chosen so that $\mathbb{E}(V_j) = 0$ and $\mathbb{E}[X_1] = \theta_0$. For example, his decision rule rejects $H_O^{(1)}$ in favour of the alternative of one change and $\Delta > 0$ if

$$T_n^* = \max_{0 \leq k \leq T} \left\{ S_k^* - \min_{0 \leq j \leq k} S_j^* \right\}$$

is too large.

Further along these lines, let $k = [T\tau] \quad 0 \leq \tau \leq 1$ and set $S_{[T\tau]} := \sum_{t=1}^{[T\tau]} X_t$. Then if θ_o is known, $H_O^{(1)}$ could be rejected in favour of $H_A^{(1)}$ with $\Delta \neq 0$ if

$$M_T = \sup_{0 \leq \tau \leq 1} \left| S_{[T\tau]} - [T\tau]\theta_0 \right|$$

is too large. We note that, under $H_O^{(1)}$

$$\frac{M_T}{T^{1/2}\sigma} \xrightarrow{\mathcal{D}} \sup_{0<t<1} |W(t)|,$$

as $T \to \infty$, where $W(t)$ is a standard Wiener process and $\sigma^2 := \mathbb{E}(X_1 - \theta_0)^2$. This result follows from Donsker's theorem restated on $D[0,1]$.

If, as is more frequently the case, θ_0 is unknown, then set $\bar{X}_T = \frac{\sum_{i=1}^{T} X_t}{T}$, which results in the following statistic

$$\widehat{M}_T = \sup_{0 \leq \tau \leq 1} \left| S_{[T\tau]} - [T\tau]\bar{X}_T \right|,$$

whose large values would reject $H_O^{(1)} : \Delta = 0$ in favour of $H_A^{(1)}$ with $\Delta \neq 0$ at some unknown time $t^\star$. We note that under $H_A^{(1)}$

$$\frac{\widehat{M}_T}{T^{1/2}\sigma} \xrightarrow{\mathcal{D}} \sup_{0 \leq \tau \leq 1} |B(\tau)|, \tag{1.4}$$

as $n \to \infty$, where $B(t)$ is a Brownian bridge. Again, this follows from Donsker's theorem on $D[0,1]$, and the same is true with σ replaced by any sequence of consistent estimators $\{\widehat{\sigma}_T\}$.

Most of the results in the statistical literature concern models that as Andrews (1993) points out are too *simple*[1] for economic applications. Most, but not all, deal with the framework introduced by Page which are location models with independent and identically distributed random variables. Andrews[2] notes that 'few econometric models are covered by such results' and hence there is need for broader class of tests that can accommodate the settings routinely found in the economics.

Chow (1960) made the first successful attempt to addresses this gap in the statistical literature by developing a test for structural breaks in linear regression models. His test is

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[1]Andrews 1993, page 822.
[2]Ibid. page 822

based on a split-sample test for breaks in the following model:

$$\begin{aligned} Y_1 &= \boldsymbol{X}_1\boldsymbol{\gamma}_1 + \boldsymbol{W}_1\boldsymbol{\delta}_1 + \varepsilon_1 \\ Y_2 &= \boldsymbol{X}_2\boldsymbol{\gamma}_2 + \boldsymbol{W}_2\boldsymbol{\delta}_2 + \varepsilon_2 \end{aligned} \tag{1.5}$$

Let $\boldsymbol{\gamma}_1$ and $\boldsymbol{\gamma}_2$ be column vectors of q elements each; and $\boldsymbol{\delta}_1$ and $\boldsymbol{\delta}_2$ be column vectors of $p-q$ elements each $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$ are row vectors containing p elements each, while $\boldsymbol{W}_1$ and $\boldsymbol{W}_2$ are row vectors containing $p-q$ elements each. The null hypothesis Chow (1960) considers is $H_O^{(2)} : \boldsymbol{\gamma}_1 = \boldsymbol{\gamma}_2 = \boldsymbol{\gamma}$ versus $H_A^{(2)} : \boldsymbol{\gamma}_1 \neq \boldsymbol{\gamma}_2$. Using this framework, he also shows that the prediction based testing method for structural breaks is equivalent to the analysis of covariance approach that he develops.

A problem arises with Chow's test due to the need to select the timing of the structural break that occurs under $H_A^{(2)}$ but not the $H_O^{(2)}$. This problem is that the time of change in $\boldsymbol{\gamma}$ is not defined under $H_O^{(2)}$ and standard testing theory does not apply. Chow resolves this problem by selecting the timing of this structural break by appealing to events known a priori. If this method is used to select the break date, it is important that the researcher argue that the events are selected exogenously of the data generating process.

Recognizing the need for a test that reveals model instability of a more general form, Brown, Durban and Evans (1975) proposed the CUSUM tests which was become widely implemented in econometric programs. The CUSUM tests are based on partial sums of recursive residuals formed from the estimated regression (1.5) when $\boldsymbol{\gamma}_1 = \boldsymbol{\gamma}_2$ and $\boldsymbol{\delta}_1 = \boldsymbol{\delta}_2$. Theoretical investigations have revealed that the CUSUM tests can be considered as tests for detecting a change in the intercept alone - in the case of the CUSUM test based on the recursive residual - or for a test for detecting instability in variance of the regression error - in the case of the CUSUM test based on the squared recursive residuals - in linear regression models. These results regarding the CUSUM tests, as well as other results, are detailed by Kramer, Ploberger and Alt (1988).

Andrews and Ploberger (1994) also consider the nonstandard problem of testing whether a sub-vector of the parameters $(\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2) \in \Gamma \subset R^p$ are equal $(\boldsymbol{\gamma}_1 = \boldsymbol{\gamma}_2)$ when the likelihood function depends on an additional parameter $\pi \in \Pi$ under the alternative hypothesis $H_A^{(2)}$. This general framework includes test statistic for one-time structural change in linear regression as well as many other econometric models. Their contribution is to derive asymptotically optimal test for this and other settings because the classical asymptotic optimum properties of Lagrange Multiplier (LM), Wald and Likelihood ratio (LR) test do not hold in these non-standard problems.

Using a weighted average power criterion function, these tests are an average exponential form and are based on the statistic

$$Exp - LM_T = (1+c)^{-p/2} \int_0^1 \exp\left(\frac{1}{2}\frac{c}{1+c} LM_T(\pi)\right) dJ(\pi), \tag{1.6}$$

p is the dimension of $(\boldsymbol{\gamma}, \boldsymbol{\delta})$, $J(\cdot)$ is a weight function over values of $\pi \in [\pi_o, 1-\pi]$ for $1 > \pi_0 > 0$. The constant c is a scalar constant that depends on the chosen weight functions over $(\boldsymbol{\gamma}, \boldsymbol{\delta})$. They also define exponential Wald and LR tests analogously to $Exp - LM_T$ with the standard Wald $W_T(\pi)$ and $LR_T(\pi)$ test statistic replacing $LM_T(\pi)$ in integral (1.6).

1.0.4 Conclusion

The family of partial sum processes developed in this thesis have many attractive features; just how attractive, will become apparent as the chapters of this thesis unfold. A convincing case will be made for their benefits as well as some of their properties relative to existing statistical methods. An important consideration in developing these methods is to advertise the ease of their implementation. It is hoped this thesis will provide a means to speak of their benefits.

Chapter 2

Early Detection Techniques for Market Risk Failure

In the aftermath of a series of bank failures that occurred during the seventies a group of ten countries (G-10) decided to create a committee to set up a regulatory framework to be observed by internationally active banks operating in these member countries. This committee coined as Basel Committee on Banking Supervision (BCBS) was intended to prevent financial institutions, in particular banks, from operating without effective supervision. The subsequent documents derived from this commitment focused on the imposition of capital requirements for internationally active banks which would serve as provisions for losses from adverse market fluctuations, concentration of risks or simply bad management of institutions. The risk measure agreed upon was the Value-at-Risk (VaR). In financial terms, this is the maximum loss on a trading portfolio for a period of time given a confidence level, and in practice, determines restrictions on the minimum amount of capital held as reserves by financial institutions. In statistical terms, VaR is a (conditional) quantile of the conditional distribution of returns on the portfolio given agent's information set.

The computation of these VaR measures has become of paramount importance in risk management since financial institutions are monitored to ensure the accuracy of the quantile measures reported. This implies that banks with sufficiently highly developed risk management systems can decide on their own internal risk models as long as these satisfy requirements set by the Basel Accord (1996) for computing capital reserves. The main toolkit for measuring and testing the performance of different VaR methodologies proposed in the Basel Accord was a statistical device denoted backtesting that consisted of out-of-sample comparisons between the actual trading results with internally model-generated risk measures. The magnitude and sign of the difference between the model-generated measure and trading results indicate whether the VaR model reported by an institution is correct for forecasting the underlying market risk and if this is not so, whether the departures are due to over- or under-risk exposure of the institution. The implications of over- or under- risk exposure being diametrically different: either extra penalties on the level of capital requirements or bad management of the outstanding equity by the institution. These backtesting techniques are usually interpreted as statistical non-parametric tests for the coverage probability α defining the conditional quantile VaR measure. The seminal papers in this area of research are due to Kupiec (1995) and Christoffersen (1998) who proposed asymptotic standard Gaussian tests and likelihood ratio tests, respectively. Escanciano and Olmo (201a, 2010b) show that these backtesting methods can be unreliable when the parameters of the risk model are estimated and under the presence of model misspecification. Another deficiency of these tests to monitor risk is that they do not provide any information about the timing of the rejection of VaR.

A rather unexplored alternative to monitor VaR performance is to use change-point detection techniques. This topic has been long studied in statistics and econometrics. Chow (1960) was the first to develop a test for detecting a one-time change in regression parameters at a known time. Work by Brown, Durbin and Evans (1975) and Dufour (1988) extended Chow's test to accommodate multiple changes in regression parameters that may occur at unknown times. Other statistical methods for detecting structural change are within the framework developed by Andrews (1993) and Andrews and Ploberger (1994). In particular, Andrews (1993) considers Wald, Lagrange multiplier and likelihood-ratio tests for parameter stability in nonlinear parametric models that are optimal in certain compact set within the (0,1) interval. Outside this interval, however, the optimality properties or even the validity of the methods is not clear. Further, the parametric nature of the tests implies that these methods are very sensitive to the adequacy of the distribution function assumed for the data. Other tests, called fluctuation tests, such as that of Ploberger, Kramer and Kontrus (1989) have also been developed for linear regression models. In a financial econometrics context Kuan and Hornik (1995) and Leisch, Hornik and Kuan (2000) propose fluctuation tests for change-point detection. In a related paper Andreou and Ghysels (2006) review this and other monitoring techniques for the volatility process when this is estimated using high-frequency data. These authors explore in detail this alternative and discuss the relative power of CUSUM-type tests when computed at different frequencies.

This article takes a non-parametric approach on the detection of change-points in the conditional VaR process. We propose test statistics constructed as the supremum of weighted U-statistic type processes, similar in spirit to the CUSUM test. These methods are based on the pioneering works of Page (1954, 1955), Gombay, Horváth and Hušková (1996, hereafter GHH) or Csörgő and Horváth (1988a, 1988b). One of the appealing properties of our test statistics is that they are constructed as a combination of Kupiec test evaluated at different times of the evaluation period, and as such, computation is straightforward. Unlike for the standard backtesting methods our U-statistic type tests do not exhibit estimation risk and can accommodate weight functions to enhance their ability to reject the null hypothesis for specific regions of the evaluation period. In this risk monitoring context this is particularly relevant for early detection. For this purpose, we construct a U-statistic weighted by a function that is optimal within the family introduced by GHH for detecting change-points early and late on in the evaluation period. This new function is an extension of that proposed by Orasch and Pouliot (2004, hereafter OP) for detecting structural breaks in the mean parameter. To show the power of this and other weighted U-statistics to detect structural breaks in the risk process, we also compare this method against likelihood ratio (LR) tests for change point detection. Our results suggest the outperformance of U-statistic type tests over LR tests, particularly for early detection. These simulations complement the findings in Worsley (1983) on relative optimality between CUSUM and LR tests for change point detection for binomial random variables.

Finally, an application to equity and commodity data shows the advantages of this risk monitoring technique. Our results reveal the breakdown of the GARCH(1,1) risk model around important announcements and occurrence of bad news worldwide. The risk model is out of control for equity more times than for commodities. The choice of the coverage probability to compute VaR and the rolling scheme to develop the re-estimation procedure are important factors for risk monitoring.

The rest of the paper is structured as follows. Section 2.1 introduces the standard backtesting monitoring techniques used to assess the validity of conditional VaR models. The section also discusses change-point tests and shows that standard CUSUM tests can complement the standard backtesting techniques by providing information about the timing of rejection of the VaR model. We exploit this result in Section 2.1.2 and introduce a U-statistic type process indexed by a family of weight functions devised to outperform in terms of statistical power standard methods for detecting structural breaks, in particular for early change-point detection. These results are illustrated in Section 2.2 that derives the asymptotic power of the different tests and introduces a new family of weight functions more sensitive to early detection. Section 2.3 complements this analysis by studying in a Monte-Carlo simulation exercise the finite-sample performance of the new family of weighted U-statistic type tests stressing the early detection property exhibited by our test statistic with respect to other CUSUM-type competitors. An application of these methods to detecting structural breaks in the dynamics of risk in commodity and equity markets is studied in Section 2.4. Section 2.5 concludes; and proofs are gathered in the Mathematical Appendix 2.6. The last Section **??** following the Mathematical Appendix 2.6 contains the Tabulated cumulative distribution functions (*cdf*) of the weighted statistics considered here.

2.1 Risk Monitoring: Backtesting techniques

We will start this section by formally defining the Value-at-Risk at an α coverage probability. Denote the real-valued time series of portfolio returns by Y_t, and assume that at time $t-1$ the agent's information set is given by $\Im_{t-1}$, which may contain past values of Y_t and other relevant explanatory variables. Also, by assuming that the conditional distribution of Y_t given $\Im_{t-1}$ is continuous, we can define the α-th conditional VaR of Y_t given $\Im_{t-1}$ as the $\mathcal{F}_{t-1}$-measurable function $q_\alpha(\Im_{t-1})$ satisfying the equation

$$\mathbb{P}(Y_t \leq q_\alpha(\Im_{t-1}) \mid \Im_{t-1}) = \alpha, \text{ almost surely (a.s.)}, \alpha \in (0,1), \ \forall t \in \mathbb{Z}, \tag{2.1}$$

with $\mathcal{F}_{t-1}$ the sigma-algebra generated by the set of information available at $t-1$.

In *parametric* VaR inference one assumes the existence of a parametric family of functions $\mathcal{M} = \{m_\alpha(\theta;\cdot) : \theta \in \Theta \subset \mathbb{R}^p\}$ and proceeds to make VaR out-of-sample forecasts using model $\mathcal{M}$. In these parametric VaR models the nuisance parameter θ belongs to a compact set Θ embedded in a finite-dimensional Euclidean space $\mathbb{R}^p$, and can be estimated by a $\sqrt{R}$-consistent estimator, with R denoting the (in-)sample size in the backtesting exercise. The most popular parametric VaR models are those derived from traditional location-scale models such as ARMA-GARCH models, but other models include quantile regression models such as those of Koenker and Xiao (2006), autoregressive quantile regression models of Engle and Manganelli (2004) and models that specify the dynamics of higher moments of the conditional distribution of Y_t. Under the null hypothesis of correct specification of the conditional VaR by a parametric model $m_\alpha(\theta;\cdot)$, expression (2.1) reads as

$$\mathbb{E}[I_{t,\alpha}(\theta) \mid \Im_{t-1}] = \alpha \text{ a.s. for some } \theta \in \Theta, \tag{2.2}$$

with $I_{t,\alpha}(\theta) := 1(Y_t \leq m_\alpha(\theta;\Im_{t-1}))$, and $1(\cdot)$ an indicator function that takes the value one if $Y_t \leq m_\alpha(\theta;\Im_{t-1})$ and zero otherwise. It is well known in the backtesting literature that this null hypothesis implies the hypothesis of serial independence of the indicator variables.

In our framework this result will be fundamental to developing the asymptotic theory for our test statistics. For sake of completeness we state this result as a corollary.

Corollary 2.1.1. *Let Y_t be a stationary time series describing the dynamics of portfolio returns, and let $\{m_\alpha(\theta;\Im_{t-1})\}_{t=1}^P$ be the associated parametric conditional quantile process satisfying* (2.2)*. Then, the sequence $\{I_{t,\alpha}(\theta)\}_{t=1}^P$ of indicator functions are IID.*

It is worth noting that Corollary 2.1.1 is not an if and only if condition, that is, $\{I_{t,\alpha}(\theta)\}_{t=1}^P$ can be IID without the underlying risk process satisfying condition (2.2). This result is referred to as model risk in the literature, see Engle and Manganelli (2004), Kuester, Mittnik and Paolella (2006) or Escanciano and Olmo (2010b), among others for a discussion on this. A proper test for condition (2.2) is Christoffersen, Hahn and Inoue (2001). Nevertheless, given that the interest of regulators is in testing whether $\{I_{t,\alpha}(\theta)\}_{t=1}^P$ is a Bernoulli IID random sequence we concentrate on testing departures of this assumption. The pioneering backtesting tests are due to Kupiec (1995) and Christoffersen (1998). These authors developed different, although asymptotically equivalent, tests for the unconditional coverage of the VaR model. In particular, Kupiec's test statistic takes this form;

$$K_P \equiv K(P,R) := \frac{1}{\sqrt{P}} \sum_{t=R+1}^{R+P} (I_{t,\alpha}(\theta) - \alpha), \tag{2.3}$$

where R is the in-sample period used to estimate the model parameters, and P is the out-of-sample evaluation period. Escanciano and Olmo (2010a) show, however, that a correction in the asymptotic distribution is needed in the case the risk model's parameters are estimated. These tests are designed to evaluate the specification of the conditional VaR measure after P out-of-sample periods. Neither method, however, is devised to exhibit power against the timing of the rejection of the null hypothesis. A potential solution to this is the use of CUSUM-type tests.

2.1.1 Change-point detection techniques

In this section we study more refined and powerful versions of the backtesting tests introduced above for the correct specification of the VaR model. We start with the standard CUSUM test as benchmark and show that this test can be built as a simple combination of the Kupiec test for two different sample periods. Now we are not only interested in detecting failure of the risk model but also in the timing of the failure. In order to do this we will use different change-point detection techniques. For sake of exposition we will assume in this and next subsection that the vector of parameters θ is known, and therefore there is no need to use an in-sample period, hence $R = 0$.

Let us consider the following process:

$$X_t = \begin{cases} 1(Y_t \leq m_\alpha(\theta; \Im_{t-1})), & 1 \leq t \leq k^\star, \\ 1(Y_t \leq m_\alpha^\star(\theta^\star; \Im_{t-1})), & k^\star < t \leq P, \end{cases} \tag{2.4}$$

with $m_\alpha^\star(\theta^\star; \Im_{t-1}) \neq m_\alpha(\theta; \Im_{t-1})$, with $\theta^\star$ the parameter set corresponding to the alternative risk model. The process $m_\alpha^\star(\theta^\star; \Im_{t-1})$ describes the actual conditional VaR process with α coverage probability after the structural break. Thus, $\mathbb{E}[X_t \mid \Im_{t-1}] = \alpha$ for all t. In contrast to standard backtesting tests the interest of this exercise is in detecting k^*, if $k^* < P$. The relevant hypothesis test is

$$H_O : k^\star \geq P$$

versus the alternative of wrong specification of the risk model given by

$$H_A : 1 \leq k^\star < P.$$

The CUSUM test for detecting a structural break in the sequence $\{X_t\}_{t=1}^P$ is based on deviations of the partial sum $X_P(\tau) = \frac{1}{\sqrt{P}} \left(\sum_{t=1}^{\lceil \tau P \rceil} X_t - \lceil \tau P \rceil \alpha \right)$ from the total sum $X_P(1) = \frac{1}{\sqrt{P}} (\sum_{t=1}^P X_t - P\alpha)$. The CUSUM process takes this form;

$$M_P^{(CS)}(\tau) := X_P(\tau) - \tau X_P(1). \tag{2.5}$$

For $X_t = I_{t,\alpha}(\theta)$, the total sum $X_P(1)$ is Kupiec's test. Moreover, we can construct a process $\widetilde{K}_P$ indexed by a parameter τ, with $0 \leq \tau \leq 1$, as follows:

$$\widetilde{K}_P(\tau) = \frac{1}{\sqrt{P}} \left(\sum_{t=1}^{\lceil \tau P \rceil} I_{t,\alpha}(\theta) - \lceil \tau P \rceil \alpha \right), \tag{2.6}$$

with $\lceil \cdot \rceil$ denoting the integer part of τP. Under H_O, Donsker's (1951) theorem restated on D$[0,1]$ now applies and implies the following weak convergence result:

$$\widetilde{K}_P(\cdot) \Rightarrow \sqrt{\alpha(1-\alpha)} W(\cdot), \quad 0 \leq \tau \leq 1, \tag{2.7}$$

with $\{W(\tau);\, 0 \leq \tau \leq 1\}$ a standard Wiener process and $\Rightarrow$ refers to weak convergence.

We now consider the class of CUSUM change-point tests characterized by the process $\widetilde{K}_P(\tau)$;

$$M_P^{(CS,O)}(\tau) := \widetilde{K}_P(\tau) - \tau \widetilde{K}_P(1), \tag{2.8}$$

with $\widetilde{K}_P(1) \equiv K_P$. This is a piece-wise continuous partial sum process with jump points at $\tau = \frac{k}{P}$, that satisfies under H_O, as $P \to \infty$,

$$\sup_{0 \leq \tau \leq 1} M_P^{(CS,O)}(\tau) \xrightarrow{\mathcal{D}} \sqrt{\alpha(1-\alpha)} \sup_{0 \leq \tau \leq 1} (W(\tau) - \tau W(1)) \equiv \sqrt{\alpha(1-\alpha)} \sup_{0 \leq \tau \leq 1} B(\tau), \tag{2.9}$$

where $B(\tau)$ denotes a Brownian bridge, and $\xrightarrow{\mathcal{D}}$ denotes convergence in distribution. Under proper standardization the critical values of the distribution of the asymptotic process are parameter-free and can be tabulated.

2.1.2 An Alternative Change-Point Detection Test for Deviations in the Tails

Section 2.1.1 introduced the CUSUM test and then showed how the Kupiec test could be considered as a special case of this test. As the CUSUM figures prominently in this research, it is interesting to make a few observations regarding the CUSUM test; one regarding the similarity of this test with that of the Kolmogorov-Smirnov (K-S) statistic; and the other regarding the consistency of the power function of the K-S statistic to deviations from the hypothesized distribution that may occur in the tails.

When the summand in (2.5) is set to the indicator of some random event, i.e. $X_t =$

$I(Y_t < \ y)$, where $Y_t, t = 1, \ldots, P$ are IID, this results in the empirical distribution function. This function is an important ingredient from which the K-S statistic is fashioned. Indeed, the K-S statistic calculates, uniformly, the distance between the empirical distribution function and the distribution function specified under the null hypothesis, and rejects said null hypothesis when this distance is too great. Mason and Schuenemeyer (1983, hereafter M-S) have shown that, both in finite and large sample theory, the K-S statistic exhibits poor sensitivity to deviations that may occur in the tails: M-S establish that the K-S statistic is inconsistent against such deviations. A similar fate holds true here for our above mentioned CUSUM test: the CUSUM test is also insensitive to deviations of this nature. In an attempt to rectify this apparent insensitivity of the K-S test to deviations from the hypothesized distribution that may occur in the tails, M-S apply weights to their statistics and find that these weighted statistics perform much better than the K-S statistic - they are consistent against such deviations. Moreover, they state that, while they are unable to find uniformly good weight functions, there do exist weight functions, dependent upon P, that make weighted versions of the K-S statistic consistent with respect to deviations that may occur in the tails. Hence, just as in the K-S setting where it was shown to be useful to employ weights, it is also desirable in our setting to introduce weights that may remedy this situation somewhat on the tails, i.e., in particular for early detection of deviations of VaR models.

More interestingly, OP study the empirical power of statistics constructed from the CUSUM, as well as the CUSUM statistic itself, that test for a change in the location parameter that occurs early on in the sample. They find that the CUSUM test is completely insensitive to such deviations. More specifically, the family of partial sum processes defined in (2.5) is more powerful for detecting changes in the distribution that occur near $P/2$ than noticing changes near the endpoints, 1 and P of the sample. These observations would indicate the value of constructing test statistics that are more sensitive to tail alternatives or, in the case of this research, early detection, yet remain sensitive to departures that may occur later on as well.

There remains, however, the heretofore unanswered question of what form the selected weights should take and how to weight the partial sum processes detailed in (2.5) and (2.8). More information on the conditions that these weight functions must satisfy will be provided in this section and some examples will follow in Section 2.2. The choice of weight functions remains an active area of research, work by Csörgő and Horváth (1988a, 1988b, 1997) provide a detailed account of the use of weight functions and site some of the many interesting properties of weighted statistics. For our purposes, however, we focus on Theorem 2.1. in Szyszkowicz (1991), which is referred to here as Theorem S. Before this we need to define some basic properties of these functions, and further notation.

Definition 1.

1.) Let Q be the class of positive functions on $(0, 1)$ which are non-decreasing in a neighborhood of zero and non-increasing in a neighborhood of one, where a function $q(\cdot)$ defined on (0,1) is called positive if

$$\inf_{\delta \le \tau \le 1-\delta} q(\tau) > 0 \qquad for\ all\ \delta \in (0, 1/2). \tag{2.10}$$

2.) Let $c > 0$ be a constant value. Then for $q \ \epsilon \ Q$,

$$\Psi(q, c) \ := \int_0^1 \frac{1}{\tau(1-\tau)} \exp\left(-\frac{c}{\tau(1-\tau)q^2(\tau)}\right) d\tau. \tag{2.11}$$

Let X_1 and X_2 be independent random variables, and $h(x, y)$ be a kernel that satisfies the following property: $h(x, y) = -h(y, x)$, $i.e.$, the kernel is antisymmetric. We have under H_O that $\mathbb{E}h(X_1, X_2) = 0$. Let $\tilde{h}(t) = \mathbb{E}h(X_1, t)$, assume that

$$\mathbb{E}h^2(X_1, X_2) \ < \ \infty \tag{2.12}$$

$$0 < \sigma^2 := \mathbb{E}\tilde{h}^2(X_2), \tag{2.13}$$

and set

$$Z_k := \sum_{i=1}^{k} \sum_{j=k+1}^{P} h(X_i, X_j), \qquad 1 \leq k < P. \tag{2.14}$$

Theorem S. *Assume that X_j for $j = 1, \ldots, P$ are IID random variables, $h(x,y) = -h(y,x)$, (2.12) and (2.13) are satisfied. Then a sequence of Brownian bridges $\{B_P(\tau), 0 \leq \tau \leq 1\}$ can be defined such that, as $P \to \infty$,*

(i)
$$\sup_{0<\tau<1} \frac{\left|\frac{P^{-3/2}}{\sigma} Z_{\lceil \tau P \rceil} - B_P(\tau)\right|}{q(\tau)} = \begin{cases} o_P(1), & \text{if and only if } \Psi(q,c) < \infty \quad \text{for all } c > 0 \\ O_P(1), & \text{if and only if } \Psi(q,c) < \infty \quad \text{for some } c > 0. \end{cases}$$

(ii) *Let $\{B(\tau);\ 0 \leq \tau \leq 1\}$ be a Brownian bridge. Then*

$$\sup_{0<\tau<1} \frac{P^{-3/2}}{\sigma} \frac{|Z_{\lceil \tau P \rceil}|}{q(\tau)} \xrightarrow{\mathcal{D}} \sup_{0<\tau<1} \frac{|B(\tau)|}{q(\tau)}$$

if and only if $\Psi(q,c) < \infty$ for some $c > 0$.

To connect (2.14) to our family of partial sum processes $M_P^{(CS,O)}(\cdot)$ we will assume $\tau = \frac{k}{P}$, set $h(x,y) = x - y$ and note that this kernel is antisymmetric. Replace $h(x,y)$ in (2.14) with $x - y$ which, after some algebra, reduces to the following;

$$Z_k = P\sum_{i=1}^{k} X_i - k\sum_{j=1}^{P} X_j. \tag{2.15}$$

Set $X_i = I_{i,\alpha}(\theta)$ in (2.15) and normalize by $P^{3/2}$; after which we arrive at the following representation:

$$\frac{Z_k}{P^{3/2}} = \frac{\sum\limits_{i=1}^{k} I_{i,\alpha}(\theta) - \frac{k}{P}\sum\limits_{j=1}^{P} I_{j,\alpha}(\theta)}{P^{1/2}} \tag{2.16}$$

which corresponds to $M_P^{(CS,O)}(\tau)$, with $\tau = \frac{k}{P}$ and appropriate subscript t, as detailed in (2.8). Using Theorem S, we are now able to make the following statements regarding weighted versions of $M_P^{(CS,O)}(\tau)$, the nature of which are detailed in Proposition 2.1.1.

Proposition 2.1.1. *Let H_O hold and $q \in Q$. Then we can define a sequence of Brownian bridges $\{B_P(\tau);\ 0 \leq \tau \leq 1\}$ such that, as $P \to \infty$, the following hold:*

i) $$\sup_{0<\tau<1} \frac{\left|\frac{1}{(\alpha(1-\alpha))^{1/2}} M_P^{(CS,O)}(\tau) - B_P(\tau)\right|}{q(\tau)} = \begin{cases} o_P(1), & \text{if and only if } \Psi(q,c) < \infty \quad \text{for all } c > 0 \\ O_P(1), & \text{if and only if } \Psi(q,c) < \infty \quad \text{for some } c > 0, \end{cases}$$

and

ii)

$$\sup_{0<\tau<1} \frac{\left|\frac{1}{(\alpha(1-\alpha))^{1/2}} M_P^{(CS,O)}(\tau)\right|}{q(\tau)} \xrightarrow{\mathcal{D}} \sup_{0<\tau<1} \frac{\left|B(\tau)\right|}{q(\tau)}$$

if only if $\Psi(q,c) < \infty$ for some c.

Remark 2.1.1. *Let $\left\{B_P(\tau) := \frac{W(P\tau)-\tau W(P)}{\sqrt{P}}; 0 \leq \tau \leq 1\right\}$ be a version of a Brownian Bridge. Then, for $P = 1, 2, \ldots$, we have*

$$\{B_P(\tau);\ 0 \leq \tau \leq 1\} \stackrel{\mathcal{D}}{=} \{B(\tau);\ 0 \leq \tau \leq 1\}.$$

2.1.3 Population Parameter Unknown

The weighted partial sum process developed in (2.8) depends on an unknown vector of population parameters θ that in practice is usually unknown. A natural solution is to replace θ by any consistent estimator. As we are interested in functionals of CUSUM test statistics, it would be of interest to know if such substitutions affect their limiting distribution. Such substitutions, as can be the case, increase the randomness of such functionals of these processes, and then cause the thus altered process to have a limiting distribution different from that of the functional of the original partial sum process. In what follows, however, we show the convergence of the estimated CUSUM process to the same limiting distribution as the original CUSUM statistic, and with it the absence of the so-called estimation risk. For simplicity, we focus on the estimated version of the test statistic $M_P^{(CS,O)}(\tau)$ computed over the out-of-sample period P and given by

$$\widehat{M}_P^{(CS,O)}(\tau) \ := \ \frac{\sum_{t=1}^{\lceil \tau P \rceil} I_{t,\alpha}(\widehat{\theta}_{t,R}) - \tau \sum_{t=1}^{P} I_{t,\alpha}(\widehat{\theta}_{t,R})}{P^{1/2}}, \quad 0 \leq \tau \leq 1, \tag{2.17}$$

with $\{\widehat{\theta}_{t,R}\}_{t=1}^{P}$ any sequence of consistent estimators of the vector of parameters θ encompassing the three different schemes used in the backtesting literature, namely, the recursive, fixed and rolling forecasting schemes. They differ in how the parameter θ is estimated. In the recursive scheme the sequence $\{\widehat{\theta}_{t,R}\}_{t=1}^{P}$, $R = 1, 2, \ldots$ is computed with all the sample available up to $R+t-1$ for $t = 1, \ldots, P$, and R denoting the in-sample size. For the fixed forecasting scheme, on the other hand, the estimator is not updated when new observations become available, and therefore leaves $\{\widehat{\theta}_{t,R}\}_{t=1}^{P} = \widehat{\theta}_R$. Finally, for the rolling estimator the subscript R denotes the number of observations used in the estimation process, in this case the sequence of estimators $\{\widehat{\theta}_{t,R}\}_{t=1}^{P}$ is constructed from the sample $t, \ldots, t+R-1$, for each $t = 1, \ldots, P$.

Proposition 2.1.2. *Let* $\widehat{M}_P^{(CS,O)}(\cdot)$ *be the estimated version of the process* $M_P^{(CS,O)}(\cdot)$*. Let* $q \in Q$ *satisfy the integral condition* $\Psi(q,c) < \infty$ *for some* $c > 0$*, and let* $\{\widehat{\theta}_{t,R}\}_{t=1}^{P}$ *be the sequence of recursive, fixed or rolling estimators of the parameter vector* θ*. Under* H_O*, as* $R, P \to \infty$*, with* $0 < \lim_{R,P\to\infty} \frac{P}{R} < \infty$*,*

$$\sup_{0<\tau<1} \frac{\left| \widehat{M}_P^{(CS,O)}(\tau) - M_P^{(CS,O)}(\tau) \right|}{q(\tau)} = o_P(1).$$

Using Proposition 2.1.2, we are able to make the following statements concerning (2.17). The nature of these statements include one concerning approximation in probability and one regarding the asymptotic distribution of the supremum over τ of these processes. These are all detailed in Proposition 2.1.3, and are similar in nature to those statements made regarding the partial sum process (2.8) and detailed in Proposition 2.1.1.

Proposition 2.1.3. *Let H_O hold and $q \in Q$. Then, we can define a sequence of Brownian Bridges $\{B_P(\tau);\ 0 \leq \tau \leq 1\}$ such that, as $R, P \to \infty$, the following hold:*

i) $$\sup_{0<\tau<1} \frac{\left|\frac{1}{(\alpha(1-\alpha))^{1/2}} \widehat{M}_P^{(CS,O)}(\tau) - B_P(\tau)\right|}{q(\tau)} = \begin{cases} o_P(1), & \text{if and only if } \Psi(q,c) < \infty \text{ for all } c > 0 \\ O_P(1), & \text{if and only if } \Psi(q,c) < \infty \text{ for some } c > 0, \end{cases}$$

and

ii)

$$\sup_{0<\tau<1} \frac{\left|\frac{1}{(\alpha(1-\alpha))^{1/2}} \widehat{M}_P^{(CS,O)}(\tau)\right|}{q(\tau)} \xrightarrow{\mathcal{D}} \sup_{0<\tau<1} \frac{\left|B(\tau)\right|}{q(\tau)}$$

only if $\Psi(q,c) < \infty$ for some c.

Under the alternative, H_A, there remains one additional parameter to estimate, $k^\star$. A number of estimators of this parameter have been proposed in the literature but we provide only one, as it is intuitive and some of its large sample properties have been detailed in the literature.

$$\widehat{k^\star} := \min\left\{ k : \frac{|M_P^{(CS,O)}(\frac{k}{P})|}{q(\frac{k}{P})} = \max_{1 \leq i < P} \frac{|M_P^{(CS,O)}(\frac{i}{P})|}{q(\frac{i}{P})} \right\}. \tag{2.18}$$

This estimator of $k^\star$ is based on the fact that the statistic should be largest at the time of failure of the risk model; hence an estimator of the time of change should be value of k for which the statistic is largest. As the k where the statistic is largest may not be unique, we take the minimum of all such k to arrive at the unique estimator, $\widehat{k^\star}$.

The asymptotic properties of this estimator have been studied by Antoch, Hušková and Veraverbeke (1995). They also show that the bootstrap approximation to this distribution is asymptotically valid. For more on this, we refer those interested to their paper. This estimator of location of change in the VaR model should greatly assist risk managers in understanding the reasons for changes in their VaR model and to propose alternative models after the break point to assess properly market risk.

2.2 Power of U-statistic type tests for detecting market risk failure

To obtain the asymptotic results detailed in Sections 2.1.2 and 2.1.3, $q(\cdot)$ was required to satisfy the integral equation (2.11) but there has been neither discussion on the form these weight functions may take nor from among those weight functions that satisfy $\Psi(q,c)<\infty$ in (2.11) - for some or all c - which ones are optimal. This section explores this by comparing the asymptotic statistical power of the standard CUSUM test, see (2.8), against different versions of the weighted U-statistic type tests introduced in this paper, see Proposition 2.1.1. The section studies, in particular, a novel class of weight functions that encompasses the family of weight functions in GHH and OP devised to be sensitive to deviations of the risk model in the tails. The aim of our new family of functions is to gain statistical power against rejections of H_O early on in the out-of-sample backtesting period. This is detailed as follows.

Remember that

$$X_t = \begin{cases} 1(Y_t \leq m_\alpha(\theta;\Im_{t-1})), & 1 \leq t \leq k^\star, \\ 1(Y_t \leq m^\star_\alpha(\theta^\star;\Im_{t-1})), & k^\star < t \leq P, \end{cases}$$

with $m^\star_\alpha(\theta^\star;\Im_{t-1}) \neq m_\alpha(\theta;\Im_{t-1})$, and both belonging to $\mathcal{M}$ and CUSUM process defined by

$$M_P^{(CS)}(\tau) := X_P(\tau) - \tau X_P(1).$$

Under H_A, the process $M_P^{(CS)}(\cdot)$ is related to $M_P^{(CS,O)}(\cdot)$ by the following expression;

$$\begin{aligned} M_P^{(CS,O)}(\tau) &= M_P^{(CS)}(\tau) + \tau\Big(1-\frac{\lceil \tau^\star P\rceil}{P}\Big)(\alpha-\widetilde{\alpha})\sqrt{P} - \frac{\tau}{\sqrt{P}}\sum_{t=\lceil \tau^\star P\rceil+1}^{P}(I_{t,\alpha}(\theta)-\widetilde{\alpha}) \\ &\quad + \frac{\tau}{\sqrt{P}}\sum_{t=\lceil \tau^\star P\rceil+1}^{P}(I^\star_{t,\alpha}(\theta)-\alpha) + \frac{1}{\sqrt{P}}\sum_{t=\lceil \tau^\star P\rceil+1}^{\lceil \tau P\rceil}\big(I_{t,\alpha}(\theta)-I^\star_{t,\alpha}(\theta)\big), \end{aligned} \tag{2.19}$$

with $I^\star_{t,\alpha}(\theta^\star) = 1(Y_t \leq m^\star_\alpha(\theta^\star;\Im_{t-1}))$.

To obtain equation (2.19) we have assumed without loss of generality that $\tau \geq \frac{k^\star}{P}$. In what follows we use the above relationship between processes to derive the power of the CUSUM test in (2.8) and of the weighted U-statistic type tests in Proposition 2.1.1. We need first to make two additional assumptions:

$$\delta(P) \to 0 \quad \text{as} \quad P \to \infty, \tag{2.20}$$

$$\delta(P)P \to 0 \quad \text{as} \quad P \to \infty. \tag{2.21}$$

Using (2.19), we have the following result for the weighted version of the statistics;

$$\begin{aligned}
\sup_{\frac{k^\star}{P}-\delta(P)\le\tau\le\frac{k^\star}{P}+\delta(P)} \frac{|M_P^{(CS,O)}(\tau)|}{q(\tau)} &= \sup_{\frac{k^\star}{P}-\delta(P)\le\tau\le\frac{k^\star}{P}+\delta(P)} \Bigg| \frac{M_P^{(CS)}(\tau)}{q(\tau)} + \frac{\tau(1-\frac{\lceil\tau^\star P\rceil}{P})(\alpha-\widetilde{\alpha})}{q(\tau)}\sqrt{P} \\
&- \frac{\tau}{\sqrt{P}}\frac{\sum_{t=\lceil\tau^\star P\rceil+1}^{P}(I_{t,\alpha}(\theta)-\widetilde{\alpha})}{q(\tau)} + \frac{\tau}{\sqrt{P}}\frac{\sum_{t=\lceil\tau^\star P\rceil+1}^{P}(I^\star_{t,\alpha}(\theta^\star)-\alpha)}{q(\tau)} \\
&+ \frac{1}{\sqrt{P}}\frac{\sum_{t=\lceil\tau^\star P\rceil+1}^{\lceil\tau P\rceil}\left(I_{t,\alpha}(\theta)-I^\star_{t,\alpha}(\theta^\star)\right)}{q(\tau)}\Bigg|,
\end{aligned} \tag{2.22}$$

which leads to the following simplification of (2.22);

$$\begin{aligned}
\sup_{\frac{k^\star}{P}-\delta(P)\le\tau\le\frac{k^\star}{P}+\delta(P)} \frac{|M_P^{(CS,O)}(\tau)|}{q(\tau)} &= \Bigg| \frac{M_P^{(CS)}(\frac{k^\star}{P})}{q(\frac{k^\star}{P})} + \frac{k^\star}{P}\frac{(1-\frac{\lceil\tau^\star P\rceil}{P})(\alpha-\widetilde{\alpha})}{q(\frac{k^\star}{P})}\sqrt{P} \\
&- \frac{\frac{k^\star}{P}}{\sqrt{P}}\frac{\sum_{t=\lceil\tau^\star P\rceil+1}^{P}(I_{t,\alpha}(\theta)-\widetilde{\alpha})}{q(\frac{k^\star}{P})} + \frac{\frac{k^\star}{P}}{\sqrt{P}}\frac{\sum_{t=\lceil\tau^\star P\rceil+1}^{P}(I^\star_{t,\alpha}(\theta^\star)-\alpha)}{q(\frac{k^\star}{P})} \\
&+ \frac{1}{\sqrt{P}}\frac{\sum_{t=k^\star+1}^{k^\star}\left(I_{t,\alpha}(\theta)-I^\star_{t,\alpha}(\theta^\star)\right)}{q(\frac{k^\star}{P})}\Bigg|.
\end{aligned} \tag{2.23}$$

The last term in (2.23) is zero which leaves only the first four terms, i.e.,

$$\begin{aligned}
\sup_{\frac{k^\star}{P}-\delta(P)\le\tau\le\frac{k^\star}{P}+\delta(P)} \frac{|M_P^{(CS,O)}(\tau)|}{q(\tau)} &= \Bigg| \frac{M_P^{(CS)}(\frac{k^\star}{P})}{q(\frac{k^\star}{P})} + \frac{k^\star}{P}\frac{(1-\frac{\lceil\tau^\star P\rceil}{P})(\alpha-\widetilde{\alpha})}{q(\frac{k^\star}{P})}\sqrt{P} \\
&- \frac{\frac{k^\star}{P}}{\sqrt{P}}\frac{\sum_{t=\lceil\tau^\star P\rceil+1}^{P}(I_{t,\alpha}(\theta)-\widetilde{\alpha})}{q(\frac{k^\star}{P})} + \frac{\frac{k^\star}{P}}{\sqrt{P}}\frac{\sum_{t=\lceil\tau^\star P\rceil+1}^{P}(I^\star_{t,\alpha}(\theta^\star)-\alpha)}{q(\frac{k^\star}{P})}\Bigg|.
\end{aligned} \tag{2.24}$$

We can remove the absolute value sign in (2.24) by noting that if the sum is negative, then multiply by -1. Thus, expression (2.24) provides the rate at which the mean of $\frac{|M_P^{(CS,O)}(\frac{k^\star}{P})|}{q(\frac{k^\star}{P})}$ is increasing; it is precisely $\frac{k^\star}{P}\frac{(1-\frac{\lceil\tau^\star P\rceil}{P})|\alpha-\widetilde{\alpha}|}{q(\frac{k^\star}{P})}\sqrt{P}$. Note that after demeaning the test statistic the different terms on the right of the previous expression converge by the central limit theorem to univariate normal distributions. More specifically,

Proposition 2.2.1. *Under H_A, (2.4), (2.20), (2.21) and (2.24), and as $P \to \infty$,*

$$\begin{bmatrix} \frac{1}{\sqrt{P}}\sum_{t=1}^{k^\star}(I_{t,\alpha}(\theta)-\alpha) \\ \frac{1}{\sqrt{P}}\sum_{t=1}^{k^\star}(I_{t,\alpha}(\theta)-\alpha) \ + \ \frac{1}{\sqrt{P}}\sum_{t=k^\star+1}^{P}(I^\star_{t,\alpha}(\theta^\star)-\alpha) \\ \frac{1}{\sqrt{P}}\sum_{t=k^\star+1}^{P}(I_{t,\alpha}(\theta)-\widetilde{\alpha}) \\ \frac{1}{\sqrt{P}}\sum_{t=k^\star+1}^{P}(I^\star_{t,\alpha}(\theta^\star)-\alpha) \end{bmatrix} \xrightarrow{\mathcal{D}} N(0,\Sigma), \tag{2.25}$$

where

$$\Sigma = \begin{bmatrix} \tau^\star\alpha(1-\alpha) & \tau^\star\alpha(1-\alpha) & 0 & 0 \\ & \alpha(1-\alpha) & (1-\tau^\star)COV(I_{k^\star+1,\alpha}(\theta),I^\star_{k^\star+1,\alpha}(\theta^\star)) & (1-\tau^\star)\alpha(1-\alpha) \\ & & (1-\tau^\star)\widetilde{\alpha}(1-\widetilde{\alpha}) & (1-\tau^\star)COV(I_{k^\star+1,\alpha}(\theta),I^\star_{k^\star+1,\alpha}(\theta^\star)) \\ & & & (1-\tau^\star)\alpha(1-\alpha) \end{bmatrix}. \tag{2.26}$$

The matrix Σ depends on α $\widetilde{\alpha}$, $\tau^\star$ and $COV(I_{k^\star+1,\alpha}(\theta), I^\star_{k^\star+1,\alpha}(\theta^\star))$. Note that

$$\mathbb{E}\left[I_{k^\star+1,\alpha}(\theta)I^\star_{k^\star+1,\alpha}(\theta^\star)\right] = \mathbb{P}\left\{Y_{k^\star+1} \leq \min\{m_\alpha(\theta;\Im_{k^\star}), m^\star_\alpha(\theta^\star;\Im_{k^\star})\} \mid \Im_{k^\star}\right\}, \tag{2.27}$$

and hence

$$COV(I_{k^\star+1,\alpha}(\theta), I^\star_{k^\star+1,\alpha}(\theta^\star)) = \begin{cases} \alpha(1-\widetilde{\alpha}), & \text{if } m_\alpha(\theta;\Im_{k^\star}) > m^\star_\alpha(\theta^\star;\Im_{k^\star}), \\ \widetilde{\alpha}(1-\alpha), & \text{otherwise.} \end{cases} \tag{2.28}$$

Now, using Proposition 2.2.1 and the Cramer-Wold device the following statement can be made regarding the statistic detailed in (2.8). Define $h' = [1, -\tau^\star, -\tau^\star, \tau^\star]$.

Theorem 2.2.1. *Assume* H_A, (2.20), (2.21) *hold, then as* $P \to \infty$,

$$\frac{q(\tau^\star)}{\sqrt{h'\Sigma h}}\left[\sup_{0<\tau<1}\frac{|M_P^{(CS,O)}(\tau)|}{q(\tau)} - \frac{\frac{[\tau^\star P]}{P}(1-\frac{[\tau^\star P]}{P})|\alpha-\widetilde{\alpha}|}{q(\tau^\star)}\sqrt{P}\right] \xrightarrow{\mathcal{D}} N(0,1), \tag{2.29}$$

with $h'\Sigma h = \tau^\star(1-\tau^\star)\left\{\alpha(1-\alpha)+\tau^\star\left[\widetilde{\alpha}(1-\widetilde{\alpha})-\alpha(1-\alpha)\right]\right\}$.

The next result follows as a consequence of the above theorem.

Corollary 2.2.1. *Theorem* 2.2.1 *establishes that*

$$\sup_{0<\tau<1}\frac{|M^{(CS,O)}(\tau)|}{P^{1/2}q(\tau)} \xrightarrow{P} \frac{\tau^\star(1-\tau^\star)|\alpha-\widetilde{\alpha}|}{q(\tau^\star)},$$

as $P \to \infty$ and, as a result, the consistency of each weighted U-statistic type test.

With these results in place we can study the power function of the different change-point detection tests discussed before. In order to do this we need first to obtain the critical value at a β significance level corresponding to each test. From Propositions 2.1.1 and 2.1.3 the critical value is obtained via simulation of the distribution of the asymptotic process $\sup\limits_{0<\tau<1} \frac{|B(\tau)|}{q(\tau)}$, and is therefore idiosyncratic to the weight function $q(\cdot)$ chosen. More specifically, the critical value $C^q_{1-\beta}$ is defined as

$$\lim_{P\to\infty} P_{H_O} \left\{ \frac{1}{\sqrt{\alpha(1-\alpha)}} \sup_{0<\tau<1} \frac{\left| M_P^{(CS,O)}(\tau) \right|}{q(\tau)} > C^q_{1-\beta} \right\} = \beta, \tag{2.30}$$

where $\lim\limits_{P\to\infty} P_{H_0}$ is the distribution of the supremum of the weighted Brownian bridge.

Theorem 2.2.2. *Assume a set of local alternative hypotheses defined by a VaR model with a coverage probability $\widetilde{\alpha}$ starting from $\tau^\star$, and that satisfies $\alpha - \widetilde{\alpha} = \frac{a}{P^\gamma}$, with $a \neq 0$, and $\gamma \geq 1/2$ constant values. Then, the power function (pf) at a β significance level is defined by*

$$\lim_{P\to\infty} pf_\beta = 1 - \Phi\left(\sqrt{\frac{\alpha(1-\alpha)}{h'\Sigma h}} q(\tau^\star) C^q_{1-\beta} - |a| \frac{\tau^\star(1-\tau^\star)}{\sqrt{h'\Sigma h}} \right), \tag{2.31}$$

for $\gamma = 1/2$, with $\Phi(\cdot)$ the cdf of a standard normal distribution, and

$$\lim_{P\to\infty} pf_\beta = 1 - \Phi\left(\sqrt{\frac{\alpha(1-\alpha)}{h'\Sigma h}} q(\tau^\star) C^q_{1-\beta} \right), \tag{2.32}$$

for $\gamma > 1/2$.

The power of the change point test statistic $\frac{\left| M_P^{(CS,O)}(\tau^\star) \right|}{q(\tau^\star)}$ for detecting a break in the risk model at $\tau^\star$ is a function of the coverage probability α of interest, the distance between this coverage probability and that reported by the wrong model $m_\alpha(\theta; \Im_{t-1})$ after the break, the timing of the break $\tau^\star$ and also the weight function $q(\cdot)$.

2.2.1 A Comparative Power Analysis of Weighted U-Statistics

The statistical power of the different change-point tests to reject the correct specification of the VaR model and detect the timing of the failure is an important issue when choosing an appropriate method to monitor risk. Theorem 2.2.2 derives the power of the CUSUM test and weighted versions, and allows us to compare the different weighted U-statistic type tests in terms of the statistical power to local alternatives.

Definition 2. A uniformly optimal weighted U-statistic type process defined by a weight function $q^{\star}(\tau)$ within the class in Definition 1, satisfies

$$q^{\star}(\tau) < q(\tau)\frac{C^{q}_{1-\beta}}{C^{q^{\star}}_{1-\beta}}, \tag{2.33}$$

for all $\tau, \beta \in (0,1)$.

The dependence of the critical values on the form of the weight function and the non-monotonic nature of the latter implies that it is very difficult to find a function that satisfies the above optimality condition globally. Instead, one can devise weight functions suited to be optimal for certain regions within $(0,1)$. In order to enhance the power against deviations in the tails, we introduce the following family of weight functions.

Definition 3. Let $q^{step}_{\nu}(\tau)$ be defined as

$$q^{step}_{\nu}(\tau) := \begin{cases} (\tau(1-\tau))^{\nu} & \text{if } \tau \ \varepsilon \ (a,b) \\ \left(\tau(1-\tau)\log\log\frac{1}{\tau(1-\tau)}\right)^{\nu} & \text{if } \tau \ \varepsilon \ \ [0,a] \ \cup \ [b,1), \end{cases} \tag{2.34}$$

where $a = 0.071033$ and $b = 0.92896$, is a step function satisfying $\Psi(q^{step}_{\nu}, c) < \infty$ for some $c > 0$.

For hypothesis tests for at most one change in the mean, OP proposed the case $\nu = 1/2$ to improve the power of CUSUM type tests with respect to the family introduced in $q^{step}_{\nu}(\dot{})$.

One might expect $\nu = 1/2$ to be a reasonable candidate for the weight functions introduced by GHH;

$$q_\nu(\tau) = \{(\tau(1-\tau))^\nu;\ 0 \le \nu \le 1/2\}. \tag{2.35}$$

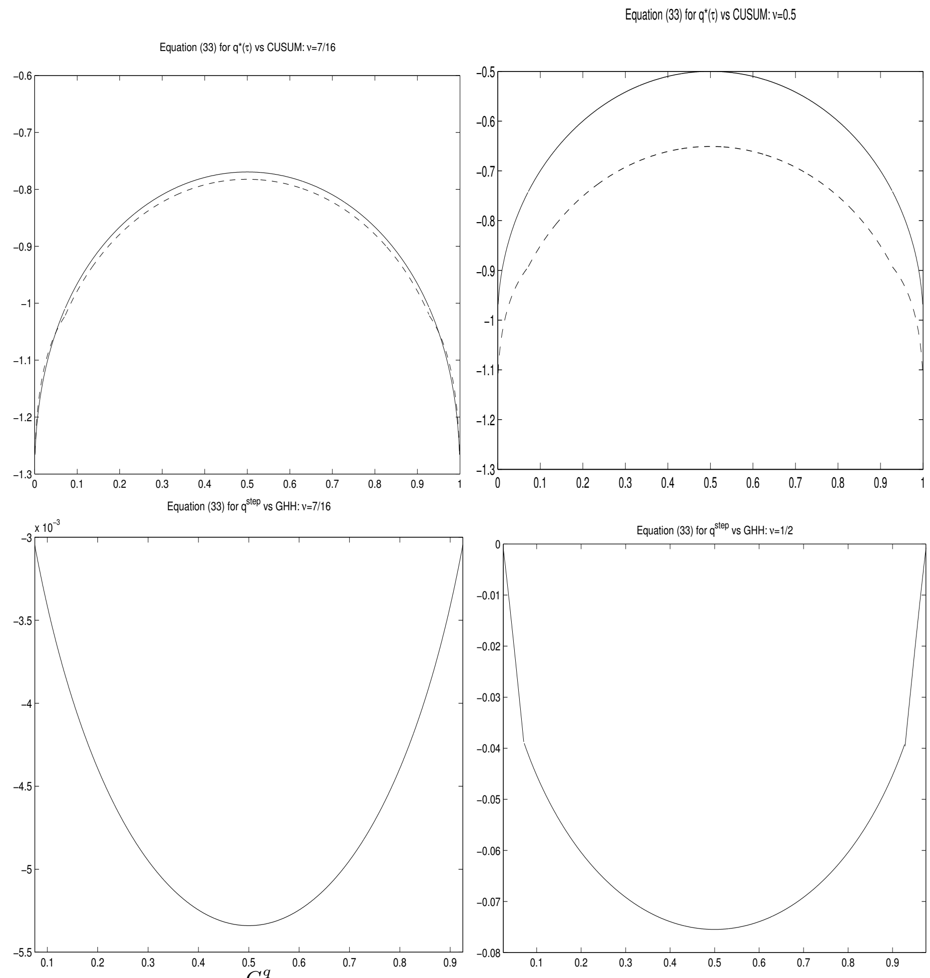
These families of weight functions are in the class of functions in Definition 1. The only exception is $\nu = 1/2$ for the GHH function. The asymptotic distribution of the supremum of the weighted U-statistics is tabulated. Thus, the critical values for the U-statistic q_ν^{step} for $\nu = 0$ and $1/2$ are found in OP, and for $\nu = 1/16$, $3/16$, $5/16$ and $7/16$ in Section 2.7. This section also contains the tables for the distribution corresponding to q_ν, for $\nu = 1/16$, $3/16$, $5/16$ and $7/16$. For $\nu = 1/2$ the relevant critical values can be easily calculated as the limiting distribution is a Double Exponential. These are [2.943, 3.660, 5.293] corresponding to 10%, 5% and 1% significance level. For the standard CUSUM test ($\nu = 0$) these are [1.232, 1.366, 1.640].

Figure 1 plots $q^\star(\tau) - q(\tau)\frac{C^q_{1-\beta}}{C^{q^\star}_{1-\beta}}$ corresponding to condition (2.33). The upper panels show evidence of a better performance of the weighted U-statistics with respect to the CUSUM test for $\nu = 7/6$ and $\nu = 1/2$, respectively. The lower panels plot the difference corresponding to $q^\star = q_\nu^{step}$ and $q = q_\nu$ for $\nu = 7/6$ and $\nu = 1/2$. For values below $\nu = 7/6$ the global optimality of q_ν^{step} over q_ν no longer holds, and there is little reason to consider weight functions with $\nu < 7/16$ unless the interest is gaining more power in specific regions.

2.2.2 A Likelihood Ratio Test for Change Point Detection in VaR models

The optimality of change-point tests is widely studied in statistics and econometrics. The non-parametric approach has been discussed in previous sections. In a parametric regression framework this problem is studied by Andrews (1993) and Andrews and Ploberger (1994). These authors use results from Davies (1977) to show that likelihood ratio tests, Lagrange Multipliers and Wald tests possess certain asymptotic optimality properties against local

Figure 2.1: Power of Weighted Statistics



Upper panels for $q^{\star}(\tau) - q(\tau)\frac{C^{q}_{1-\beta}}{C^{q^{\star}}_{1-\beta}}$ at $\beta = 0.05$ significance level. Solid line for $q^{\star} = q_{\nu}$ and dashed line for $q^{\star} = q_{\nu}^{step}$. Lower panels for $q_{\nu}^{step}(\tau) - q_{\nu}(\tau)\frac{C^{q_{\nu}}_{1-\beta}}{C^{q_{\nu}^{step}}_{1-\beta}}$. Left panels for $\nu = 7/16$ and right panels for $\nu = 1/2$.

alternatives for large sample size and small significance level. These methods, though, depend on the correct specification of the parametric model and are not well suited to detecting breaks near the end points of $[\pi_0, 1-\pi_0]$.

In this section we introduce a LR test for detecting the timing of the failure of VaR models. The Lagrange Multiplier and Wald test are asymptotically equivalent so it is enough to restrict to the first type of test. The testing regression equation is

$$I_{t,\alpha}(\theta) = \alpha I(t \leq t_0) + \widetilde{\alpha} I(t > t_0) + \varepsilon_t, \text{ for } t = 1, \ldots, P, \tag{2.36}$$

with ε_t a zero-mean random error, $t_0 \in \Omega \subset \mathbb{R}$, with Ω a compact set, and $\alpha \neq \widetilde{\alpha}$ the coverage probabilities under the null and alternative hypotheses. If the model is correctly specified we have $E[I_{t,\alpha}(\theta)|\Im_{t-1}] = \alpha$ for $t = 1, \ldots, P$ and the null hypothesis reads $H_O : \widetilde{\alpha} = \alpha$.

The estimation of this regression equation can be done by standard OLS methods. Under H_O the parameter t_0 is not identified implying that the asymptotic distribution of these tests is no longer a χ^2, but needs to be approximated by simulation or bootstrap techniques. In our context the LR test takes this form:

$$\sup_{t_0 \in \Omega} F(t_0). \tag{2.37}$$

The F-test is now a process indexed by t_0 and defined as $F(t_0) = (P-3)\frac{\widehat{\sigma}_0^2 - \widehat{\sigma}^2(t_0)}{\widehat{\sigma}^2(t_0)}$, with $\widehat{\sigma}^2(t_0)$ the residual variance estimated from the regression equation determined by t_0 and $\widehat{\sigma}_0^2$ the residual variance from the model under H_0. The finite-sample distribution of this test can be approximated by simulation or bootstrap methods. For the Monte-Carlo section we choose the bootstrap method very well described in Hansen (1997).

The following section compares this parametric method against CUSUM and weighted U-statistic type tests in terms of power. Similar studies are carried out by Worsley (1983) for IID binomial random variables and McCabe (1988) for elliptical distributions. The first author finds a statistical power tradeoff between CUSUM type tests and LR tests that depends on the location of the break in the evaluation period, that is, the CUSUM test outperforms LR tests in the middle of the sample but is beat for breaks early and late in the evaluation period.

2.3 Monte-Carlo Experiments

The aim of this section is to see the significance of our theoretical findings in finite samples. In order to do this we carry out a study of the finite-sample size and power of the different CUSUM tests developed in the previous section. We will concentrate on three features of these change-point tests: first, we will see that the three tests are consistent under the null hypothesis. The second aim will be the study of potential finite-sample distorting effects on the empirical size of the test derived from estimating the relevant parameters of the VaR model. Finally, we will gauge the power of the different versions of our tests against departures from the null hypothesis given by $\alpha - \widetilde{\alpha} = \frac{a}{\sqrt{P}}$ with $a \neq 0$, and stress the out performance of weighted versions of the U-statistic type processes introduced here when compared against standard CUSUM tests. In this power study we will also investigate the performance of the LR test (2.37) for change point detection and Christoffersen's (1998) unconditional LR test for hypothesis (2.2). This is an important exercise; for if the method developed here cannot out perform Christoffersen's non-parametric test then there would little need for it. For completeness, we also introduce this test:

$$LR = 2log \frac{L(\widehat{\alpha}; \{I_{t,\widehat{\alpha}}(\theta_0)\}_{t=1}^{P})}{L(\alpha; \{I_{t,\alpha}(\theta_0)\}_{t=1}^{P})}$$

with

$$L(\widehat{\alpha}; \{I_{t,\widehat{\alpha}}(\theta_0)\}_{t=1}^{P}) = (1-\hat{\alpha})^{P_0} \hat{\alpha}^{P_1},$$

and $\hat{\alpha} = \frac{P_1}{P}$, where P_1 is the number of violations of VaR at α coverage probability and $P_0 = P - P_1$. The asymptotic distribution of this test is χ_1^2.

The null hypothesis assumes no structural break in a conditional VaR modeled by a location-scale process. In particular the process considered here is a pure GARCH(1,1) data generating process with Gaussian innovations;

$$Y_t = \mu(\gamma_0; \Im_{t-1}) + u_t, \tag{2.38}$$

where $\mu(\gamma_0; \Im_{t-1}) = 0$ and $u_t = \sigma(\gamma_0; \Im_{t-1})\varepsilon_t$, with

$$\sigma(\gamma_0; \Im_{t-1}) = \sqrt{0.05 + 0.10u_{t-1}^2 + 0.85\sigma^2(\gamma_0; \Im_{t-2})},$$

and ε_t are the standardized innovations which are usually assumed to be *IID,* and independent of $\Im_{t-1}$. Note that under such assumptions the α-th conditional VaR is given by

$$m_\alpha(\theta; \Im_{t-1}) = \sigma(\gamma_0; \Im_{t-1})\Phi_\varepsilon^{-1}(\alpha), \tag{2.39}$$

where $\Phi_\varepsilon(\cdot)$ denotes the cumulative normal distribution function and $\Phi_\varepsilon^{-1}(\alpha)$ the corresponding $\alpha-$quantile function of ε_t. The parameter vector is

$$\theta = (\gamma_0, \Phi_\varepsilon^{-1}(\alpha)) = \left((0.05, 0.10, 0.85), \Phi_\varepsilon^{-1}(\alpha)\right).$$

Table 2.1 reports the empirical size for the standard CUSUM, ($M_P^{(CS,O)}$), for the GHH family of tests ($M_{P,GHH_\nu}^{(CS,O)}$) and for the new alternative U-statistic family of tests ($M_{P,q_\nu^{step}}^{(CS,O)}$), the last two tests computed for $\nu = 7/16$. The choice of this parameter is for comparison purposes across weighted functions. As discussed before, the GHH family of functions indexed at $\nu = 1/2$ does not satisfy the integral condition and one needs to use an alternative standardization to obtain the asymptotic critical values. The GHH weight functions corresponding to $\nu < 1/2$ do satisfy the integrability of (2.11) for all $c > 0$; see Proposition 1 of GHH. Further, even though Figure 1 suggests that $\nu = 1/2$ provides the largest power when compared to the CUSUM test and GHH's choice of weight function, non-reported results on the size of the asymptotic test for this case indicate the presence of large inflation of the nominal significance level. This reason, and the one aforementioned, led us to the selection of $\nu = 7/16$. The simulations revealed that this choice does not sacrifice much power but achieves however significant reduction in the inflation of nominal coverage probabilities. Further additional simulations not reported here indicate that $\nu < 7/18$ reduce empirical power when compared to the case $\nu = 7/16$, hence the reason for not exploring lower levels of ν.

Table 2.1: Empirical Power (EP)

θ Assumed to be Known									
	$M_P^{(CS,O)}$			$M_{P,GHH_{7/16}}^{(CS,O)}$			$M_{P,q_{7/16}^{step}}^{(CS,O)}$		
$\alpha = 0.01$	0.10	0.05	0.01	0.10	0.05	0.01	0.10	0.05	0.01
P=100	0.049	0.035	0.018	0.076	0.059	0.040	0.089	0.081	0.056
P=300	0.040	0.019	0.006	0.084	0.060	0.034	0.095	0.077	0.052
P=500	0.035	0.027	0.007	0.085	0.057	0.036	0.088	0.070	0.041
$\alpha = 0.05$	0.10	0.05	0.01	0.10	0.05	0.01	0.10	0.05	0.01
P=100	0.046	0.027	0.004	0.094	0.076	0.007	0.106	0.096	0.024
P=300	0.038	0.016	0.006	0.077	0.068	0.008	0.087	0.074	0.019
P=500	0.042	0.019	0.001	0.082	0.067	0.010	0.090	0.065	0.017
θ Estimated									
$\alpha = 0.01$	0.10	0.05	0.01	0.10	0.05	0.01	0.10	0.05	0.01
P=100	0.057	0.037	0.018	0.087	0.072	0.048	0.112	0.094	0.063
P=300	0.072	0.048	0.029	0.127	0.095	0.060	0.140	0.113	0.078
P=500	0.072	0.043	0.017	0.109	0.087	0.049	0.131	0.099	0.070
$\alpha = 0.05$	0.10	0.05	0.01	0.10	0.05	0.01	0.10	0.05	0.01
P=100	0.047	0.023	0.007	0.071	0.063	0.009	0.081	0.074	0.030
P=300	0.067	0.039	0.010	0.109	0.089	0.018	0.121	0.098	0.031
P=500	0.061	0.029	0.013	0.109	0.089	0.018	0.121	0.098	0.031

Table 2.1. Empirical size of different change-point detection tests for (2.39). θ parameters are assumed to be known and then estimated under H_O.

The hypothesis test is computed for VaR measures computed at $\alpha = 0.01$ and $\alpha = 0.05$ coverage probabilities, and derived from model (2.39). Whereas the weighted CUSUM tests approximate rather well the different nominal sizes for both $\alpha = 0.01$, 0.05 the standard CUSUM test exhibits poor results. For these sample sizes the empirical size underestimates considerably the nominal size in the tails. The lower half of Table 2.1 studies the impact of estimation effects on the size of the test. For this we compute empirical sizes of the relevant out-of-sample tests using P observations, and assume a previous in-sample period of $R = 500$ observations to estimate, by quasi-maximum likelihood, the parameters of the GARCH(1,1) model (fixed forecasting scheme). In contrast to standard backtesting tests, see Escanciano and Olmo (2010a), the results of this table lend support to the hypothesis of no estimation risk for CUSUM-type tests. Similar analysis have been carried out for null hypotheses given by heavy-tailed error distributions. The results are available from the authors upon request.

Tables 2.2 - 2.5 illustrate the power of the different tests at a $\beta = 0.05$ significance level against departures from the null hypothesis. Tables 2.2 and 2.3 consider the non-parametric case and Tables 2.4 and 2.5 the comparison of the LR tests against our U-statistic candidate introduced in Definition 3.

Under the alternative hypothesis, we assume after $k^\star$, with $1 < k^\star < P$, an alternative data generating process

$$Y_t = \frac{a^*}{\sqrt{P}}\sigma(\gamma_0; \Im_{t-1}) + u_t, \tag{2.40}$$

with $a^\star$ a constant. After $k^\star$, this process yields a true VaR process defined by

$$m_\alpha^\star(\theta^\star; \Im_{t-1}) = \frac{a^\star}{\sqrt{P}}\sigma(\gamma_0; \Im_{t-1}) + \sigma(\gamma_0; \Im_{t-1})\Phi_\varepsilon^{-1}(\alpha), \tag{2.41}$$

with $\theta^\star = \theta$ in this case. This change implies that the process $m_\alpha(\theta; \Im_{t-1})$ is misspecified after $k^\star$ since

$$E\left[I(Y_t \leq m_\alpha(\theta; \Im_{t-1})) \mid \Im_{t-1}\right] = \Phi_\varepsilon\left(\Phi_\varepsilon^{-1}(\alpha) - \frac{a^\star}{\sqrt{P}}\right) := \widetilde{\alpha} \quad \text{for } t > k^\star. \tag{2.42}$$

Using a first-order Taylor expansion we know that

$$\Phi_\varepsilon\left(\Phi_\varepsilon^{-1}(\alpha) - \frac{a^\star}{\sqrt{P}}\right) = \alpha - \frac{a^\star\phi\left(\Phi_\varepsilon^{-1}(\alpha)\right)}{\sqrt{P}} + O\left(\frac{1}{\sqrt{P}}\right), \tag{2.43}$$

with $\phi(\cdot)$ the density function of a normal distribution.

In our simulation study we analyze single structural breaks occurring at four fractions of the out-of-sample period: $\tau^\star = 0.05$, 0.10, 0.30 and 0.50, for $P = 100$, 300 and 500. The coverage probabilities are $\alpha = 0.01, 0.05$ and $|a^\star| = 1, 5$. Finally, to avoid in-sample distorting

Table 2.2: Empirical Power (EP)

$\alpha = 0.01$	$M_P^{(CS,O)}$				$M_{P,GHH_{7/16}}^{(CS,O)}$				$M_{P,q_{7/16}^{step}}^{(CS,O)}$			
$\tau^\star$	0.05	0.10	0.30	0.50	0.05	0.10	0.30	0.50	0.05	0.10	0.30	0.50
$\mathbf{a}^\star = \mathbf{1}$,P=100, EP	0.051	0.05	0.05	0.065	0.12	0.131	0.121	0.127	0.146	0.168	0.15	0.158
$\hat{\tau}^\star$	0.402	0.418	0.397	0.379	0.443	0.457	0.431	0.333	0.409	0.442	0.428	0.348
se($\hat{\tau}^\star$)	(0.223)	(0.231)	(0.235)	(0.202)	(0.223)	(0.231)	(0.235)	(0.202)	(0.399)	(0.409)	(0.406)	(0.385)
P=300	0.044	0.039	0.056	0.038	0.129	0.122	0.145	0.137	0.15	0.142	0.167	0.155
$\hat{\tau}^\star$	0.427	0.425	0.4	0.374	0.431	0.376	0.448	0.41	0.441	0.379	0.429	0.414
se($\hat{\tau}^\star$)	(0.2)	(0.222)	(0.222)	(0.21)	(0.200)	(0.222)	(0.222)	(0.21)	(0.441)	(0.421)	(0.429)	(0.435)
P=500	0.036	0.037	0.047	0.046	0.142	0.139	0.139	0.141	0.152	0.155	0.156	0.164
$\hat{\tau}^\star$	0.463	0.472	0.403	0.373	0.407	0.44	0.458	0.394	0.401	0.441	0.43	0.406
se($\hat{\tau}^\star$)	(0.195)	(0.232)	(0.212)	(0.154)	(0.195)	(0.232)	(0.212)	(0.154)	(0.441)	(0.443)	(0.439)	(0.438)
$\mathbf{a}^\star = \mathbf{5}$, P=100, EP	0.008	0.012	0.033	0.047	0.064	0.095	0.094	0.104	0.076	0.118	0.11	0.129
$\hat{\tau}^\star$	0.378	0.235	0.245	0.271	0.209	0.297	0.241	0.188	0.196	0.265	0.237	0.197
se($\hat{\tau}^\star$)	(0.273)	(0.181)	(0.143)	(0.135)	(0.273)	(0.181)	(0.143)	(0.135)	(0.326)	(0.369)	(0.341)	(0.296)
P=300	0.01	0.019	0.037	0.041	0.096	0.103	0.119	0.109	0.102	0.109	0.136	0.127
$\hat{\tau}^\star$	0.343	0.292	0.264	0.328	0.295	0.215	0.296	0.258	0.304	0.224	0.292	0.268
se($\hat{\tau}^\star$)	(0.228)	(0.208)	(0.162)	(0.183)	(0.228)	(0.208)	(0.162)	(0.183)	(0.426)	(0.373)	(0.399)	(0.381)
P=500	0.013	0.01	0.031	0.043	0.119	0.103	0.118	0.121	0.124	0.112	0.138	0.138
$\hat{\tau}^\star$	0.424	0.354	0.279	0.36	0.323	0.26	0.344	0.28	0.322	0.286	0.34	0.279
se($\hat{\tau}^\star$)	(0.198)	(0.247)	(0.137)	(0.142)	(0.198)	(0.247)	(0.137)	(0.142)	(0.435)	(0.423)	(0.424)	(0.395)
$\mathbf{a}^\star = -\mathbf{1}$, P=100, EP	0.12	0.119	0.102	0.102	0.178	0.188	0.16	0.164	0.222	0.234	0.203	0.196
$\hat{\tau}^\star$	0.426	0.437	0.47	0.472	0.521	0.546	0.541	0.453	0.492	0.529	0.523	0.437
se($\hat{\tau}^\star$)	(0.220)	(0.211)	(0.204)	(0.218)	(0.220)	(0.211)	(0.204)	(0.218)	(0.381)	(0.383)	(0.384)	(0.387)
P=300	0.086	0.066	0.084	0.061	0.162	0.153	0.181	0.161	0.191	0.182	0.209	0.187
$\hat{\tau}^\star$	0.455	0.454	0.461	0.47	0.509	0.485	0.535	0.507	0.495	0.476	0.504	0.49
se($\hat{\tau}^\star$)	(0.204)	(0.211)	(0.21)	(0.223)	(0.204)	(0.211)	(0.21)	(0.223)	(0.424)	(0.419)	(0.415)	(0.431)
P=500	0.065	0.068	0.064	0.069	0.153	0.162	0.154	0.175	0.191	0.184	0.173	0.198
$\hat{\tau}^\star$	0.472	0.477	0.429	0.463	0.472	0.506	0.506	0.496	0.454	0.486	0.493	0.494
se($\hat{\tau}^\star$)	(0.178)	(0.200)	(0.183)	(0.198)	(0.178)	(0.200)	(0.183)	(0.198)	(0.427)	(0.426)	(0.438)	(0.436)
$\mathbf{a}^\star = -\mathbf{5}$, P=100, EP	0.459	0.417	0.375	0.335	0.406	0.39	0.348	0.328	0.497	0.476	0.427	0.418
$\hat{\tau}^\star$	0.397	0.405	0.432	0.487	0.481	0.53	0.549	0.556	0.424	0.483	0.499	0.507
se($\hat{\tau}^\star$)	(0.194)	(0.185)	(0.17)	(0.179)	(0.194)	(0.185)	(0.17)	(0.179)	(0.298)	-0.313	-0.295	(0.308)
P=300	0.256	0.219	0.227	0.218	0.268	0.244	0.267	0.277	0.357	0.317	0.335	0.338
$\hat{\tau}^\star$	0.444	0.462	0.441	0.498	0.552	0.58	0.592	0.615	0.501	0.529	0.545	0.572
se($\hat{\tau}^\star$)	(0.19)	(0.177)	(0.167)	(0.161)	(0.19)	(0.177)	(0.167)	(0.161)	(0.35)	(0.356)	(0.354)	(0.347)
P=500	0.185	0.193	0.164	0.2	0.229	0.225	0.219	0.256	0.293	0.29	0.274	0.306
$\hat{\hat{\tau}}^\star$	0.448	0.421	0.46	0.493	0.557	0.571	0.603	0.594	0.499	0.513	0.565	0.564
se($\hat{\tau}^\star$)	(0.156)	(0.157)	(0.152)	-0.165	(0.156)	(0.157)	(0.152)	(0.165)	(0.379)	(0.372)	(0.382)	(0.365)

Table 2.3: Empirical Power (EP)

$\alpha = 0.05$	$M_P^{(CS,O)}$				$M_{P,GHH_{7/16}}^{(CS,O)}$				$M_{P,q_{7/16}^{step}}^{(CS,O)}$			
$\tau^\star$	0.05	0.10	0.30	0.50	0.05	0.10	0.30	0.50	0.05	0.10	0.30	0.50
$\mathbf{a}^\star = \mathbf{1}$, P=100, EP	0.024	0.032	0.029	0.035	0.111	0.120	0.114	0.116	0.135	0.140	0.140	0.138
$\hat{\tau}^\star$	0.493	0.411	0.354	0.369	0.405	0.530	0.427	0.450	0.404	0.507	0.417	0.424
se($\hat{\tau}^\star$)	(0.224)	(0.145)	(0.189)	(0.136)	(0.224)	(0.145)	(0.189)	(0.136)	(0.435)	(0.446)	(0.439)	(0.443)
P=300	0.033	0.019	0.042	0.032	0.128	0.118	0.117	0.104	0.144	0.143	0.145	0.124
$\hat{\tau}^\star$	0.435	0.408	0.436	0.388	0.472	0.419	0.421	0.462	0.464	0.421	0.409	0.420
se($\hat{\tau}^\star$)	(0.168)	(0.191)	(0.175)	(0.182)	(0.168)	(0.191)	(0.175)	(0.182)	(0.463)	(0.463)	(0.440)	(0.453)
P=500	0.034	0.027	0.04	0.036	0.141	0.141	0.114	0.116	0.152	0.161	0.136	0.136
$\hat{\tau}^\star$	0.445	0.448	0.415	0.439	0.505	0.387	0.485	0.455	0.497	0.383	0.450	0.440
se($\hat{\tau}^\star$)	(0.171)	(0.200)	(0.179)	(0.143)	(0.171)	(0.200)	(0.179)	(0.143)	(0.469)	(0.449)	(0.453)	(0.462)
$\mathbf{a}^\star = \mathbf{5}$, P=100, EP	0.002	0.006	0.030	0.046	0.078	0.084	0.091	0.086	0.082	0.100	0.127	0.108
$\hat{\tau}^\star$	0.275	0.175	0.257	0.339	0.192	0.337	0.238	0.233	0.195	0.304	0.218	0.219
se($\hat{\tau}^\star$)	(0.078)	(0.023)	(0.105)	(0.124)	(0.078)	(0.023)	(0.105)	(0.124)	(0.371)	(0.425)	(0.348)	(0.351)
P=300	0.007	0.007	0.047	0.071	0.101	0.098	0.105	0.096	0.111	0.114	0.135	0.124
$\hat{\tau}^\star$	0.305	0.279	0.311	0.383	0.311	0.243	0.294	0.314	0.309	0.247	0.265	0.287
se($\hat{\tau}^\star$)	(0.195)	(0.218)	(0.126)	(0.131)	(0.195)	(0.218)	(0.126)	(0.131)	(0.444)	(0.409)	(0.384)	(0.386)
P=500	0.014	0.01	0.058	0.087	0.113	0.125	0.103	0.110	0.119	0.132	0.141	0.147
$\hat{\tau}^\star$	0.447	0.355	0.289	0.393	0.358	0.25	0.326	0.361	0.367	0.222	0.257	0.335
se($\hat{\tau}^\star$)	(0.182)	(0.181)	(0.112)	(0.109)	(0.182)	(0.181)	(0.112)	(0.109)	(0.466)	(0.393)	(0.359)	(0.404)
$\mathbf{a}^\star = -\mathbf{1}$, P=100, EP	0.059	0.076	0.057	0.068	0.140	0.152	0.146	0.156	0.180	0.184	0.183	0.193
$\hat{\tau}^\star$	0.444	0.461	0.49	0.479	0.501	0.597	0.556	0.569	0.498	0.564	0.533	0.558
se($\hat{\tau}^\star$)	(0.203)	(0.188)	(0.178)	(0.169)	(0.203)	(0.188)	(0.178)	(0.169)	(0.427)	(0.417)	(0.426)	(0.422)
P=300	0.07	0.05	0.058	0.05	0.143	0.137	0.134	0.124	0.170	0.162	0.161	0.142
$\tau^\star$	0.458	0.467	0.441	0.496	0.513	0.482	0.484	0.541	0.508	0.497	0.480	0.520
se($\hat{\tau}^\star$)	(0.168)	(0.162)	(0.153)	(0.200)	(0.168)	(0.162)	(0.153)	(0.200)	(0.443)	(0.448)	(0.432)	(0.446)
P=500	0.052	0.052	0.054	0.06	0.159	0.164	0.126	0.143	0.173	0.171	0.154	0.165
$\hat{\tau}^\star$	0.421	0.456	0.45	0.477	0.574	0.459	0.557	0.539	0.529	0.446	0.503	0.525
se($\hat{\tau}^\star$)	(0.165)	(0.170)	(0.161)	(0.144)	(0.165)	(0.170)	(0.161)	(0.144)	(0.455)	(0.440)	(0.435)	(0.438)
$\mathbf{a}^\star = -\mathbf{5}$, P=100, EP	0.292	0.305	0.388	0.392	0.315	0.307	0.346	0.370	0.383	0.401	0.440	0.468
$\hat{\tau}^\star$	0.419	0.416	0.384	0.442	0.600	0.643	0.562	0.587	0.526	0.547	0.478	0.536
se($\hat{\tau}^\star$)	(0.182)	(0.174)	(0.141)	(0.132)	(0.182)	(0.174)	(0.141)	(0.132)	(0.341)	(0.340)	(0.314)	(0.308)
P=300	0.201	0.158	0.269	0.289	0.207	0.203	0.246	0.257	0.246	0.222	0.326	0.343
$\hat{\tau}^\star$	0.447	0.432	0.396	0.450	0.555	0.578	0.539	0.607	0.525	0.507	0.485	0.539
se($\hat{\tau}^\star$)	(0.181)	(0.161)	(0.162)	(0.135)	(0.181)	(0.161)	(0.162)	(0.135)	(0.344)	(0.379)	(0.338)	(0.320)
P=500	0.123	0.138	0.235	0.269	0.200	0.226	0.211	0.237	0.168	0.197	0.247	0.289
$\hat{\tau}^\star$	0.433	0.463	0.385	0.433	0.614	0.565	0.548	0.560	0.535	0.540	0.469	0.554
se($\hat{\tau}^\star$)	(0.160)	(0.174)	(0.153)	(0.125)	(0.160)	(0.174)	(0.153)	(0.125)	(0.372)	(0.374)	(0.321)	(0.305)

Table 2.4: Empirical Power (EP)

$\alpha = 0.01$	LR				$\sup F(\tau)$				$M^{(CS,O)}_{P,q^{step}_{7/16}}$			
$\tau^\star$	0.05	0.10	0.30	0.50	0.05	0.10	0.30	0.50	0.05	0.10	0.30	0.50
$\mathbf{a}^\star = \mathbf{1}$,P=100, EP	0.086	0.140	0.101	0.094	0.000	0.050	0.100	0.100	0.146	0.168	0.151	0.158
$\hat{\tau}^\star$					0.191	0.218	0.198	0.205	0.409	0.442	0.428	0.348
se($\hat{\tau}^\star$)					(0.240)	(0.260)	(0.246)	(0.255)	(0.399)	(0.409)	(0.406)	(0.385)
P=300	0.083	0.144	0.117	0.110	0.010	0.101	0.110	0.140	0.152	0.142	0.167	0.155
$\hat{\tau}^\star$					0.326	0.291	0.271	0.295	0.441	0.379	0.429	0.414
se($\hat{\tau}^\star$)					(0.230)	(0.247)	(0.248)	(0.247)	(0.441)	(0.421)	(0.429)	(0.435)
P=500	0.07	0.121	0.095	0.096	0.020	0.064	0.070	0.078	0.152	0.155	0.156	0.164
$\hat{\tau}^\star$					0.361	0.347	0.334	0.295	0.401	0.441	0.43	0.406
se($\hat{\tau}^\star$)					(0.234)	(0.233)	(0.238)	(0.228)	(0.441)	(0.443)	(0.439)	(0.438)
$\mathbf{a}^\star = \mathbf{5}$, P=100, EP	0.087	0.604	0.496	0.403	0.020	0.083	0.089	0.124	0.076	0.118	0.11	0.129
$\hat{\tau}^\star$					0.054	0.059	0.086	0.093	0.196	0.265	0.237	0.197
se($\hat{\tau}^\star$)					(0.150)	(0.162)	(0.176)	(0.161)	(0.326)	(0.369)	(0.341)	(0.296)
P=300	0.097	0.531	0.439	0.304	0.010	0.091	0.067	0.090	0.102	0.109	0.136	0.127
$\hat{\tau}^\star$					0.253	0.238	0.235	0.235	0.304	0.224	0.292	0.268
se($\hat{\tau}^\star$)					(0.239)	(0.257)	(0.250)	(0.232)	(0.426)	(0.373)	(0.399)	(0.381)
P=500	0.055	0.483	0.345	0.242	0.010	0.086	0.075	0.092	0.124	0.112	0.138	0.138
$\hat{\tau}^\star$					0.307	0.315	0.271	0.236	0.322	0.286	0.34	0.279
se($\hat{\tau}^\star$)					(0.239)	(0.256)	(0.236)	(0.197)	(0.435)	(0.423)	(0.424)	(0.395)
$\mathbf{a}^\star = -\mathbf{1}$, P=100, EP	0.07	0.042	0.04	0.051	0.000	0.048	0.037	0.110	0.222	0.234	0.203	0.196
$\hat{\tau}^\star$					0.426	0.448	0.501	0.402	0.492	0.529	0.523	0.437
se($\hat{\tau}^\star$)					(0.232)	(0.190)	(0.216)	(0.203)	(0.381)	(0.383)	(0.384)	(0.387)
P=300	0.084	0.046	0.062	0.078	0.031	0.050	0.056	0.078	0.191	0.182	0.209	0.187
$\hat{\tau}^\star$					0.522	0.498	0.506	0.528	0.495	0.476	0.504	0.49
se($\hat{\tau}^\star$)					(0.214)	(0.226)	(0.210)	(0.198)	(0.424)	(0.419)	(0.415)	(0.431)
P=500	0.069	0.052	0.062	0.047	0.030	0.113	0.090	0.120	0.191	0.184	0.173	0.198
$\hat{\tau}^\star$					0.472	0.506	0.506	0.496	0.454	0.486	0.493	0.494
se($\hat{\tau}^\star$)					(0.188)	(0.180)	(0.196)	(0.201)	(0.427)	(0.426)	(0.438)	(0.436)
$\mathbf{a}^\star = -\mathbf{5}$, P=100, EP	0.083	0.002	0.014	0.018	0.057	0.107	0.155	0.172	0.497	0.476	0.427	0.418
$\hat{\tau}^\star$					0.325	0.371	0.387	0.424	0.424	0.483	0.499	0.507
se($\hat{\tau}^\star$)					(0.232)	(0.224)	(0.206)	(0.227)	(0.298)	(-0.313)	(-0.295)	(0.308)
P=300	0.081	0.005	0.012	0.029	0.010	0.062	0.190	0.198	0.357	0.317	0.335	0.338
$\hat{\tau}^\star$					0.354	0.349	0.375	0.404	0.501	0.529	0.545	0.572
se($\hat{\tau}^\star$)					(0.211)	(0.217)	(0.211)	(0.216)	(0.35)	(0.356)	(0.354)	(0.347)
P=500	0.068	0.056	0.043	0.04	0.000	0.047	0.159	0.162	0.293	0.290	0.274	0.306
$\hat{\hat{\tau}}^\star$					0.367	0.348	0.395	0.374	0.499	0.513	0.565	0.564
se($\hat{\tau}^\star$)					(0.221)	(0.225)	(0.221)	(0.223)	(0.379)	(0.372)	(0.382)	(0.365)

Table 2.5: Empirical Power (EP)

$\alpha = 0.05$	LR				$\sup F(\tau)$				$M^{(CS,O)}_{P,q^{step}_{7/16}}$			
$\tau^\star$	0.05	0.10	0.30	0.50	0.05	0.10	0.30	0.50	0.05	0.10	0.30	0.50
$\mathbf{a}^\star = \mathbf{1}$, P=100, EP	0.068	0.142	0.129	0.116	0.020	0.051	0.049	0.036	0.135	0.140	0.140	0.138
$\hat{\tau}^\star$					0.325	0.325	0.324	0.326	0.404	0.507	0.417	0.424
se($\hat{\tau}^\star$)					(0.230)	(0.227)	(0.216)	(0.227)	(0.435)	(0.446)	(0.439)	(0.443)
P=300	0.061	0.111	0.101	0.088	0.020	0.087	0.101	0.096	0.144	0.143	0.145	0.124
$\hat{\tau}^\star$					0.325	0.321	0.330	0.329	0.464	0.421	0.409	0.420
se($\hat{\tau}^\star$)					(0.240)	(0.231)	(0.262)	(0.237)	(0.463)	(0.463)	(0.440)	(0.453)
P=500	0.049	0.088	0.065	0.055	0.020	0.076	0.065	0.068	0.152	0.161	0.136	0.136
$\hat{\tau}^\star$					0.342	0.322	0.356	0.353	0.497	0.383	0.450	0.440
se($\hat{\tau}^\star$)					(0.238)	(0.238)	(0.239)	(0.220)	(0.469)	(0.449)	(0.453)	(0.462)
$\mathbf{a}^\star = \mathbf{5}$, P=100, EP	0.064	0.842	0.708	0.524	0.040	0.112	0.123	0.109	0.082	0.100	0.127	0.108
$\hat{\tau}^\star$					0.272	0.257	0.221	0.262	0.195	0.304	0.218	0.219
se($\hat{\tau}^\star$)					(0.252)	(0.269)	(0.210)	(0.198)	(0.371)	(0.425)	(0.348)	(0.351)
P=300	0.065	0.755	0.602	0.422	0.021	0.038	0.078	0.091	0.111	0.114	0.135	0.124
$\hat{\tau}^\star$					0.316	0.312	0.214	0.300	0.309	0.247	0.265	0.287
se($\hat{\tau}^\star$)					(0.240)	(0.248)	(0.217)	(0.204)	(0.444)	(0.409)	(0.384)	(0.386)
P=500	0.045	0.657	0.508	0.303	0.010	0.031	0.099	0.120	0.119	0.132	0.141	0.147
$\hat{\tau}^\star$					0.343	0.327	0.268	0.318	0.367	0.222	0.257	0.335
se($\hat{\tau}^\star$)					(0.236)	(0.245)	(0.217)	(0.200)	(0.466)	(0.393)	(0.359)	(0.404)
$\mathbf{a}^\star = -\mathbf{1}$, P=100, EP	0.049	0.03	0.035	0.048	0.023	0.067	0.120	0.132	0.180	0.184	0.183	0.193
$\hat{\tau}^\star$					0.380	0.349	0.374	0.364	0.498	0.564	0.533	0.558
se($\hat{\tau}^\star$)					(0.219)	(0.234)	(0.207)	(0.232)	(0.427)	(0.417)	(0.426)	(0.422)
P=300	0.054	0.057	0.048	0.069	0.020	0.065	0.100	0.111	0.170	0.162	0.161	0.142
$\tau^\star$					0.324	0.348	0.363	0.319	0.508	0.497	0.480	0.520
se($\hat{\tau}^\star$)					(0.247)	(0.238)	(0.244)	(0.226)	(0.443)	(0.448)	(0.432)	(0.446)
P=500	0.056	0.061	0.050	0.058	0.010	0.060	0.072	0.081	0.173	0.171	0.154	0.165
$\hat{\tau}^\star$					0.332	0.340	0.366	0.383	0.529	0.446	0.503	0.525
se($\hat{\tau}^\star$)					(0.231)	(0.227)	(0.228)	(0.228)	(0.455)	(0.440)	(0.435)	(0.438)
$\mathbf{a}^\star = -\mathbf{5}$, P=100, EP	0.059	0.137	0.064	0.044	0.055	0.150	0.124	0.156	0.383	0.401	0.440	0.468
$\hat{\tau}^\star$					0.348	0.374	0.329	0.445	0.526	0.547	0.478	0.536
se($\hat{\tau}^\star$)					(0.237)	(0.236)	(0.199)	(0.186)	(0.341)	(0.340)	(0.314)	(0.308)
P=300	0.057	0.381	0.196	0.119	0.042	0.055	0.134	0.119	0.246	0.222	0.326	0.343
$\hat{\tau}^\star$					0.307	0.379	0.352	0.340	0.525	0.507	0.485	0.539
se($\hat{\tau}^\star$)					(0.224)	(0.248)	(0.228)	(0.197)	(0.344)	(0.379)	(0.338)	(0.320)
P=500	0.041	0.408	0.274	0.151	0.045	0.052	0.138	0.122	0.159	0.197	0.247	0.289
$\hat{\tau}^\star$					0.368	0.330	0.286	0.391	0.535	0.540	0.469	0.554
se($\hat{\tau}^\star$)					(0.229)	(0.222)	(0.216)	(0.199)	(0.372)	(0.374)	(0.321)	(0.305)

effects for the Christoffersen test, we assume the parameter vector θ known and therefore no in-sample period ($R = 0$). The results of this experiment are reported for the family of local alternatives defined by $a^\star = 1, 5$ indicating over-exposure of the null VaR measure, and the most interesting case defined by $a^\star = -1, -5$ indicating under-exposure of the VaR model. $M = 1000$ Monte-Carlo replications were considered in this simulation.

Most of the discussions of the Monte-Carlo exercise will focus on Tables 2.2 and 2.4 as the results there are best and most likely of interest to regulators as well as practitioners due to the choice of $\alpha = 0.01$, the coverage probability recommended for risk management purposes by Basel Committee (1996) in banking supervision. The simulation focuses on unusually small local departures from the null hypothesis coverage probability. In particular, the empirical power for the CUSUM test statistics when $a^\star = 1$ and $P = 100$ details an empirical power of 6%, whereas the test statistic of GHH is successful 13% of the time while the statistic $M^{(CS,O)}_{P,q^{step}_{7/16}}$ has a success rate of 17%; these results hold uniformly on $\tau^\star$. As the out-of-sample size increases this pattern continues: $M^{(CS,O)}_{P,q^{step}_{7/16}}$ detects a change in the quantile process 15% of the time, while the CUSUM's is only able to detect this departure less than 5% of the time; the statistic of GHH is in between these two. The different versions of the LR test, on the other hand, perform poorly. The LR Christoffersen test yields result comparable to the CUSUM test, however, the change point test based on regression model (2.36) hardly exhibits power against deviations from the null hypothesis. Table 2.6 will show that this method improves considerably for larger departures from the null hypothesis. Nevertheless, even for these alternatives the method is not able to capture failures of the risk model early in the evaluation period.

In contrast to what intuition suggests, allowing for a larger positive departure from the null process, *i.e.* $a^\star = 5$, entails falls in the empirical power for all test statistics entertained (below 5% for the CUSUM and to around 12% for $M^{(CS,O)}_{P,q^{step}_{7/16}}$) in this simulation. A possible explanation for this reduction for alternatives determined by a positive a could be that in these cases $m_\alpha(\theta;\cdot)$ yields a very conservative VaR process implying no violations of the risk model, and in turn, no meaningful observations for the test statistic. Asymptotically, however, Theorem 2.2.2 shows that the different tests are consistent against this family of alternatives as well. For studying the power of alternatives defined by negative values of a the conclusions are rather opposed. The alternative VaR model in this case under-estimates risk exposure. We observe in this case that even for small departures characterized by $a^\star = -1$, all test statistics see an improved ability to detect the change in the risk model. The statistic that performs best is $M^{(CS,O)}_{P,q^{step}_{7/16}}$ which exhibits a power of 20%. The remaining two statistics have empirical power in the range of 15% and 8% for GHH's statistic and the CUSUM respectively. As intuition suggests, when $a^\star = -5$ the power of all statistics rises: the CUSUM detects this departure 40% to 45% of the time, while $M^{(CS,O)}_{P,q^{step}_{7/16}}$ detects this departure 50%. The results in Tables 2.3 and 2.5 for a coverage probability $\alpha = 0.05$ are consistent with these findings. Interestingly, the results in the tables are consistent with the formula for the asymptotic power obtained in Theorem 2.2.2. Thus, we observe that as $\tau^\star$ approaches the middle of the sample the power increases in all cases. More importantly for regulatory purposes, and as stated in (2.31), the power of the test increases for $\alpha = 0.01$ compared to $\alpha = 0.05$. This is an interesting result that provides further evidence about the importance of these change points as opposed to standard Christoffersen and Kupiec type tests. Escanciano and Olmo (2010a,b) show that the $\alpha = 0.01$ case is rather problematic when approximating the correct finite-sampling and asymptotic distributions of the LR test statistic.

For the analysis of the location of the break parameter it is fairly obvious that the results obtained are very poor, this being confirmed by estimates close to 0.5 and large standard errors. Our conjecture for such poor results is that the family of local alternatives studied is very close to the null hypothesis. In order to see if the different tests in (2.18) are able to capture the location of the break, we have simulated an alternative hypothesis defined by $a^\star = -30$ for $\alpha = 0.01$. Table 2.6 confirms this conjecture by showing that for $P = 100$ the estimates of the location, defined by $k^\star = \lceil \tau^\star P \rceil$, report accurately the theoretical location of the break. More importantly, the estimates of $\tau^\star$ based on $M^{(CS,O)}_{P,q^{step}_{7/16}}$ are very close to the actual location of the break when this occurs early on in the evaluation period. This is in contrast to standard CUSUM and LR tests; Table 2.6 provides clear evidence of their failure to detect these early departures from the null coverage probability.

2.4 Empirical Application

A very popular technique to monitor risk is the use of GARCH type models. These methods are widely implemented in statistical software packages and are simple to estimate, produce Value-at-Risk forecasts and to interpret. This section analyzes in particular the usefulness of the GARCH(1,1) model to gauge risk in an out-of-sample exercise for equity and commodities data, in particular, the Dow-Jones Industrial Average (DJIA) Index and the Commodity Index Report from the Commodity Research Bureau Inc. (CRB), over the period August 2002 to May 2010. The choice of these markets and period is due to our interest in observing whether the current financial crisis has had similar effects in both markets and whether risk models valid before 2007 were still of use after it. We use a rolling scheme to analyze the data, that is, data is divided into six different periods subsequently subdivided into an in-sample period where model parameters are estimated and an out-of-sample period to evaluate the model. The total number of observations is 2038 for DJIA and 2021 for CRB. Thus, the first period considers the first 1000 observations to estimate by QML a GARCH(1,1) model with Student-t innovations. The test statistics are computed using the following 500 observations. The sample period under study is rolled over 100 observations and the experiment is run again[1]. For the second window, the in-sample period considers observations between 100 and

[1] In the literature on backtesting, there is no definitive answer to the size of the rolling window to use in practical applications. The size of the rolling window is sometime selected so not to require frequent re-estimation of the VaR model. Using a smaller roll increases the number of model estimation which uses computing resources and takes additional time. This, in part, contributed to selecting 100 as the roll.

Table 2.6: Empirical Power (EP)

$\alpha = 0.01$	$\sup F(\tau)$			$M_P^{(CS,O)}$			$M_{P,GHH_{7/16}}^{(CS,O)}$			$M_{P,q_{7/16}^{step}}^{(CS,O)}$		
$\tau^\star$	0.05	0.10	0.50	0.05	0.10	0.50	0.05	0.10	0.50	0.05	0.10	0.50
$\mathbf{a}^\star = -\mathbf{30}$, P=100, EP	0.200	0.900	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
$\tau^\star$	0.163	0.044	0.355	0.253	0.198	0.505	0.071	0.113	0.506	0.082	0.113	0.506
se($\hat{\tau}^\star$)	(0.209)	(0.065)	(0.013)	(0.200)	(0.122)	(0.013)	(0.200)	(0.122)	(0.013)	(0.070)	(0.033)	(0.015)
P=300	0.000	0.150	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
$\tau^\star$	0.261	0.133	0.379	0.403	0.268	0.516	0.233	0.166	0.520	0.240	0.164	0.520
se($\hat{\tau}^\star$)	(0.255)	(0.176)	(0.050)	(0.225)	(0.170)	(0.024)	(0.225)	(0.170)	(0.024)	(0.260)	(0.128)	(0.034)
P=500	0.000	0.050	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
$\tau^\star$	0.313	0.134	0.378	0.412	0.283	0.518	0.292	0.193	0.523	0.280	0.186	0.523
se($\hat{\tau}^\star$)	(0.246)	(0.193)	(0.049)	(0.230)	(0.182)	(0.028)	(0.230)	(0.182)	(0.028)	(0.290)	(0.165)	(0.042)

1100 and for the out-of-sample the following 500 observations. The choice of this method is to be consistent with common practice in the risk management industry. Practitioners reevaluate periodically their risk management models to gain robustness against structural breaks in the model parameters producing failures of the risk model. It is therefore very important to have in place mechanisms that allow a risk manager to detect early the failure of the risk model even after automatic periodic readjustment of the models.

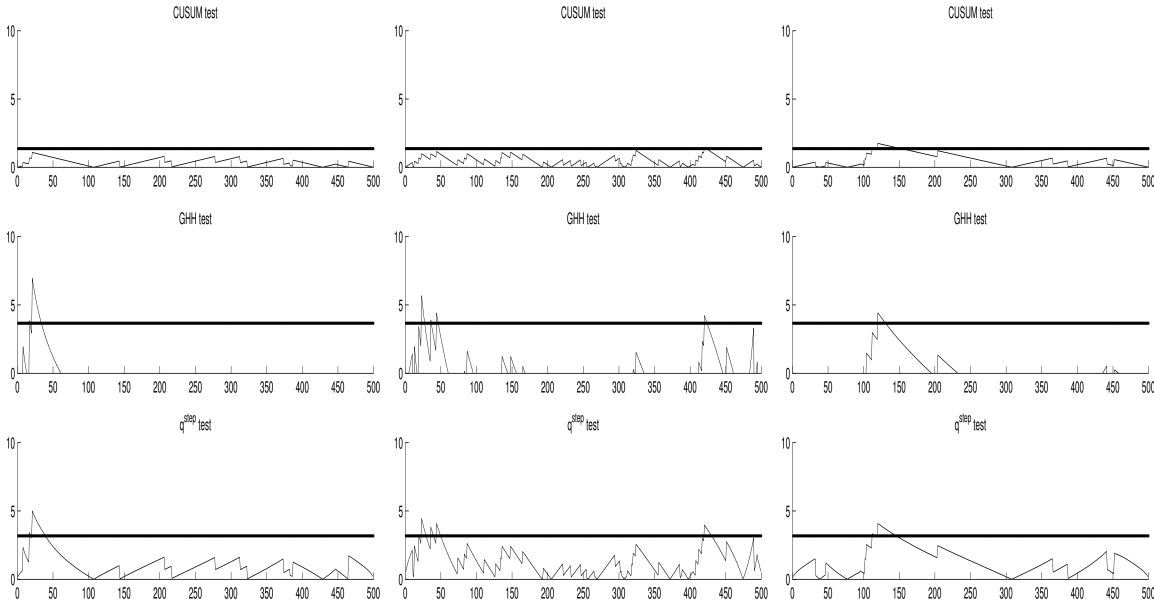
Our aim in this section is to detect the timing of the breakdown of the GARCH(1,1) risk model as soon as it occurs. In order to do this we implement the standard CUSUM test based on $M_P^{(CS,O)}$, and two U-statistic type processes defined by the GHH weight function, $M_{P,GHH_{7/16}}^{(CS,O)}$, and the refinement discussed in (2.34) and denoted $M_{P,q_{7/16}^{step}}^{(CS,O)}$. Table 2.7 reports the results. The main findings of this table are threefold. First, the location of breaks is more concentrated for the commodities market than for the DJIA index. This implies that whereas our method detects four clearly differentiated breaks for the equity index, it only detects one or two at most for the commodities index. The transmission mechanism is not clear. Some of these breaks affect first the equity index and are later transmitted to the commodities market, but the reverse effect is also observed in the last period. The test statistic based on $M_{P,q_{7/16}^{step}}^{(CS,O)}$ detects deviations very early in the evaluation period. The CUSUM test does capture these breaks though. Finally, the failure of the risk models is different for $\alpha = 0.01$ than for $\alpha = 0.05$. This is more significant in magnitude for those periods in which both processes fail.

Figures 2.2 and 2.3 also show the dynamics of the different U-statistic processes over the evaluation period for three different sub-periods. For the equity market we report the periods that show financial distress, that is, periods 1, 4 and 6. Since the breaks occur early (within the first 100 observations in the out-of-sample period) the automatic readjustment of the model obtained from rolling the window is sufficient to absorb the break and produce a new risk model under control. Thus, after period 1 the next break is in period 4 and the

Table 2.7: Estimation of model and location of break

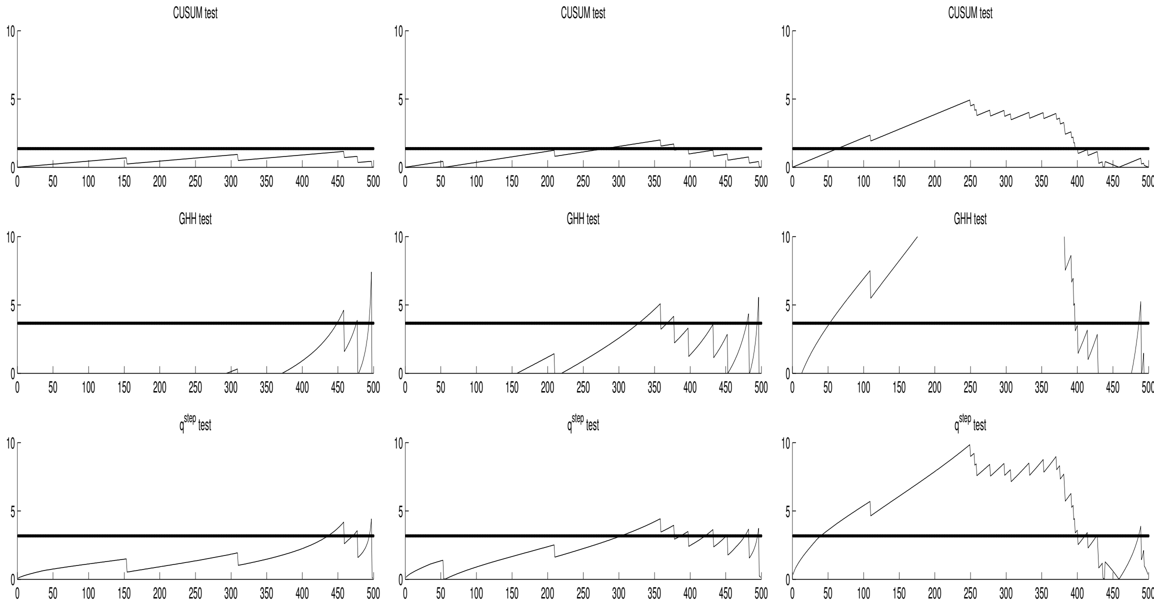
	DJIA Index						Commodity Index					
	$\alpha = 0.01$											
	$\hat{\beta}_0$	$\hat{\beta}_1$	$\hat{\beta}_2$	$M_P^{(CS,O)}$	$M_{P,GHH_{7/16}}^{(CS,O)}$	$M_{P,q_{7/16}^{step}}^{(CS,O)}$	$\hat{\beta}_0$	β_1	$\hat{\beta}_2$	$M_P^{(CS,O)}$	$M_{P,GHH_{7/16}}^{(CS,O)}$	$M_{P,q_{7/16}^{step}}^{(CS,O)}$
1	0.006	0.053	0.937	0	1017	1017	0.002	0.034	0.953	0	1449	1437
2	0.008	0.048	0.937	0	0	0	0.002	0.031	0.955	1380	1428	1404
3	0.009	0.049	0.930	0	0	1217	0.023	0.052	0.818	1264	1253	1240
4	0.032	0.040	0.884	1720	1323	1319	0.044	0.035	0.724	1362	1354	1347
5	0.017	0.051	0.916	0	0	0	0.049	0.043	0.697	1447	1430	1425
6	0.010	0.052	0.931	1612	1620	1612	0.010	0.024	0.929	1592	1595	1592
	$\alpha = 0.05$											
1	0.006	0.053	0.937	1180	1008	1008	0.002	0.034	0.953	0	0	0
2	0.008	0.048	0.937	1199	1237	1174	0.002	0.031	0.955	1210	1264	1205
3	0.008	0.049	0.930	0	0	0	0.023	0.052	0.818	1364	1383	1300
4	0.032	0.040	0.884	0	1301	1326	0.045	0.036	0.724	1394	1310	1310
5	0.016	0.051	0.916	1620	1717	1624	0.049	0.043	0.697	1615	1623	1441
6	0.010	0.052	0.931	1620	1624	1620	0.010	0.024	0.929	1605	1605	1604

Figure 2.2: DJIA data for period April 2002-May 2010. Dynamics of weighted U-statistics for periods 1, 4 and 6.



last in period 6. On the other hand, the study of the commodity market reveals a different picture. The distress periods in this case are 1,2 and 3. Figure 2.3 clearly shows how the q^{step} U-statistic starts early picking up the break; as the break becomes more evident (further sub-periods) the other methods reflect it as well. It is noteworthy observing that the re-estimation of the GARCH process is not sufficient to absorb the break until the in-sample evaluation period contains the breaking period. Thus, for the sub-periods 4, 5 and 6 the risk model is again under control. These periods are not reported in the figure but the effect is reflected in the location of breaks in Table 2.7 and significant changes in the parameter estimates of the GARCH(1,1) model.

Figure 2.3: Commodity index data for period April 2002-May 2010. Dynamics of weighted U-statistics for periods 1, 2 and 3.



2.5 Conclusion

Backtesting techniques are of paramount importance for risk managers and regulators concerned with assessing the risk exposure of a financial institution to market risk. We have shown in this paper that by combining the standard backtesting Kupiec test statistic computed over different subsamples of the evaluation period one can develop alternative backtesting procedures that not only allow detection of deviations of the risk model from the actual risk exposure but also the calculation of the timing of these departures. Also, the paper concludes that weighted versions of U-statistic type tests exhibit more power to detect the presence of breaks in the conditional VaR model when these occur early on in the out-of-sample evaluation period. In particular, the novel family of U-statistics developed in the paper has proven to be the most powerful test statistic within an extensive group of weight functions existing in the literature. Interestingly, this test is very powerful against deviations for coverage probabilities of $\alpha = 0.01$ and more importantly for regulatory purposes, for detecting under-exposure of the risk model.

The application shows the existence of different breaks in the risk model for the equity market. A first break in August 2007, followed by two breaks in March and September 2008, and a final break in October 2009. For the commodities index there are three breaks at most, around March 2008, January 2009 and October 2009. These periods coincide with important announcements and news in financial markets worldwide, as the turmoil of October 2008 and 2009. As the theory predicts, the weight function developed here is capable of detecting a break in the underlying risk faced by both markets before the rest of CUSUM methods does. This finding has implications in the choice of the statistical method to detect the break and in the design by risk managers of optimal rolling windows to monitor risk.

2.6 Mathematical Appendix to Chapter 2

Proof of Corollary 2.1.1. Under H_O $\mathbb{E}[I_{t,\alpha}(\theta)] = \alpha$. This will be proved by induction. Set $N = 2$ and consider $I_{1,\alpha}(\theta)$ and $I_{2,\alpha}(\theta)$.

$$\begin{aligned}
\mathbb{P}\left\{I_{1,\alpha}(\theta) \leq a_1, I_{2,\alpha}(\theta) \leq a_2\right\} &= \mathbb{E}\{1(I_{1,\alpha}(\theta) \leq a_1, I_{2,\alpha}(\theta) \leq a_2)\} && (2.44)\\
&= \mathbb{E}\{\mathbb{E}\{1(I_{1,\alpha}(\theta) \leq a_1, I_{2,\alpha}(\theta) \leq a_2)|\Im_1\}\} && (2.45)\\
&= \mathbb{E}\{\mathbb{E}\{1(I_{1,\alpha}(\theta) \leq a_1)1(I_{2,\alpha}(\theta) \leq a_2)|\Im_1\}\} && (2.46)\\
&= \mathbb{E}\{\mathbb{E}\{1(I_{2,\alpha}(\theta) \leq a_2)|\Im_1\}1(I_{1,\alpha}(\theta) \leq a_1)\} && (2.47)\\
&= \mathbb{P}\{I_{2,\alpha}(\theta) \leq a_2\}\mathbb{E}\{1(I_{1,\alpha}(\theta) \leq a_1)\} && (2.48)\\
&= \mathbb{P}\{I_{2,\alpha}(\theta) \leq a_2\}\mathbb{P}\{I_{1,\alpha}(\theta) \leq a_1\}. && (2.49)
\end{aligned}$$

Equation (2.44) is a basic result from probability theory, equation (2.45) follows form the law of iterative expectations; equation (2.46) follows from the result that $1(\{A \bigcap B\}) = 1(\{A\})1(\{B\})$ - for a proof of this see Goldberg (1976), page 15. Equation (2.47) follows from Theorem 34.3 Billingsley (1995), page 447 and recognizing that $I_{1,\alpha}(\theta)$ is measurable with respect to $\Im_j$ for $j = 1, 2$. Equation (2.48) follows since

$$\mathbb{P}\{I_{2,\alpha}(\theta) \leq a_2 | \Im_1\} = \begin{cases} 1-\alpha, & a_2 < 1 \\ 1, & a_2 \geq 1. \end{cases} \tag{2.50}$$

This result implies the following:

$$\mathbb{E}\{\mathbb{E}\{I_{2,\alpha}(\theta) \leq a_2 | \Im_1\} 1(I_{1,\alpha}(\theta) \leq a_1)\} = \mathbb{P}\{I_{2,\alpha}(\theta) \leq a_2 | \Im_1\} \mathbb{P}\{I_{1,\alpha}(\theta) \leq a_1\}. \tag{2.51}$$

But $\mathbb{P}\{I_{2,\alpha}(\theta) \leq a_2 | \Im_1\} = \mathbb{P}\{I_{2,\alpha}(\theta) \leq a_2\}$ since $\mathbb{P}\{I_{2,\alpha}(\theta) \leq a_2 | \Im_1\}$ depends upon α only. Hence taking expectations of the latter term, *i.e.* $\mathbb{E}\{\mathbb{P}\{I_{2,\alpha}(\theta) \leq a_2 | \Im_1\}\}$ leads to;

$$\mathbb{P}\{I_{2,\alpha}(\theta) \leq a_2\} = \begin{cases} 1-\alpha, & a_2 < 1 \\ 1, & a_2 \geq 1. \end{cases} \tag{2.52}$$

A similar statement holds for $\mathbb{E}\{1(I_{1,\alpha}(\theta) \leq a_1)\}$, i.e.,

$$\mathbb{P}\{I_{1,\alpha}(\theta) \leq a_1\} = \mathbb{E}\{1(I_{1,\alpha}(\theta) \leq a_1)\} = \mathbb{E}\{\mathbb{E}\{1(I_{1,\alpha}(\theta) \leq a_1) | \Im_0\}\} = \begin{cases} 1-\alpha, & a_2 < 1 \\ 1, & a_2 \geq 1. \end{cases} \tag{2.53}$$

This establishes equation (2.49) and the independence of $I_{1,\alpha}$ and $I_{2,\alpha}$ follows.

Assume that $I_{1,\alpha}(\theta), \ldots, I_{N-1,\alpha}(\theta)$ are independent, we show this holds for $N = P$.

$$\begin{aligned} \mathbb{P}\{I_{1,\alpha}(\theta) \leq a_1, \ldots, I_{P,\alpha}(\theta) \leq a_P\} &= \mathbb{E}\{1(I_{1,\alpha}(\theta) \leq a_1, \ldots, I_{P,\alpha}(\theta) \leq a_P)\} \\ &= \mathbb{E}\{1(I_{1,\alpha}(\theta) \leq a_1, \ldots, I_{P-1,\alpha}(\theta) \leq a_{P-1}) 1(I_{P,\alpha}(\theta) \leq a_P)\} \\ &= \mathbb{E}\{1(I_{1,\alpha}(\theta) \leq a_1, \ldots, I_{P-1,\alpha}(\theta) \leq a_{P-1})\} \mathbb{E}\{I_{P,\alpha}(\theta) \leq a_P | \Im_{P-1}\} \\ &= \mathbb{P}\{I_{1,\alpha}(\theta) \leq a_1\} \cdots \mathbb{P}\{I_{P-1,\alpha}(\theta) \leq a_{P-1}\} \mathbb{P}\{I_{P,\alpha}(\theta) \leq a_P\}. \end{aligned}$$

This establishes the independence of the sequence of indicator functions.

Proof of Proposition 2.1.2. Under H_O, by Corollary 1, the sequence $I_{t,\alpha}(\theta)$ for $t = 1, \ldots, P$ are IID random variables. In the case of the fixed, recursive and rolling forecasting schemes both R and P need to go to infinity. The proof is as follows.

$$
\begin{aligned}
\sup_{0<\tau<1} \frac{\left|\bar{M}_n(t_1,t_2) - M_n(t_1,t_2)\right|}{q(\tau)} &\leq \sup_{0<\tau<\frac{1}{P+1}} \frac{\left|\bar{M}_n(t_1,t_2) - M_n(t_1,t_2)\right|}{q(\tau)} \\
&\quad + \sup_{\frac{1}{P+1}<\tau<\frac{P}{P+1}} \frac{\left|\bar{M}_n(t_1,t_2) - M_n(t_1,t_2)\right|}{q(\tau)} + \sup_{\frac{P}{P+1}<\tau<1} \frac{\left|\bar{M}_n(t_1,t_2) - M_n(t_1,t_2)\right|}{q(\tau)} \\
&= J_1(P,R) + J_2(P,R) + J_3(P,R) \qquad (2.54) \\
&= o_P(1), \qquad R, P \to \infty. \qquad (2.55)
\end{aligned}
$$

The result claimed in (2.55) can be established by verifying that each term in (2.54) is $o_P(1)$, as $R, P \to \infty$. Let $\{\widehat{\theta}_{t,R}\}_{t=1}^{P}$ be a sequence of consistent estimators for θ. We note that $I_{t,\alpha}(\hat{\theta}_{t,R})$ converges in probability to $I_{t,\alpha}(\theta)$ as $R \to \infty$, at all points of continuity.

We first consider $J_2(P,R) = o_P(1)$, as $R, P \to \infty$. For every $\epsilon > 0$,

$$
\begin{aligned}
\mathbb{P}\left\{ \frac{|\widehat{M}_P^{(CS,O)}(\tau) - M_P^{(CS,O)}(\tau)|}{q(\tau)} > \epsilon \right\} &\leq \mathbb{P}\left\{ P^{-1/2} \frac{|\sum_{t=1}^{\lceil \tau P \rceil} I_{t,\alpha}(\widehat{\theta}_{t,R}) - \sum_{t=1}^{\lceil \tau P \rceil} I_{t,\alpha}(\theta)|}{q(\tau)} > \frac{\epsilon}{2} \right\} \qquad (2.56) \\
&\quad + \mathbb{P}\left\{ P^{-1/2} \frac{\tau|\sum_{t=1}^{P} I_{t,\alpha}(\widehat{\theta}_{t,R}) - \sum_{t=1}^{P} I_{t,\alpha}(\theta)|}{q(\tau)} > \frac{\epsilon}{2} \right\} \\
&\leq \frac{8}{\epsilon} \mathbb{E} \sum_{t=1}^{P} |I_{t,\alpha}(\widehat{\theta}_{t,R}) - I_{t,\alpha}(\theta)| \sup_{\frac{1}{P+1}<\tau<\frac{P}{P+1}} \frac{\tau^{1/2}}{q(\tau)} \qquad (2.57)
\end{aligned}
$$

The inequality (2.56) follows from basic results in probability theory, while (2.57) follows from Markov's inequality. (2.57) implies immediately the following inequality:

$$\lim_{P\to\infty}\lim_{R\to\infty}\mathbb{P}\left\{\sup_{\frac{1}{P+1}<\tau<\frac{P}{P+1}}\frac{|\widehat{M}_P^{(CS,O)}(\tau)\ -\ M_P^{(CS),O}(\tau)|}{q(\tau)}\ >\ \epsilon\right\}\ \leq$$

$$\frac{8}{\epsilon}\lim_{P\to\infty}\lim_{R\to\infty}\left[\mathbb{E}\left[\sum_{t=1}^{P}|I_{t,\alpha}(\widehat{\theta}_{t,R})-I_{t,\alpha}(\theta)|\right]\sup_{\frac{1}{P+1}<\tau<\frac{P}{P+1}}\frac{\tau^{1/2}}{q(\tau)}\right]=$$

$$\frac{8}{\epsilon}\lim_{P\to\infty}\lim_{R\to\infty}\left[\mathbb{E}\left[\sum_{t=1}^{P}|I_{t,\alpha}(\widehat{\theta}_{t,R})-I_{t,\alpha}(\theta)|\right]\sup_{\frac{1}{P+1}<\tau<\frac{P}{P+1}}\frac{\tau^{1/2}}{q(\tau)}\right]=$$

$$\frac{8}{\epsilon}\lim_{P\to\infty}\lim_{R\to\infty}\left[\sum_{t=1}^{P}\mathbb{E}|I_{t,\alpha}(\widehat{\theta}_{t,R})-I_{t,\alpha}(\theta)|\lim_{P\to\infty}\sup_{\frac{1}{P+1}<\tau<\frac{P}{P+1}}\frac{\tau^{1/2}}{q(\tau)}\right]= \tag{2.58}$$

$$\frac{8}{\epsilon}\left[\lim_{P\to\infty}\left[\sum_{t=1}^{P}\lim_{R\to\infty}\mathbb{E}|I_{t,\alpha}(\widehat{\theta}_{t,R})-I_{t,\alpha}(\theta)|\right]\sup_{\frac{1}{P+1}<\tau<\frac{P}{P+1}}\frac{\tau^{1/2}}{q(\tau)}\right]= \tag{2.59}$$

$$\frac{8}{\epsilon}\left[\lim_{P\to\infty}\left[\sum_{t=1}^{P}\lim_{R\to\infty}\mathbb{E}|I_{t,\alpha}(\widehat{\theta}_{t,R})-I_{t,\alpha}(\theta)|\right]\lim_{P\to\infty}\sup_{\frac{1}{P+1}<\tau<\frac{P}{P+1}}\frac{\tau^{1/2}}{q(\tau)}\right]\leq$$

$$\frac{8}{\epsilon}\lim_{P\to\infty}\sum_{t=1}^{P}\frac{\epsilon^2}{8C\cdot P}\lim_{P\to\infty}\sup_{\frac{1}{P+1}<\tau<\frac{P}{P+1}}\frac{\tau^{1/2}}{q(\tau)}=$$

$$\frac{8}{\epsilon}\lim_{P\to\infty}P\frac{\epsilon^2}{8C\cdot P}\lim_{P\to\infty}\sup_{\frac{1}{P+1}<\tau<\frac{P}{P+1}}\frac{\tau^{1/2}}{q(\tau)}=$$

$$\frac{\epsilon}{C}\lim_{P\to\infty}\sup_{\frac{1}{P+1}<\tau<\frac{P}{P+1}}\frac{\tau^{1/2}}{q(\tau)}=\epsilon, \tag{2.60}$$

for sufficiently large R and P. $\lim_{R\to\infty}\mathbb{E}|I_{t,\alpha}(\widehat{\theta}_{t,R})-I_{t,\alpha}(\theta)|\ =\ 0$ which follows from the consistency of the sequence of estimators $\{\widehat{\theta}_{t,R}\}_{t=1}^{P}$ and from a proposition in Cohn [Proposition 3.1.5, page 89]. As a result of these statements, it is possible to pick a sufficiently large R such that $\mathbb{E}|I_{t,\alpha}(\widehat{\theta}_{t,R})-I_{t,\alpha}(\theta)|\leq \frac{\epsilon^2}{8C\cdot P}$, where $C=\sup_{0<\tau<1}\frac{\tau^{1/2}}{q(\tau)}$. There remains to argue that the term $\lim_{P\to\infty}\sup_{\frac{1}{P+1}<\tau<\frac{P}{P+1}}\frac{\tau}{q(\tau)}=C$ but this follows along the lines of the argument used in GHH (1996) [equation 3.4, page 155]. With these arguments now provided, we conclude with the result stated in (2.60).

The remaining two terms in (2.54), $J_1(P,R)$ and $J_3(P,R)$, are also $o_P(1)$ a result which can be established using the same argument applied to obtain $J_2(P,R)=o_P(1)$. In these two additional cases, however, one must now argue $\lim_{P\to\infty}\sup_{0<\tau<\frac{1}{P+1}}\frac{\tau}{P^{1/2}q(\tau)}=0$ and $\lim_{P\to\infty}\sup_{\frac{P}{P+1}<\tau<1}\frac{\tau}{P^{1/2}q(\tau)}=0$, respectively, rather than $\lim_{P\to\infty}\sup_{\frac{1}{P+1}<\tau<\frac{P}{P+1}}\frac{\tau}{P^{1/2}q(\tau)}=0$. We provide this argument here. By assumption $\Psi(q,c)<\infty$ from some $c>0$ which implies $\lim_{\tau\to 0} q(\tau)/\tau^{1/2}=\infty$ and $\lim_{\tau\to 1} q(\tau)/(1-\tau)^{1/2}=\infty$; for a detailed proof and discussion of this result consult Csörgő and Horváth [1993, pp. 188-189]. As a result of this, we conclude (2.55).

Proof of Proposition 2.1.3: This follows from statement ii) of Theorem S and Proposition 2.1.2 (Proposition 2.1.2 requires the integral condition to hold only for some $c>0$). This proposition in conjunction with Theorem S, as $R,P\to\infty$, provide the following result;

$$\left|\mathbb{P}\left\{\sup_{0<\tau<1}\frac{|\widehat{M}_P^{(CS,O)}(\tau)|}{q(\tau)}\le x\right\}-\mathbb{P}\left\{\sup_{0<\tau<1}\frac{|B(\tau)|}{q(\tau)}\le x\right\}\right|\le \tag{2.61}$$

$$\le\left|\mathbb{P}\left\{\sup_{0<\tau<1}\frac{|\widehat{M}_P^{(CS,O)}(\tau)|}{q(\tau)}\right\}-\mathbb{P}\left\{\sup_{0<\tau<1}\frac{|M_P^{(CS,O)}(\tau)|}{q(\tau)}\le x\right\}\right|+$$

$$+\left|\mathbb{P}\left\{\sup_{0<\tau<1}\frac{|M_P^{(CS,O)}(\tau)|}{q(\tau)}\le x\right\}-\mathbb{P}\left\{\sup_{0<\tau<1}\frac{|B(\tau)|}{q(\tau)}\le x\right\}\right|=0,$$

for all $x\in\mathbb{R}$. The last line establishes statement ii) of Proposition 2.1.3.

Proof of Proposition 2.2.1: This is a direct consequence of the multivariate version of the Lindeberg-Lévy CLT.

Proof of Theorem 2.2.1: This follows as a result of Proposition 2.2.1, equation (2.24) and the continuous mapping theorem.

Proof of Theorem 2.2.2: Assume a set of local alternative hypotheses defined by a VaR model with coverage probability $\widetilde{\alpha}$ from $\tau^\star$, that satisfies $\alpha-\widetilde{\alpha}=\frac{a}{P^\gamma}$, with $a\neq 0$, and $\gamma\ge 1/2$

constant values. The asymptotic power of the test is given by

$$P\left\{\frac{1}{\sqrt{\alpha(1-\alpha)}}\sup_{0<\tau<1}\frac{\left|M_P^{(CS,O)}(\tau)\right|}{q(\tau)} > C_{1-\beta}^{q}\right\}.$$

After some algebra we obtain

$$P\left\{\frac{q(\tau^\star)}{\sqrt{h'\Sigma h}}\left[\sup_{0<\tau<1}\frac{\left|M_P^{(CS,O)}(\tau)\right|}{q(\tau)} - \frac{\tau^\star(1-\tau^\star)|\alpha-\widetilde{\alpha}|}{q(\tau^\star)}\sqrt{P}\right] > \frac{q(\tau^\star)}{\sqrt{h'\Sigma h}}\left[C_{1-\beta}^{q}\sqrt{\alpha(1-\alpha)} - \frac{\tau^\star(1-\tau^\star)|\alpha-\widetilde{\alpha}|}{q(\tau^\star)}\sqrt{P}\right]\right\}.$$

Now, using that $\alpha-\widetilde{\alpha}=\frac{a}{P^\gamma}$ note that

$$h'\Sigma h = \tau^\star(1-\tau^\star)\left\{\alpha(1-\alpha)+\tau^\star\left[\widetilde{\alpha}(1-\widetilde{\alpha})-\alpha(1-\alpha)\right]\right\} = \tau^\star(1-\tau^\star)\left\{\alpha(1-\alpha)+\tau^\star\frac{|a|}{P^\gamma}\left(1+\widetilde{\alpha}+\alpha\right)\right\}.$$

Therefore, after further algebra, and by Theorem 2.2.1 we obtain

$$\lim_{P\to\infty} pf_\beta = 1-\Phi\left(\frac{q(\tau^\star)}{\sqrt{\tau^\star(1-\tau^\star)}}C_{1-\beta}^{q} - |a|\sqrt{\frac{\tau^\star(1-\tau^\star)}{\alpha(1-\alpha)}}\right),$$

if $\gamma=1/2$.

The proof for $\gamma>1/2$ follows immediately from the preceding arguments.

2.7 Tabulated CDFs for Weighted Statistics

$G(x) \stackrel{\text{def}}{=} \mathbb{P}\left\{ \sup_{0<t<1} \frac{\lvert W(t)-tW(1)\rvert}{(t(1-t))^{7/16}} \rvert \le x \right\}$					
x	$G(x)$	x	$G(x)$	x	$G(x)$
1.146	0.01	1.734	0.34	2.112	0.67
1.206	0.02	1.744	0.35	2.123	0.68
1.251	0.03	1.754	0.36	2.136	0.69
1.284	0.04	1.764	0.37	2.149	0.70
1.315	0.05	1.776	0.38	2.164	0.71
1.345	0.06	1.787	0.39	2.180	0.72
1.370	0.07	1.797	0.40	2.194	0.73
1.390	0.08	1.809	0.41	2.209	0.74
1.408	0.09	1.822	0.42	2.227	0.75
1.426	0.10	1.833	0.43	2.244	0.76
1.443	0.11	1.844	0.44	2.261	0.77
1.458	0.12	1.855	0.45	2.278	0.78
1.473	0.13	1.865	0.46	2.297	0.79
1.489	0.14	1.876	0.47	2.315	0.80
1.504	0.15	1.887	0.48	2.333	0.81
1.517	0.16	1.897	0.49	2.353	0.82
1.531	0.17	1.909	0.50	2.375	0.83
1.545	0.18	1.921	0.51	2.397	0.84
1.558	0.19	1.931	0.52	2.421	0.85
1.572	0.20	1.941	0.53	2.446	0.86
1.586	0.21	1.953	0.54	2.472	0.87
1.599	0.22	1.964	0.55	2.502	0.88
1.611	0.23	1.976	0.56	2.532	0.89
1.623	0.24	1.988	0.57	2.563	0.90
1.633	0.25	2.001	0.58	2.595	0.91
1.644	0.26	2.014	0.59	2.631	0.92
1.655	0.27	2.026	0.60	2.675	0.93
1.666	0.28	2.037	0.61	2.727	0.94
1.676	0.29	2.048	0.62	2.784	0.95
1.688	0.30	2.060	0.63	2.856	0.96
1.700	0.31	2.073	0.64	2.931	0.97
1.711	0.32	2.086	0.65	3.080	0.98
1.722	0.33	2.101	0.66	3.282	0.99

$G(x) \stackrel{\text{def}}{=} \mathbb{P}\left\{ \sup_{0<t<1} \frac{\lvert W(t)-tW(1)\rvert}{(t(1-t))^{5/16}} \rvert \le x \right\}$					
x	$G(x)$	x	$G(x)$	x	$G(x)$
0.818	0.01	1.27	0.34	1.593	0.67
0.857	0.02	1.28	0.35	1.604	0.68
0.887	0.03	1.29	0.36	1.616	0.69
0.913	0.04	1.30	0.37	1.629	0.70
0.936	0.05	1.31	0.38	1.642	0.71
0.957	0.06	1.32	0.39	1.655	0.72
0.975	0.07	1.33	0.40	1.669	0.73
0.993	0.08	1.34	0.41	1.684	0.74
1.008	0.09	1.34	0.42	1.697	0.75
1.022	0.10	1.35	0.43	1.711	0.76
1.039	0.11	1.36	0.44	1.726	0.77
1.051	0.12	1.37	0.45	1.743	0.78
1.062	0.13	1.38	0.46	1.758	0.79
1.074	0.14	1.39	0.47	1.772	0.80
1.086	0.15	1.40	0.48	1.787	0.81
1.097	0.16	1.41	0.49	1.803	0.82
1.109	0.17	1.42	0.50	1.819	0.83
1.121	0.18	1.43	0.51	1.840	0.84
1.131	0.19	1.44	0.52	1.862	0.85
1.142	0.20	1.45	0.53	1.886	0.86
1.152	0.21	1.46	0.54	1.908	0.87
1.162	0.22	1.47	0.55	1.932	0.88
1.171	0.23	1.48	0.56	1.959	0.89
1.180	0.24	1.49	0.57	1.987	0.90
1.189	0.25	1.50	0.58	2.021	0.91
1.197	0.26	1.51	0.59	2.065	0.92
1.206	0.27	1.52	0.60	2.109	0.93
1.215	0.28	1.53	0.61	2.148	0.94
1.225	0.29	1.54	0.62	2.201	0.95
1.234	0.30	1.55	0.63	2.268	0.96
1.244	0.31	1.56	0.64	2.345	0.97
1.253	0.32	1.57	0.65	2.449	0.98
1.262	0.33	1.58	0.66	2.624	0.99

$G(x) \stackrel{\text{def}}{=} \mathbb{P}\left\{\sup\limits_{0<t<1} \frac{\lvert W(t)-tW(1)\rvert}{(t(1-t))^{3/16}} \le x\right\}$					
x	$G(x)$	x	$G(x)$	x	$G(x)$
0.620	0.01	0.991	0.34	1.273	0.67
0.654	0.02	0.998	0.35	1.283	0.68
0.681	0.03	1.005	0.36	1.292	0.69
0.698	0.04	1.014	0.37	1.302	0.70
0.714	0.05	1.022	0.38	1.311	0.71
0.730	0.06	1.029	0.39	1.321	0.72
0.745	0.07	1.036	0.40	1.333	0.73
0.759	0.08	1.044	0.41	1.345	0.74
0.771	0.09	1.053	0.42	1.357	0.75
0.783	0.10	1.062	0.43	1.370	0.76
0.794	0.11	1.070	0.44	1.384	0.77
0.806	0.12	1.078	0.45	1.397	0.78
0.816	0.13	1.086	0.46	1.411	0.79
0.826	0.14	1.095	0.47	1.424	0.80
0.836	0.15	1.103	0.48	1.438	0.81
0.845	0.16	1.111	0.49	1.454	0.82
0.855	0.17	1.120	0.50	1.469	0.83
0.864	0.18	1.130	0.51	1.484	0.84
0.873	0.19	1.140	0.52	1.500	0.85
0.881	0.20	1.148	0.53	1.519	0.86
0.889	0.21	1.155	0.54	1.540	0.87
0.897	0.22	1.163	0.55	1.563	0.88
0.906	0.23	1.172	0.56	1.589	0.89
0.915	0.24	1.180	0.57	1.621	0.90
0.924	0.25	1.188	0.58	1.649	0.91
0.932	0.26	1.197	0.59	1.679	0.92
0.941	0.27	1.207	0.60	1.713	0.93
0.949	0.28	1.217	0.61	1.755	0.94
0.957	0.29	1.226	0.62	1.798	0.95
0.964	0.30	1.236	0.63	1.852	0.96
0.971	0.31	1.245	0.64	1.920	0.97
0.978	0.32	1.253	0.65	2.005	0.98
0.985	0.33	1.263	0.66	2.166	0.99

$G(x) \stackrel{\text{def}}{=} \mathbb{P}\left\{\sup\limits_{0<t<1} \frac{\lvert W(t)-tW(1)\rvert}{(t(1-t))^{1/16}} \le x\right\}$					
x	$G(x)$	x	$G(x)$	x	$G(x)$
0.482	0.01	0.795	0.34	1.032	0.67
0.514	0.02	0.802	0.35	1.040	0.68
0.534	0.03	0.808	0.36	1.049	0.69
0.551	0.04	0.814	0.37	1.058	0.70
0.566	0.05	0.820	0.38	1.067	0.71
0.578	0.06	0.826	0.39	1.076	0.72
0.589	0.07	0.833	0.40	1.086	0.73
0.600	0.08	0.839	0.41	1.095	0.74
0.610	0.09	0.845	0.42	1.105	0.75
0.619	0.10	0.852	0.43	1.116	0.76
0.629	0.11	0.858	0.44	1.128	0.77
0.638	0.12	0.865	0.45	1.140	0.78
0.646	0.13	0.873	0.46	1.152	0.79
0.655	0.14	0.880	0.47	1.165	0.80
0.663	0.15	0.887	0.48	1.177	0.81
0.671	0.16	0.893	0.49	1.190	0.82
0.678	0.17	0.900	0.50	1.203	0.83
0.686	0.18	0.908	0.51	1.218	0.84
0.693	0.19	0.916	0.52	1.232	0.85
0.700	0.20	0.924	0.53	1.248	0.86
0.707	0.21	0.932	0.54	1.265	0.87
0.714	0.22	0.940	0.55	1.284	0.88
0.722	0.23	0.948	0.56	1.306	0.89
0.729	0.24	0.955	0.57	1.330	0.90
0.735	0.25	0.962	0.58	1.355	0.91
0.742	0.26	0.969	0.59	1.381	0.92
0.749	0.27	0.977	0.60	1.410	0.93
0.756	0.28	0.984	0.61	1.443	0.94
0.763	0.29	0.992	0.62	1.483	0.95
0.769	0.30	0.999	0.63	1.531	0.96
0.775	0.31	1.008	0.64	1.591	0.97
0.781	0.32	1.016	0.65	1.653	0.98
0.788	0.33	1.024	0.66	1.795	0.99

$G(x) \stackrel{\text{def}}{=} \mathbb{P}\left\{\sup_{0<t<1} \frac{\lvert W(t)-tW(1)\rvert}{q^{step}_{7/16}(t)} \le x\right\}$					
x	$G(x)$	x	$G(x)$	x	$G(x)$
1.118	0.01	1.682	0.34	2.080	0.67
1.189	0.02	1.693	0.35	2.092	0.68
1.225	0.03	1.704	0.36	2.107	0.69
1.255	0.04	1.716	0.37	2.125	0.70
1.281	0.05	1.727	0.38	2.139	0.71
1.304	0.06	1.738	0.39	2.155	0.72
1.325	0.07	1.748	0.40	2.169	0.73
1.348	0.08	1.758	0.41	2.182	0.74
1.367	0.09	1.769	0.42	2.198	0.75
1.383	0.10	1.781	0.43	2.217	0.76
1.400	0.11	1.793	0.44	2.236	0.77
1.417	0.12	1.805	0.45	2.256	0.78
1.431	0.13	1.817	0.46	2.273	0.79
1.445	0.14	1.831	0.47	2.292	0.80
1.459	0.15	1.844	0.48	2.314	0.81
1.471	0.16	1.856	0.49	2.335	0.82
1.484	0.17	1.868	0.50	2.355	0.83
1.498	0.18	1.880	0.51	2.374	0.84
1.512	0.19	1.892	0.52	2.398	0.85
1.524	0.20	1.904	0.53	2.426	0.86
1.535	0.21	1.917	0.54	2.457	0.87
1.547	0.22	1.927	0.55	2.484	0.88
1.560	0.23	1.937	0.56	2.512	0.89
1.575	0.24	1.947	0.57	2.546	0.90
1.588	0.25	1.958	0.58	2.575	0.91
1.600	0.26	1.971	0.59	2.614	0.92
1.610	0.27	1.985	0.60	2.662	0.93
1.620	0.28	1.997	0.61	2.705	0.94
1.630	0.29	2.009	0.62	2.757	0.95
1.639	0.30	2.022	0.63	2.824	0.96
1.648	0.31	2.036	0.64	2.933	0.97
1.658	0.32	2.050	0.65	3.032	0.98
1.670	0.33	2.064	0.66	3.264	0.99

$G(x) \stackrel{\text{def}}{=} \mathbb{P}\left\{\sup_{0<t<1} \frac{\lvert W(t)-tW(1)\rvert}{q^{step}_{5/16}(t)} \le x\right\}$					
x	$G(x)$	x	$G(x)$	x	$G(x)$
0.821	0.01	1.277	0.34	1.599	0.67
0.874	0.02	1.285	0.35	1.611	0.68
0.906	0.03	1.294	0.36	1.623	0.69
0.930	0.04	1.302	0.37	1.638	0.7
0.948	0.05	1.314	0.38	1.65	0.71
0.966	0.06	1.323	0.39	1.661	0.72
0.983	0.07	1.331	0.40	1.674	0.73
0.999	0.08	1.338	0.41	1.687	0.74
1.014	0.09	1.347	0.42	1.701	0.75
1.027	0.10	1.357	0.43	1.716	0.76
1.040	0.11	1.366	0.44	1.732	0.77
1.054	0.12	1.375	0.45	1.747	0.78
1.068	0.13	1.384	0.46	1.762	0.79
1.081	0.14	1.394	0.47	1.777	0.80
1.092	0.15	1.403	0.48	1.797	0.81
1.102	0.16	1.413	0.49	1.815	0.82
1.113	0.17	1.423	0.50	1.836	0.83
1.123	0.18	1.433	0.51	1.859	0.84
1.133	0.19	1.442	0.52	1.880	0.85
1.144	0.20	1.451	0.53	1.904	0.86
1.155	0.21	1.461	0.54	1.928	0.87
1.165	0.22	1.472	0.55	1.957	0.88
1.176	0.23	1.483	0.56	1.984	0.89
1.185	0.24	1.493	0.57	2.014	0.90
1.194	0.25	1.503	0.58	2.041	0.91
1.203	0.26	1.515	0.59	2.070	0.92
1.212	0.27	1.525	0.60	2.108	0.93
1.221	0.28	1.535	0.61	2.149	0.94
1.229	0.29	1.544	0.62	2.194	0.95
1.238	0.30	1.553	0.63	2.259	0.96
1.247	0.31	1.563	0.64	2.329	0.97
1.257	0.32	1.574	0.65	2.454	0.98
1.267	0.33	1.586	0.66	2.67	0.99

$G(x) \overset{\text{def}}{=} \mathbb{P}\left\{\sup_{0<t<1} \frac{\lvert W(t)-tW(1)\rvert}{q_{3/16}^{step}(t)} \le x\right\}$					
x	$G(x)$	x	$G(x)$	x	$G(x)$
0.629	0.01	1.007	0.34	1.277	0.67
0.666	0.02	1.013	0.35	1.289	0.68
0.690	0.03	1.020	0.36	1.302	0.69
0.712	0.04	1.027	0.37	1.313	0.70
0.732	0.05	1.034	0.38	1.324	0.71
0.751	0.06	1.042	0.39	1.338	0.72
0.767	0.07	1.050	0.40	1.349	0.73
0.782	0.08	1.058	0.41	1.360	0.74
0.793	0.09	1.065	0.42	1.372	0.75
0.803	0.10	1.072	0.43	1.386	0.76
0.813	0.11	1.079	0.44	1.402	0.77
0.823	0.12	1.086	0.45	1.415	0.78
0.832	0.13	1.092	0.46	1.429	0.79
0.842	0.14	1.099	0.47	1.443	0.80
0.852	0.15	1.107	0.48	1.455	0.81
0.862	0.16	1.116	0.49	1.470	0.82
0.872	0.17	1.124	0.50	1.486	0.83
0.881	0.18	1.131	0.51	1.503	0.84
0.890	0.19	1.138	0.52	1.520	0.85
0.898	0.20	1.147	0.53	1.540	0.86
0.907	0.21	1.156	0.54	1.558	0.87
0.916	0.22	1.165	0.55	1.580	0.88
0.924	0.23	1.174	0.56	1.599	0.89
0.933	0.24	1.183	0.57	1.621	0.90
0.941	0.25	1.192	0.58	1.648	0.91
0.949	0.26	1.201	0.59	1.682	0.92
0.957	0.27	1.209	0.60	1.717	0.93
0.964	0.28	1.217	0.61	1.752	0.94
0.971	0.29	1.226	0.62	1.796	0.95
0.977	0.30	1.235	0.63	1.851	0.96
0.984	0.31	1.245	0.64	1.907	0.97
0.992	0.32	1.256	0.65	1.978	0.98
1.000	0.33	1.266	0.66	2.145	0.99

$G(x) \overset{\text{def}}{=} \mathbb{P}\left\{\sup \frac{\lvert(t)-tW(1)\rvert}{q_{1/16}^{step}(t)} \le x\right\}$					
x	$G(x)$	x	$G(x)$	x	$G(x)$
0.497	0.01	0.806	0.34	1.049	0.67
0.525	0.02	0.813	0.35	1.058	0.68
0.548	0.03	0.820	0.36	1.066	0.69
0.565	0.04	0.826	0.37	1.075	0.70
0.580	0.05	0.833	0.38	1.083	0.71
0.591	0.06	0.840	0.39	1.093	0.72
0.601	0.07	0.847	0.40	1.103	0.73
0.613	0.08	0.854	0.41	1.113	0.74
0.625	0.09	0.861	0.42	1.124	0.75
0.636	0.10	0.868	0.43	1.135	0.76
0.645	0.11	0.875	0.44	1.147	0.77
0.653	0.12	0.882	0.45	1.159	0.78
0.662	0.13	0.889	0.46	1.171	0.79
0.671	0.14	0.897	0.47	1.184	0.80
0.681	0.15	0.903	0.48	1.198	0.81
0.689	0.16	0.910	0.49	1.213	0.82
0.696	0.17	0.916	0.50	1.228	0.83
0.704	0.18	0.923	0.51	1.243	0.84
0.711	0.19	0.930	0.52	1.260	0.85
0.718	0.20	0.937	0.53	1.275	0.86
0.724	0.21	0.944	0.54	1.290	0.87
0.730	0.22	0.951	0.55	1.307	0.88
0.736	0.23	0.958	0.56	1.324	0.89
0.743	0.24	0.966	0.57	1.347	0.90
0.749	0.25	0.973	0.58	1.371	0.91
0.756	0.26	0.981	0.59	1.399	0.92
0.762	0.27	0.989	0.60	1.424	0.93
0.768	0.28	0.998	0.61	1.455	0.94
0.774	0.29	1.006	0.62	1.494	0.95
0.780	0.30	1.015	0.63	1.534	0.96
0.786	0.31	1.023	0.64	1.591	0.97
0.793	0.32	1.031	0.65	1.682	0.98
0.799	0.33	1.039	0.66	0.990	0.99

Chapter 3

A U-statistic Type Test to Disentangle Breaks in Intercept from Slope in Linear Regression Models

3.1 Introduction

Economics and finance frequently consider linear regression models (hereafter LRMs) with coefficients that are assumed to be constant for all time periods. It is well-known that these parameters can, and do, change over time due, for example, to abrupt policy changes, wars, oil price or technology shocks. This has led to considerable econometric research into methods that can detect if such exogenous events have caused parameters of linear regression models to change. One of the first papers published on this matter was by Chow (1960). He constructed two test statistics; one based on prediction errors and the other on the difference between the restricted and unrestricted sum of squared residuals, that are capable of detecting a one-time change in regression parameters at a known time. Work by Brown, Durbin and Evans (1975, hereafter BDE) and Dufour (1988) extended Chow's test to accommodate multiple changes in regression parameters that may occur at unknown times.

Other tests, called fluctuation tests, such as that of Ploberger, Kramer and Kontrus (1989) (hereafter PKK) have also been developed. An interesting contribution to this literature is that of Altissimo and Corradi (2003) who developed a sequentially consistent test statistic which is capable of testing for any number of break-points in LRMs. A different econometric approach for detecting structural breaks in LRMs has been developed by Andrews (1993), Andrews and Ploberger (1994, hereafter AP) and Andrews (2003). In particular, Andrews (1993) considers Wald (W_T), Lagrange multiplier (LM) and likelihood-ratio (LR)-like tests for parameter stability in nonlinear parametric models. These tests have power against local alternatives of the form $\boldsymbol{\beta}_t = \boldsymbol{\beta}_0 + \eta(t/T)/\sqrt{T}$, with $\eta(\cdot)$ a bounded function on $[0,1]$, as long as $\eta(\cdot)$ is not almost surely constant. It is important to note that these tests are not optimal for the class of alternatives he considers: the tests have only nontrivial power. Local optimality of these tests was established by AP, and depends on correct specification of the likelihood function. An interesting additional feature of the tests developed by AP is their use of weight functions. Optimality of their tests when considering the entire interval $(0,1)$, or a broad class of weight functions or even more general alternatives no longer holds.

Hansen (1991 and 1996) makes two interesting contributions to the literature on testing for structural breaks in linear regression models. His first contribution (1991) develops test statistics which detect change/changes in individual parameters of linear regression models. The individual test statistics are then combined to form one test statistic which is capable of testing for a structural break in any of the regression parameters. There is, however, one flaw with his method, it cannot be used to estimate the timing of the break (see page 520). It would seem that this is an important oversight in his method for detecting structural breaks. Hansen (1996) also considers the nonstandard problem of testing whether a sub-vector of $\theta \in \Theta \subset \mathbb{R}^s$ equals zero when the likelihood function depends on an additional parameter $\pi \in \Pi$, Π a compact subset of the interval $(0, 1)$, that is not identified under the null hypothesis. The significant contribution here is his method of simulating the asymptotic limiting distribution of many of the test statistics considered in this larger literature such as those developed by AP. His method for simulating the asymptotic distribution is employed in the simulation undertaken in Section 3.4 of this paper.

The CUSUM and fluctuation tests, when applied to regression models, are not devised to distinguish between changes in intercept or slope and, in turn, although informative about the number of break points are not very informative about the nature of the rejection. The latter tests, based on regression analysis, have statistical power against changes in the intercept but are, however, inconsistent against changes close to the boundary of the (0,1) interval. These gaps of the structural break tests literature are not without importance. The knowledge of the break can be of fundamental importance in different areas of interest in finance and economics. The application of this paper, for example, illustrates this in empirical asset pricing setting. In particular, this method provides a method that can measure manager's performance in the mutual fund industry by disentangling changes in the α parameter, often referred to as Jensen's alpha - see Jensen (1968) - from changes in the risk factor parameter given by the betas, the slope parameters. Changes to Jensen's alpha reflect the changes in managerial stock selecting abilities which can be useful for compensation or investing purposes.

A second application where detecting changes only in the intercept can be useful is in detecting insider trading. Research by Olmo, Pilbeam and Pouliot (2009, hereafter OPP) apply some of the techniques developed here to detecting insider trading on a dataset studied in two Occasion Papers Series produced by the Financial Services Authority. OPP show that abrupt changes in the intercept parameter of an extended capital asset pricing model before unscheduled corporate announcements can be an indication of insider-trading. Regulatory bodies who are mandated to maintain integrity of financial markets can use the methods discussed therein to study price moments for insider trading should movements look unusual.

The contribution of this paper is to fill in these gaps. In order to do so, we introduce a composite test that can disentangle breaks in the intercept from breaks in the slope of linear regression models. The test statistic is constructed from a bivariate U-statistic type process that can accommodate the presence of weight functions that improve the power of structural break tests against changes that occur early and later on in the evaluation period. As a byproduct, our test also exhibits power against changes in the skewness of the error distribution. The asymptotic theory is based on functionals of a Brownian bridge and therefore critical values used in our tests of change in intercept or slope can be easily tabulated.

The paper is structured as follows; Section 3.2 designs simultaneous and joint tests for detecting a change in intercept/slope in LRMs under the assumption that the parameters of the LRMs are known and Section 3.2.2 when the parameters of LRMs are replaced by estimates; Section 3.3 explores the power of the statistics; Section 3.4 subjects our test as well as others to a detailed Monte Carlo experiment, studying nominal size and power of the test against alternatives that include a one-time change in intercept and/or slope. The application to investigate mutual funds manager's performance is in Section 3.5. Section 3.6 concludes. Any tables referred to in this paper can be found in Section 3.7.

3.2 A New Test to Disentangle Breaks in Intercept from Slope

The purpose of this section is to design tests with statistical power to detect a change in intercept and slope. A novel and interesting feature of these tests is their ability to distinguish changes among the parameters: i.e. slope from intercept. The mutual independence between the test for a change in intercept and that corresponding to a change in slope permit control of global error rates. The flexibility of U-statistic type processes also permits improvement to the power of these tests against parameter changes that occur early and later on in the evaluation period. In this regard our tests complement and improves the approaches of Andrews (1993), AP and Hansen (1996). We divide our study into two cases; i) model parameters are known and ii) model parameters are estimated.

The entertained piecewise linear regression model is

$$Y_t = \begin{cases} \beta_0^{(1)} + \boldsymbol{\beta}'^{(1)}\mathbf{X}_t + \sigma\varepsilon_t, & 1 \le t \le t^*, \\ \beta_0^{(2)} + \boldsymbol{\beta}'^{(2)}\mathbf{X}_t + \sigma\varepsilon_t, & t^* < t \le T, \end{cases} \tag{3.1}$$

where ε_t are independent and identically distributed (iid) random variables (rvs) with

$$\mathbb{E}\varepsilon_t = 0,\ \mathbb{E}\varepsilon_t^2 = 1 \text{ and } \mathbb{E}|\varepsilon_t|^4 < \infty,\ t = 1, \dots, T \tag{3.2}$$

and $'$ refers to the vector transpose. In addition, under the alternative hypothesis of a break in either intercept or slope parameters, we assume that at least one of the following holds: $\beta_0^{(1)} \neq \beta_0^{(2)}$ or $\boldsymbol{\beta}^{(1)} \neq \boldsymbol{\beta}^{(2)}$ with $\boldsymbol{\beta}^{(1)}$ and $\boldsymbol{\beta}^{(2)}$ $K \times 1$ parameter vectors, and $\mathbf{X}_t$ the corresponding vector of explanatory variables. It is assumed that all components of the vector of explanatory variables, $\mathbf{X}_t$, and dependent variable Y_t are stationary. This assumption is required in order to establish weak convergence results regarding the processes to be considered in the coming sections.

This model can be considered as a regime switching model with threshold variable given by time (t). Hypothesis tests for detecting the nonlinearity of this model are introduced by Andrews (1993), AP and Hansen (1996) among others. Under homoskedasticity in the data, these tests are based on likelihood ratio tests, as Andrews (1993) and Hansen (1997), under conditional heteroskedasticity Hansen (1996) develops Wald type and Lagrange multiplier tests. In order to maximize the power of these tests AP propose an exponential average test. The problem of all these tests is that they need to be considered over a compact set within $(0,1)$, in particular Andrews (1993) proposes $[0.15, 0.85]$ and Hansen (1996) the interval $[0.20, 0.80]$.

Alternatively, the problem of detecting parameter instability in the linear regression framework can also be represented with the following hypothesis;

$$H_O : t^\star \geq T$$

versus the alternative hypothesis of at-most-one change (AMOC) in intercept or slope;

$$H_A : 1 \leq t^\star < T.$$

3.2.1 Parameters Known

For illustration purposes we first construct a test based on U-statistics to determine if process (3.1) is linear or piecewise linear. Unlike the F-test, $M_T^{(O)}(\tau)$ only compares the residual sum of squares under the null hypothesis not under the alternative hypothesis as well. The process defined on $\tau \in (0,1)$ is given by

$$M_T^{(O)}(\tau) \quad := \quad T^{-1/2}\left\{\sum_{t=1}^{[(T+1)\tau]}(Y_t - \beta_0^{(1)} - \boldsymbol{\beta}'^{(1)}\mathbf{X}_t)^2 - \tau\sum_{t=1}^{T}(Y_t - \beta_0^{(1)} - \boldsymbol{\beta}'^{(1)}\mathbf{X}_t)^2\right\} \quad (3.3)$$

Under the null hypothesis the kernel function of this U-statistic type process satisfies that $\mathbb{E}(Y_1 - \beta_0^{(1)} - \boldsymbol{\beta}'^{(1)}\mathbf{X}_1)^2 = \sigma^2$. Csörgő and Horváth (1987) and later Gombay, Horváth and Hušková (1996, hereafter GHH) explore these processes to detect deviations in the mean/variance parameter respectively.

This process remains a function of τ, and as such cannot be used in its present form to test the null hypothesis of no change in intercept or slope: that is, it is not yet a statistic because of its dependency on τ. Here, interest centers on how large this process can be for

$0 < \tau < 1$. A suitable test statistic that indirectly also yields an estimator of the break point $\tau^\star$ is

$$\sup_{0<\tau<1} |M_T^{(O)}(\tau)|. \tag{3.4}$$

GHH show that this statistic converges to the supremum of a Brownian Bridge. This statistic does not distinguish between rejections in the intercept or the slope and as such it does not serve our purpose. Our interest is not in this statistic but in using the structure of the statistic to construct tests that detect deviations in process (3.1) that are also able to determine the type of rejection. In order to do this, two auxiliary U-statistic type processes are required, each process depends on a function that is unbiased, under null and alternative hypotheses, for intercept and slope parameters of the LRM. The first kernel function sets $(y_1 - \beta_0^{(1)} - \boldsymbol{\beta}'^{(1)}\mathbf{x}_1)^2$ and is used to detect structural breaks in the slope parameter because $\mathbb{E}(Y_1 - \beta_0^{(1)} - \boldsymbol{\beta}'^{(1)}\mathbf{X}_1)^2 = \sigma^2$. Although this function is unbiased for the variance, it is used to detect deviations in the slope parameter for two reasons. One, it closely corresponds to CUSUM of squares test of BDE and secondly, it can be shown that a change in slope translates into a change in the variance of the residuals as long as a particular condition holds. Remark 3.3.1 details said condition. The second kernel function selected sets $(y_1 - \boldsymbol{\beta}'^{(1)}\mathbf{x}_1)$ because $\mathbb{E}(Y_1 - \boldsymbol{\beta}'^{(1)}\mathbf{X}_1) = \beta_0^{(1)}$.

The first function can be used to fashion a statistic that is sensitive to a one-time change in the slope parameters and robust to changes in the intercept, while the second function can be used to fashion a statistic sensitive to a one-time change in the intercept, and desirably, also robust to a one-time change in slope when it occurs. With this in mind, the following processes are now defined;

$$M_T^{(1)}(\tau) \;:=\; T^{-1/2}\left\{\sum_{t=1}^{[(T+1)\tau]}(Y_t-\beta_0-\boldsymbol{\beta}'^{(1)}\mathbf{X}_t)^2-\tau\sum_{t=1}^{T}(Y_t-\beta_0-\boldsymbol{\beta}'^{(1)}\mathbf{X}_t)^2\right\} \tag{3.5}$$

$$M_T^{(2)}(\tau) \;:=\; T^{-1/2}\left\{\sum_{t=1}^{[(T+1)\tau]}(Y_t-\boldsymbol{\beta}'\mathbf{X}_t)-\tau\sum_{t=1}^{T}(Y_t-\boldsymbol{\beta}'\mathbf{X}_t)\right\}, \tag{3.6}$$

where in $M_T^{(1)}(\tau)$ set

$$\beta_0 = \begin{cases} \beta_0^{(1)}, & t \le t^\star \\ \beta_0^{(2)}, & t > t^\star \end{cases}$$

and; in $M_T^{(2)}(\tau)$ set

$$\boldsymbol{\beta} = \begin{cases} \boldsymbol{\beta}^{(1)}, & t \le t^\star \\ \boldsymbol{\beta}^{(2)}, & t > t^\star. \end{cases}$$

This choice of parametrization makes $M_T^{(1)}(\tau)$ robust to a change in intercept and $M_T^{(2)}(\tau)$ robust to a change in slope. In order to maximize the power of this test to detect a break we construct the following test statistics:

$$\sup_{0<\tau<1}\frac{|M_T^{(i)}(\tau)|}{q(\tau)}, \tag{3.7}$$

for $i = 1,\ 2$, in which $q(\cdot)$ is a weight function devised to improve the power of the tests for breaks that occur in specific subsamples of the evaluation period. In the same spirit, Andrews (1993) and AP develop distribution functions defined on the real domain that are devised to increase the sensitivity of the test to detect deviations in certain regions of interest. The use of this function $q(\tau)$ is particularly important when compared to the framework studied by these authors that propose optimal tests for structural breaks in a compact interval within

$(0,1)$. Our interest, therefore, will be to develop weight functions that can be used to improve the power of the test over the whole $(0,1)$ range, in particular close to the points 0 and 1, and therefore, that are sensitive to change points that occur early and late in the evaluation period. In order to do this, these functions need to satisfy the following two assumptions:

A.1: The function $q(\cdot)$ defined on (0,1) is such that $\inf_{\delta \le \tau \le 1-\delta} q(\tau) > 0$ for all $\tau \in (0,1)$ and $\delta \in (0, 1/2)$.

A.2: $I(q,c) = \int_0^1 \frac{1}{\tau(1-\tau)} \exp^{-\frac{c}{(\tau(1-\tau))q^2(\tau)}} d\tau < \infty$ for some constant $c > 0$.

Csörgő, Csörgő, Horváth and Mason (1988) show that $I(q,c) < \infty$ for all $c > 0$, if and only if

$$\lim_{\tau \downarrow 0} \frac{|W(\tau)|}{q(\tau)} = \lim_{\tau \uparrow 1} \frac{|W(\tau)|}{q(1-\tau)} = 0,$$

almost surely, with $W(\tau)$ a standard Wiener process. One family of weight functions that has received some attention is due to GHH. This family of functions depends on a tuning parameter ν, and is given by

$$q(\tau) = q(\tau;\nu) := \{(\tau(1-\tau))^{\nu};\ 0 \le \nu < 1/2\}. \tag{3.8}$$

This class of functions satisfies A.1 and A.2 for all $c > 0$, and has been shown to be sensitive to a change that occurs both early and later on in the sample. We exploit this class of functions to construct a statistic that is well defined for $\tau \in (0,1)$ and that improves the power near the boundary of $(0,1)$.

Proposition 3.2.1. *Assume H_O; let the process* (3.1) *satisfy conditions detailed in* (3.2)*; and let $q(\cdot)$ satisfy* A.1 *and* A.2*. Then, as $T \to \infty$,*

(i) $\sup_{0<\tau<1} \frac{\left|\frac{1}{\Delta^{(i)}} M_T^{(i)}(\tau) - B_T(\tau)\right|}{q(\tau)} = O_P(1).$

Further, if in A.2 *the integral holds for all $c > 0$ rather than for some $c > 0$, then* $\sup\limits_{0<\tau<1} \frac{\left|\frac{1}{\Delta^{(i)}} M_T^{(i)}(\tau) - B_T(\tau)\right|}{q(\tau)} = o_P(1)$.

(ii) $\sup\limits_{0<\tau<1} \frac{\left|\frac{1}{\Delta^{(i)}} M_T^{(i)}(\tau)\right|}{q(\tau)} \xrightarrow{\mathcal{D}} \sup\limits_{0<\tau<1} \frac{|B(\tau)|}{q(\tau)}$, *with* $\Delta^{(1)} = \sigma^2 \sqrt{Var(\varepsilon_1^2)}$ *and* $\Delta^{(2)} = \sigma$.

Proof. Under the LRM model detailed in (3.1), $\frac{M_T^{(1)}(\tau)}{q(\tau)} = T^{-1/2} \left(\frac{\sum^{[(T+1)\tau]} \epsilon_t^2 - \tau \sum_{t=1}^T \epsilon_t^2}{q(\tau)} \right)$ and $\frac{M_T^{(2)}(\tau)}{q(\tau)} = T^{-1/2} \left(\frac{\sum^{[(T+1)\tau]} \epsilon_t - \tau \sum_{t=1}^T \epsilon_t}{q(\tau)} \right)$. Then statements *(i)* and *(ii)* follow as a direct consequence of Theorem 2.1 of Szyszkowicz (1991).

Model (3.1) requires the errors of the regression equation ε_t to be identically distributed. This is not really necessary to establish Proposition 3.2.1 if we appeal to Corollary 4.1 of Csörgő and Horváth (1988) rather than to Theorem 2.1 of Szyszkowicz (1991). This, however, would require the ϵ_is to have $4 + \delta$ moments rather than only 4 moments. For a complete justification of this, we refer those interested to Remark 2.2 statement (i) of Ferger (2001). Hence our processes can easily accommodate conditional heteroskedasticity of the variance of ε_t, they do not need to be identically distributed, and the same results established in Proposition 3.2.1 remain unaltered.

As interest here is with a bivariate process formed out of the two processes given in (3.5) and (3.6), we need to introduce an appropriate metric as well as some additional notation. Define $D[0, 1]$ to represent the space of functions $x(\cdot)$ on $[0, 1]$ that are right-continuous and have left-hand limits (cf. Billingsley (1968), p. 109); let $D^2[0, 1] = D[0, 1] \times D[0, 1]$ and let the metric associated with this space be given by

$$\sup_{0<\tau<1} |x_1(\tau) - y_1(\tau)| + \sup_{0<\tau<1} |x_2(\tau) - y_2(\tau)|, \tag{3.9}$$

where $[x_1(\tau), x_2(\tau)]'$ and $[y_1(\tau), y_2(\tau)]'$ are elements of $D^2[0,\ 1]$. With the appropriate metric defined, the behaviour of the bivariate process is detailed in Proposition 3.2.2.

Proposition 3.2.2. *Assume H_O; let the process* (3.1) *satisfy conditions detailed in* (3.2)*; and let $q(\cdot)$ satisfy* A.1*. Then, as $T \to \infty$,*

$$\mathbf{M}_T := \begin{bmatrix} \frac{1}{\sigma^2 \sqrt{Var(\varepsilon_1^2)}} \frac{M_T^{(1)}(\cdot)}{q(\cdot)} \\ \frac{1}{\sigma} \frac{M_T^{(2)}(\cdot)}{q(\cdot)} \end{bmatrix} \Rightarrow \begin{bmatrix} \frac{B^{(1)}(\cdot)}{q(\cdot)} \\ \frac{\rho B^{(1)}(\cdot) \ + \ (1-\rho^2)^{1/2} B^{(2)}(\cdot)}{q(\cdot)} \end{bmatrix},$$

only if $I(q, c) < \infty$ for all $c > 0$. $B^{(1)}(\tau)$ and $B^{(2)}(\tau)$ are independent Brownian bridges, $\rho = \frac{\mathbb{E}[\varepsilon_1^3]}{\sqrt{Var(\varepsilon_1^2)}}$, and $\Rightarrow$ refers to weak convergence.

By the continuous mapping theorem,

$$\begin{bmatrix} \sup\limits_{0<\tau<1} \frac{1}{\sigma^2 \sqrt{Var(\varepsilon_1^2)}} \frac{|M_T^{(1)}(\tau)|}{q(\tau)} \\ \sup\limits_{0<\tau<1} \frac{1}{\sigma} \frac{M_T^{(2)}(\tau)}{q(\tau)} \end{bmatrix} \xrightarrow{\mathcal{D}} \begin{bmatrix} \sup\limits_{0<\tau<1} \frac{|B^{(1)}(\tau)|}{q(\tau)} \\ \sup\limits_{0<\tau<1} \frac{\rho B^{(1)}(\tau) \ + \ (1-\rho^2)^{1/2} B^{(2)}(\tau)}{q(\tau)} \end{bmatrix}, \tag{3.10}$$

with $\xrightarrow{\mathcal{D}}$ denoting convergence in distribution.

Proof. Let $||\cdot||$ be the metric on $D^2[0,\ 1]$ as defined in (3.9). Define two sequences of Brownian bridges $\{B_T^{(i)}(\tau);\ 0 \leq \tau \leq 1\}$ for $i = 1,\ 2$. Then, via statement (i) of Proposition 3.2.1, $||\mathbf{M}_T - \mathbf{B}_T(\tau)|| \ = \ o_P(1)$, as $T \to \infty$, where $\mathbf{B}_T(\tau) = [B_T^{(1)}(\tau), B_T^{(2)}(\tau)]'$, is a sequence of bivariate Brownian Bridges.

Proposition 3.2.2 characterizes the limiting behaviour of the processes (3.5) and (3.6) in terms of a vector of Brownian bridges that depend on unknown parameters: variance, skewness and kurtosis of the error term. Under symmetry of the distribution error, $\rho = 0$, the vector $\mathbf{M}_T$ converges to two identical and independent copies of a weighted Brownian bridge. In this case our testing framework for parameter changes in intercept or slope boils down to comparing the absolute value of the test statistics in (3.10) against the critical value at an α significance level, b_α, obtained from the corresponding tabulated asymptotic distribution. For example, if

$$\frac{1}{\sigma^2\sqrt{Var(\varepsilon_1^2)}}\sup_{0<\tau<1}\frac{|M_T^{(1)}(\tau)|}{q(\tau)} > b_\alpha > \sup_{0<\tau<1}\frac{1}{\sigma}\frac{|M_T^{(2)}(\tau)|}{q(\tau)},$$

the test detects a break only in the slope parameter. If

$$\frac{1}{\sigma^2\sqrt{Var(\varepsilon_1^2)}}\sup_{0<\tau<1}\frac{|M_T^{(1)}(\tau)|}{q(\tau)} > b_\alpha \text{ and } \frac{1}{\sigma}\frac{|M_T^{(2)}(\tau)|}{q(\tau)} > b_\alpha$$

both intercept and slope parameters have changed. Finally, if the critical value is greater than the two statistics the process is under the null hypothesis of no structural break.

For sake of generality we study the asymmetric case for the distribution error as well. Here, given the dependence between the two marginal asymptotic distributions, it is not clear how to construct the relevant asymptotic critical values. The following corollary introduces an alternative reformulation of the above proposition that solves this problem.

Corollary 3.2.1. *Under the same assumptions of Proposition* 3.2.2 *the following holds, as* $T \to \infty$,

$$\begin{bmatrix} \frac{1}{\sigma^2\sqrt{Var(\varepsilon_1^2)}}\frac{M_T^{(1)}(\cdot)}{q(\cdot)} \\ \frac{-\rho((1-\rho^2)\sigma^4 Var(\varepsilon_1^2))^{-\frac{1}{2}}M_T^{(1)}(\cdot)\ +\ ((1-\rho^2)\sigma^2)^{-\frac{1}{2}}M_T^{(2)}(\cdot)}{q(\cdot)} \end{bmatrix} \Rightarrow \begin{bmatrix} \frac{B^{(1)}(\cdot)}{q(\cdot)} \\ \frac{B^{(2)}(\cdot)}{q(\cdot)} \end{bmatrix}. \tag{3.11}$$

By the continuous mapping theorem,

$$\begin{bmatrix} \sup_{0<\tau<1}\frac{1}{\sigma^2\sqrt{Var(\varepsilon_1^2)}}\frac{|M_T^{(1)}(\tau)|}{q(\tau)} \\ \sup_{0<\tau<1}\frac{|-\rho((1-\rho^2)\sigma^4 Var(\varepsilon_1^2))^{-\frac{1}{2}}M_T^{(1)}(\tau)\ +\ ((1-\rho^2)\sigma^2)^{-\frac{1}{2}}M_T^{(2)}(\tau)|}{q(\tau)} \end{bmatrix} \xrightarrow{\mathcal{D}} \begin{bmatrix} \sup_{0<\tau<1}\frac{|B^{(1)}(\tau)|}{q(\tau)} \\ \sup_{0<\tau<1}\frac{|B^{(2)}(\tau)|}{q(\tau)} \end{bmatrix}, \tag{3.12}$$

with $\xrightarrow{\mathcal{D}}$ *denoting convergence in distribution.*

This bivariate process and the corresponding asymptotic theory enable us to introduce two different test statistics for the null hypothesis of no change in the above linear regression model (3.1). The first component remains unchanged; it is a robust test for the hypothesis of a change in the slope of LRM. The second test, however, is sensitive to changes in either intercept or slope and is used to define a joint hypothesis

$$H_{OJ} : \beta_0^{(1)} = \beta_0^{(2)} \text{ and } \boldsymbol{\beta}^{(1)} = \boldsymbol{\beta}^{(2)}. \tag{3.13}$$

The alternative hypothesis in the joint test corresponds to a change in at least one parameter of the model. A value of the test statistic greater than b_α implies the rejection of H_{OJ}. Further, the two tests in (3.12) can be combined in order to determine whether slope or intercept, if any, have changed. This new test, called simultaneous test, runs simultaneously the test statistic in the upper row of (3.12) to test for $H_{0,slope} : \boldsymbol{\beta}^{(1)} = \boldsymbol{\beta}^{(2)}$ and the second row test statistic for $H_{0,J}$. This composite test has power to detect a break only in the intercept if

$$\sup_{0<\tau<1} \frac{|-\rho((1-\rho^2)\sigma^4 Var(\varepsilon_1^2))^{-\frac{1}{2}} M_T^{(1)}(\tau) \; + \; ((1-\rho^2)\sigma^2)^{-\frac{1}{2}} M_T^{(2)}(\tau)|}{q(\tau)} > b_\alpha > \sup_{0<\tau<1} \frac{1}{\sigma^2\sqrt{Var(\varepsilon_1^2)}} \frac{|M_T^{(1)}(\tau)|}{q(\tau)}. \tag{3.14}$$

Likewise, if both statistics are below b_α the model is under the null simultaneous (also joint) hypothesis of no change in either intercept or slope. Note that if both test statistics are above b_α there is a break in slope, but the test is inconclusive about whether there is a break in the intercept or not. In this scenario we recommend the test based on $\frac{M_T^{(2)}(\tau)}{q(\tau)}$ discussed before, that is robust to possible changes in slope. Its asymptotic distribution is detailed in Proposition 3.2.1 statement ii). If this test rejects the null hypothesis of no change in intercept then one concludes there was a change in both slope and intercept; otherwise one concludes only the slope has changed.

The simultaneous test has several interesting features. First, the standardization provided in Corollary 3.2.1 guarantees that the marginal asymptotic Brownian bridges are independent and identically distributed. This implies that the critical values at the same significance level for each test in (3.12) are identical. Furthermore, one can control the global error rate by calculating the product of one minus the error rate idiosyncratic to each marginal test and noting the global error rate is one minus this product; i.e, if the idiosyncratic error is 5% then the global error rate is $1-(1-0.05)^2 \;=\; 0.0975$. Unfortunately in the inconclusive case in which we also have to run the marginal test for the intercept we lose independence between the corresponding Brownian bridge and that corresponding to the joint test statistic. This implies that in this case we lose control of the global error rate.

There remains the issue of more than one change in intercept/slope parameters and whether the statistics proposed here can be applied in such situations. The statistics defined by equations (3.5) and (3.6) can still be applied in this situation but only the parameter with the largest change in will be detected. For example, if there are multiple changes in the intercept but no more than one change in slope, then the process $\sup_{0<\tau<1}|\frac{M^{(2)}(\tau)}{q(\tau)}|$ will detect only the change in intercept which is largest and will neglect the other smaller changes. A similar statement can be made regarding changes to the slope parameter. If there are no more than one change in intercept, but more than one change in slope parameters, then statistic $\sup_{0<\tau<1}|\frac{M^{(2)}(\tau)}{q(\tau)}|$ will detect said change but only when the change in slope is largest. Moreover, Orasch (1998) has shown in the case of scale-location class of models that tests based on U-statistic type processes such as the ones developed here remain consistent regardless of the number of changes in location parameter. With some minor modifications, his theorem applies here as well. If there should be a change in both slope and intercept each occurring at different times Propositions 3.3.1 and 3.3.2 hold and each statistic has non-trivial power. Moreover, an estimate the timing of each rejection can be obtained using the estimator of $\widehat{t}^{\star}$ recommended in 3.17 by replacing the appropriate process for $\widehat{M}_T(\frac{k}{T})$.

3.2.2 Parameters Unknown

The processes defined in (3.5) and (3.6) depend on unknown parameters. Ordinary Least Squares (OLS) will produce consistent estimators of $\beta_0^{(i)}$ and $\boldsymbol{\beta}^{(i)}$ for $i = 1,\ 2$ under H_O and H_A; let these sequences of estimators be denoted $\{\widehat{\beta}_{T,0}^{(i)}\}_{T=1}^{\infty}$ and $\{\widehat{\boldsymbol{\beta}}_T^{(i)}\}_{T=1}^{\infty}$ for $i = 1,\ 2$. When these sample estimates are substituted for the population parameters, this produces the following slightly altered sequence of partial sum processes;

$$\widehat{M}_T^{(1)}(\tau) := T^{-1/2}\left\{\sum_{t=1}^{[(T+1)\tau]}(Y_t - \widehat{\beta}_{T,0} - \widehat{\boldsymbol{\beta}}_T'\mathbf{X}_t)^2 - \tau\sum_{t=1}^{T}(Y_t - \widehat{\beta}_{T,0} - \widehat{\boldsymbol{\beta}}_T'\mathbf{X}_t)^2\right\} \tag{3.15}$$

$$\widehat{M}_T^{(2)}(\tau) := T^{-1/2}\left\{\sum_{t=1}^{[(T+1)\tau]}(Y_t - \widehat{\boldsymbol{\beta}}_T'\mathbf{X}_t) - \tau\sum_{t=1}^{T}(Y_t - \widehat{\boldsymbol{\beta}}_T'\mathbf{X}_t)\right\}. \tag{3.16}$$

In $\widehat{M}_T^{(1)}(\tau)$ set

$$\widehat{\beta}_{T,0} = \begin{cases} \widehat{\beta}_{T,0}^{(1)}, & t \leq \widehat{t}^{\star} \\ \widehat{\beta}_{T,0}^{(2)}, & t > \widehat{t}^{\star}, \end{cases}$$

and $\widehat{\boldsymbol{\beta}}_T = \widehat{\boldsymbol{\beta}}_T^{(1)}$; and in $\widehat{M}_T^{(2)}(\tau)$ set

$$\widehat{\boldsymbol{\beta}}_T = \begin{cases} \widehat{\beta}_T^{(1)}, & t \leq \widehat{t}^{\star} \\ \widehat{\boldsymbol{\beta}}_T^{(2)}, & t > \widehat{t}^{\star}, \end{cases}$$

where $\widehat{t}^{\star}$ is some consistent estimator of $t^{\star}$. One consistent estimator of $t^{\star}$ that has been widely studied in the literature is defined as follows:

$$\widehat{t}^{\star} := \frac{1}{T}\min\left\{k : \frac{|\widehat{M}_T(\frac{k}{T})|}{q(\frac{k}{T})} = \max_{1\leq i<PT}\frac{|\widehat{M}_T(\frac{i}{T})|}{q(\frac{i}{T})}\right\} \tag{3.17}$$

$$\widehat{M}_T(t) = T^{-1/2}\left\{\sum_{t=1}^{[(T+1)\tau]}(Y_t - \widehat{\beta}_{LS,0} - \widehat{\boldsymbol{\beta}}_{LS}'\mathbf{X}_t)^2 - \tau\sum_{t=1}^{T}(Y_t - \widehat{\beta}_{LS,0} - \widehat{\boldsymbol{\beta}}_{LS}'\mathbf{X}_t)^2\right\},$$

where the subscript LS refers to the least squares estimator of β_0 and $\boldsymbol{\beta}$ using all T observations.

The asymptotic properties of this estimator have been studied by Antoch, Hušková and Veraverbeke (1995). They also show that the bootstrap approximation to this distribution is asymptotically valid. For more on this, we refer those interested to their paper.

The following lemma shows the absence of estimation risk for the test statistics based on $\widehat{M}_T^{(i)}(\tau)$, with $i = 1, 2$.

Lemma 3.2.1. *Assume* $\{\widehat{\beta}_{T,0}\}_{T=1}^{\infty}$ *and* $\{\widehat{\boldsymbol{\beta}}_T\}_{T=1}^{\infty}$ *are sequences of consistent estimators of the parameters in* (3.1)*. Then, under the same conditions of Proposition* 3.2.1*,*

$$\sup_{0<\tau<1} \frac{|M_T^{(i)}(\tau) \; - \; \widehat{M}_T^{(i)}(\tau)|}{q(\tau)} \;=\; o_P(1),$$

for $i = 1, 2$*, as* $T \to \infty$*.*

Proof: We only show this result for (3.15), as a similar argument applies to process (3.16).

$$\begin{aligned}
T^{1/2}\widehat{M}_T^{(1)}(\tau) &= T^{1/2}M_T^{(1)}(\tau) + (\widehat{\boldsymbol{\beta}}_T' - \boldsymbol{\beta}')\left(\sum_{t=1}^{[(T+1)\tau]} X_t - \tau\sum_{t=1}^{T} X_t\right) \\
\frac{|\widehat{M}_T^{(1)}(\tau) - M_T^{(1)}(\tau)|}{q(\tau)} &\leq \left|\frac{(\widehat{\boldsymbol{\beta}}_T - \boldsymbol{\beta})'(\sum_{t=1}^{[(T+1)\tau]} X_t - \tau\sum_{t=1}^{T} X_t)}{q(\tau)T^{1/2}}\right| \\
&\leq ||\widehat{\boldsymbol{\beta}}_T' - \boldsymbol{\beta}'||^E \left|\left|\frac{\sum_{t=1}^{[(T+1)\tau]} X_t - \tau\sum_{t=1}^{T} X_t}{T^{1/2}q(\tau)}\right|\right|^E \qquad (3.18) \\
\frac{|\widehat{M}_T^{(1)}(\tau) - M_T^{(1)}(\tau)|}{q(\tau)} &\leq ||\widehat{\boldsymbol{\beta}}_T' - \boldsymbol{\beta}'||^E \left|\left|\sup_{0<\tau<1}\frac{|\sum_{t=1}^{[(T+1)\tau]} X_t - \tau\sum_{t=1}^{T} X_t|}{T^{1/2}q(\tau)}\right|\right|^E \\
\sup_{0<\tau<1}\frac{|\widehat{M}_T^{(1)}(\tau) - M_T^{(1)}(\tau)|}{q(\tau)} &\leq ||\widehat{\boldsymbol{\beta}}_T' - \boldsymbol{\beta}'||^E \left|\left|\sup_{0<\tau<1}\frac{|\sum_{t=1}^{[(T+1)\tau]} X_t - \tau\sum_{t=1}^{T} X_t|}{T^{1/2}q(\tau)}\right|\right|^E \\
&= o_P(1)O_P(1) = o_P(1),
\end{aligned}$$

where $||\cdot||^E$ refers to the Euclidean norm on $\mathbb{R}^{\mathrm{k}}$. The Cauchy-Swarchz Inequality was used to obtian the result in line 3.18.

Lemma 3.2.1 shows that the results in Propositions 3.2.1 and 3.2.2 and Corollary 3.2.1 continue to hold when the parameters are replaced by the above estimators. This result is obviously extended to the case given by the sample versions of the parameters ρ, σ and $Var(\varepsilon_1^2)$.

3.3 Asymptotics Under the Alternative Hypothesis

Here, the asymptotics of statistics defined as supremum of (3.5) and (3.6) are studied. The first of two Propositions to follow describes the distribution of statistic (3.5) under local alternatives of at-most-one change in the slope.

Proposition 3.3.1. *Assume H_A in equation (3.1), moment conditions (3.2), $t^\star = [T\tau^\star]$, $\tau^\star \in (0,\ 1)$ hold, set $\boldsymbol{\beta}^{(2)} = \boldsymbol{\beta}^{(1)} + \boldsymbol{\delta}$ where $\boldsymbol{\delta} = (\delta_1, \ldots, \delta_K)$. Then $\sigma^{2\star} = \mathbb{E}(Y_1 - \beta_0^{(1)} - \boldsymbol{\beta}^{(1)'}\mathbf{X}_1)^2 + \boldsymbol{\delta}'\mathbb{E}[\mathbf{X}_1\mathbf{X}_1']\boldsymbol{\delta}$, with $\boldsymbol{\delta} = \boldsymbol{\delta}(T) \to \mathbf{0}$ and $\Lambda = \Lambda(T) \to 0$ as $T \to \infty$. Let $q(\cdot)$ satisfy* A.1 *and* A.2*, then as $T \to \infty$,*

$$\frac{q(\tau^\star)}{\sqrt{\tau^\star(1-\tau^\star)}}\frac{1}{\sigma^2\sqrt{Var(\varepsilon_1^2)}}\left\{\sup_{0<\tau<1}\frac{|M_T^{(1)}(\tau)|}{q(\tau)} - T^{1/2}\left(\boldsymbol{\delta}'\mathbb{E}[\mathbf{X}_1\mathbf{X}_1']\boldsymbol{\delta}'\frac{t^\star}{T}\frac{\left(1-\frac{t^\star}{T}\right)}{q(\frac{t^\star}{T})}\right)\right\} \xrightarrow{\mathcal{D}} N(0,1).$$

Proof. This follows from Theorem 1.4 of GHH.

Remark 3.3.1. *Proposition 3.3.1 reveals that a one-time change in slope parameters will cause a one-time change in variance only if the following condition holds:*

$$\boldsymbol{\delta}'\mathbf{X}_1 \neq 0.$$

A direct result of Proposition 3.3.1 is the consistency of this test for a one-time change in slope. This result is formally introduced in the next corollary.

Corollary 3.3.1. *Under the conditions of Proposition* 3.3.1*, then, as* $T \to \infty$*,*

$$\frac{1}{T^{1/2}(\boldsymbol{\delta}' I\!\!E[\mathbf{X}_1\mathbf{X}_1']\boldsymbol{\delta})} \sup_{0<\tau<1} \frac{|M_T^{(1)}(\tau)|}{q(\tau)} \xrightarrow{P} \frac{\tau^\star(1-\tau^\star)}{q(\tau^\star)}.$$

The next proposition details the asymptotic distribution of the AMOC in intercept statistic (cf. (3.6)).

Proposition 3.3.2. *Assume* H_A *in equation* (3.1)*, moment conditions* (3.2)*,* $t^\star = [T\tau^\star]$*,* $\tau^\star \in (0,\ 1)$ *and* $\beta_0^{(2)} = \beta_0^{(1)} + \Lambda$ *hold. Then for* $q(\cdot)$ *satisfying* A.1 *and* A.2 *and as* $T \to \infty$*,*

$$\frac{q(\tau^\star)}{\sigma\sqrt{\tau^\star(1-\tau^\star)}} \left\{ \sup_{0<\tau<1} \frac{|M_T^{(2)}(\tau)|}{q(\tau)} - T^{1/2}\Lambda \frac{t^\star}{T} \frac{\left(1-\frac{t^\star}{T}\right)}{q(\frac{t^\star}{T})} \right\} \xrightarrow{\mathcal{D}} N(0,1).$$

Proof. Without loss of generality, let $\frac{t^\star}{T} > \tau$, $t^\star = [(T+1)\tau^\star]$ and assume $\delta_0(T) \to 0$, as $T \to 0$ and $\delta_0(T)T \to 0$, as $T \to \infty$. Then

$$\begin{aligned}
\sup_{\frac{t^\star}{T}-\delta_0(T)<\tau<\frac{t^\star}{T}+\delta_o(T)} \frac{1}{q(\tau)}\frac{|M_T^{(2)}(\tau)|}{q(\tau)} &= T^{-1/2} \sup_{\frac{t^\star}{T}-\delta_0(T)<\tau<\frac{t^\star}{T}+\delta_o(T)} \Bigg| \sum_{t=1}^{[(T+1)\tau]} (Y_t - \boldsymbol{\beta}^{'} X_t) - \tau \sum_{t=1}^{t^\star} (Y_t - \boldsymbol{\beta}^{'} X_t) \\
&\qquad - \tau \sum_{t=t^\star+1}^{T} (Y_t - \boldsymbol{\beta}^{'} X_t) \Bigg| \\
&= T^{-1/2} \sup_{\frac{t^\star}{T}-\delta_0(T)<\tau<\frac{t^\star}{T}+\delta_0(T)} \frac{1}{q(\tau)} \Bigg| \sum_{t=1}^{[(T+1)\tau]} (Y_t - \boldsymbol{\beta}^{'} X_t) - \tau \sum_{t=1}^{t^\star} (Y_t - \boldsymbol{\beta}^{'} X_t) \\
&\qquad - \tau \sum_{t=t^\star+1}^{T} (Y_t - \boldsymbol{\beta}^{'} X_t) \Bigg| \\
&= T^{-1/2} \sup_{\frac{t^\star}{T}-\delta_0(T)<\tau<\frac{t^\star}{T}+\delta_0(T)} \frac{1}{q(\tau)} \Bigg| \sum_{t=1}^{[(T+1)\tau]} ((Y_t - \boldsymbol{\beta}^{'} X_t) - \beta_0^{(1)}) - \tau \sum_{t=1}^{t^\star} ((Y_t - \boldsymbol{\beta}^{'} X_t) - \beta_0^{(1)}) \\
&\qquad - \tau \sum_{t=t^\star+1}^{T} ((Y_t - \boldsymbol{\beta}^{'} X_t) - \beta_0^{(2)}) \; + \; ([(T+1)\tau] - \tau t^\star)\beta_0^{(1)} - \tau(T - t^\star)\beta_0^{(2)} \Bigg| \\
&= T^{-1/2} \sup_{\frac{t^\star}{T}-\delta_0(T)<\tau<\frac{t^\star}{T}+\delta_0(T)} \left| \sigma \left(\sum_{t=1}^{[(T+1)\tau]} \varepsilon_t - \tau \sum_{t=1}^{T} \varepsilon_t \right) \; - \; \tau(T - t^\star)\Lambda \right| \\
&= \left| \frac{\sigma}{T^{1/2}} \left(\sum_{t=1}^{t^\star} \varepsilon_t - \tau \sum_{t=1}^{T} \varepsilon_i \right) \; - \; T^{1/2} \frac{t^\star}{T} \left(1 - \frac{t^\star}{T} \right) \Lambda \right| \qquad (3.19)
\end{aligned}$$

Lemma 3.3.1, found below, will be needed to establish the proposition. The absolute value in equation (3.19) can be removed as it has no effect on the limiting distribution: that is, when inside the absolute value is negative, simply multiply by -1 and remove the absolute value. Hence, we have

$$\frac{\sigma}{T^{1/2}} \left(\sum_{t=1}^{t^\star} \varepsilon_t - \tau \sum_{t=1}^{T} \varepsilon_t \right) \; - \; T^{1/2} \frac{t^\star}{T} \left(1 - \frac{t^\star}{T} \right) \Lambda. \qquad (3.20)$$

Now, Lemma 3.3.1 and (3.19) establish the above proposition.

Lemma 3.3.1. *Under the same conditions as specified in Proposition* 3.3.1, *and as* $T \to \infty$,

$$\begin{bmatrix} \frac{\sum_{t=1}^{t^\star} \varepsilon_t}{T^{1/2}} \\ \frac{\sum_{t=1}^{T} \varepsilon_t}{T^{1/2}} \end{bmatrix} \xrightarrow{\mathcal{D}} N \left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \ \Psi \right],$$

where

$$\Psi = \sigma_1^2 \begin{bmatrix} \tau^{\star 2} & \tau^\star \\ \tau^\star & 1 \end{bmatrix}.$$

Proof. This follows from the bivariate version of the Lindberg-Levy Central Limit Theorem.

A corollary similar to Corollary 3.3.1 holds here as well and is a direct consequence of Proposition 3.3.2.

Corollary 3.3.2. *Under the conditions of Proposition* 3.3.2*, and as* $T \to \infty$*,*

$$\frac{1}{\sigma\ T^{1/2}} \sup_{0<\tau<1} \frac{|M_T^{(2)}(\tau)|}{q(\tau)} \xrightarrow{P} \frac{\tau^\star(1-\tau^\star)}{q(\tau^\star)}.$$

3.4 Monte Carlo Simulation

In this section the LRM (cf. (3.1)), with $K = 1$, is estimated and the statistic calculated first under the assumption that $t^\star \geq T$ which provides an estimate of nominal coverage of these tests and next considers two further simulations: one considering a one-time change in the intercept β_0 and the other considering a one time change in slope β. This will allow a more realistic assessment of the ability of the newly fashioned statistics to detect a change in intercept/slope and follows closely the criteria used by PKA which requires tests to obtain a nominal coverage consistent with the corresponding significance level: 5% in the simulation undertaken here. As the second criterion, the tests will be compared on their empirical power. Under the two scenarios just discussed, the simulation considered here first sets the distribution of the error term in the LRM to a χ_1^2 with one degree of freedom and then a second simulation sets the distribution of the errors to a standard normal, i.e. $N(0,1)$ random variable.

More specifically, for the purpose of this Monte Carlo study the entertained model is given by

$$Y_t = \begin{cases} \beta_0^{(1)} + \beta^{(1)} X_t + \sigma\varepsilon_t, & 1 \leq t \leq t^\star, \\ \beta_0^{(2)} + \beta^{(2)} X_t + \sigma\varepsilon_t, & t^\star < t \leq T, \end{cases} \tag{3.21}$$

where the ε_t's satisfy conditions detailed in (3.2), and the corresponding change point hypothesis test is

$$H_{OJ} : \beta_0^{(1)} = \beta_0^{(2)} \text{ and } \beta^{(1)} = \beta^{(2)}$$

versus the one-time change alternative,

$$H_{AJ} : \beta_0^{(1)} \neq \beta_0^{(2)} \text{ or } \beta^{(1)} \neq \beta^{(2)} \text{ for some } t^\star \text{ satisfying } 1 < t^\star < T.$$

For the first, of two simulation studies undertaken here, the slope parameter was fixed at $\beta^{(1)} = 3$ and the intercept parameter, β_0, was allowed to change from $\beta_0^{(1)} = 1$ under H_O to one of the three values of $\beta_0^{(2)} = 1.25,\ 1.5,\ 1.75,\ 2$ under H_A. The second simulation study fixed $\beta_0^{(1)} = 1$ and allowed only the slope parameter to change from $\beta_{(1)} = 3$ under H_O to $\beta^{(2)} = 3.75,\ 4.5,\ 5.25,\ 6$.

The results from the simulation are recorded in Section 3.7. Tables 3.3 and 3.4 tabulate the empirical power under the null hypothesis H_{OJ} of no change in either intercept and slope. Table 3.3 records results for the case when $\varepsilon_t \sim \chi_1^2$, for $t = 1, \ldots, T$ with one degree of freedom; while Table 3.4 records results for the case when $\varepsilon_t \sim N(0,1)$, for $t = 1, \ldots, T$.

Tables 3.3 and 3.4 indicate that the nominal coverage of all the test statistics under study, except the fluctuation test of PKK which has a nominal coverage probability of around 20%, have nominal coverage less than 8% - the significance level throughout these simulations was chosen to be 5%. Both tables provide clear evidence that the two statistics fashioned from the U-statistic type process and weighted by the function $q(\tau, \nu = \frac{15}{128})$[1] perform very well in terms of nominal coverage; the coverage is less than or equal 8%. A similar statement can be made regarding the test statistics used by AP and Andrews (1993); the exponential average Wald (EXPW) test reports a nominal coverage just below 5%, while the supremum Wald (MAXW) test, reports a nominal coverage well below 5%. As the PKK fluctuation test does not meet the first criteria of our adopted PKA criteria - nominal coverage consistent with adopted significance level, the fluctuation test is not appropriate for the sample sizes considered here.

Tables 3.5 and 3.6 detail the empirical power of the newly fashioned test statistics as well as the competitors under the alternative hypothesis of a one-time change in intercept. The lack of symmetry of the χ_1^2 has a positive effect on the JOINT test (JOINT refers to the test statistic in the second element of the vector detailed in 3.12 of Corollary 3.2.1 with sample estimators replacing population parameters. For clarity of exposition, this test statistic is provided below and will be referred to as M_T^{JOINT} for the sake of this simulation;

$$M_T^{JOINT} := \sup_{0<\tau<1} \frac{\left| -\widehat{\rho}((1-\widehat{\rho}^2)\widehat{\sigma}^4 \widehat{Var(\varepsilon_1^2)})^{-\frac{1}{2}} \widehat{M}_T^{(1)}(\tau) \ + \ ((1-\widehat{\rho}^2)\widehat{\sigma}^2)^{-\frac{1}{2}} \widehat{M}_T^{(2)}(\tau) \right|}{q(\tau)}.$$

The symmetry of the standard normal distribution decreases the empirical power of the JOINT test but increases the empirical power of the CUSUM/EXPW/MAXW test which is consistent with the optimal results of the exponential average test: the EXPW test of

[1]Similar results are obtained for alternative choices of ν. The value $\nu = \frac{15}{128}$ is observed to maximize the power of the test for this data generating process.

Andrews and Ploberger (1994) and Andrews, Lee and Ploberger (1996) is locally optimal in large samples for all distributions of the errors and optimal in finite samples when the errors are normally distributed. Notwithstanding the optimality of EXPW, the JOINT test (M_T^{JOINT} Table 3.6) displays much higher empirical power. In particular, for a change in the middle of the sample, the power reaches a value of almost 1. All tests entertained here lack power to detect a change in intercept when it occurs either late or early in the sample.

The last simulation undertaken was to determine the ability of the simultaneous test to detect a one-time change in slope of the LRM detailed in (3.21). Tables 3.7 and 3.8 record results from the simulation when the residuals were distributed as a χ_1^2 with 1 degree of freedom and when the errors were normally distributed (N(0,1)), respectively. Under this alternative hypothesis, the JOINT test and $\sup\limits_{0<\tau<1} \frac{|M_T^{(1)}(\tau)|}{q(\tau,\nu=\frac{15}{128})}$ continue to perform very well for a one-time change in slope regardless of where the change occurs. The only departure now is that both tests constructed here must be employed; the JOINT test performs well when the change occurs in the middle of the sample, while $\sup\limits_{0<\tau<1} \frac{|M_T^{(1)}(\tau)|}{q(\tau,\nu=\frac{15}{128})}$ performs well for changes that occur early or later on in the sample.

3.5 Application to Manager's Performance in Mutual Fund Industry

The Capital Asset Pricing model (CAPM) and its many variants has been used by academics for applied financial research and by practitioners. One notable use of the CAPM was by some investment houses to evaluate managers' performance; by subtracting predicted portfolio returns from realized returns an estimate of a manager's alpha could be obtained. If this estimated alpha happened to be positive, the manager was said to have produced a positive alpha and compensated accordingly. Such simple compensation schemes are seldom used today but they do suggest the importance of this model and provide an interesting interpretation of alpha.

More recently, academic research conducted by Barras, Scaillet and Wermers (2010, hereafter BSW) has searched for outperforming mutual fund managers via extended versions of the CAPM but in such a way that they control for 'false discoveries': a false discovery occur when the estimated alpha is statistically significant and less/greater than 0 when the true alpha is zero. BSW is a significant and current paper that uses well-established financial and econometrics methods and data sources. Because of their recent contribution and use of well-established methods, it is natural then to adopt their econometric model and data sources in order to provide a sound application of the methods developed in previous sections. Our interest here, however, is not to evaluate or even suggest further refinements to their methods but rather to apply the statistics developed and outlined in the previous sections of this paper to see whether we can detect a change in the alpha; a change in alpha can be suggestive of an improvement/impairment in a fund manager's stock selection skills.

The method of estimating fund manager's performance follows that of BSW ; they estimate a four factor model proposed by Carhart (1997);

$$r_{i,t} = \alpha_i + b_i \cdot r_{m,t} + s_i \cdot r_{smb,t} + h_i \cdot r_{hml,t} + m_i \cdot r_{mom,t} + \varepsilon_{i,t}. \tag{3.22}$$

Here, $r_{i,t}$ is the monthly tth excess return of fund i over the risk free rate (proxied by the monthly 30-day T-bill beginning-of-month yield); $r_{m,t}$ is the month t excess return on the NASDAQ and $r_{smbt}, r_{hml,t}$ and $r_{mom,t}$ are the month t returns on zero-investment factor mimicking portfolios for size, book-to-market, and momentum obtained from Kenneth French's website. The mutual fund price data was obtained from yahoofinance.com and consisted of 20 funds listed in Table 3.10 which can be found in Section 3.7. In order to account for distributions that mutual funds regularly pay and possible stock splits, the adjusted closed price[2]was used to calculate monthly return rather than the close price. In order to keep the analysis current, the evaluation period ran form January 2001 to August 2010.

Table 3.9 provides results of the tests for intercept or slope when applied to the residuals of Carhart's four factor model. The results provide some interesting findings regarding how to implement the tests developed here. For example, Fund 17 (UBS GLOBAL Equity) reports a result from the JOINT test of 1.50 which is significant at a 10% significance level. The test value for change in slope is 2.14 which is statistically significant at a 5% significance level. Hence, there appears to be a change in slope for this mutual fund. To determine whether there happens to be a change in intercept, $M_T^{(2)}(\tau)$ must be used. The test value is 1.8 which is also statistically significant at 5%. Table 3.1 found below records the estimated parameters for Carhart's four factor model. We see that the alpha was negative before January of 2003, then become statistically insignificant. The estimated parameter of excess return on the NASDAQ changed the most - increased from 0.408 to 0.99.

Summing up then, for this mutual fund there appears to be an increase in alpha and the beta corresponding to the NASDAQ[3] of the betas, that occurred in January, 2003. This structural break in both intercept and slope suggests that there are two types of changes; one associated with a change in market risk described by the four factor model, and a second effect due to idiosyncratic component of the fund. We attribute the increase in alpha to an improvement in managerial ability.

A second interesting example is Fund 10 (Goldman Sachs Growth Opportunities), here

[2]The adjusted close price adjusts the closing price for the requested day, week, or month, adjusted for all applicable splits and distributions/dividend payments. Data is adjusted using appropriate split and dividend multipliers, adhering to Center for Research in Security Prices (CRSP) standards. Split multipliers are determined by the split ratio. For instance, in a 2 for 1 split, the pre-split data is multiplied by 0.5. Dividend multipliers are calculated based on dividend as a percentage of price, primarily to avoid negative historical pricing. For example, when a \$0.08 cash dividend is distributed on Feb 19 (ex-date), and the Feb 18 closing price was \$24.96, the pre-dividend data is multiplied by (1-0.08/24.96) = 0.9968.

[3]The size of $\sup_{0<\tau<0} \frac{|M_T^{(2)}(\tau)|}{q(\tau)}$ is driven by the largest change in one of the slope parameters. In this case the estimated change in the beta corresponding to the NASDAQ is 0.12, while the estimated change in the beta of the book-to-market is only 0.0375 which is very small. This leads to the conclusion made here that the beta of NASDAQ changed.

Table 3.1: Estimated Version of Carhart's Model

4 Factor Model Before Change - Fund 17			
Coefficient	Estimates	Strd Error	P-value
Intercept	-0.016	0.0068	0.028
Size	-0.0068	0.0038	0.0986
Book-to-Market	0.014	0.039	0.0026
Momentum	0.00144	0.100148	0.349
NASDAQ	0.41	0.13715	0.0100
4 Factor Model After Change			
Coefficient	Estimates	Strd Error	P-value
Intercept	0.00346	0.00244	0.1602
Size	-0.09	0.0038	0.000
Book-to-Market	0.0089	0.00153	0.000
Momentum	-0.00123	0.0004	0.009
NASDAQ	0.995	0.02	0.000

we note that the Simultaneous test reports a value of only 1.43 which is almost statistically significant at the 10% level. Both marginal tests reject the null hypothesis at a 5% significance level. The low value of the simultaneous test even when there appears to be a change in slope and/or intercept was noted in the simulation section; we found the simultaneous test to be somewhat weak at detecting a change in the slope when it occurred early in the evaluation period or when the underlying distribution of the residuals was asymmetric. As a possible remedy, we propose the use of large significance levels when employing the simultaneous test, perhaps using a 15% to 20% significance level for this test, while maintaining a 5% significance level when testing for change in slope or intercept. The regression results for Carhart's four factor model are provided in Table 3.2 located below. Again, we see an increase in alpha which we attribute to improvement in managerial ability.

The results for the rest of the funds are mixed; we find funds with no change in either alpha or beta parameters, funds with a change in betas only and funds with a change in both slope and intercept. For those funds in which no change in any parameters was reported, we conclude there was no change in market risk or idiosyncratic components of the fund. For funds that our tests reported a change in betas only, we conclude there was a change in market risk, and for those funds that reported a change in alpha and betas, a similar

Table 3.2: Estimated Version of Carhart's Model

4 Factor Model Before Change - Fund 10			
Coefficient	Estimates	Strd Error	P-value
Intercept	-0.039	0.012	0.079
Size	-0.00532	0.0036	0.1680
Book-to-Market	0.003	0.0038	0.443
Momentum	0.00204	0.0031	0.522
NASDAQ	0.40	0.194	0.0684
4 Factor Model After Change			
Coefficient	Estimates	Strd Error	P-value
Intercept	0.00128	0.00221	0.5639
Size	-0.017	0.00126	0.18
Book-to-Market	0.00169	0.00144	0.24
Momentum	-0.0004	0.0004	0.3095
NASDAQ	0.99	0.02	0.000

conclusion to that made above can be made; there was a change in both market risk and in manager's stock selection abilities.

3.6 Conclusion

This paper has introduced a statistical methodology to disentangle between structural breaks in the intercept and slope of linear regression models. The test statistic is constructed from a bivariate U-statistic type process that can accommodate the presence of weight functions that improve the power of structural break tests against changes that occur early and later on in the evaluation period. We have shown that the test also exhibits power against changes in the skewness of the error distribution.

The application to uncover time varying mutual fund manager's performance has shown a change in mutual fund performance starting in January 2001. This phenomenon is robust to the possible changes in market risk. In fact, we also observe during the period under analysis a change in market risk.

3.7 Tables

Table 3.3: Nominal Coverage

No Change in Parameters - Errors χ_1^2 Distributed			
	T = 75	T = 100	T =125
$\sup\limits_{0<\tau<1} \frac{\|M_T^{(1)}(\tau)\|}{q(\tau,\nu=\frac{15}{128})}$	0.02	0.02	0.02
CUSUM Test	0.044	0.026	0.044
EXPW	0.02	0.02	0.02
MAXW	0.01	0.01	0.01
FLUCT	**0.192**	**0.18**	**0.152**
M_T^{JOINT}	0.08	0.08	0.07

Table 3.4: Nominal Coverage

No Change in Parameters - Errors $N(0,1)$ Distributed			
	T = 75	T = 100	T=125
$\sup\limits_{0<\tau<1} \frac{\|M_T^{(1)}(\tau)\|}{q(\tau,\nu=\frac{15}{128})}$	0.030	0.028	0.026
CUSUM Test	0.050	0.056	0.040
EXPW	0.08	0.05	0.06
MAXW	0.03	0.02	0.02
FLUCT	**0.224**	**0.224**	**0.208**
M_T^{JOINT}	0.052	0.046	0.048

$G(x) = \mathbb{P}\left\{\sup\limits_{0<\tau<1} \frac{\|B(\tau)\|}{q(\tau,\nu=\frac{15}{128})<x}\right\}$							
x	$G(x)$	x	$G(x)$	x	$G(x)$	x	$G(x)$
0.53	0.01	0.83	0.26	1.01	0.51	1.23	0.76
0.57	0.02	0.83	0.27	1.01	0.52	1.24	0.77
0.60	0.03	0.84	0.28	1.02	0.53	1.25	0.78
0.62	0.04	0.85	0.29	1.03	0.54	1.26	0.79
0.63	0.05	0.85	0.30	1.04	0.55	1.28	0.80
0.65	0.06	0.86	0.31	1.05	0.56	1.29	0.81
0.66	0.07	0.87	0.32	1.05	0.57	1.30	0.82
0.67	0.08	0.88	0.33	1.06	0.58	1.32	0.83
0.68	0.09	0.88	0.34	1.07	0.59	1.33	0.84
0.70	0.10	0.89	0.35	1.08	0.60	1.35	0.85
0.71	0.11	0.90	0.36	1.09	0.61	1.37	0.86
0.72	0.12	0.90	0.37	1.10	0.62	1.39	0.87
0.73	0.13	0.91	0.38	1.10	0.63	1.40	0.88
0.74	0.14	0.92	0.39	1.11	0.64	1.42	0.89
0.75	0.15	0.93	0.40	1.12	0.65	1.45	0.90
0.75	0.16	0.93	0.41	1.13	0.66	1.47	0.91
0.76	0.17	0.94	0.42	1.14	0.67	1.50	0.92
0.77	0.18	0.95	0.43	1.14	0.68	1.53	0.93
0.78	0.19	0.96	0.44	1.15	0.69	1.56	0.94
0.79	0.20	0.97	0.45	1.16	0.70	1.61	0.95
0.79	0.21	0.97	0.46	1.18	0.71	1.67	0.96
0.80	0.22	0.98	0.47	1.19	0.72	1.72	0.97
0.81	0.23	0.99	0.48	1.20	0.73	1.80	0.98
0.81	0.24	0.99	0.49	1.21	0.74	1.92	0.99
0.82	0.25	1.00	0.50	1.22	0.75	2.04	1.00

Table 3.5: Empirical Power

Change in Intercept - β_0												
Errors χ_1^2 Distributed												
MIDDLE OF SAMPLE ($\tau^\star = 0.5$)												
	$\beta_0^{(2)} = 1.25$			$\beta_0^{(2)} = 1.5$			$\beta_0^{(2)} = 1.75$			$\beta_0^{(2)} = 2$		
Statistic	T = 75	T = 100	T=125	T=75	T = 100	T=125	T=75	T = 100	T=125	T = 75	T = 100	T=125
$\sup_{0<\tau<1} \frac{\lvert M_T^{(1)}(\tau)\rvert}{q(\tau,\nu=\frac{15}{218})}$	0.024	0.027	0.021	0.027	0.02	0.023	0.037	0.025	0.032	0.033	0.039	0.04
CUSUM	0.057	0.06	0.057	0.104	0.128	0.161	0.218	0.273	0.336	0.379	0.479	0.578
EXPW	0.043	0.036	0.04	0.167	0.173	0.215	0.338	0.397	0.427	0.517	0.623	0.714
MAXW	0.017	0.007	0.015	0.093	0.089	0.11	0.234	0.269	0.287	0.41	0.512	0.596
FLUC	0.265	0.211	0.227	0.382	0.372	0.38	0.518	0.564	0.636	0.663	0.752	0.838
M_T^{JOINT}	0.09	0.152	0.206	0.428	0.584	0.733	0.798	0.907	0.976	0.961	0.993	0.999
LATE DETECTION ($\tau^\star = 0.9$)												
	T = 75	T = 100	T=125	T=75	T = 100	T=125	T=75	T = 100	T=125	T = 75	T = 100	T=125
$\sup_{0<\tau<1} \frac{\lvert M_T^{(1)}(\tau)\rvert}{q(\tau,\nu=\frac{15}{218})}$	0.021	0.031	0.021	0.025	0.024	0.033	0.028	0.017	0.023	0.028	0.035	0.048
CUSUM	0.044	0.044	0.036	0.035	0.052	0.051	0.198	0.034	0.054	0.035	0.045	0.047
EXPW	0.037	0.037	0.027	0.05	0.039	0.058	0.077	0.075	0.087	0.137	0.152	0.185
MAXW	0.015	0.015	0.012	0.035	0.021	0.03	0.051	0.045	0.051	0.094	0.097	0.114
FLUC	0.225	0.201	0.164	0.234	0.223	0.226	0.493	0.276	0.232	0.289	0.258	0.363
M_T^{JOINT}	0.063	0.053	0.067	0.059	0.054	0.072	0.07	0.077	0.129	0.091	0.155	0.254
EARLY DETECTION ($\tau^\star = 0.1$)												
	T = 75	T = 100	T=125	T=75	T = 100	T=125	T=75	T = 100	T=125	T = 75	T = 100	T=125
$\sup_{0<\tau<1} \frac{\lvert M_T^{(1)}(\tau)\rvert}{q(\tau,\nu=\frac{15}{218})}$	0.031	0.023	0.028	0.033	0.029	0.024	0.031	0.026	0.025	0.033	0.044	0.045
CUSUM	0.04	0.059	0.051	0.112	0.126	0.166	0.239	0.253	0.337	0.348	0.455	0.5
EXPW	0.024	0.023	0.016	0.03	0.035	0.035	0.053	0.069	0.06	0.104	0.157	0.18
MAXW	0.011	0.014	0.012	0.009	0.013	0.007	0.017	0.02	0.016	0.038	0.084	0.038
FLUC	0.241	0.199	0.171	0.238	0.197	0.185	0.257	0.197	0.186	0.262	0.259	0.219
M_T^{JOINT}	0.082	0.064	0.044	0.066	0.072	0.104	0.089	0.171	0.269	0.19	0.411	0.523

Table 3.6: Empirical Power

Change in Intercept - β_0												
Errors $N(0,1)$ Distributed												
MIDDLE OF SAMPLE $(\tau^\star = 0.5)$												
	$\beta_0^{(2)} = 1.25$			$\beta_0^{(2)} = 1.5$			$\beta_0^{(2)} = 1.75$			$\beta_0^{(2)} = 2$		
Statistic	T = 75	T = 100	T=125	T=75	T = 100	T=125	T=75	T = 100	T=125	T = 75	T = 100	T=125
$\sup_{0<\tau<1} \frac{\lvert M_T^{(1)}(\tau)\rvert}{q(\tau,\nu=\frac{15}{218})}$	0.04	0.029	0.037	0.042	0.026	0.027	0.017	0.021	0.024	0.022	0.015	0.022
CUSUM	0.072	0.075	0.096	0.144	0.185	0.243	0.345	0.439	0.596	0.57	0.746	0.848
EXPW	0.064	0.051	0.043	0.142	0.168	0.172	0.346	0.379	0.44	0.527	0.616	0.707
MAXW	0.028	0.018	0.012	0.072	0.043	0.08	0.227	0.245	0.289	0.404	0.487	0.591
FLUC	0.297	0.28	0.277	0.431	0.509	0.564	0.721	0.788	0.885	0.896	0.971	0.996
M_T^{JOINT}	0.072	0.096	0.134	0.268	0.416	0.531	0.641	0.822	0.913	0.899	0.981	0.998
LATE DETECTION $(\tau^\star = 0.1)$												
	T = 75	T = 100	T=125	T=75	T = 100	T=125	T=75	T = 100	T=125	T = 75	T = 100	T=125
$\sup_{0<\tau<1} \frac{\lvert M_T^{(1)}(\tau)\rvert}{q(\tau,\nu=\frac{15}{218})}$	0.037	0.027	0.038	0.044	0.037	0.038	0.027	0.041	0.053	0.053	0.076	0.108
CUSUM	0.042	0.045	0.055	0.054	0.034	0.041	0.0349	0.043	0.049	0.039	0.049	0.049
EXPW	0.081	0.081	0.066	0.103	0.135	0.142	0.166	0.256	0.256	0.341	0.418	0.511
MAXW	0.081	0.035	0.019	0.055	0.063	0.062	0.095	0.155	0.156	0.244	0.309	0.386
FLUC	0.261	0.21	0.181	0.26	0.249	0.214	0.704	0.331	0.276	0.302	0.302	0.452
M_T^{JOINT}	0.035	0.034	0.04	0.036	0.047	0.064	0.075	0.085	0.134	0.091	0.174	0.255
EARLY DETECTION $(\tau^\star = 0.9)$												
	T = 75	T = 100	T=125	T=75	T = 100	T=125	T=75	T = 100	T=125	T = 75	T = 100	T=125
$\sup_{0<\tau<1} \frac{\lvert M_T^{(1)}(\tau)\rvert}{q(\tau,\nu=\frac{15}{218})}$	0.036	0.038	0.033	0.045	0.043	0.054	0.074	0.069	0.068	0.116	0.16	0.143
CUSUM	0.09	0.085	0.085	0.14	0.178	0.243	0.288	0.377	0.447	0.483	0.631	0.723
EXPW	0.088	0.064	0.06	0.103	0.119	0.13	0.172	0.239	0.3	0.295	0.441	0.519
MAXW	0.044	0.064	0.017	0.045	0.048	0.06	0.172	0.138	0.185	0.205	0.319	0.405
FLUC	0.257	0.245	0.206	0.289	0.254	0.215	0.311	0.267	0.276	0.329	0.37	0.389
M_T^{JOINT}	0.034	0.037	0.041	0.023	0.048	0.052	0.053	0.079	0.098	0.073	0.103	0.194

Table 3.7: Empirical Power

Change in Slope - β												
Errors χ^2_1 Distributed												
MIDDLE OF SAMPLE $(\tau^\star = 0.5)$												
	$\beta^{(2)} = 3.75$			$\beta^{(2)} = 4.5$			$\beta^{(2)} = 5.25$			$\beta^{(2)} = 6$		
Statistic	T = 75	T = 100	T=125	T=75	T = 100	T=125	T=75	T = 100	T=125	T = 75	T = 100	T=125
$\sup\limits_{0<\tau<1} \frac{\lvert M_T^{(1)}(\tau)\rvert}{q(\tau,\nu=\frac{15}{128})}$	0.03	0.02	0.022	0.033	0.027	0.024	0.042	0.027	0.042	0.032	0.023	0.025
CUSUM	0.182	0.235	0.306	0.616	0.713	0.818	0.822	0.929	0.973	0.919	0.979	0.992
EXPW	0.527	0.664	0.726	0.956	0.98	0.989	0.998	1	1	1	1	1
MAXW	0.399	0.548	0.612	0.945	0.973	0.988	0.998	1	1	1	1	1
FLUC	0.768	0.832	0.911	0.995	1	1	1	1	1	1	1	1
M_T^{JOINT}	0.726	0.868	0.949	0.986	0.999	1	0.999	1	1	1	1	1
LATE DETECTION $(\tau^\star = 0.9)$												
	T = 75	T = 100	T=125	T=75	T = 100	T=125	T=75	T = 100	T=125	T = 75	T = 100	T=125
$\sup\limits_{0<\tau<1} \frac{\lvert M_T^{(1)}(\tau)\rvert}{q(\tau,\nu=\frac{15}{128})}$	0.038	0.036	0.043	0.113	0.137	0.164	0.335	0.419	0.483	0.58	0.694	0.788
CUSUM	0.031	0.03	0.037	0.029	0.042	0.044	0.04	0.052	0.074	0.04	0.075	0.099
EXPW	0.138	0.178	0.19	0.513	0.659	0.73	0.831	0.927	0.934	0.955	0.98	0.994
MAXW	0.1	0.108	0.124	0.471	0.613	0.674	0.802	0.92	0.94	0.938	0.978	0.991
FLUC	0.31	0.277	0.3	0.507	0.535	0.607	0.665	0.853	0.861	0.795	0.866	0.965
M_T^{JOINT}	0.06	0.072	0.107	0.091	0.123	0.235	0.072	0.152	0.25	0.077	0.11	0.185
EARLY DETECTION $(\tau^\star = 0.1)$												
	T = 75	T = 100	T=125	T=75	T = 100	T=125	T=75	T = 100	T=125	T = 75	T = 100	T=125
$\sup\limits_{0<\tau<1} \frac{\lvert M_T^{(1)}(\tau)\rvert}{q(\tau,\nu=\frac{15}{128})}$	0.046	0.054	0.039	0.159	0.18	0.232	0.375	0.489	0.56	0.655	0.765	0.822
CUSUM	0.205	0.272	0.312	0.58	0.7	0.752	0.791	0.896	0.93	0.903	0.959	0.986
EXPW	0.098	0.166	0.18	0.478	0.624	0.706	0.761	0.887	0.916	0.92	0.963	0.98
MAXW	0.045	0.099	0.093	0.409	0.572	0.662	0.75	0.876	0.91	0.901	0.959	0.97
FLUC	0.298	0.305	0.296	0.589	0.76	0.783	0.849	0.955	0.979	0.977	0.999	0.999
M_T^{JOINT}	0.129	0.24	0.316	0.397	0.701	0.811	0.504	0.762	0.877	0.383	0.624	0.785

Table 3.8: Empirical Power

Change in Slope - β												
Errors $N(0,1)$ **Distributed**												
MIDDLE OF SAMPLE $(\tau^\star = 0.5)$												
	$\beta^{(2)} = 3.75$			$\beta^{(2)} = 4.5$			$\beta^{(2)} = 5.25$			$\beta^{(2)} = 6$		
Statistic	T = 75	T = 100	T=125	T=75	T = 100	T=125	T=75	T = 100	T=125	T = 75	T = 100	T=125
$\sup_{0<\tau<1} \frac{\lvert M_T^{(1)}(\tau)\rvert}{q(\tau,\nu=\frac{15}{128})}$	0.007	0.016	0.02	0.016	0.012	0.011	0.013	0.009	0.022	0.016	0.027	0.019
CUSUM	0.28	0.374	0.512	0.794	0.905	0.962	0.94	0.995	1	0.985	0.997	1
EXPW	0.924	0.984	0.996	1	1	1	1	1	1	1	1	1
MAXW	0.86	0.954	0.984	1	1	1	1	1	1	1	1	1
FLUC	0.949	0.983	0.998	0.999	1	1	1	1	1	1	1	1
M_T^{JOINT}	0.595	0.779	0.9	0.988	0.997	1	1	1	1	0.999	1	1
LATE DETECTION $(\tau^\star = 0.1)$												
	T = 75	T = 100	T=125	T=75	T = 100	T=125	T=75	T = 100	T=125	T = 75	T = 100	T=125
$\sup_{0<\tau<1} \frac{\lvert M_T^{(1)}(\tau)\rvert}{q(\tau,\nu=\frac{15}{128})}$	0.065	0.087	0.105	0.437	0.579	0.799	0.821	0.927	0.987	0.909	0.983	0.998
CUSUM	0.035	0.048	0.05	0.038	0.052	0.06	0.05	0.063	0.108	0.047	0.087	0.176
EXPW	0.324	0.458	0.521	0.829	0.939	0.967	0.947	0.989	0.999	0.994	1	1
MAXW	0.22	0.341	0.38	0.806	0.916	0.959	0.946	0.987	0.999	0.991	1	1
FLUC	0.355	0.358	0.4	0.652	0.701	0.864	0.817	0.944	0.957	0.889	0.941	0.992
M_T^{JOINT}	0.047	0.087	0.126	0.095	0.162	0.259	0.091	0.116	0.251	0.066	0.125	0.227
EARLY DETECTION $(\tau^\star = 0.9)$												
	T = 75	T = 100	T=125	T=75	T = 100	T=125	T=75	T = 100	T=125	T = 75	T = 100	T=125
$\sup_{0<\tau<1} \frac{\lvert M_T^{(1)}(\tau)\rvert}{q(\tau,\nu=\frac{15}{128})}$	0.12	0.149	0.184	0.63	0.762	0.823	0.899	0.967	0.988	0.958	0.996	0.998
CUSUM	0.278	0.403	0.414	0.736	0.846	0.914	0.928	0.971	0.99	0.962	0.996	0.999
EXPW	0.289	0.406	0.487	0.79	0.941	0.962	0.961	0.993	0.997	0.989	1	1
MAXW	0.21	0.315	0.376	0.768	0.927	0.945	0.957	0.991	0.997	0.981	1	1
FLUC	0.387	0.428	0.44	0.826	0.949	0.978	0.99	1	1	0.998	1	1
M_T^{JOINT}	0.037	0.072	0.115	0.068	0.11	0.201	0.054	0.116	0.175	0.041	0.093	0.15

Table 3.9: U-statistic Type Test Statistics

Statistic	Fund 1	Fund 2	Fund 3	Fund 4	Fund 5	Fund 6	Fund 7
Intercept Test	1.17	1.56	1.57	0.91	1.44	1.16	1.4
Slope Test	1.91	1.91	1.91	1.19	2.4	1.58	1.77
M_T^{JOINT} Test	1.11	1.25	1.27	0.91	1.39	1.1	1.22
Change (Date)	04/2009	12/2002	12/2009	10/2002	09/2002	01/2009	01/2003
———	Fund 8	Fund 9	Fund 10	Fund 11	Fund 12	Fund 13	Fund 14
Intercept Test	1.41	1.6	1.69	1.03	1.4	1.52	1.13
Slope Test	1.78	1.81	2.54	2.54	2.2	1.92	2.22
M_T^{JOINT} Test	1.1	1.15	1.43	1.42	1.38	1.22	1.09
Change (Date)	01/2003	12/2002	09/2002	03/2003	10/2002	07/2002	05/2002
———	Fund 15	Fund 16	Fund 17	Fund 18	Fund 19	Fund 20	———
Intercept Test	1.43	1.63	1.80	1.64	1.80	1.58	———
Slope Test	1.41	2.07	2.14	2.06	2.11	1.66	———
M_T^{JOINT} Test	1.3	1.34	1.50	1.34	1.51	1.58	———
Change (Date)	11/2002	03/2003	01/2003	09/2003	03/2003	02/2002	———

Table 3.10: Mutual Funds

Fund 1	Burnham Financial Industry C
Fund 2	Prudential Mid Cap Value M
Fund 3	Prudential Mid Cap Value X
Fund 4	Templeton Developing Market
Fund 5	Franklin Large Cap A
Fund 6	John Hancock Large Cap Equity A
Fund 7	John Hancock Large Cap Equity J
Fund 8	John Hancock Global Leader
Fund 9	Goldman Sachs Mid Cap Value Services
Fund 10	Goldman Sachs Growth Opportunities
Fund 11	Goldman Sachs Asia Equity B
Fund 12	Wells Fargo Omega Growth C
Fund 13	Wells Fargo Global Opportunities C
Fund 14	Wells Fargo Precious Metals A
Fund 15	UBS US Small Cap Growth A
Fund 16	UBS Global Equity C
Fund 17	UBS GLOBAL Equity (BPGEX)
Fund 18	UBS GLOBAL Equity (BNEBX)
Fund 19	UBS Global Equity (BNGEX)
Fund 20	Van Kampen Equity Growth B

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